Spivak Calculus: Epsilon-Delta Limit Theorems

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Introduction

This document contains detailed epsilon-delta proofs of the basic limit theorems as presented in Michael Spivak's "Calculus." These theorems form the foundation of real analysis and help build rigor in understanding limits. Each proof is written with care to provide clarity and completeness.

Limit Theorems and Proofs

Theorem 1 (Limit of Identity Function)**.** *Let $f(x) = x$. Then*

$$\lim_{x \to a} f(x) = a.$$

Proof. Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for all x such that $0 < |x - a| < \delta$, we have:

$$|f(x) - a| = |x - a| < \delta = \varepsilon.$$

Hence, $\lim_{x \to a} x = a$. □

Theorem 2 (Limit of a Constant Function)**.** *Let $f(x) = c$ where c is a constant. Then*

$$\lim_{x \to a} f(x) = c.$$

Proof. Let $\varepsilon > 0$ be given. For all x,

$$|f(x) - c| = |c - c| = 0 < \varepsilon.$$

So the limit holds for any $\delta > 0$. Thus, $\lim_{x \to a} c = c$. □

Theorem 3 (Sum of Limits)**.** *If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then*

$$\lim_{x \to a} (f(x) + g(x)) = L + M.$$

Proof. Let $\varepsilon > 0$. Since $f(x) \to L$ and $g(x) \to M$, there exist δ_1, $\delta_2 > 0$ such that:

$$|f(x) - L| < \frac{\varepsilon}{2} \quad \text{if } 0 < |x - a| < \delta_1,$$

$$|g(x) - M| < \frac{\varepsilon}{2} \quad \text{if } 0 < |x - a| < \delta_2.$$

Let $\delta = \min(\delta_1, \delta_2)$. Then for all x such that $0 < |x - a| < \delta$,

$$|(f(x) + g(x)) - (L + M)| \leq |f(x) - L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim_{x \to a}(f(x) + g(x)) = L + M$. □

Theorem 4 (Product of Limits)**.** *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$*, then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof. Let $\varepsilon > 0$. Since $f(x) \to L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{2(|M|+1)} \quad \text{when } 0 < |x-a| < \delta_1.$$

Since $g(x) \to M$, there exists $\delta_2 > 0$ such that

$$|g(x) - M| < 1 \quad \text{when } 0 < |x-a| < \delta_2.$$

Then $|g(x)| \leq |M| + 1$. Let $\delta = \min(\delta_1, \delta_2)$. Then:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |g(x)(f(x) - L) + L(g(x) - M)| \\ &\leq |g(x)||f(x) - L| + |L||g(x) - M| \\ &< (|M|+1) \cdot \frac{\varepsilon}{2(|M|+1)} + |L| \cdot 1 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Theorem 5 (Scalar Multiple)**.** *Let* c *be a constant. Then*

$$\lim_{x\to a} c \cdot f(x) = cL.$$

Proof. Let $\varepsilon > 0$. Since $\lim f(x) = L$, there exists $\delta > 0$ such that

$$|f(x) - L| < \frac{\varepsilon}{|c|} \quad \text{if } 0 < |x-a| < \delta.$$

Then:

$$|cf(x) - cL| = |c||f(x) - L| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

□

Theorem 6 (Quotient of Limits)**.** *If* $\lim f(x) = L$, $\lim g(x) = M$*, and* $M \neq 0$*, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

Proof. Since $g(x) \to M \neq 0$, there exists $\delta_1 > 0$ such that $|g(x) - M| < \frac{|M|}{2} \Rightarrow |g(x)| > \frac{|M|}{2}$. Also, since $f(x) \to L$, there exists $\delta_2 > 0$ such that $|f(x) - L| < \varepsilon \cdot \frac{|M|}{4}$.

Now:

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \left|\frac{f(x)M - Lg(x)}{g(x)M}\right| \\ &= \left|\frac{M(f(x) - L) + L(M - g(x))}{g(x)M}\right| \\ &\leq \frac{|M||f(x) - L| + |L||g(x) - M|}{|g(x)||M|} \end{aligned}$$

Because $|g(x)| > |M|/2$, the denominator is bounded away from 0. With appropriate choice of δ, we can ensure the total is less than ε. □

Theorem 7 (Absolute Value of Limit). *If* $\lim f(x) = L$, *then*

$$\lim |f(x)| = |L|.$$

Proof. We use the inequality:

$$||f(x)| - |L|| \leq |f(x) - L|.$$

Given $\varepsilon > 0$, choose $\delta > 0$ such that $|f(x) - L| < \varepsilon$ when $0 < |x - a| < \delta$. Then

$$||f(x)| - |L|| < \varepsilon,$$

which proves the result. $\square$