Non-convex Robust PCA: Provable Bounds

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Joint work with Praneeth Netrapalli, U.N. Niranjan, Prateek Jain and Sujay Sanghavi.

Learning with Big Data

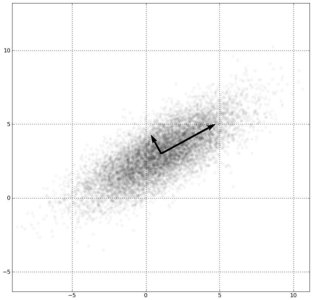
High Dimensional Regime

- Missing observations, gross corruptions, outliers, ill-posed problems.
- Needle in a haystack: finding low dimensional structures in high dimensional data.

Principled approaches for finding low dimensional structures?

PCA: Classical Method

- Denoising: find hidden low rank structures in data.
- Efficient computation, perturbation analysis.



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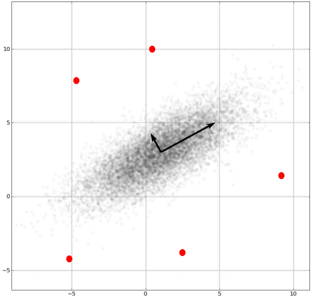

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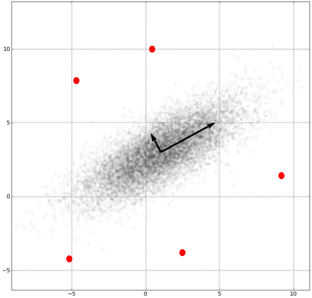
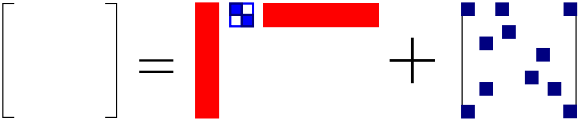


Not robust to even a few outliers

Robust PCA Problem

- Find low rank structure after removing sparse corruptions.
- Decompose input matrix as low rank + sparse matrices.



M L^* S^*

- $M \in \mathbb{R}^{n\times n}$, L^* is low rank and S^* is sparse.
- Applications in computer vision, topic and community modeling.

History

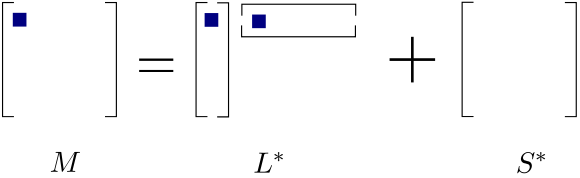
Heuristics without guarantees

- Multivariate trimming [Gnanadeskian+ Kettering 72]
- Random sampling [Fischler+ Bolles81].
- Alternating minimization [Ke+ Kanade03].
- Influence functions [de la Torre + Black 03]

Convex methods with Guarantees

- Chandrasekharan et. al, Candes et. al '11: seminal guarantees.
- Hsu et. al '11, Agarwal et. al '12: further guarantees.
- (Variants) Xu et. al '11: Outlier pursuit, Chen et. al '12: community detection.

Why is Robust PCA difficult?



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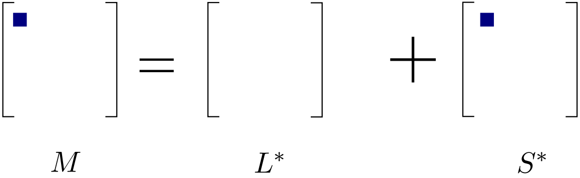

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Convex Relaxation Techniques

(Hard) Optimization Problem, given $M \in \mathbb{R}^{n \times n}$

$$\boxed{\min_{L,S} \mathrm{Rank}(L) + \gamma \|S\|_0, \quad M = L + S.}$$

- $\mathrm{Rank}(L) = \{\#\sigma_i(L) : \sigma_i(L) \neq 0\}$, $\|S\|_0 = \{\#S(i,j) : S(i,j) \neq 0\}$ are not tractable.

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Convex Relaxation

$$\boxed{\min_{L,S} \|L\|_* + \gamma\|S\|_1, \quad M = L + S.}$$

- $\|L\|_* = \sum_i \sigma_i(L)$, $\|S\|_1 = \sum_{i,j} |S(i,j)|$ are convex sets.

- Chandrasekharan et. al, Candes et. al '11: seminal works.

Other Alternatives for Robust PCA?

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- Computational cost: $O(n^3/\epsilon)$ to achieve error of ϵ
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Non-convex alternatives?

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A non-convex heuristic (AltProj)

- Initialize $L, S = 0$ and iterate:
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Any hope for proving guarantees?

Observations regarding non-convex analysis

Challenges

- Multiple stable points: bad local optima, solution depends on initialization.
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Non-convex Projections vs. Convex Projections

- Projections on to non-convex sets: NP-hard in general.
  - Projections on to rank and sparse sets: tractable.
- Less information than convex projections: zero-order conditions.

$$\|P(M) - M\| \leq \|Y - M\|, \quad \forall\, Y \in C(\text{Non-convex}),$$

$$\|P(M) - M\|^2 \leq \langle Y - M, P(M) - M \rangle, \quad \forall\, Y \in C(\text{Convex}).$$

Non-convex success stories

Classical Result

- PCA: Convergence to global optima!

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(Somewhat) common theme

- Characterize basin of attraction for global optimum.
- Obtain a good initialization to "land in the ball".

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Our results for (a variant of) AltProj

- Guaranteed recovery of low rank L^* and sparse part S^*.
- Match the bounds for convex methods (deterministic sparsity).
- Reduced computation: only require low rank SVDs!

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Best of both worlds: reduced computation with guarantees!

Outline

Toy example: Rank-1 case

$$\boxed{M = L^* + S^*, \quad L^* = u^*(u^*)^\top}$$

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Exploit incoherence of L^*?

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Solution for handling large $\|S^*\|$

- First threshold M before rank-1 projection.
- Ensures large entries of S^* are identified.

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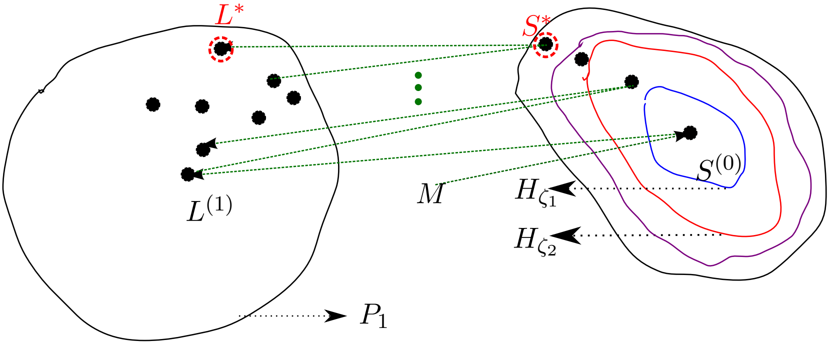

- To analyze progress, track $E^{(t+1)} := S^* - S^{(t+1)}$

Rank-1 Analysis Contd.

One iteration of AltProj

$$L^{(0)} = 0, S^{(0)} = H_{\zeta_0}(M), \boxed{L^{(1)} \leftarrow P_1(M - S^{(0)}),\ S^{(1)} \leftarrow H_{\zeta}(M - L^{(1)})}.$$

Analyze $E^{(1)} := S^* - S^{(1)}$

- Thresholding is element-wise operation: require $\|L^{(1)} - L^*\|_\infty$.
- In general, no special bound for $\|L^{(1)} - L^*\|_\infty$.
- Exploit sparsity of S^* and incoherence of L^*?

Rank-1 Analysis Contd.

- $L^{(1)} = uu^\top = P_1(M - S^{(0)})$ and $E^{(0)} = S^* - S^{(0)}$.

Fixed point equation for eigenvectors $(M - S^{(0)})u = \lambda u$

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- $\langle u^*, u\rangle u^* + (S^* - S^{(0)})u = \lambda u$ or $u = \lambda\langle u^*, u\rangle \left(I - \frac{E^{(0)}}{\lambda}\right)^{-1} u^*$

Taylor Series

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- Counting walks in sparse graphs.
- In addition, u^* is incoherent: $\|u^*\|_\infty < \frac{\mu}{\sqrt{n}}$.

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Contraction of error $E^{(t)}$ when degree d is bounded.

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Perturbation analysis in general rank case

- Small $\lambda^*_{\min}(L^*)$: no recovery of lower eigenvectors.

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- Reduce perturbation before recovering lower eigenvectors!

Improved Algorithm for General Rank Setting

Stage-wise Projections

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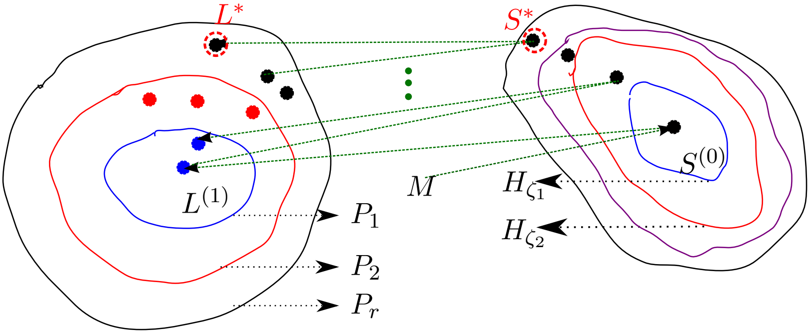

Summary of Results

- Low rank part: $L^* = U^* \Lambda^* (V^*)^\top$ has rank r.
- Incoherence: $\|U^*(i,:)\|_2, \|V^*(i,:)\|_2 \leq \frac{\mu\sqrt{r}}{\sqrt{n}}$.
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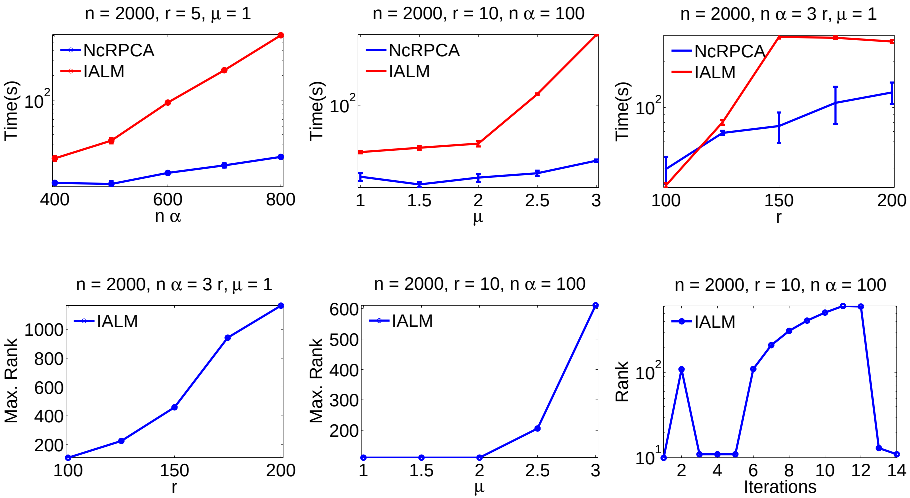
Best of both worlds: reduced computation with guarantees!

"Non-convex Robust PCA," P. Netrapalli, U.N. Niranjan, S. Sanghavi, A. , P. Jain, NIPS '14.

Outline

Synthetic Results

- NcRPCA: Non-convex Robust PCA.
- IALM: Inexact augmented Lagrange multipliers.



Real data: Foreground/background Separation

Original

Rank-10 PCA

AltProj

IALM

Real data: Foreground/background Separation

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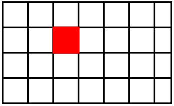
Outline

Robust Tensor PCA



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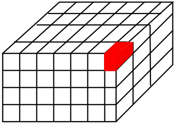

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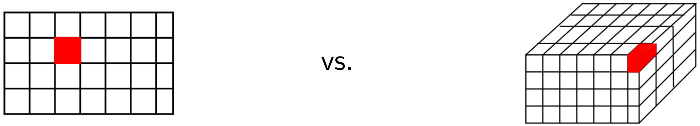
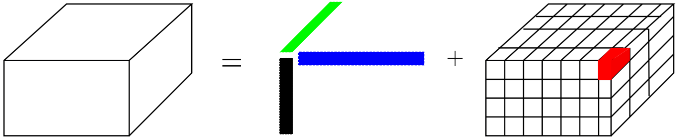


Robust Tensor Problem



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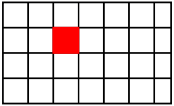

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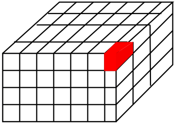

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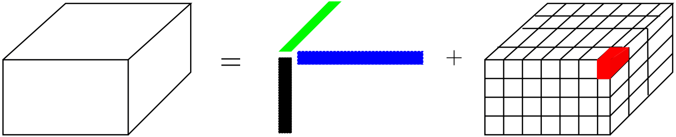

Applications: Robust Learning of Latent Variable Models.

A. , R. Ge, D. Hsu, S.M. Kakade and M. Telgarsky "Tensor Decompositions for Learning Latent Variable Models," Preprint, Oct. '12.

Challenges and Preliminary Observations

$$\boxed{T = L^* + S^* \in \mathbb{R}^{n \times n \times n}, \quad L^* = \sum_{i \in [r]} a_i^{\otimes 3}.}$$

Convex methods

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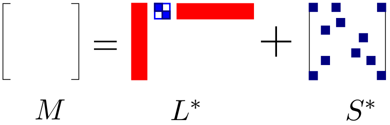
Guaranteed recovery possible!

Outline

Conclusion

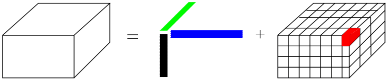


Guaranteed Non-Convex Robust PCA

- Simple non-convex method for robust PCA.
- Alternating rank projections and thresholding.
- Estimates for low rank and sparse parts “grown gradually”.
- Guarantees match convex methods.
- Low computational complexity: scalable to large matrices.

Possible to have both: guarantees and low computation!

Outlook



- Reduce computational complexity? Skip stages in rank projections? Tight bounds for incoherent row-column subspaces?
- Extendable to the tensor setting with tight scaling guarantees.
- Other problems where non-convex methods have guarantees?
  - Csiszar's alternating minimization framework.
- (Laserre) hierarchy for convex methods: increasing complexity for "harder" problems.
- Analogous unified thinking for non-convex methods?

Holy grail: A general framework for non-convex methods?