derive posterior distribution

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The suggested Bayesian model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\mu}$ and covariance matrix Σ.

We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\mu}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

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Let $\mathbf{B} = (\boldsymbol{\beta}_1^\top, ..., \boldsymbol{\beta}_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model, let

$$\epsilon_i(t) = Y_i(t+6) - \Big(\beta_{0,i} + \beta_{1,i} Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\Big) \overset{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \sim N(\boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top, \sigma^2)$$

where $\boldsymbol{X}_i(t) = (1, Y_i(t), \Delta_{i,1}(t), \Delta_{i,2}(t), \Delta_{i,3}(t))$, and $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}, \beta_{4,i})$. Therefore, the wind speed of i^{th} hurricane at time t follows the normal distribution with the pdf below

$$f_{Y_i(t+6)}(y_i(t+6) \mid \boldsymbol{X}_i(t), \boldsymbol{\beta}_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{ -\frac{1}{2\sigma^2}\Big(y_i(t+6) - \boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top\Big)^2\Big\}$$

Therefore, the conditional distribution of Y_i, the wind speed of i^{th} hurricane follows the multivariate normal distribution below, (since $Y_i(t)$'s are independent across t)

$$(\boldsymbol{Y}_i \mid \boldsymbol{X}_i, \boldsymbol{\beta}_i, \sigma^2) \sim \mathcal{N}(\boldsymbol{X}_i\boldsymbol{\beta}_i^\top, \sigma^2 I)$$

where Y_i is an m_i-dimensional vector and $\boldsymbol{X}_i$ is a $m_i \times d$ matrix.

Hence, the joint likelihood function of all $i's$ hurricanes can be expresses as

$$L_Y(\mathbf{B}, \sigma^2) = \prod_{i=1}^n \Big\{ \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\Big)\Big\}$$

where I is an identical matrix with dimension consistent with Y_i.

From Bayesian theorem, the posterior distribution for Θ is

$$\pi(\boldsymbol{\Theta}|\boldsymbol{Y}) = \pi(\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma \mid \boldsymbol{Y}) \propto L_Y(\mathbf{B}, \sigma^2) \times \pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \times \pi(\boldsymbol{\mu}) \times \pi(\sigma^2) \times \pi(\boldsymbol{\Sigma}),$$

where $\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma)$ is the joint multivariate normal density of β, since

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Therefore

$$\pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \prod_{i=1}^n \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top)\Big\}.$$

So we have the following posterior distribution:

$$\begin{aligned}\pi(\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma \mid Y) \propto & \prod_{i=1}^n \Big\{(2\pi\sigma^2)^{-m_i/2} \exp\big\{ -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\big\}\Big\} \\ & \times \prod_{i=1}^n \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\big\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\big\}\Big\} \times \frac{1}{\sigma^2} \times \det(\boldsymbol{\Sigma})^{-(d+1)} \exp\big\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\big\}\end{aligned}$$

```r
# betavec: 1*5
# sigma2: 1*1
# mu: 1*5
# Sigma: 5*5

# d = 5
# n = 702


loglik_sigma2 <- function(sigma2){
    if (sigma2 <= 0) {
        return(-Inf)
    } else {
        return(log(1/sigma2^2))
    }

}

loglik_Sigma <- function(Sigma, d){
    if (min(eigen(Sigma)$values) <= 0) {
        return(-Inf)
    } else {
        loglik = log(det(Sigma)^(-(d+1))*exp(-(1/2)*solve(Sigma)))
        return(loglik)
    }
}

# log likelihood for y given a specific hurricane's data given sigma2 and betavec
loglik_Y <- function(y, X, betavec, sigma2){
    m = length(y)
    loglik = (2*pi*sigma2^2)^(-(m/2))*exp(-(1/2)*t(X %*% t(betavec)) %*% solve(diag(sigma2, m)) %*% (X %
    loglik = log(loglik)
    return(loglik)

}

# log likelihood for betavec given a specific hurricane's mu and Sigma

loglik_betavec <- function(betavec, mu, Sigma){
    if (min(eigen(Sigma)$values) <= 0) {
        return(-Inf)
    } else {
        loglik = log(det(2*pi*Sigma)^(-(1/2))*exp(-(1/2)*(betavec - mu) %*% solve(Sigma) %*% t(betavec -
        return(loglik)
    }
}
```

```r
library(tidyverse)
```

```
## Warning: package 'tidyverse' was built under R version 4.0.5

## -- Attaching packages --------------------------------------- tidyverse 1.3.1 --

## v ggplot2 3.3.5     v purrr   0.3.4
## v tibble  3.1.6     v dplyr   1.0.8
```

```
## v tidyr   1.2.0     v stringr 1.4.0
## v readr   1.4.0     v forcats 0.5.1
## Warning: package 'ggplot2' was built under R version 4.0.5
## Warning: package 'tibble' was built under R version 4.0.5
## Warning: package 'tidyr' was built under R version 4.0.5
## Warning: package 'dplyr' was built under R version 4.0.5
## Warning: package 'forcats' was built under R version 4.0.5
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```r
load("./dt_long.RData")
dt_windspe <- dt_long %>% select(c("ID", "Wind.kt", "Wind_prev", "Lat_change", "Long_change", "Wind_cha

IDs <- unique(dt_windspe$ID)

length(IDs)
```

```
## [1] 700
```

```r
nrow(dt_windspe[which(dt_windspe$ID == IDs[1]),])
```

```
## [1] 50
```

```r
dt_mtx <- NULL

for(i in 1:length(IDs)){
    dt_i = dt_windspe[which(dt_windspe$ID == IDs[i]),]
    y = dt_i$Wind.kt
    X = dt_i[3:6]
    X = data.matrix(X)
    dt = list(y = y, X = X)
}
```