HMMA 307 : Advanced Linear Modeling

Chapter 1 : Linear regression

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https://github.com/MegDie/advanced_lm_introduction

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Model

Suppose the data consists of n observations $(yi, xi)_{i=1}^{n}$ with p features. The model can be written in matrix notation as :

$$y = X\beta + \epsilon$$

where

- X is an $n \times p$ matrix of regressors
- β is a $p \times 1$ vector of unknown parameters
- ϵ is a vector of normal random errors with mean 0

The OLS estimator is any coefficient vector $\hat{\beta}^{LS} \in \mathbb{R}^p$ such that :

$$\hat{\beta}^{LS} \in \text{argmin} \frac{1}{2n} \|y - X\beta\|^2$$

with,

$$f(\beta) = \frac{1}{2n} \sum_{i=1}^{n} (y_i - \frac{1}{2n}(X\beta)_i)^2$$

$$= \beta^T \frac{X^T X}{2n} \beta + \frac{1}{2n} \|y\|^2 - \langle y, X\beta \rangle$$

where, $\langle y, X\beta \rangle = y^T X\beta = \beta^T X^T y = \langle \beta, X^T y \rangle$

The matrix $\hat{\Sigma} = \frac{X^T X}{n}$ matrix is called the Gram matrix.

$$X^T X = \begin{pmatrix} x_1^T \\ . \\ . \\ x_p^T \end{pmatrix} (x_1^T ... x_p^T),$$

The Gram matrix is equivalent to :

$$[X^T X]_{j,j'} = [\langle x_j, x_{j'} \rangle]_{(j,j') \in [1,p]^2}$$

Remark

Most of the times, we scale features.

We have : $\bar{X}_j = \frac{1}{n} \sum_{i=1}^{n} x_{ij}$

First Order Optimality Condition

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Let $\Sigma \in \mathbb{R}^{p*p}$.
If $\Sigma^T = \Sigma$ then Σ is diagonalizable.

Theorem

For all matrix $M \in \mathbb{R}^{m1*m2}$ of rank r, there exist two orthogonal matrix $U \in \mathbb{R}^{m1*r}$ and $V \in \mathbb{R}^{m2*r}$ such that :

$$M = U diag(s_1...s_r)U^T$$

where $s_1 \geq s_2 \geq ... \geq s_r \geq 0$ are the singular values of M.

Note that : $M = \sum_{j=1}^{r} s_j u_j v_j^T$ with : $U = [u_1, ..., u_r]$ et $V = [v_1...v_r]$

Definition

For $M \in \mathbb{R}^{m1*m2}$, a pseudoinverse of M is defined as a matrix M^+ satisfying :

$$M^+ = V diag(\tfrac{1}{s_1}...\tfrac{1}{s_r})U^T = \sum_{j=1}^{r} \tfrac{1}{s_j} v_j u_j^T$$

Remark : If M is invertible, its pseudoinverse is its inverse. That is, $A^+ = A^{-1}$

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