# Detection of Gauss Markov Random Fields under Routing Energy Constraint

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# Detection-Energy Tradeoff

## Distributed Detection

- Quantization rule @ sensors
- Inference rule @ fusion center

## Classical Routing

- Generic Performance Metric
- Layered architecture

## Shortcomings of Classical Detection

- For sensors in a large field, multi-hop routing is needed
- For energy-constrained networks, loss in detection performance

## Shortcomings of Classical Routing

Need only likelihood ratio for inference, not raw data at fusion center

Tradeoff between Routing and Detection in Wireless Sensor Networks

# Tradeoff: Optimal Detection under Energy Constraint

## Optimal Detection of Binary Hypothesis

Neyman Pearson: Min. miss detection subject to false alarm

## Large Networks: $n \to \infty$

$\max -\frac{1}{n}\log P_M$ subject to false alarm and avg. routing energy $\bar{E}$

## Optimal Node Density $\lambda_*$

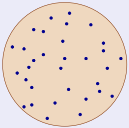
$$\lambda_* \stackrel{\Delta}{=} \arg\max_{\lambda>0} D_\lambda \quad \text{subject to } \bar{\mathsf{C}} \leq \bar{E}$$

$D_\lambda$: Neyman-Pearson error exponent

$\bar{\mathsf{C}}$: Average Routing Energy per node

# Node Deployment Strategies for Optimal Tradeoff

## Node deployments



Random

## Setup

- Random: Nodes drawn from uniform or Poisson distribution
- Constant density $\lambda$: $n$ nodes in area $\frac{n}{\lambda}$

## Factors

- Signal & Energy Model
- Nature of Tradeoff

## Implications

- $\lambda_* \to 0$ or $\infty$: Large/Small area
- $\lambda_* \in (0, \infty)$: Careful Deployment

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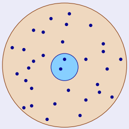

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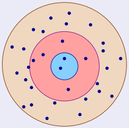

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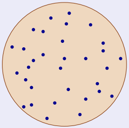

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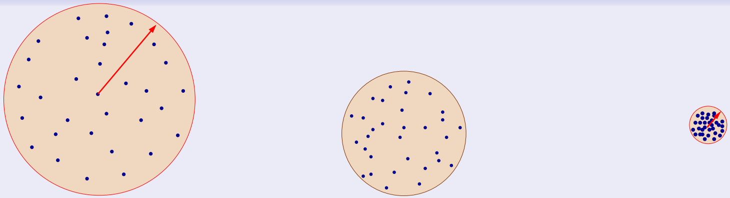

$\lambda^* \to 0$ or $R \to \infty$ $\qquad$ $\lambda^* \in (0, \infty)$ $\qquad$ $\lambda^* \to \infty$ or $R \to 0$

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# Example: Same Variances, No Energy Constraint

**Detection of Correlation**

$\mathcal{H}_1$: Correlated data vs. $\mathcal{H}_0$: Independent observations

**Assumptions**

- Uniform signal field: same variance at every node, under $\mathcal{H}_0$ and $\mathcal{H}_1$
- Correlation decays with distance under $\mathcal{H}_1$

Only way to distinguish $\mathcal{H}_0$ and $\mathcal{H}_1$: Correlation

**Intuition: to maximize correlation : Minimize inter-node distance**

In this case, $\lambda_* \to \infty$. What happens when variances are different?

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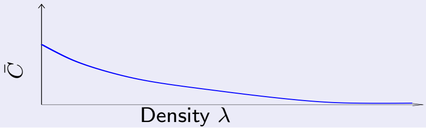
## Variance Ratio $K$

$K$ is ratio of variances under alternative and null hypotheses

## Avg. Energy vs. $\lambda$

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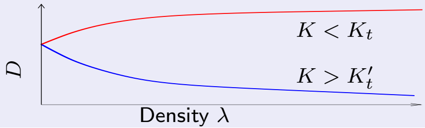


## No Energy constraint

## Feasible Energy

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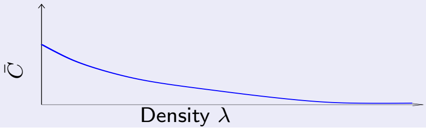


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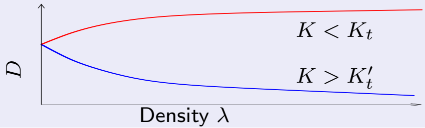


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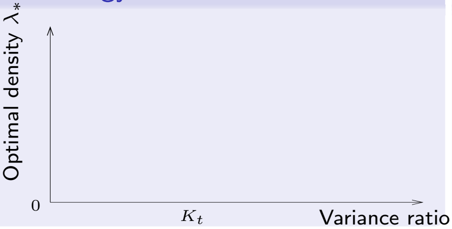


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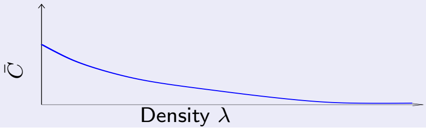


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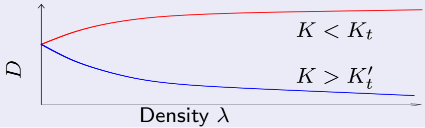


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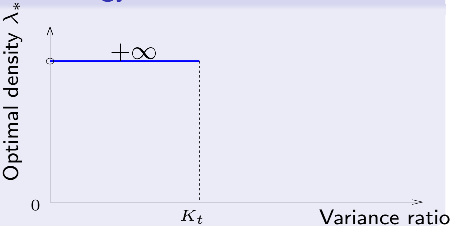


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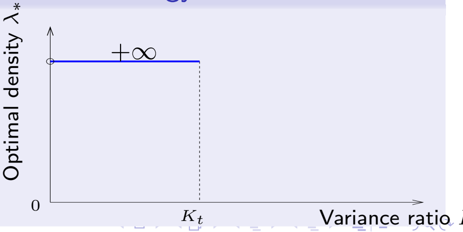

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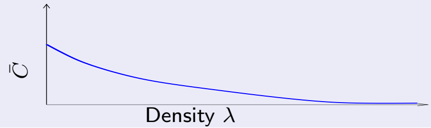


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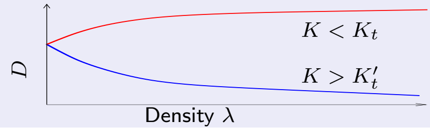


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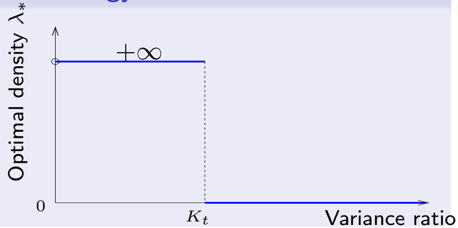


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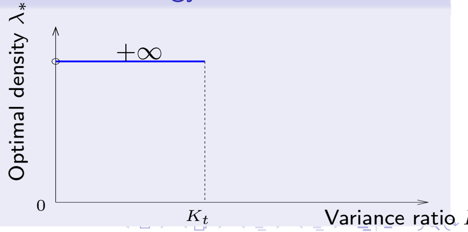

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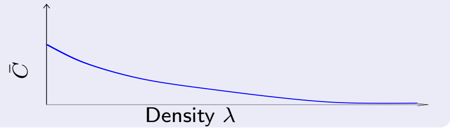


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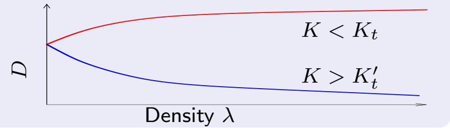


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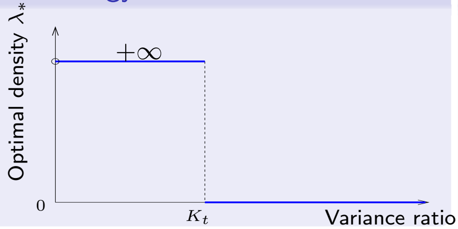


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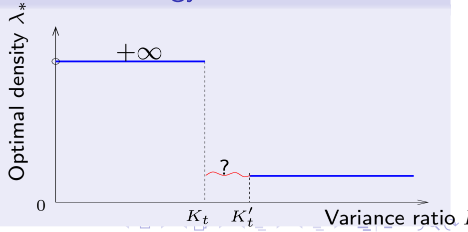

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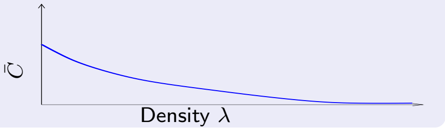


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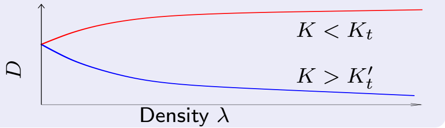


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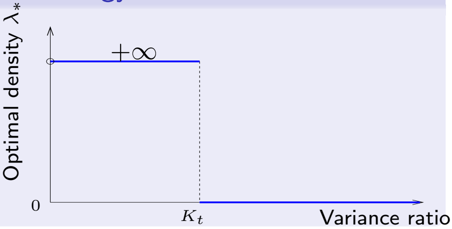


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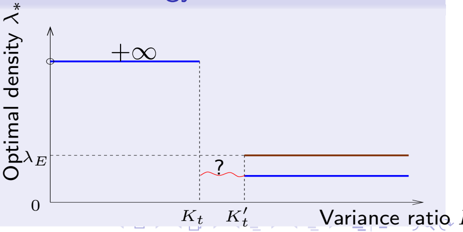

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## Detection of Correlation

$\mathcal{H}_1$: Correlated data vs. $\mathcal{H}_0$: Independent observations

## Modeling correlation

- Gauss-Markov random field
- Correlation decays with dist.
- Partial correlation at 0
- Nearest-neighbor dependency

## Min. energy routing

- NP-hard (CISS 07)
- 2-approx. algo DFMRF
- Closed-form average energy
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Closed-form Neyman Pearson error exponent (ICASSP 07)

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Analyze error exponent

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# Previous Results on Detection & Routing

## Routing with Aggregation

- Cond indep: Intangoniwat *et al.* 00, Krishnachari *et al.* 02
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- Stationary Gaussian process (Donsker & Varadhan, 85)
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- Exponent for Gauss-Markov process (Sung *et al.* 06)

## Detection-Routing

- Independent Measurements: (Yang & Blum 07, Appadwedula *et al.* 05, Yu & Ephremides 06)
- 1-D Gauss-Markov process:
  - Chernoff Routing (Sung *et al.* 06): Link-metric for detection
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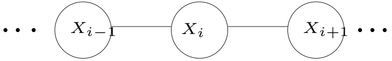
# Model for Correlated Data : Graphical Model

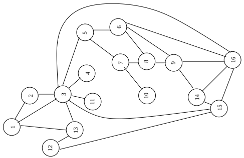



$X_{i-1} \perp X_{i+1} | X_i$

Linear graph corresponding to autoregressive process of order 1

## Temporal signals

- Conditional independence based on ordering
- Fixed number of neighbors
- Causal (random processes)




Graph of German states and states with common borders are neighbors

## Spatial signals

- Conditional independence based on (undirected) Dependency Graph
- Variable set of neighbors
- Maybe acausal

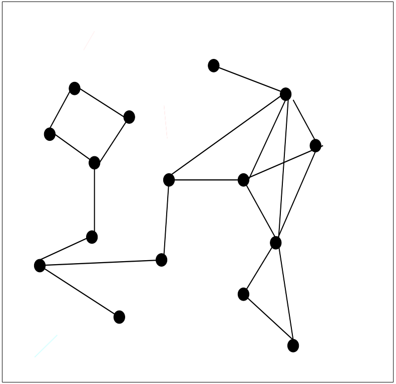
## Remark

Dependency graph is NOT related to communication capabilities, but to the correlation structure of data!

# Markov Random Field

**Definition : MRF with Dependency Graph $\mathcal{G}_d(\mathcal{V},\mathcal{E})$**

$\mathbf{Y}(\mathcal{V}) = \{Y_i : i \in \mathcal{V}\}$ is MRF with $\mathcal{G}_d(\mathcal{V},\mathcal{E})$ if PDF satisfies positivity condition and Markov property



**Markov Property**

- $A$, $B$, $C$ are disjoint
- $A$, $B$ non-empty
- $C$ separates $A, B$

$$\mathbf{Y}_A \perp \mathbf{Y}_B | \mathbf{Y}_C$$

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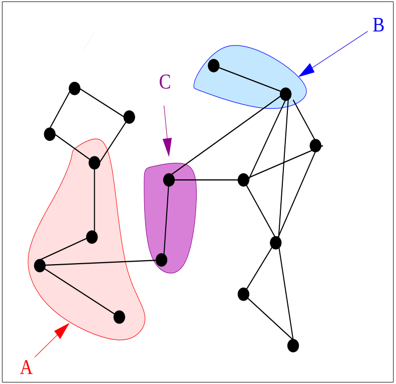


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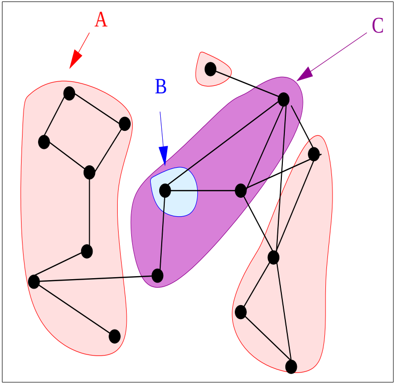


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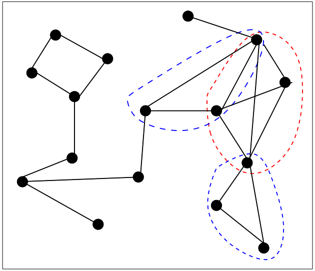
# Likelihood Function of MRF

## Hammersley-Clifford Theorem (1971)

For a MRF $\mathbf{Y}$ with dependency graph $\mathcal{G}_d(\mathcal{V}, \mathcal{E}_d)$,

$$\log \mathbb{P}(\mathbf{Y}; \mathcal{G}_d) = Z + \sum_{c \in \mathcal{C}} \Psi_c(\mathbf{Y}_c), \;\; Z \triangleq e^{-\int_{\mathbf{Y}} \prod_{c \in \mathcal{C}} \Psi_c(\mathbf{Y}_c)},$$

where $\mathcal{C}$ is the set of all cliques in $\mathcal{G}_d$ and $\Psi_C$ the clique potential

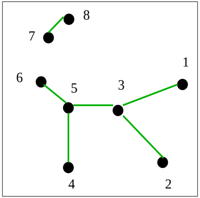


Dependency Graph

# Potential Matrix of GMRF

## Potential Matrix

- Inverse of covariance matrix of a GMRF
- Non-zero elements of Potential matrix correspond to graph edges




Dependency Graph

$$\begin{bmatrix} \times & & \times & & & & & \\ & \times & \times & & & & & \\ \times & \times & \times & & \times & & & \\ & & & \times & \times & & & \\ & & \times & \times & \times & \times & & \\ & & & & \times & \times & & \\ & & & & & & \times & \times \\ & & & & & & \times & \times \end{bmatrix}$$

$\times$ : Non-zero element of Potential Matrix

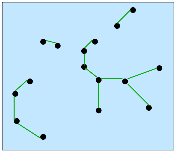
## Form of Log-Likelihood of zero-mean GMRF with potential matrix $\mathbf{A}$

$$-\log P(\mathbf{Y}_n; \mathcal{G}_d, \mathbf{A}) = \frac{1}{2}\Big(-n \log 2\pi + \log|\mathbf{A}| + \sum_{(i,j)\in\mathcal{E}_d} A(i,j) Y_i Y_j + \sum_{i\in\mathcal{V}} A(i,i) Y_i^2\Big)$$

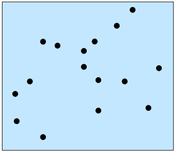
## Acyclic Dependency Graph

Given Covariance matrix, closed-form expression of likelihood

# Hypothesis Testing for Independence



$\mathcal{H}_1$ : GMRF with dep. graph $\mathcal{G}_d$



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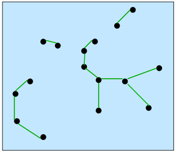

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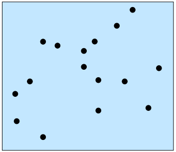

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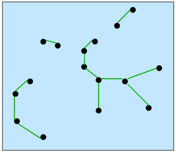

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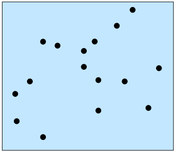

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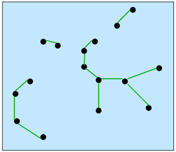

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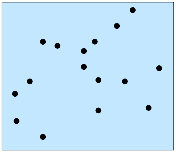

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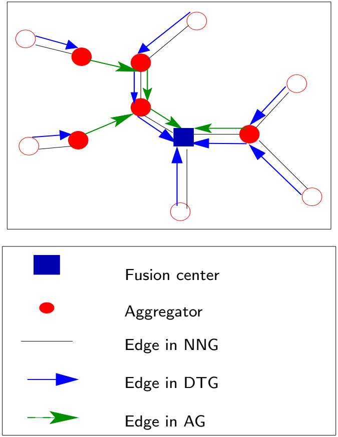
# Outline

# Minimum Energy Routing for Optimal Inference

## Minimum Energy Routing for Inference

Minimize total energy of routing such that LLR is delivered to fusion center




Transmission scheme delivering LLR

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2-Approximation: $\dfrac{\text{C(DFMRF)}}{\text{C}(\mathcal{G}_*)} \leq 2$

## Network and Energy Model

- Connected UDG, Power control
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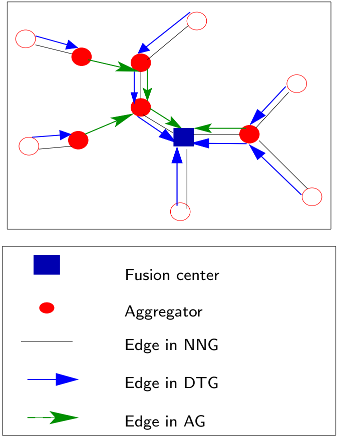


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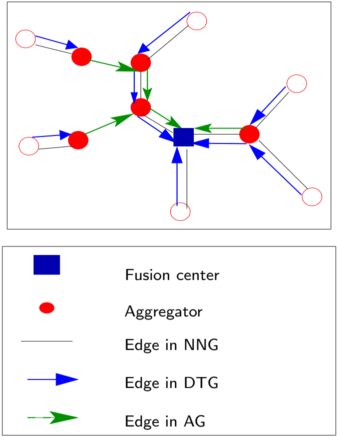


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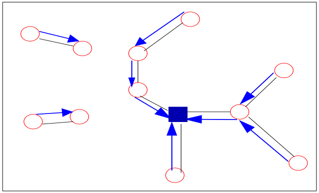

## Raw-data transmission phase

Tx raw data over NNG, compute edge potential locally

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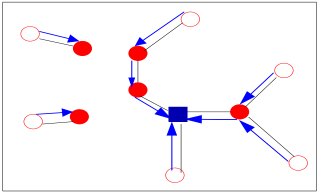

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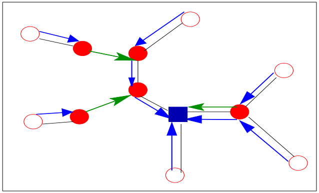

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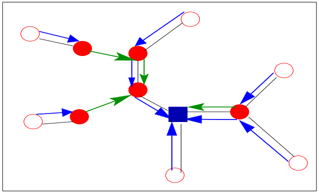

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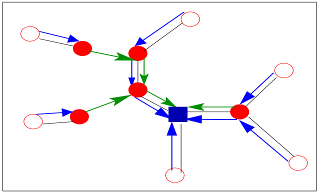

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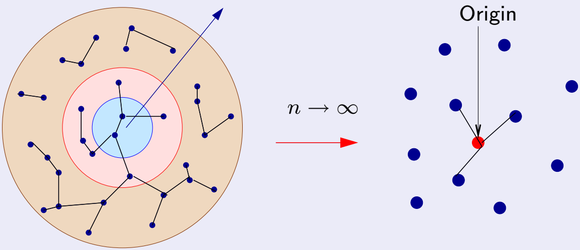
# Outline

# LLN for graph functionals (Penrose & Yukich, 02)

## Pictorial Representation of result




Normalized sum of edge weights

Expectation of edges
of origin of Poisson process

$$\frac{\sum_{e\in\mathcal{E}} \Phi(R_e)}{n} \longrightarrow \mathbb{E}\sum_{\mathbf{x}\in\mathcal{P}_\lambda} \phi(R_{\mathbf{0},\mathbf{x}})$$

## Remarks

LLN states that limit is a localized effect around origin

# Result on Error Exponent $D$

## Use LLN to find error exponent

$$D = \lim_{n\to\infty} \frac{1}{n}\mathsf{LLR}(\mathbf{Y}_n; \mathcal{G}_d) = \lim_{n\to\infty} \frac{1}{n}\Big[\sum_{i\in\mathcal{V}} \Phi_i + \sum_{(i,j)\in\mathcal{E}_d} \Phi_{i,j}\Big] \quad \mathbf{Y}_n \sim \mathcal{H}_0$$

## Closed-form $D$: Correlation + IID terms

$$D(\lambda, K; g) = \tfrac{1}{2}\mathbb{E}_\lambda\, h\big(Z\lambda^{-0.5}, K; g\big) + D_{IID}(K)$$

## Variance Ratio $K$ of Signal Model

$K$ is ratio of mean signal powers under alternative and null hypotheses

## Avg. energy for DFMRF

Tran. + Proc. Energies

$$\bar{\mathsf{C}} = \lambda^{-\frac{\nu}{2}} C_t c_e(\nu) + C_p$$

## Constraint leads to bound on $\lambda$

$$\bar{\mathsf{C}} \le \bar{E} \Rightarrow \lambda \ge \lambda_{\bar{E}} \stackrel{\Delta}{=} \left(\frac{(\bar{E} - C_p)^+}{C_t c_e(\nu)}\right)^{\frac{2}{\nu}}$$

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# Results on Optimal Node Density

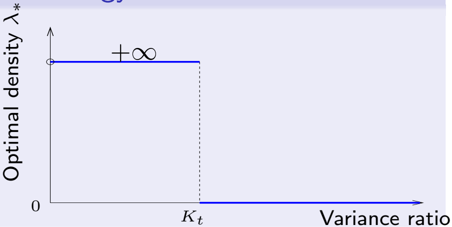
## Modified Optimization

$\lambda_* \stackrel{\Delta}{=} \arg\max_{\lambda>0} D_\lambda$ subject to $\bar{\mathsf{C}} \leq \bar{E}$ becomes $\lambda_* = \arg\max D_\lambda,\ \lambda \geq \lambda_E$

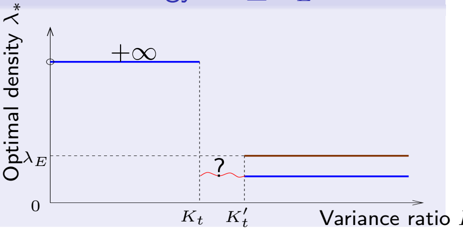
## Thresholds in terms of $M$: correlation at zero

$$K_t(M) = -\frac{1}{\log(1-M^2)}\frac{2M^2}{1-M^2}, \quad K_t'(M) = \frac{2}{1-M^2}$$

## No Energy constraint




## Feasible Energy: $\lambda \geq \lambda_E$



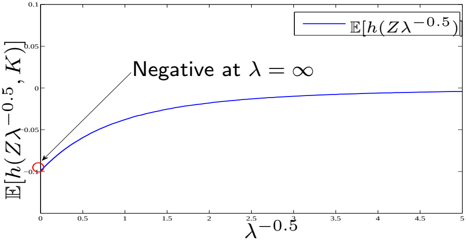
# Idea of Proof: Behavior at $\lambda = \infty$

## Tight Energy Constraint: $\bar{E} \to 0$

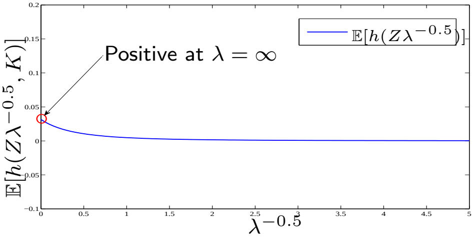
Energy constraint satisfied when $\lambda \to \infty$ and Max. correlation at $\lambda = \infty$

## At $\lambda = \infty$: Contribution from corr. has a threshold

$$\text{Contribution from correlation at } \lambda = \infty \begin{cases} < 0, & \text{for } K > K_t(M) \\ \geq 0, & \text{for } K < K_t(M) \end{cases}$$




Corr. has -ve contribution $K = 2K_t$




Corr. has +ve contribution $K = 2 < K_t$

# Conclusion

## Summary

- Characterized node density $\lambda_*$ that maximizes detection error exponent subject to a average energy constraint
- Measurement variance ratio is crucial
  - Determines whether energy constraint limits detection performance
  - Optimal density displays a threshold behavior
- Derived threshold value analytically and verified it with simulations

## Outlook

- Selection of nodes with "useful" data, node and link failures
- Extend to other dependency models
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Thank You !

# LLR

$$\mathsf{LLR}(\mathbf{Y}_n, \mathcal{V}) \triangleq \log \frac{p[\mathbf{Y}_n, \mathcal{V}; \mathcal{H}_0]}{p[\mathbf{Y}_n, \mathcal{V}; \mathcal{H}_1]} = \log \frac{p[\mathbf{Y}_n; \mathcal{H}_0]}{p[\mathbf{Y}_n | \mathcal{V}; \mathcal{H}_1]},$$

$$= \frac{1}{2}\left(\log \frac{|\mathbf{\Sigma}_{1,\mathcal{V}}|}{|\sigma_0^2 \mathbf{I}|} + \mathbf{Y}_n^T [\mathbf{\Sigma}_{1,\mathcal{V}}^{-1} - (\sigma_0^2 \mathbf{I})^{-1}] \mathbf{Y}_n\right),$$

$$\mathsf{LLR}(\mathbf{Y}_n; \mathcal{G}_d) = \sum_{i \in \mathcal{V}} \phi_i(Y_i) + \sum_{(i,j) \in \mathcal{E}_d} \phi_{i,j}(Y_i, Y_j)$$

$$\phi_{i,j}(i,j) \triangleq \frac{1}{2} \log[1 - g^2(R_{ij})] - \frac{g(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i Y_j}{\sigma_1^2}$$

$$+ \frac{g^2(R_{ij})}{1 - g^2(R_{ij})} \frac{Y_i^2 + Y_j^2}{2\sigma_1^2}$$

$$\phi_i(Y_i) \triangleq \log \frac{\sigma_1}{\sigma_0} + \frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) Y_i^2 \rightarrow D_{IID}(K)$$