Feedback Message Passing for Inference in Gaussian Graphical Models

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Gaussian Graphical Models

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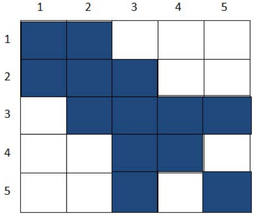
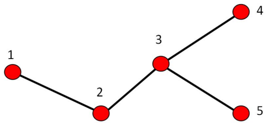
- An information matrix J is sparse or Markov with respect to a graph if $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$: $\forall (i,j) \notin \mathcal{E}$, $J_{ij} = 0$.

Matrix Structure



Graph Structure



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- Applications: gene regulatory networks, medical diagnostics, oceanography, and communication systems

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- Generalized BP (Yedidia et al.), embedded trees (Sudderth et al.), inference by tractable subgraphs (Chandrasekaran et al.)

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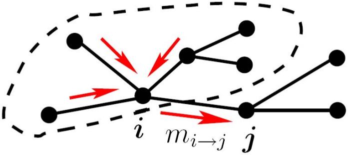
  1. Obtain inference results for feedback nodes first.
  2. Make corrections for the non-feedback nodes afterward.

Gaussian Belief Propagation

1. Message Passing

$$\forall j \in \mathcal{N}(i), \quad \Delta J_{i\to j}$$
$$\Delta h_{i\to j}$$



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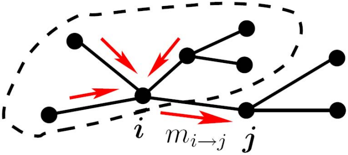

2. Marginal Computation

$$\forall i \in \mathcal{V}, \quad \hat{J}_i = J_{ii} + \sum_{k \in \mathcal{N}(i)} \Delta J_{k \to i} \qquad \hat{h}_i = h_i + \sum_{k \in \mathcal{N}(i)} \Delta h_{k \to i}$$

$$\mu_i = \hat{J}_i^{-1} \hat{h}_i \qquad \mathbf{Var}\{i\} = \hat{J}_i^{-1}$$

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- Some special nodes?

Feedback Vertex Set

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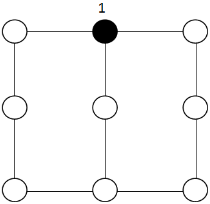
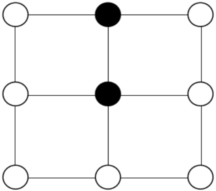
- Feedback vertex set (FVS) is a set of nodes whose removal results in a cycle-free graph.





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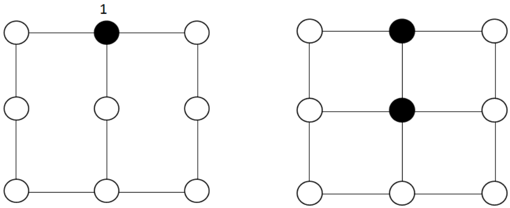
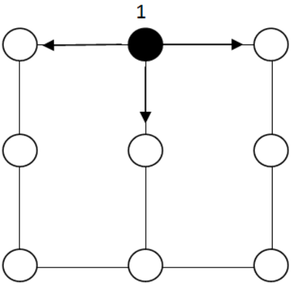


- In practice, a pseudo-FVS (a small subset of the FVS) may be sufficient for convergence and accuracy.

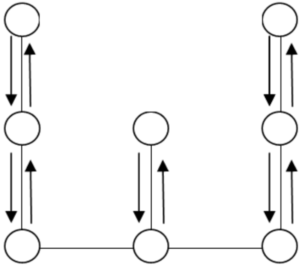
Exact Inference: a Single Feedback Node Case



- Extra potential vector $\mathbf{h}^1$,

$$h_j^1 = \begin{cases} 0 & j \notin \mathcal{N}(1) \\ J_{1j} & j \in \mathcal{N}(1) \end{cases}$$

Exact Inference: a Single Feedback Node Case (cont')



- Run belief propagation on $\mathcal{T}$. The messages are

$$\Delta J_{i\to j}^{\mathcal{T}} \quad \Delta h_{i\to j}^{\mathcal{T}} \quad \Delta h_{i\to j}^{1}$$

We obtain partial variance, partial mean, and feedback gain:

$$\mathrm{Var}^{\mathcal{T}}\{i\} \quad \mu_i^{\mathcal{T}} \quad g_i^1$$

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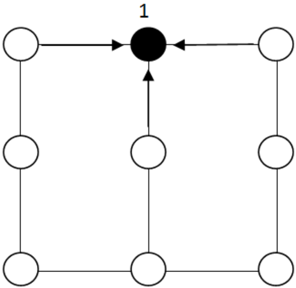

$$\mathrm{Var}\{1\} = (J_{11} - \sum_{k \in \mathcal{N}(1)} J_{1k} g_k^1)^{-1}$$

$$\mu_1 = \mathrm{Var}\{1\}(h_1 - \sum_{j \in \mathcal{N}(1)} J_{1j} \mu_j^{\mathcal{T}})$$

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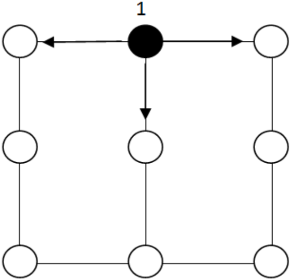
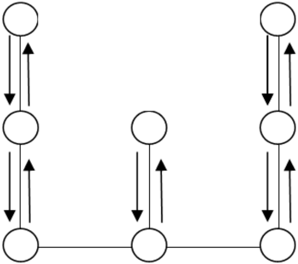


- Node 1 tells its neighbors to make revisions on their node potentials.

$$\widetilde{h}_j = h_j - J_{1j}\mu_1,\ \forall j \in \mathcal{N}(1) \qquad \widetilde{h}_j = h_j,\ \forall j \notin \mathcal{N}(1)$$

Exact Inference: a Single Feedback Node Case (cont')



- Run BP on $\mathcal{T}$ with revised node potentials $\widetilde{\mathbf{h}}$ to obtain exact means.
- The exact variances can be achieved as

$$\mathrm{Var}\{i\} = \mathrm{Var}^{\mathcal{T}}\{i\} + \mathrm{Var}\{1\}(g_i^1)^2, \quad \forall i \in \mathcal{T}.$$

Exact Inference: Multiple Feedback Nodes Case

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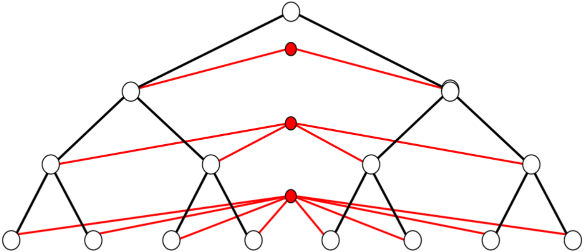
- Example:
  Exact Inference: $\mathcal{O}((\log n)^2 n)$



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- Exact inference among the feedback nodes.

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- When it converges, feedback nodes get exact means and variances.
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- For attractive models (where $J_{ij} \leq 0$ for $i \neq j$), better lower bounds of the variances.

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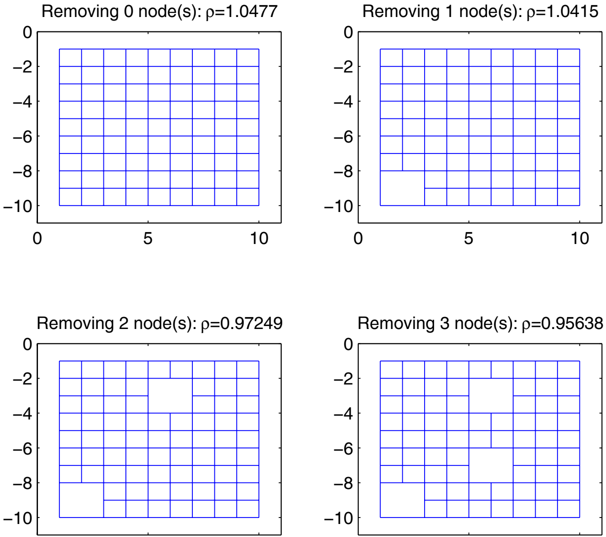
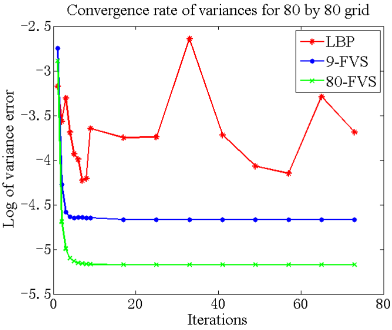
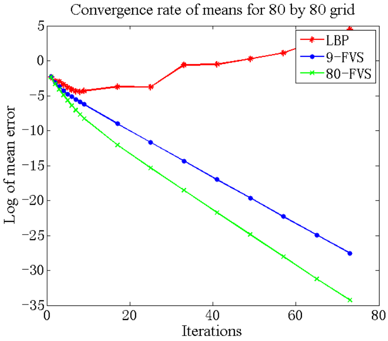
  2. $s(i) = \sum_{l,k \in \mathcal{N}(i), l<k} |J_{il} J_{ik}|$
- Pick up one node with the largest score $s(i)$ at one step and continue with the remaining graph

Numerical Results





(o) Iterations versus variance errors



(p) Iterations versus mean errors

Figure: Inference errors of a 80×80 grid graph

- Empirically, $k = \mathcal{O}(\log n)$ seems to be sufficient.

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- Corresponding structural learning problem

Questions and Comments?

- Thank you!