derive posterior distribution

Renjie Wei

2022-05-02

The suggested Bayesian model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i} Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\beta}$ and covariance matrix Σ.

We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

1

Let $\mathbf{B} = (\boldsymbol{\beta}_1^\top, ..., \boldsymbol{\beta}_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \boldsymbol{\beta}^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model, let

$$\epsilon_i(t) = Y_i(t+6) - \Big(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\Big) \overset{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \sim N(\boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top, \sigma^2)$$

where $\boldsymbol{X}_i(t) = (1, Y_i(t), \Delta_{i,1}(t), \Delta_{i,2}(t), \Delta_{i,3}(t))$, and $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}, \beta_{4,i})$. Therefore, the wind speed of i^{th} hurricane at time t follows the normal distribution with the pdf below

$$f_{Y_i(t+6)}(y_i(t+6) \mid \boldsymbol{X}_i(t), \boldsymbol{\beta}_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\Big\{ -\frac{1}{2\sigma^2}\Big(y_i(t+6) - \boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top\Big)^2 \Big\}$$

Therefore, the conditional distribution of Y_i, the wind speed of i^{th} hurricane follows the multivariate normal distribution below

$$(\boldsymbol{Y}_i \mid \boldsymbol{X}_i, \boldsymbol{\beta}_i, \sigma^2) \sim \mathcal{N}(\boldsymbol{X}_i\boldsymbol{\beta}_i^\top, \sigma^2 I)$$

where Y_i is an m_i-dimensional vector and $\boldsymbol{X}_i$ is a $m_i \times d$ matrix.

Hence, the joint likelihood function of all $i's$ hurricanes can be expresses as

$$L_Y(\mathbf{B}, \sigma^2) = \prod_{i=1}^n \Big\{ \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\Big)\Big\}$$

where I is an identical matrix with dimension consistent with Y_i.

From Bayesian theorem, the posterior distribution for Θ is

$$\pi(\boldsymbol{\Theta}|\boldsymbol{Y}) = \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \boldsymbol{\Sigma} \mid \boldsymbol{Y}) \propto L_Y(\mathbf{B}, \sigma^2) \times \pi(\mathbf{B} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}) \times \pi(\boldsymbol{\beta}) \times \pi(\sigma^2) \times \pi(\boldsymbol{\Sigma}),$$

where $\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma)$ is the joint multivariate normal density of β, since

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

Therefore

$$\pi(\mathbf{B} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}) = \prod_{i=1}^n \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top)\Big\}.$$

So we have the following posterior distribution:

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \boldsymbol{\Sigma} \mid Y) \propto \prod_{i=1}^n \Big\{(2\pi\sigma^2)^{-m_i/2} \exp\big\{ -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\big\}\Big\}$$

$$\times \prod_{i=1}^n \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\big\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top\big\}\Big\} \times \frac{1}{\sigma^2} \times |\boldsymbol{\Sigma}|^{-(d+1)} \exp\big\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\big\}.$$

By rearranging

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \boldsymbol{\Sigma} \mid Y) \propto \frac{1}{\sigma^2} \times \prod_{i=1}^n (2\pi\sigma^2)^{-m_i/2} \times |\boldsymbol{\Sigma}|^{-(d+1)} \exp\big\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\big\} \times \det(2\pi\boldsymbol{\Sigma})^{-n/2}$$

$$\times \prod_{i=1}^{n} \Big\{ \exp\big\{ -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\big\}\Big\} \times \prod_{i=1}^{n} \Big\{ \exp\big\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \big\}\Big\}.$$

For a given Y_i,

$$\log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \sum_{i=1}^{n} -\frac{1}{2}\log 2\pi - \log\sigma - \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mu_i\Big)^2 +$$

$$\sum_{i=1}^{n} -\frac{1}{2}\log(\det(2\pi\Sigma)) - \frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top\Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta}) - 2\log\sigma - (d+1)\log|\Sigma| - \frac{1}{2}\Sigma^{-1}.$$