derive posterior distribution

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The suggested Bayesian model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{5,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\mu}$ and covariance matrix Σ.

We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\mu}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Posterior Distributions

Let $\mathbf{B} = (\boldsymbol{\beta}_1^\top, ..., \boldsymbol{\beta}_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model, let

$$\epsilon_i(t) = Y_i(t+6) - \Big(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\Big) \overset{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \sim N(\boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top, \sigma^2)$$

where $\boldsymbol{X}_i(t) = (1, Y_i(t), \Delta_{i,1}(t), \Delta_{i,2}(t), \Delta_{i,3}(t))$, and $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}, \beta_{4,i})$. Therefore, the wind speed of i^{th} hurricane at time t follows the normal distribution with the pdf below

$$f_{Y_i(t+6)}(y_i(t+6) \mid \boldsymbol{X}_i(t), \boldsymbol{\beta}_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{ -\frac{1}{2\sigma^2}\Big(y_i(t+6) - \boldsymbol{X}_i(t)\boldsymbol{\beta}_i^\top\Big)^2 \Big\}$$

Therefore, the conditional distribution of Y_i, the wind speed of i^{th} hurricane follows the multivariate normal distribution below, (since $Y_i(t)$'s are independent across t)

$$(\boldsymbol{Y}_i \mid \boldsymbol{X}_i, \boldsymbol{\beta}_i, \sigma^2) \sim \mathcal{N}(\boldsymbol{X}_i\boldsymbol{\beta}_i^\top, \sigma^2 I)$$

where Y_i is an m_i-dimensional vector and $\boldsymbol{X}_i$ is a $m_i \times d$ matrix.

Hence, the joint likelihood function of all $i's$ hurricanes can be expresses as

$$L_Y(\mathbf{B}^\top, \sigma^2) = \prod_{i=1}^{n} \Big\{ \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\Big)\Big\}$$

where I is an identical matrix with dimension consistent with Y_i.

From Bayesian theorem, the posterior distribution for Θ is

$$\pi(\boldsymbol{\Theta}|\boldsymbol{Y}) = \pi(\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma \mid \boldsymbol{Y}) \propto L_Y(\mathbf{B}^\top, \sigma^2) \times \pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \times \pi(\boldsymbol{\mu}) \times \pi(\sigma^2) \times \pi(\boldsymbol{\Sigma}),$$

where $\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma)$ is the joint multivariate normal density of β, since

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Therefore

$$\pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \prod_{i=1}^{n} \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top)\Big\}.$$

So we have the following posterior distribution:

$$\begin{aligned}\pi(\mathbf{B}^\top, \boldsymbol{\mu}^\top, \sigma^2, \Sigma \mid Y) \propto & \prod_{i=1}^{n} \Big\{ (2\pi\sigma^2)^{-m_i/2} \exp\big\{ -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\big\}\Big\} \\ & \times \prod_{i=1}^{n} \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp\big\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\big\}\Big\} \times \frac{1}{\sigma^2} \times \det(\boldsymbol{\Sigma})^{-(d+1)} \exp\big\{ -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\big\}\end{aligned}$$

To apply MCMC, we need to find conditional posterior distribution of each parameter.

1. For $\pi(\mathbf{B}|.)$

$$
\begin{aligned}
\pi(\mathbf{B}|.) &\propto L_Y(\mathbf{B}^\top, \sigma^2) \times \pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
&\propto \prod_{i=1}^{n} \Big\{ (2\pi\sigma^2)^{-m_i/2} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top (\sigma^2 I)^{-1} (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top) \Big) \Big\} \\
&\times \prod_{i=1}^{n} \Big\{ \det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top) \Big\} \\
&\propto \prod_{i=1}^{n} \exp\{-\frac{1}{2}\Big((\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top (\sigma^2 I)^{-1} (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top) + (\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top) \Big)\}
\end{aligned}
$$

Considering the exponential term in each component in the product,

$$
\begin{aligned}
&(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top (\sigma^2 I)^{-1} (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top) + (\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top) \\
&= \boldsymbol{Y}_i^\top (\sigma^2 I)^{-1} \boldsymbol{Y}_i + \boldsymbol{\beta}_i \boldsymbol{X}_i^\top (\sigma^2 I)^{-1} \boldsymbol{X}_i \boldsymbol{\beta}_i^\top - 2\boldsymbol{Y}_i^\top (\sigma^2 I)^{-1} \boldsymbol{X}_i \boldsymbol{\beta}_i^\top \\
&+ \boldsymbol{\beta}_i \boldsymbol{\Sigma}^{-1} \boldsymbol{\beta}_i^\top + \boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^\top - 2\boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}_i^\top \\
&= \boldsymbol{\beta}_i \boldsymbol{V} \boldsymbol{\beta}_i^\top - 2\boldsymbol{M}\boldsymbol{\beta}_i^\top + \boldsymbol{R}
\end{aligned}
$$

where,

$$
\begin{aligned}
\boldsymbol{V} &= \boldsymbol{\Sigma}^{-1} + \boldsymbol{X}_i^\top (\sigma^2 I)^{-1} \boldsymbol{X}_i \\
\boldsymbol{M} &= \boldsymbol{Y}_i^\top (\sigma^2 I)^{-1} \boldsymbol{X}_i + \boldsymbol{\mu}\boldsymbol{\Sigma}^{-1} \\
\boldsymbol{R} &= \boldsymbol{Y}_i^\top (\sigma^2 I)^{-1} \boldsymbol{Y}_i + \boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^\top
\end{aligned}
$$

re-writing the conditional posterior distribution, and ignoring some constant terms

$$
\pi(\mathbf{B}|.) \propto \prod_{i=1}^{n} \exp\{(\boldsymbol{\beta}_i^\top - \boldsymbol{V}^{-1}\boldsymbol{M})^\top \boldsymbol{V} (\boldsymbol{\beta}_i^\top - \boldsymbol{V}^{-1}\boldsymbol{M})\}
$$

Hence, each $\boldsymbol{\beta}_i$ has a conditional posterior multivariate normal distribution

$$
\pi(\boldsymbol{\beta}_i|.) \sim N(\boldsymbol{V}^{-1}\boldsymbol{M}, \boldsymbol{V}^{-1})
$$

2. For $\pi(\sigma^2|.)$

$$
\begin{aligned}
\pi(\sigma^2|.) &\propto L_Y(\mathbf{B}^\top, \sigma^2) \times \pi(\boldsymbol{\sigma}^2) \\
&\propto \frac{1}{\sigma^2} \prod_{i=1}^{n} \Big\{ (2\pi\sigma^2)^{-m_i/2} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top (\sigma^2 I)^{-1} (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top) \Big) \Big\} \\
&\propto \frac{1}{\sigma^2}^{(\frac{\sum_{i=1}^{n} m_i}{2} + 1)} \exp\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top (\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)\}
\end{aligned}
$$

which follows the form of pdf of inverse gamma distribution

$$
f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{x} \exp\{-\frac{\beta}{x}\}
$$

in this case, x is replaced by σ^2, α is replaced by $\frac{1}{2}\sum\limits_{i=1}^{n} m_i$, β is replaced by $\frac{1}{2}\sum\limits_{i=1}^{n}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)$

i.e.

$$\pi(\sigma^2|.) \sim IG(\frac{1}{2}\sum_{i=1}^{n} m_i, \frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top)^\top(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^\top))$$

3. For $\pi(\boldsymbol{\Sigma}|.)$

$$\begin{aligned}
\pi(\boldsymbol{\Sigma}|.) \propto& \pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})\pi(\boldsymbol{\Sigma}^{-1}) \\
\propto& \prod_{i=1}^{n}\Big\{\det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}}\exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top)\Big\}|\Sigma|^{-(d+1)}\exp(-\frac{1}{2}\Sigma^{-1}) \\
\propto& |\Sigma|^{-(n+d+1+d+1)/2}\exp\{-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top - \frac{1}{2}\Sigma^{-1}\} \\
\propto& |\Sigma|^{-(n+d+1+d+1)/2}\exp\{-\frac{1}{2}tr(\boldsymbol{S}\boldsymbol{\Sigma}^{-1})\}
\end{aligned}$$

where

$$\boldsymbol{S} = \boldsymbol{I} + \sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top$$

which is the form of pdf of the inverse wishart distribution Inverse Wishart$(\mathbf{V}, \boldsymbol{S})$, where $\mathbf{V} = n + d + 1$, i.e.

$$\pi(\boldsymbol{\Sigma}|.) \sim IW(n + d + 1,\ \boldsymbol{I} + \sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top)$$

4. For $\pi(\boldsymbol{\mu}|.)$

$$\begin{aligned}
\pi(\boldsymbol{\mu}|.) \propto& \pi(\mathbf{B} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma})\pi(\boldsymbol{\mu}) \\
=& \prod_{i=1}^{n}\Big\{\det(2\pi\boldsymbol{\Sigma})^{-\frac{1}{2}}\exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top)\Big\} \\
\propto& \exp\{-\frac{1}{2}\sum_{i=1}^{n}(\boldsymbol{\beta}_i - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\mu})^\top\} \\
\propto& \exp\{-\frac{1}{2}\Big(\sum_{i=1}^{n}\boldsymbol{\beta}_i\ \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}_i^\top + n\boldsymbol{\mu}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^\top - 2\sum_{i=1}^{n}\boldsymbol{\beta}_i\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}^\top\Big)\} \\
=& \exp\{-\frac{1}{2}\Big(\boldsymbol{\mu}\boldsymbol{V'}\boldsymbol{\mu}^\top - 2\boldsymbol{M'}\boldsymbol{\mu}^\top + \boldsymbol{R'}\Big)\} \\
\propto& \exp\{-\frac{1}{2}(\boldsymbol{\mu} - \boldsymbol{V'}^{-1}\boldsymbol{M'})\boldsymbol{V'}(\boldsymbol{\mu} - \boldsymbol{V'}^{-1}\boldsymbol{M'})^\top\}
\end{aligned}$$

where

$$\boldsymbol{V'} = n\boldsymbol{\Sigma}^{-1},\ \boldsymbol{M'} = \sum_{i=1}^{n}\boldsymbol{\beta}_i\boldsymbol{\Sigma}^{-1},\ \boldsymbol{R'} = \sum_{i=1}^{n}\boldsymbol{\beta}_i\ \boldsymbol{\Sigma}^{-1}\boldsymbol{\beta}_i^\top$$

Hence

$$\pi(\boldsymbol{\Sigma}|.) \sim N(\boldsymbol{V'}^{-1}\boldsymbol{M'}, \boldsymbol{V'}^{-1})$$

Markov Chain Monte Carlo

Because our hierarchical Bayesian Model exploited non-informative priors for four parameters, the Gibbs Sampling method would be implemented, updating parameters in the following order from their conditional posteriors distributions, $\boldsymbol{B}$, σ^2, $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$.