Final Project

Computational Methods

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This exercise is motivated by the work Investment, Capacity Utilization, and the Real Business Cycle by Greenwood, Hercowitz, and Huffman (1988), published in the American Economic Review. Please refer to the paper for a detailed understanding of the research. This exercise focuses on numerical implementation.

Here is the recursive social planner problem. Capital is denoted by k, the technological shift factor by ε, and l represents the labor supply.

$$V(k, \varepsilon_r) = \max_{c,k'} \frac{1}{1-\gamma}(c - \frac{\hat{l}^{1+\theta}}{1+\theta})^{1-\gamma} + \beta \sum_{s=1}^{2} \pi_{rs} V(k', \varepsilon_s),$$

subject to

$$c = A(k_i\hat{h})^{\alpha}\hat{l}^{1-\alpha} - k'e^{-\varepsilon_r} + k_i(1 - B\hat{h}^{\omega}/\omega)e^{-\varepsilon_r}$$

where

$$\hat{h}, \hat{l} = \arg\max[A(k_i\hat{h})^{\alpha}\hat{l}^{1-\alpha} + k_i(1 - B\hat{h}^{\omega}/\omega)e^{-\varepsilon_r} - \frac{\hat{l}^{1+\theta}}{1+\theta}]$$

Calibration: Set $\beta = 0.96$, $\alpha = 0.33$, $\theta = 1.0$, $\gamma = 2$, $B = 0.075$ and $\omega = 2$. The value of $A = 0.592$ and the grid for k should be of 100 points.

In terms of the stochastic process, set $\pi_{11} = \pi_{22} = \pi$ and $\varepsilon_1 = -\varepsilon_2 = \Theta$. Note that the standard deviation $\sigma = \Theta$ and the coefficient of autocorrelation is $\lambda = 2\pi - 1$. These parameters are picked so as to make the model generate the same standard deviation and first-order serial correlation for output as is observed in the data. $\sigma = 0.051$ and $\lambda = 0.44$.

Part (a)

Read the original paper and briefly explain the roles of h and ε in the model mechanism.

Role of h: Capital Utilization Rate

h represents the intensity of capital, determining for a given capital stock the ”flow of capital services” $k_t h_t$ i.e. the speed or how many hours capital is run.

- Of all labor available l_t, some goes to production or direct capital usage h_t and the rest goes to maintenance of the capital.
- Output depends on effective capital services, meaning when h increases, capital is used more intensely, boosting current output.

- This higher utilization rate invokes something Keynes calls ”user cost”. This refers to a faster depreciation of the capital stock through either wear and tear or less labor devoted to maintenance.
- From this we have a trade off between higher current output and faster capital decay, making h a key intra-temporal margin for the planner.

Role of ε: Investment-Specific Technology Shock

ε is a shock to the marginal efficiency of investment, i.e. a productivity shock that only affects newly installed capital, not the existing stock.

- A positive ε increases the effectiveness of current investment, making new capital goods more productive.
- Higher $\varepsilon \implies$ higher investment $\implies$ more capital accumulation $\implies$ increases in labor effort and utilization (h).
- ε doesn’t directly affect the productivity of capital already installed.

Part (b)

Solve the model using ***value function iteration*** *+ discretization. Plot V and k’ as a function of k and include two lines, one for each value of the shock ε. Do the same for the policy function $K'(k_i, \varepsilon_r)$.*

To go about solving the model, I define a capital grid over a plausible range of values for the capital shock, as well as two shock states representing high and low productivity. This allows evaluation of the value function for every possible combination of k and ε.

```
epsilon_vals = np.array([0.051, -0.051])  # espilon1 = -epsilon2 = theta
pi = np.array([[0.72, 0.28], [0.28, 0.72]])  # transition matrix

# capital grid
k_min = 0.2
k_max = 1.5
```

```
nk = 100
k_grid = np.linspace(k_min, k_max, nk)
```

Then, from the social planner problem I define the utility, production, and depreciation functions:

```
# utility function
def utility(c, l):
    if c <= 0 or l <= 0:
        return -1e10
    return ((c - l**(1 + theta) / (1 + theta))**(1 - gamma)) / (1 - gamma)

# production function
def production(k, h, l):
    return A * (k * h)**alpha * l**(1 - alpha)

# depreciation function
def depreciation(h):
    return B * (h**omega) / omega
```

Then comes the joint optimization of hiring and labor. For every possible (k, ε) pair, I solve for the optimal human capital h and labor l using numerical optimization:

```
def optimize_hiring(k_i, epsilon):
    def objective(x):
        h, l = x
        if h <= 0 or l <= 0:
            return 1e10
        output = production(k_i, h, l)
        depreciation_term = k_i * (1 - depreciation(h)) * np.exp(-epsilon)
        c = output + depreciation_term
        if c <= 0:
            return 1e10
        return -utility(c, l)

    x0 = np.array([0.5, 0.5])
    bounds = [(1e-4, 1.0), (1e-4, 1.0)]
    res = minimize(objective, x0, bounds=bounds, method='L-BFGS-B')

    if res.success:
        return res.x
```

```
    else:
        return (0.1, 0.1)
```

Next, I compute consumption from the resource constraint and evaluated the Bellman equation over all possible next-period capital k'. The maximization produces the updated value function $V(k, \varepsilon$ and the optimal policy function $k'(k, \varepsilon)$. This then iterates until convergence.

Finally, I plot $V(k)$ and $k'(k)$ for both shock values to compare policy behavior across states.

Part (c)

Explain your decision of the minimum and the maximum k in your grid.

I set the capital grid to span the interval $k \in [k_{min}, k_{max}] = [0.2, 1.5]$. A lower bound of 0.2 was chosen to ensure that capital doesn't become unrealistically small or numerically unstable. Values below 0.2 typically led to negative or near-zero consumption, which is infeasible.

The upper bound of 1.5 was chosen to ensure that during high productivity shocks, the planner's optimal capital decision k' is still within the grid. After several iterations of the value function, I observed that policy function $k'(k)$ never approached the upper boundary, meaning 1.5 is sufficiently large to contain the planner's optimal saving behavior.

Part (d)

Solve the model using ***policy function iteration****. Plot same as problem (a).*

Full code using policy function iteration can be seen in the submitted code.

Part (e)

Consider using the endogenous grid method to solve this problem. Describe your comprehensive idea for applying this method. If it offers advantages, solve the model using this method and compare the results to the two previous methods along any relevant dimensions.

To use the endogenous grid method, one would have to invert the Euler equation to solve directly for the endogenous state variable (current capital k) given a grid for the next-period capital choice (k'). To do this, I would:

1. Reformulate the problem in Euler equation form

2. Choose a grid for k' (future capital)

3. Recover current capital k (the endogenous grid)

4. Interpolate

5. Iterate to convergence

There would be a few advantages to the EGM method, the first being speed. EGM would converge much faster than VFI/PFI with this model because it eliminates the maximization loop. Additionally, it would be potentially be more accurate with smoother policy functions and have greater scalability.

That being said, it does offer some benefits and so I went ahead and tried it.

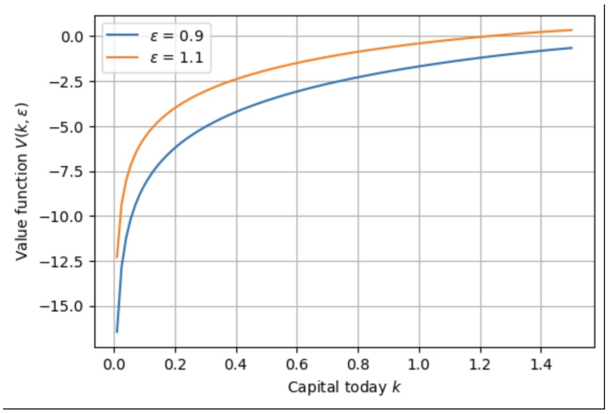


Figure 1: VFI Method

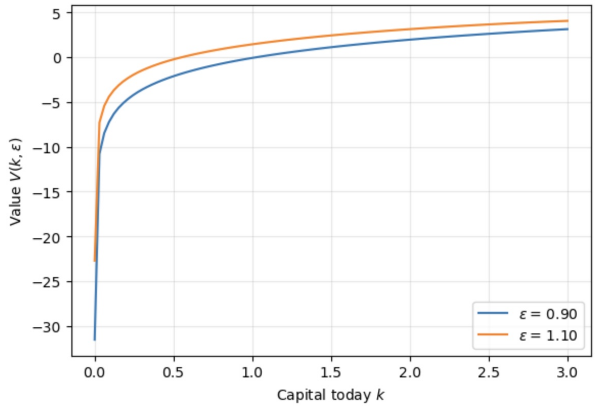


Figure 2: PFI Method

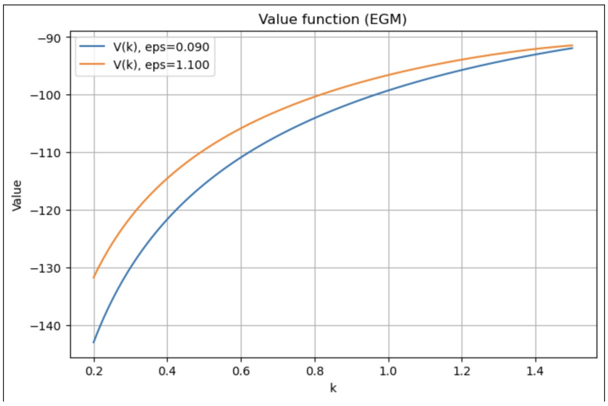


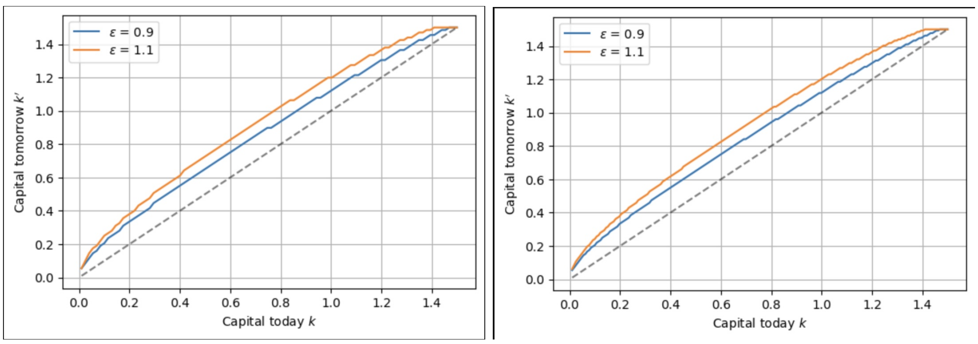
Figure 3: EG Method

When comparing the graphs of the value functions between the three, it's clear that the EGM and PFI return the same result, which is to be expected considering they are solving the same Bellman/Euler equation problem. The VFI method produces a much more jagged outcome than the others do, due to discrete grid search and linear interpolation in the maximization steps.

Part (f)

Compare the results with the solution using 200 grid points.

When I run the functions using 200 grid points as opposed to 100, I find that the resulting plots of the policy function is much smoother, particularly with the VFI method. This makes sense, as VFI's accuracy depends heavily on the resolution of the value function approximation at each steo, and interpolation errors compound over successive iterations. In comparison, PFI updates policies more directly and EGM constructs the policy using the Euler equation, making both methods less sensitive to the courseness of the grid.



(a) VFI Policy, 100 Grid Points (b) VFI Policy, 200 Grid Points

Part (h)

Look for data for time series of output, consumption, investment, hours worked, productivity for the US and Germany for the postwar period. Do a table like Table 1 below and report the sources very carefully. Before computing the statistics detrend the series using the HP method or other. Note that if you detrend log(y) instead of y, the detrended series is in percentage deviation from trend. What are the key differences and similarities across countries?

For these tables, I used data from the Penn World Table version 10.01 from the Groningen Growth and Development Center. The table documents income, output, input, and productivity from 183 countries between 1950 and 2019. The specific series used for this analysis are the following:

- Output (rgdpna): Real GDP at constant 2017 national prices (in mil. 2017US$)
- Comsumption (ccon): Real consumption of households and government, at current PPPs (in mil. 2017US$)
- Investment (cda-ccon): Real domestic absorption - Real consumption
- Hours worked (avh x emp) Average annual hours worked by persons engaged x Number of persons engaged
- Productivity (rgdpna / hours worked): Output of hours worked
- Capacity Utilization: Actual GDP / HP-filtered trend GDP

United States			
Variable	Standard deviation	Correlation with output	Autocorrelation
Output	2.018	1.000	0.536
Consumption	1.751	0.858	0.694
Investment	6.039	0.828	0.476
Hours worked	2.022	0.907	0.554
Productivity	0.871	0.211	0.485
Utilization	1.984	0.995	0.539

Germany			
Variable	Standard deviation	Correlation with output	Autocorrelation
Output	1.817	1.000	0.496
Consumption	2.489	0.579	0.796
Investment	6.329	0.837	0.449
Hours worked	1.176	0.451	0.568
Productivity	1.661	0.775	0.678
Utilization	1.734	0.893	0.445

Looking at the volatility of output and consumption, we can see the countries' differing reactions to short term changes. In the table, the standard deviations are telling the typical percentage fluctuations around trend, with the US having output stdev = 2.02%, consumption stdev = 1.75%. Because consumption is less volatile than output, this indicates households are smoothing with weaker reactions to changes to in output. Germany has the opposite effect, with output stdev = 1.82%, which is higher than consumption stdev = 2.49%.

The correlation with output is measuring how closely each variable moves with output's cycle, with high correlation (close to 1) being strongly procyclical. Results of note are that the US has strongly procyclical hours worked (0.907), meaning it the variable rises when output rises. Productivity in the US has small link to output (0.221). Germany has the exact opposite results, with weakly procyclical hours worked (0.451) and stronly procyclical productivity (0.775).

Autocorrelation results show how persistent deviations are over time, with Germany's consumption correlation (0.796) being higher than that in the US (0.694), meaning that consumption cycles in Germany last longer.