Limits exercise

Problem 1: Evaluate the following limits

(i) $\lim_{x \to 1} \frac{x^2 - 1}{x + 1}$

Algebraic Approach:

$$\frac{x^2 - 1}{x + 1} = \frac{(x - 1)(x + 1)}{x + 1} = x - 1$$
$$\lim_{x \to 1}(x - 1) = 0$$

L'Hôpital's Rule:

$$\lim_{x \to 1} \frac{x^2 - 1}{x + 1} = \frac{0}{2} = 0 \quad \text{(Direct substitution, no 0/0)}$$

Answer: $\boxed{0}$

(ii) $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

Algebraic Approach:

$$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$$
$$\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4$$
$$\lim_{x \to 2} x^2 + 2x + 4 = 4 + 4 + 4 = 12$$

L'Hôpital's Rule:

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} = \lim_{x \to 2} \frac{3x^2}{1} = 3 \cdot 4 = 12$$

Answer: $\boxed{12}$

(iii) $\lim_{x \to 3} \frac{x^3 - 8}{x - 2}$

Direct Substitution:

$$\frac{27 - 8}{3 - 2} = \frac{19}{1} = 19$$

Answer: $\boxed{19}$

(iv) $\lim_{x \to y} \dfrac{x^n - y^n}{x - y}$

Algebraic Identity:

$$\frac{x^n - y^n}{x - y} = \sum_{k=0}^{n-1} x^{n-1-k} y^k \Rightarrow \lim_{x \to y} \sum_{k=0}^{n-1} x^{n-1-k} y^k = n y^{n-1}$$

L'Hôpital's Rule:

$$\lim_{x \to y} \frac{d}{dx}\left(x^n\right) = \frac{n x^{n-1}}{1} \Rightarrow \text{at } x = y : \quad n y^{n-1}$$

Answer: $\boxed{n y^{n-1}}$

(v) $\lim_{y \to x} \dfrac{x^n - y^n}{x - y}$

This is symmetric to (iv), with x and y switched.

Using symmetry:

$$\lim_{y \to x} \frac{x^n - y^n}{x - y} = \lim_{y \to x} \frac{y^n - x^n}{y - x} = n x^{n-1}$$

L'Hôpital's Rule:

$$\frac{d}{dy}(-y^n) = -n y^{n-1} \Rightarrow \text{At } y = x : \quad n x^{n-1}$$

Answer: $\boxed{n x^{n-1}}$

(vi) $\lim_{h \to 0} \dfrac{\sqrt{a + h} - \sqrt{a}}{h}$

Multiply numerator and denominator by the conjugate:

$$\frac{\sqrt{a + h} - \sqrt{a}}{h} \cdot \frac{\sqrt{a + h} + \sqrt{a}}{\sqrt{a + h} + \sqrt{a}} = \frac{(a + h) - a}{h(\sqrt{a + h} + \sqrt{a})} = \frac{1}{\sqrt{a + h} + \sqrt{a}}$$

$$\lim_{h \to 0} \frac{1}{\sqrt{a + h} + \sqrt{a}} = \frac{1}{2\sqrt{a}}$$

L'Hôpital's Rule:

$$\frac{d}{dh}\left(\sqrt{a + h}\right) = \frac{1}{2\sqrt{a + h}} \Rightarrow \text{At } h = 0 : \frac{1}{2\sqrt{a}}$$

Answer: $\boxed{\dfrac{1}{2\sqrt{a}}}$