Control Theory Homework 3

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1 Introduction

The goal of the assignment was to linearize non-linear control system at point (0, 0, 0, 0), determine its controlability and make the system stable using state feedback controller (pole placement method) and LQR method.

2 Tasks solution

Task 1

Write equations of motion of the system in manipulator form

$$M(q)\ddot{q} + n(q, \dot{q}) = Bu$$

- *where* $u = F$, $q = \begin{bmatrix} x & \theta \end{bmatrix}^T$ *is vector of generalized coordinates;*

Answer:

$$\begin{bmatrix} M+m & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix} \ddot{q} + \begin{bmatrix} mlsin(\theta)\dot{\theta}^2 \\ -gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Task 2

Write dynamics of the system in control affine nonlinear form $\dot{z} = f(z) + g(z)u$

- *where* $z = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$ *is vector of states of the system;*

Solution:

Lets find $\ddot{x}$

$$\ddot{Q} = \frac{1}{l}(gsin(\theta) + cos(\theta)\ddot{x})$$

$$\Rightarrow (M + m(1 - cos^2(\theta)))\ddot{x} - mgsin(\theta)cos(\theta) + mlsin(\theta)\dot{\theta}^2 = F$$

$$\ddot{x} = \frac{msin(\theta)(gcos(\theta) - l\dot{\theta}^2) + F}{(M + msin^2(\theta))}$$

Lets find $\ddot{\theta}$

$$\ddot{x} = \frac{gsin(\theta) - l\ddot{\theta}}{-cos(\theta)}$$

$$\Rightarrow -g(M+m)tan(\theta) + \frac{l(M+m)}{cos(\theta)}\ddot{\theta} - mlcos(\theta)\ddot{\theta} + mlsin(\theta)\dot{\theta}^2 = F$$

$$\ddot{\theta}(l\frac{M+msin^2(\theta)}{cos(\theta)}) = g(M+m)tan(\theta) - mlsin(\theta)\dot{\theta}^2 + F$$

$$\ddot{\theta} = \frac{g(M+m)sin(\theta) - mlsin(\theta)cos(\theta)\dot{\theta}^2 + cos(\theta)F}{l(M+msin^2(\theta))}$$

Lets find dynamics of the system in control Affine nonlinear form

$$\dot{z} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{msin(\theta)(gcos(\theta)-l\dot{\theta}^2)}{(M+msin^2(\theta))}\\ \frac{g(M+m)sin(\theta)-mlsin(\theta)cos(\theta)\dot{\theta}^2}{l(M+msin^2(\theta))}\end{bmatrix} + u\begin{bmatrix}0\\ 0\\ \frac{1}{(M+msin^2(\theta))}\\ \frac{1}{l(M+msin^2(\theta))}\end{bmatrix}$$

Task 3

Linearize nonlinear dynamics of the systems around equilibrium point $\bar{z} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$

$$\delta\dot{z} = A\delta z + B\delta u$$

Solution:

$$A = \frac{\delta f}{\delta z} = \begin{bmatrix}\frac{df}{dx}\\ \frac{df}{d\theta}\\ \frac{df}{d\dot{x}}\\ \frac{df}{d\dot{\theta}}\end{bmatrix}$$

$$f = \begin{bmatrix}\dot{x}\\ \dot{\theta}\\ \frac{msin(\theta)(gcos(\theta)-l\dot{\theta}^2)}{(M+msin^2(\theta))}\\ \frac{g(M+m)sin(\theta)-mlsin(\theta)cos(\theta)\dot{\theta}^2}{l(M+msin^2(\theta))}\end{bmatrix}$$

$$\frac{df}{dx} = \begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$$

$$\frac{df}{d\theta} = \begin{bmatrix}0 & 0 & \frac{gm}{M} & \frac{g(M+m)}{lM}\end{bmatrix}^T$$

$$\frac{df}{d\dot{x}} = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^T$$

$$\frac{df}{d\dot{\theta}} = \begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}^T$$

$$A = \frac{\delta f}{\delta z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{gm}{M} & \frac{g(M+m)}{lM} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{lM} \end{bmatrix}$$

Task 4

Check stability of the linearized system using any method you like;

Solution:

Let's find the eigenvalues of A matrix:

$$det(\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & \frac{gm}{M} & -\lambda & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & -\lambda \end{bmatrix}) =$$

$$= -\lambda(-\lambda^3 + \lambda\frac{g(M+m)}{lm}) = \lambda^4 - \lambda^2(\frac{gM+m}{lM}) = 0$$

$$\lambda = 0 \lambda = \sqrt{\frac{g(M+m)}{lM}} \lambda = -\sqrt{\frac{g(M+m)}{lM}}$$

Thus there exist lambda with abs greater the zero, system without control is not stable

Task 5

Check if linearized system is controllable;

Solution: Controlability - is the possibility of forcing the system into a particular state by using an appropriate control signal.

System is controllable, when rank of matrix C is equal to 4, where $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & 0 \\ 0 & 0 & 0 & \frac{gm}{M} \\ 0 & 0 & 0 & \frac{g(M+m)}{lM} \end{bmatrix}, A^2B = \begin{bmatrix} 0 \\ 0 \\ \frac{gm}{M^2} \\ \frac{g(M+m)}{(lM)^2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & \frac{gm}{M} \\ 0 & 0 & 0 & \frac{g(M+m)}{lM} \\ 0 & \frac{g^2m(M+m)}{lM^2} & 0 & 0 \\ 0 & (\frac{g(M+m)}{lM})^2 & 0 & 0 \end{bmatrix}, A^3B = \begin{bmatrix} \frac{gm}{lM^2} \\ \frac{g(M+m)}{(lM)^2} \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \frac{1}{M} & 0 & \frac{gm}{lM^2} \\ 0 & \frac{1}{lM} & 0 & \frac{g(M+m)}{l^2M^2} \\ \frac{1}{M} & 0 & \frac{gm}{M^2} & 0 \\ \frac{1}{lM} & 0 & \frac{g(M+m)}{l^2M^2} & 0 \end{bmatrix}$$

If rank of C is less, then 4, then $\lambda = 0$ - one of its eigenvalues, thus, the determinant of matrix C is zero itself. Lets find this determinant

$$det(C) = -\frac{1}{M}(\frac{1}{M}(\frac{g(M+m)}{l^2M^2})^2 - \frac{g^2m(M+m)}{(lM)^3M^2})-$$

$$-\frac{gm}{lM^2}(\frac{gm}{(lM)^2M^2} - \frac{g(M+m)}{M(lM)^3}) =$$

$$= \frac{g^2}{M^4l^2}(-(M+m)^2 + m(M+m) - m^2 + m(M+m)) =$$

$$= \frac{g^2}{M^4l^2}((M+m)(-M+m) - m^2) = \frac{g^2}{M^4l^2}(-M^2) = \frac{-g^2}{M^2l^2} \neq 0$$

So, system is controllable

Task 6

If the system is controllable, design state feedback controller for linearized system using pole placement method. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions;

Solution:

Pole replacement method is a controller of form

$$u = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

, where k_i is some fixed number. k_is could be found using some fixes eigenvalues and initial values (M, m and l).

First set of initial conditions

$M = 0.01$ $m = 0.001$ $l = 0.1$

Desired eigenvalues are: -0.25, -0.5, -1 and -2

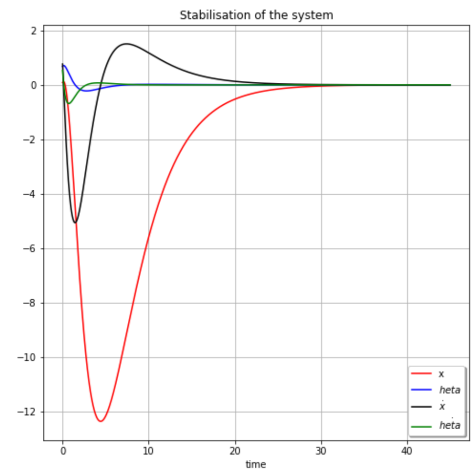

Figure 1: The first set of eigenvalues and initial conditions

Lets change the eigenvalues Desired eigenvalues are: -1.1, -1.2, -1.3, -1.4

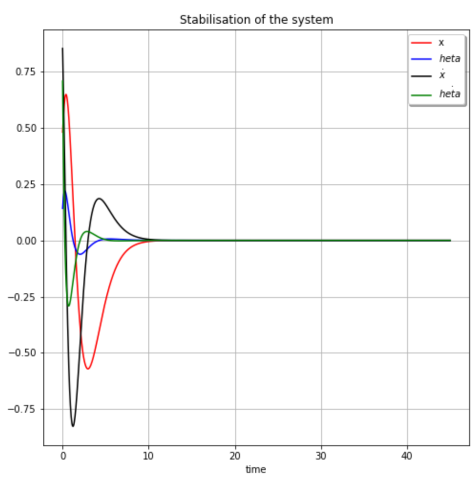

Figure 2: The first set of initial conditions and second set of eigenvalues

In can be seen, that increasing of absolute values of the eigenvalues results in increasing of the convergence velocity.

Let's consider complex eigenvalues Desired eigenvalues are: -0.1 + 1j, -0.1 -

1j, -6 + 1j, -6 - 1j

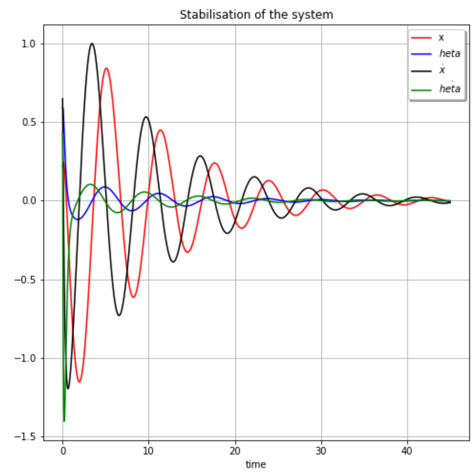

Figure 3: The first set of initial conditions and third set of eigenvalues

Now the system converges slower and oscillates.

Thus the best eigenvalues to stabilize the system are negative and with relatively-big absolute value.

Second set of initial conditions

$M = 0.02$ $m = 0.01$ $l = 0.5$

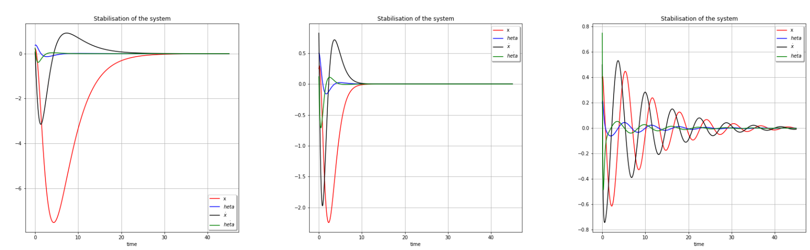

Figure 4: The second set of initial conditions and all previous sets of eigenvalues

Third set of initial conditions

$M = 0.2$ $m = 1$ $l = 0.1$

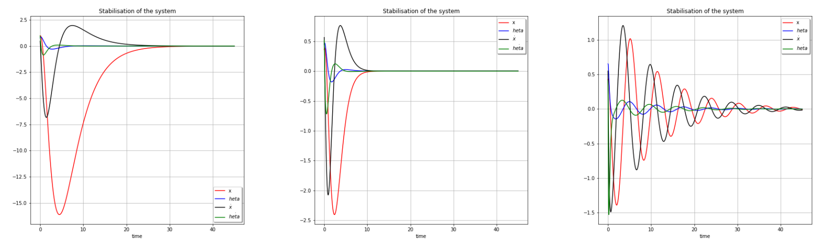

Figure 5: The third set of initial conditions and all previous sets of eigenvalues

The same conclusions could be achieved in case, when m is greater then M. (Only amplitude changes)

Task 7

If the system is controllable, design linear quadratic regulator for linearized system. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions;

Solution:

For the LQR method Q was chosen as Identity matrix 4x4 and R was set to 4.

The method was tested on the same initial conditions, as at the 6-th task.

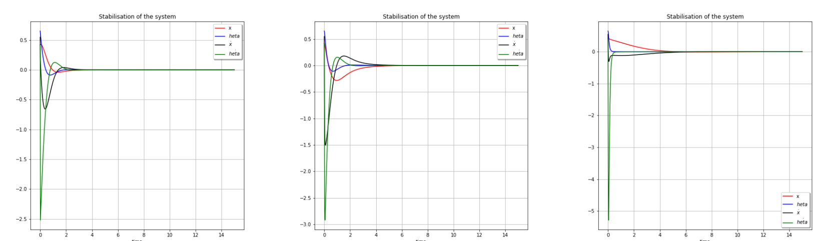

Figure 6: The all sets of initial conditions for LQR solver

It can be seen that system converges much faster (the limits on the plots are now set to 15 instead of 45 as it was at the 6-th task). Also the highest amplitude of convergence is now reached by the velocity of θ instead of its velocity. In addition, the general view of the convergence graph is not now identical or proportional as it was for state feedback controller.

Note on initial condition validity

The initial conditions were chosen in a physically-meaningful way: l, m and M are positive, relatively small (their module is less then g) and they do not differ from each other much (less the 100 times).

3 Conclusion

It can be seen that LQR controller works faster, however state feedback controller also makes sense for the task.