

# derive posterior distribution

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The suggested Bayesian model

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where  $Y_i(t)$  the wind speed at time  $t$  (i.e. 6 hours earlier),  $\Delta_{i,1}(t)$ ,  $\Delta_{i,2}(t)$  and  $\Delta_{i,3}(t)$  are the changes of latitude, longitude and wind speed between  $t$  and  $t+6$ , and  $\epsilon_{i,t}$  follows a normal distributions with mean zero and variance  $\sigma^2$ , independent across  $t$ .

In the model,  $\beta_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{7,i})$  are the random coefficients associated the  $i$ th hurricane, we assume that

$$\beta_i \sim N(\beta, \Sigma)$$

follows a multivariate normal distributions with mean  $\beta$  and covariance matrix  $\Sigma$ .

We assume the following non-informative or weak prior distributions for  $\sigma^2$ ,  $\beta$  and  $\Sigma$ .

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\beta) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

$d$  is dimension of  $\beta$ .

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Let  $\mathbf{B} = (\beta_1^\top, \dots, \beta_n^\top)^\top$ , derive the posterior distribution of the parameters  $\Theta = (\mathbf{B}^\top, \beta^\top, \sigma^2, \Sigma)$ .

Note from given Bayesian model, let

$$\epsilon_i(t) = Y_i(t+6) - \left( \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \right) \stackrel{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \sim N(\mathbf{X}_i(t)\beta_i^\top, \sigma^2)$$

where  $\mathbf{X}_i(t) = (1, Y_i(t), \Delta_{i,1}(t), \Delta_{i,2}(t), \Delta_{i,3}(t))$ , and  $\beta_i = (\beta_{0,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}, \beta_{4,i})$ . Therefore, the wind speed of  $i^{th}$  hurricane at time  $t$  follows the normal distribution with the pdf below

$$f_{Y_i(t+6)}(y_i(t+6) | \mathbf{X}_i(t), \beta_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} \left( y_i(t+6) - \mathbf{X}_i(t)\beta_i^\top \right)^2 \right\}$$

Therefore, the conditional distribution of  $Y_i$ , the wind speed of  $i^{th}$  hurricane follows the multivariate normal distribution below, (since  $Y_i(t)$ 's are independent across  $t$ )

$$(\mathbf{Y}_i | \mathbf{X}_i, \beta_i, \sigma^2) \sim \mathcal{N}(\mathbf{X}_i\beta_i^\top, \sigma^2 I)$$

where  $Y_i$  is an  $m_i$ -dimensional vector and  $\mathbf{X}_i$  is a  $m_i \times d$  matrix.

Hence, the joint likelihood function of all  $i$ 's hurricanes can be expresses as

$$L_Y(\mathbf{B}, \sigma^2) = \prod_{i=1}^n \left\{ \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp \left( -\frac{1}{2} (\mathbf{Y}_i - \mathbf{X}_i\beta_i^\top)^\top (\sigma^2 I)^{-1} (\mathbf{Y}_i - \mathbf{X}_i\beta_i^\top) \right) \right\}$$

where  $I$  is an identical matrix with dimension consistent with  $Y_i$ .

From Bayesian theorem, the posterior distribution for  $\Theta$  is

$$\pi(\Theta | \mathbf{Y}) = \pi(\mathbf{B}, \beta, \sigma^2, \Sigma | \mathbf{Y}) \propto L_Y(\mathbf{B}, \sigma^2) \times \pi(\mathbf{B} | \beta, \Sigma) \times \pi(\beta) \times \pi(\sigma^2) \times \pi(\Sigma),$$

where  $\pi(\mathbf{B} | \beta, \Sigma)$  is the joint multivariate normal density of  $\beta$ , since

$$\beta_i \sim N(\beta, \Sigma)$$

Therefore

$$\pi(\mathbf{B} | \beta, \Sigma) = \prod_{i=1}^n \left\{ \det(2\pi\Sigma)^{-\frac{1}{2}} \exp \left( -\frac{1}{2} (\beta_i - \beta)^\top \Sigma^{-1} (\beta_i - \beta) \right) \right\}.$$

So we have the following posterior distribution:

$$\begin{aligned} \pi(\mathbf{B}, \beta, \sigma^2, \Sigma | \mathbf{Y}) &\propto \prod_{i=1}^n \left\{ (2\pi\sigma^2)^{-m_i/2} \exp \left\{ -\frac{1}{2} (\mathbf{Y}_i - \mathbf{X}_i\beta_i^\top)^\top (\sigma^2 I)^{-1} (\mathbf{Y}_i - \mathbf{X}_i\beta_i^\top) \right\} \right\} \\ &\times \prod_{i=1}^n \left\{ \det(2\pi\Sigma)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\beta_i - \beta)^\top \Sigma^{-1} (\beta_i - \beta) \right\} \right\} \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp \left\{ -\frac{1}{2} \Sigma^{-1} \right\} \end{aligned}$$