P8108 Homework 2

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Problem 1

For discrete survival time, show $f(t_j) = h_j \prod_{i \le j-1} (1 - h_i)$

Proof

For discrete survival time, we have

$$\begin{aligned}
h_j &= h(t_j) \\
&= \frac{P(T = t_j \ \cap \ T \ge t_j)}{P(T \ge t_j)} \\
&= \frac{P(T = t_j)}{P(T \ge t_j)} \\
&= \frac{P(T = t_j)}{P(T > t_{j-1})} \\
&= \frac{f(t_j)}{S(t_{j-1})} \\
&= \frac{f(t_j)}{P(T > t_{j-1} \mid T \ge t_{j-1}) \times P(T > t_{j-2} \mid T \ge t_{j-2}) \times \ldots \times P(T > t_1 \mid T \ge t_1)} \\
&= \frac{f(t_j)}{(1 - h_{j-1}) \times (1 - h_{j-2}) \times \ldots \times (1 - h_1)} \\
&= \frac{f(t_j)}{\prod_{i \le j-1} (1 - h_i)}
\end{aligned}$$

By rearranging the equation, we can get

$$f(t_j) = h_j \prod_{i \le j-1} (1 - h_i)$$

□

Problem 2

Show that if $S_1(t) = \{S_0(t)\}^\lambda$, then $h_1(t) = \lambda h_0(t)$

Proof

Since in continuous survival time case, we have

$$h(t) = -\frac{d \log S(t)}{dt}$$

Hence,

$$\begin{aligned}
h_0(t) &= -\frac{d \log S_0(t)}{dt} \\
h_1(t) &= -\frac{d \log S_1(t)}{dt} = -\frac{d \log \{S_0(t)\}^\lambda}{dt} = -\lambda \frac{d \log S_0(t)}{dt} = \lambda h_0(t)
\end{aligned}$$

□

Problem 3

Show that the survival function $S(t) = \exp(-H(t))$, where $H(t)$ is a cumulative hazard function

By the definition,

$$\begin{aligned}
H(t) &= \int_0^t h(x) dx \\
&= \int_0^t -\frac{d \log S(x)}{dx} dx \\
&= -\log S(t)
\end{aligned}$$

Hence,

$$S(t) = \exp(-H(t))$$

□

Problem 4

In a two-arm randomized and controlled clinical trial, the median survival time in the control and new treatment arms are 9 months and 14 months, respectively. Assuming the survival time follows exponential distribution, what is the hazard rate for the control and new treatment arms? What is the risk reduction in the new treatment in comparison to the control arm?

For exponential distributions,

$$\begin{aligned}
f(t) &= \lambda e^{-\lambda t} \\
S(t) &= P(T > t) \\
&= 1 - F(t) \\
&= e^{-\lambda t} \\
\lambda(t) &= \frac{f(t)}{S(t)} \\
&= \lambda
\end{aligned}$$

Since the median survival time in the control(c) and new treatment(t) arms are 9 months and 14 months, respectively. That is,

$$S(t_{cm}) = 0.5 = e^{-\lambda_c t_{cm}} = e^{-9\lambda_c}$$
$$S(t_{tm}) = 0.5 = e^{-\lambda_t t_{tm}} = e^{-14\lambda_t}$$

Solving the equations above, we can get $\lambda_c = 0.077, \lambda_t = 0.050$, and the hazard ratio between control and new treatment arms is $\frac{\lambda_c}{\lambda_t} = 1.556$ or $\frac{\lambda_t}{\lambda_c} = 0.643$. So the risk reduction in the new treatment in comparison to the control arm is 35.7%

Problem 5

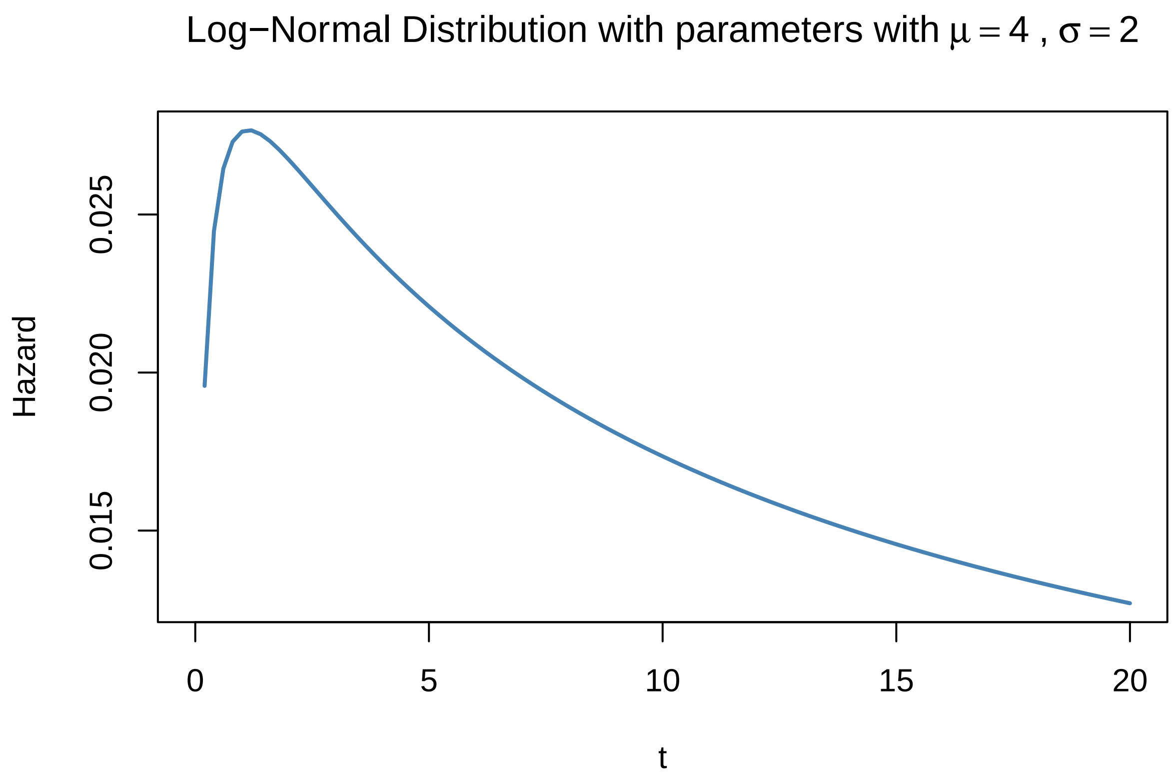
Plot $h(t)$ for log-normal, Gamma distributions with your choice of parameters

1) For log-normal distribution $LN(\mu, \sigma^2)$, in this case, I choose $\mu = 4, \sigma = 2$

For a log-normal distribution,

$$S(t) = 1 - \Phi(\frac{\ln t - \mu}{\sigma})$$
$$f(t) = \frac{\exp(-(\frac{\ln t - \mu}{\sqrt{2}\sigma})^2)}{\sqrt{2\pi}t\sigma}$$
$$h(t) = \frac{f(t)}{S(t)}$$

Here, I create a function for the hazard function of log-normal distribution, and plot the hazard function with my parameters.

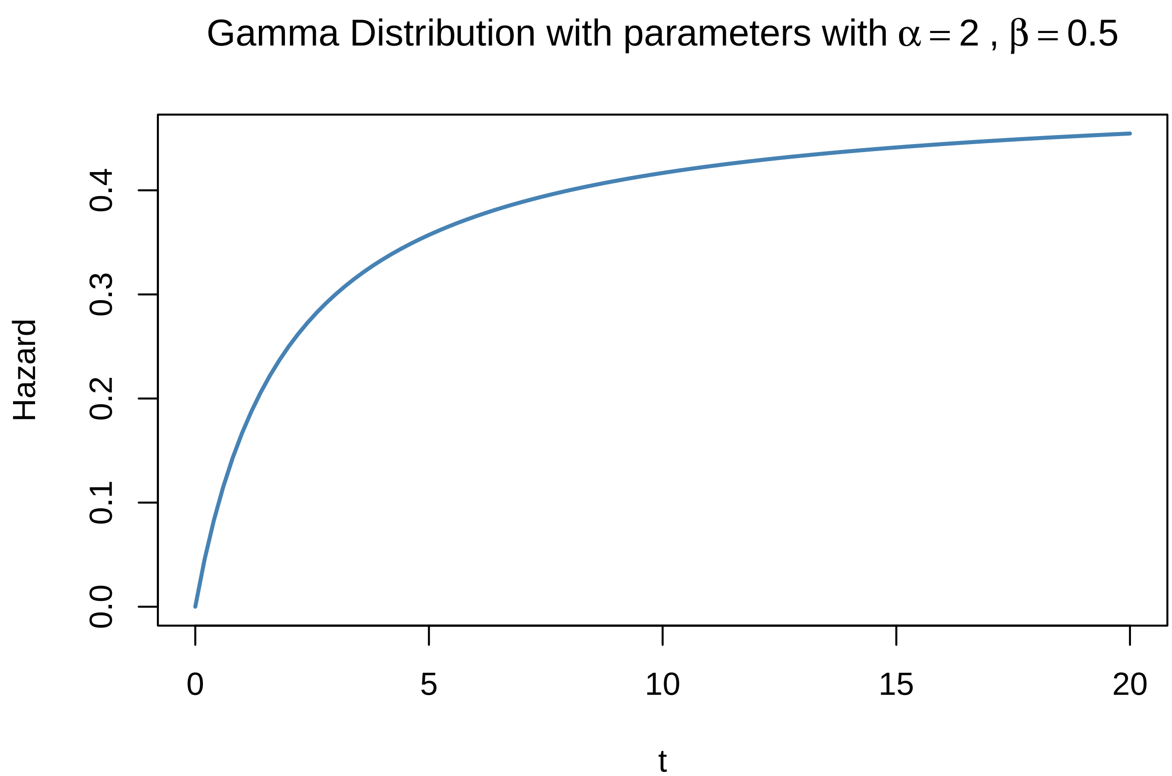


2) For Gamma distribution $Gamma(\alpha, \beta)$, in this case, I choose $\alpha = 2, \beta = 0.5$

For Gamma distribution, there is no close form of the survival function, the density function of a Gamma distribution is,

$$f(t) = \frac{\beta^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-}\beta$$

Here, I create a function for the hazard function of Gamma distribution, and plot the hazard function with my parameters.



Appendix: Code for this report

```
knitr::opts_chunk$set(echo = FALSE)
library(stats)
library(tidyverse)
hlognorm <- function(t, mu = 0, sd = 1){
  S = 1 - pnorm((log(t)-mu)/sd)
  f = exp(-(((log(t)-mu)^2)/(2*sd^2)))/ (sqrt(2*pi)*t*sd)
  h = f/S
  return(h)
}
curve(hlognorm(x,4,2), from=0, to=20,
    main = bquote("Log-Normal Distribution with parameters with" ~ mu == 4 ~ "," ~ sigma == 2),
    ylab = 'Hazard',
    xlab = 't',
```

```r
    lwd = 2,
    col = 'steelblue')
hgamma <- function(t, alpha = 0, beta = 0.5){
  f = dgamma(t, alpha, beta)
  S = 1 - pgamma(t, alpha, beta)
  h = f/S
  return(h)
}
curve(hgamma(x,2,0.5), from=0, to=20,
    main = bquote("Gamma Distribution with parameters with" ~ alpha == 2 ~ "," ~ beta == 0.5),
    ylab = 'Hazard',
    xlab = 't',
    lwd = 2,
    col = 'steelblue')
```