Problem 2: ε-δ Proofs

In each of the following cases, find a δ such that $|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.

(i) $f(x) = x^4$, $a = a$, $l = a^4$

We want:

$$|x^4 - a^4| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta$$

Factor:

$$|x^4 - a^4| = |x - a||x^3 + x^2 a + x a^2 + a^3|$$

Assume $|x - a| < 1$ so that $|x| < |a| + 1$.
Then:

$$|x^3 + x^2 a + x a^2 + a^3| \le (|a| + 1)^3 + (|a| + 1)^2 |a| + (|a| + 1)|a|^2 + |a|^3 = M$$

Thus,

$$|f(x) - l| < M|x - a| < \varepsilon \Rightarrow \delta = \min(1, \frac{\varepsilon}{M})$$

(ii) $f(x) = \frac{1}{x}$, $a = 1$, $l = 1$

$$\left|\frac{1}{x} - 1\right| = \left|\frac{1 - x}{x}\right| = \frac{|x - 1|}{|x|}$$

Let $|x - 1| < \frac{1}{2} \Rightarrow x \in (0.5, 1.5) \Rightarrow |x| > 0.5$
Then:

$$\left|\frac{1}{x} - 1\right| < \frac{|x - 1|}{0.5} = 2|x - 1| < \varepsilon \Rightarrow \delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{2}\right)$$

(iii) $f(x) = x^4 + \frac{1}{x}$, $a = 1$, $l = 2$

We want:

$$\left|x^4 + \frac{1}{x} - 2\right| = |x^4 - 1 + \frac{1}{x} - 1| \le |x^4 - 1| + \left|\frac{1}{x} - 1\right|$$

From (i): $|x^4 - 1| < \frac{\varepsilon}{2}$ for small enough δ.
From (ii): $\left|\frac{1}{x} - 1\right| < \frac{\varepsilon}{2}$ for small enough δ.
Thus:

$$|f(x) - 2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Choose:

$$\delta = \min(\delta_1, \delta_2) \text{ where each corresponds to (i) and (ii).}$$

(iv) $f(x) = \frac{x}{1 + \sin^2 x}$, $a = 0$, $l = 0$

We want:

$$\left|\frac{x}{1 + \sin^2 x}\right| < \varepsilon$$

Note: $1 \le 1 + \sin^2 x \le 2$ so

$$\left|\frac{x}{1 + \sin^2 x}\right| \le |x|$$

Thus, if $|x| < \varepsilon$, then the inequality holds.
So:

$$\delta = \varepsilon$$

(v) $f(x) = \sqrt{|x|}$, $a = 0$, $l = 0$

$$|\sqrt{|x|} - 0| = \sqrt{|x|} < \varepsilon \Rightarrow |x| < \varepsilon^2$$

Thus:

$$\delta = \varepsilon^2$$

(vi) $f(x) = \sqrt{x}$, $a = 1$, $l = 1$

We want:

$$|\sqrt{x} - 1| < \varepsilon$$

Use identity:

$$|\sqrt{x} - 1| = \frac{|x-1|}{\sqrt{x}+1}$$

If $|x - 1| < 1 \Rightarrow x \in (0, 2) \Rightarrow \sqrt{x} + 1 \leq \sqrt{2} + 1 = M$
Then:

$$|\sqrt{x} - 1| < \frac{|x-1|}{M} < \varepsilon \Rightarrow \delta = \min(1, M\varepsilon)$$