Appendix for FCD Load Balancing under Switching Costs and Imperfect Observations

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I. Proof of Proposition 1

Proof: Consider a simple example of $n=2$ servers in a super time slot. At $t=2k$, assume without loss of generality that $X_1(2k)>X_2(2k)$. At the exploration slot $t=2k+1$ with exploration probability γ_{2k}, the expected number of users is thus

$$\begin{aligned}\mathbb{E}\left[X_1(2k+1)\right] &= X_1(2k)(1-\gamma_{2k})+X_2(2k)\gamma_{2k}\\ \mathbb{E}\left[X_2(2k+1)\right] &= X_2(2k)(1-\gamma_{2k})+X_1(2k)\gamma_{2k}.\end{aligned}$$

Under the assumption that $0<\gamma_{2k}<\frac{1}{2}$, it is apparent that

$$\mathbb{E}\left[X_1(2k+1)\right]>\mathbb{E}\left[X_2\left(2k+1\right)\right].$$

On one hand, each user a who switched from server 1 to server 2 experiences lighter load and switches back with probability $f_{21}:=f(X_1(2k),X_2(2k+1),\gamma_{2k})$. On the other hand, each user b who switched from server 2 to server 1 experiences heavier load and hence, switches back with probability 1.

At $t=2k+2$, the expected load in each server is thus

$$\begin{aligned}\mathbb{E}[X_1(2k+2)] &= X_1(2k)[1-\gamma_{2k}(1-f_{21})]+m_2\gamma_{2k}(1-f_{12})\\ \mathbb{E}[X_2(2k+2)] &= X_2(2k)[1-\gamma_{2k}(1-f_{12})]+m_1\gamma_{2k}(1-f_{21})\end{aligned}$$

where $f_{12}=1$.

In order to ensure fast convergence to ϵ-Nash equilibrium, it is intuitive to have the expected loads in the two servers to be balanced after each round of the algorithm, i.e., setting $\mathbb{E}[X_1(2k+2)]=\mathbb{E}[X_2(2k+2)]$, we obtain

$$f_{21}=1-\frac{1}{2\gamma_{2k}}\left(1-\frac{X_2(2k)}{X_1(2k)}\right).$$

Extend to n-server scenario. In order for rapid convergence, the backtracking probability can be achieved by setting load balance of the next time slot. $f_{ji}\left(X_i(2k),X_j(2k)\right)$ is as follows:

$$f_{ji}=\begin{cases}\max\left\{1-\frac{n-1}{n\gamma_{2k}}\left(1-\frac{X_j(2k)}{X_i(2k)}\right),0\right\}, & X_j\left(2k\right)<X_i(2k)\\ 1, & X_j\left(2k\right)\geq X_i\left(2k\right)\end{cases}$$

In the case that the exploration probability meets the condition that $0\leq\gamma_{2k}\leq\frac{n-1}{n}$, $f_{ji}\left(X_i(2k),X_j(2k)\right)$ is

$$f_{ji}=\begin{cases}1, & \mathcal{H}_i^j(2k)\\ 1-\frac{n-1}{n\gamma_{2k}}\left(1-\frac{X_j(2k)}{X_i(2k)}\right), & \mathcal{M}_i^j(2k)\\ 0, & \mathcal{L}_i^j(2k)\end{cases}.$$

While in the case that the exploration probability meets the condition that $\frac{n-1}{n}<\gamma_{2k}<1$, $f_{ji}\left(X_i(2k),X_j(2k)\right)$ is

$$f_{ji}=\begin{cases}1-\frac{n-1}{n\gamma_{2k}}\left(1-\frac{X_j(2k)}{X_i(2k)}\right), & X_j\left(2k\right)<X_i\left(2k\right)\\ 1, & X_j\left(2k\right)\geq X_i\left(2k\right)\end{cases}.$$

In order for $f_{ji}\leq 1$, the exploration probability should be $\gamma_{2k}\geq\frac{(n-1)X_j(2k)}{nX_i(2k)}$, where $X_j(2k)\leq X_i(2k)$.

It is reasonable to assume that $0\leq\gamma_{2k}\leq\frac{n-1}{n}$. □

II. Estimation of $X_j(2k)$

Since

$$\mathbb{E}[X_j(2k+1)]=X_j(2k)(1-\gamma_{2k})+(m-X_j(2k))\,\frac{\gamma_{2k}}{n-1},$$

it is easy to obtain

$$X_j(2k)=\frac{\mathbb{E}[X_j(2k+1)]-\frac{m\gamma_{2k}}{n-1}}{1-\frac{n\gamma_{2k}}{n-1}}.$$

Now the problem is that user population m is not known to each user in distributed systems. However, an asymptotically unbiased estimation $\hat{m}=nX_j(2k+1)$ can be employed to solve this problem.

$$\begin{aligned}\hat{X}_j(2k) &= \left(1-\frac{n\gamma_{2k}}{n-1}\right)\frac{\mathbb{E}[X_j(2k+1)]}{1-\frac{n\gamma_{2k}}{n-1}}\\ \hat{X}_j(2k) &= X_j(2k+1).\end{aligned}$$

III. Proofs of Observations

We need the following observation to estimate the potential function of the system in order to assess the convergence of it.

Observation 1 Expected load on server i under the condition that the last state of the network is x is

$$\mathbb{E}[X_i(2k+2)|X\left(2k\right)=x]\leq\sum_{\substack{l=1\\ l\neq i}}^{n}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(2k)}\frac{2\gamma_{2k}}{n-1}x_l+x_i.$$

Proof:

$$\mathbb{E}[X_i(2k+2)|X\,(2k)=x]=\sum_{l=1}^{n}x_l p_{l,i}\,(x)$$
$$=\sum_{l\in\mathcal{C}_i(t)}x_l\frac{1}{n}\left(1-\frac{x_i}{x_l}\right)+\sum_{l\in\mathcal{D}_i(t)}x_l\frac{\gamma_{2k}}{n-1}$$
$$+x_i\left[1-\sum_{l\in\mathcal{B}_i(t)}\frac{1}{n}\left(1-\frac{x_i}{x_l}\right)-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}\right]$$
$$=\sum_{l\in\mathcal{B}_i(t)\bigcup\mathcal{C}_i(t)}\frac{1}{n}\left(x_l-x_i\right)+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}x_i+x_i$$
$$\leq\sum_{l\in\mathcal{B}_i(t)\bigcup\mathcal{C}_i(t)}\frac{1}{n}\left[x_l-x_l\left(1-\frac{n\gamma_{2k}}{n-1}\right)\right]+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l$$
$$-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}\frac{x_l}{1-\frac{n\gamma 2k}{n-1}}+x_i$$
$$=\sum_{\substack{l=1\\l\neq i}}^{n}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{2\gamma_{2k}}{n-1}x_l+x_i$$

□

Observation 2 Variance of load on server i under the condition that the last state of the network is x is

$$\mathrm{Var}\left[X_i(2k+2)|X\,(2k)=x\right]$$
$$\leq\sum_{l\in\mathcal{B}_i(t)\bigcup\mathcal{C}_i(t)}\frac{\gamma_{2k}}{n-1}x_l+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}x_l+x_i$$

Proof:

$$\mathrm{Var}\left[X_i(2k+2)|X\,(2k)=x\right]=\sum_{l=1}^{n}x_l p_{li}\,(x)\left(1-p_{li}\,(x)\right)$$
$$\leq\sum_{l\in\mathcal{C}_i(t)}\frac{1}{n}\left(x_l-x_i\right)+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l$$
$$+x_i\left[1-\sum_{l\in\mathcal{B}_i(t)}\frac{1}{n}\left(1-\frac{x_l}{x_i}\right)-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}\right]$$
$$\leq\sum_{l\in\mathcal{B}_i(t)\bigcup\mathcal{C}_i(t)}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}\frac{x_l}{1-\frac{n\gamma_{2k}}{n-1}}+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l$$
$$+x_i$$
$$\leq\sum_{l\in\mathcal{B}_i(t)\bigcup\mathcal{C}_i(t)}\frac{\gamma_{2k}}{n-1}x_l+\sum_{l\in\mathcal{D}_i(t)}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{\gamma_{2k}}{n-1}x_l+x_i$$

□

IV. Proof of Proposition 3

Proof: Since

$$\mathbb{E}[\Phi(X(2k+2))|X(2k)]+n\overline{X}(2k)^2$$
$$\leq\sum_{i=1}^{n}(\mathbb{E}[X_i(2k+2)|X(2k)])^2+\sum_{i=1}^{n}\mathrm{Var}[X_i(2k+2)|X(2k)]$$

we can derive the structure of $\mathbb{E}[\Phi(X(2k+2))|X(2k)=x]$ as follows:

$$\mathbb{E}[\Phi(X(2k+2))|X(2k)=x]+n\bar{x}^2$$
$$\leq\sum_{i=1}^{n}\left(\sum_{\substack{l=1\\l\neq i}}^{n}\frac{\gamma_{2k}}{n-1}x_l+\left(1-\frac{\gamma_{2k}}{n-1}\right)x_i-\sum_{l\in\mathcal{A}_i(t)}\frac{2\gamma_{2k}}{n-1}x_l\right)^2$$
$$+\sum_{i=1}^{n}\left\{\sum_{l=1}^{n}\frac{\gamma_{2k}}{n-1}x_l-\sum_{l\in\mathcal{A}_i(t)}\frac{2\gamma_{2k}}{n-1}x_l\right\}+n\overline{x}$$

Now, focus on estimating these three terms: $\sum_{i=1}^{n}\left(\sum_{l\in\mathcal{A}_i(t)}x_l\right)^2$, $\sum_{i=1}^{n}\left(\sum_{l\in\mathcal{A}_i(t)}x_l\right)$ and $\sum_{i=1}^{n}\left(x_i\sum_{l\in\mathcal{A}_i(t)}x_l\right)$.

The upper bound of these three items are as follows:

$$\begin{cases}(1)\ \sum_{i=1}^{n}\left(\sum_{l:x_l\leq x_i\left(1-\frac{n\gamma_{2k}}{n-1}\right)}x_l\right)^2\leq\frac{1}{4}\left(1-\frac{n\gamma_{2k}}{n-1}\right)^2n^2\sum_{i=1}^{n}x_i^2\\(2)\ \sum_{i=1}^{n}\left(\sum_{l:x_l\leq x_i\left(1-\frac{n\gamma_{2k}}{n-1}\right)}x_l\right)\geq n\left(1-\frac{n\gamma_{2k}}{n-1}\right)\overline{x}\\(3)\ \sum_{i=1}^{n}\left(x_i\sum_{l:x_l\leq x_i(1-\frac{n\gamma 2k}{n-1})}x_l\right)\geq\left(1-\frac{n\gamma_{2k}}{n-1}\right)\sum_{i=1}^{n}x_i^2\end{cases}$$

Note that even for $\gamma_{2k}(a)$ being different for different users a, these inequalities still hold given $\gamma_{2k}=\frac{1}{m}\sum_{a=1}^{m}\gamma_{2k}(a)$ and $\min_a\gamma_{2k}(a)\approx\max_a\gamma_{2k}(a)$.

$$\Rightarrow\mathbb{E}[\Phi(X(2k+2))|X(2k)=x]$$
$$\leq\left[\left(\frac{n-1}{\gamma_{2k}}-1\right)^2+\left(1-\frac{n\gamma_{2k}}{n-1}\right)^2n^2-4\left(\frac{n-1}{\gamma_{2k}}-1\right)\left(1-\frac{n\gamma_{2k}}{n-1}\right)\right]$$
$$\left(\frac{\gamma_{2k}}{n-1}\right)^2\sum_{i=1}^{n}x_i^2$$
$$+\left(\frac{\gamma_{2k}}{n-1}\right)^2\left[n^3-2n^2+2n^2\frac{n-1}{\gamma_{2k}}-4n^2(1-\frac{n\gamma_{2k}}{n-1})-\left(\frac{n-1}{\gamma_{2k}}\right)^2n\right]\overline{x}^2$$
$$+\left[\frac{\gamma_{2k}n^2}{n-1}-\frac{2n\gamma_{2k}}{n-1}\left(1-\frac{n\gamma_{2k}}{n-1}\right)+n-\frac{\gamma_{2k}n}{n-1}\right]\overline{x}\tag{1}$$

Now we focus on estimating the upper bound of $\mathbb{E}[\Phi(X(2k+2)|X(2k)=x)]$. We can incorporate $\Phi(X(2k))$

into equation (1).

$$\Rightarrow \mathbb{E}[\Phi(X(2k+2))|X(2k)=x]$$

$$\begin{aligned}
&\leq \left(\tfrac{\gamma_{2k}}{n-1}\right)^2 \left[\left(\tfrac{n-1}{\gamma_{2k}}-1\right)^2+\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)^2 n^2 \right.\\
&\quad \left. -4\left(\tfrac{n-1}{\gamma_{2k}}-1\right)\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)\right]\Phi(X(2k)=x)\\
&\quad +\left[\left(\tfrac{n-1}{\gamma_{2k}}-1\right)^2+\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)^2 n^2-4\left(\tfrac{n-1}{\gamma_{2k}}-1\right)\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)\right.\\
&\quad \left. +n^2-2n+2n\tfrac{n-1}{\gamma_{2k}}-4n+4n^2\tfrac{\gamma_{2k}}{n-1}-\left(\tfrac{n-1}{\gamma_{2k}}\right)^2\right]\left(\tfrac{\gamma_{2k}}{n-1}\right)^2 n\overline{x}^2\\
&\quad +\left[\tfrac{\gamma_{2k}n^2}{n-1}-\tfrac{2n\gamma_{2k}}{n-1}\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)+n-\tfrac{\gamma_{2k}n}{n-1}\right]\overline{x}
\end{aligned} \tag{2}$$

When $\Phi\left(X(2k)\right)$ small,the dominant terms in equation (2) are the second and the third ones, the system is already in converged state. However, when $\Phi\left(X(2k)\right)$ grows, the dominant term is

$$\begin{aligned}
\mathbb{E}[\Phi(X(2k+2))] \leq \left(\tfrac{\gamma_{2k}}{n-1}\right)^2 &\left[\left(\tfrac{n-1}{\gamma_{2k}}-1\right)^2+\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)^2 n^2\right.\\
&\left.-4\left(\tfrac{n-1}{\gamma_{2k}}-1\right)\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)\right]\Phi(X(2k)).
\end{aligned}$$

Note $\gamma_{2k} = \frac{1}{m}\sum_{a=1}^{m}\gamma_{2k}(a)$ if $\min_a \gamma_{2k}(a) \approx \max_a \gamma_{2k}(a)$. In other words, proposition holds even for users with different $\gamma_{2k}(a)$'s as long as they perturbations are small. $\square$

V. Proof of Lemma 1

Proof:

1) First let us derive a sufficient region of β in which convergence is feasible. $\gamma_t = \frac{1}{m}\sum_{a=1}^{m}\gamma_t(a)$. Denote $\mathfrak{Q}(t) = \left[\left(\frac{n-1}{\gamma_t}-1\right)^2+\left(1-\frac{n\gamma_t}{n-1}\right)^2 n^2 -4\left(\frac{n-1}{\gamma_t}-1\right)\left(1-\frac{n\gamma_t}{n-1}\right)\right]\left(\frac{\gamma_t}{n-1}\right)^2$. Since n is a large enough number, and $\gamma_t = t^{-\beta}$,

$$\begin{aligned}\mathfrak{Q}(t) = {}& 1-\tfrac{6}{n}t^{-\beta}+\left(\tfrac{5}{n^2}+\tfrac{4}{n}+1\right)t^{-2\beta}\\ &-\left(\tfrac{4}{n^2}+2\right)t^{-3\beta}+t^{-4\beta}.\end{aligned}$$

With the fact that $\mathbb{E}[\Phi(X(2t+2))] \leq \mathfrak{Q}(t)\mathbb{E}[\Phi(X(2t))]$,

$$\begin{aligned}
&\mathbb{E}[\Phi(X(2t))]\\
&\leq \mathfrak{Q}(t)\mathbb{E}[\Phi(X(2t-2))]\\
&\leq \prod_{\tau=1}^{t}\left[1+\tau^{-2\beta}(5-4\tau^{-\beta})(\tfrac{1}{n})^2-\tau^{-\beta}(6-4\tau^{-\beta})\tfrac{1}{n}\right.\\
&\quad\left.+\tau^{-2\beta}(1-\tau^{-\beta})^2\right]\mathbb{E}[\Phi(X(0))]
\end{aligned}$$

The sufficient region of β should satisfy

$$0<\mathfrak{Q}(t)<1 \text{ or } \log\mathfrak{Q}(t)<0 \tag{3}$$

According to [1, Theorem 3.2]: the following results hold:

a) For $\beta=0$: recall from Proposition 1 that $0\leq\gamma_t\leq\frac{n-1}{n}$, $\beta=0$ is not feasible.
b) For $\beta=1,\frac{1}{2},\frac{1}{3}$, or $\frac{1}{4}$: $\mathfrak{Q}\leq 1$ holds.
c) For $\beta>0$ and $\beta\neq 1,\frac{1}{2},\frac{1}{3},\frac{1}{4}$: feasible values of β satisfy Equation (4).

$$\begin{aligned}
&\log\prod_{\tau=1}^{t}\mathfrak{Q}(\tau)\\
&\leq t\log\left\{1+\left(\tfrac{5}{n^2}+\tfrac{4}{n}+1\right)\left[\tfrac{t^{-2\beta}}{1-2\beta}+\tfrac{\zeta(2\beta)}{t}+O(t^{-2\beta-1})\right]\right.\\
&\quad+\left[\tfrac{t^{-4\beta}}{1-4\beta}+\tfrac{\zeta(4\beta)}{t}+O(t^{-4\beta-1})\right]\\
&\quad-\tfrac{6}{n}\left[\tfrac{t^{-\beta}}{1-\beta}+\tfrac{\zeta(\beta)}{t}+O(t^{-\beta-1})\right]\\
&\quad\left.-\left(\tfrac{4}{n^2}+2\right)\left[\tfrac{t^{-3\beta}}{1-3\beta}+\tfrac{\zeta(3\beta)}{t}+O(t^{-3\beta-1})\right]\right\}\\
&\leq 0
\end{aligned}\tag{4}$$

are difficult to obtain. Still we can derive some sufficient regions where β satisfies Equation (3). The dominant terms are $\frac{t^{-2\beta}}{1-2\beta}-2\frac{t^{-3\beta}}{1-3\beta}+\frac{t^{-4\beta}}{1-4\beta}$, we can see that if $-1<1-2\beta<0$, then the decaying condition satisfies. Which means that $\frac{1}{2}<\beta<1$ is a sufficient region for convergence.

From (a), (b) and (c), a sufficient condition on β for convergence is $\beta\in[\frac{1}{2},1]$.

2) Now, under the condition that the decaying condition on β (Equation (3))is satisfied, the following result holds according to Jensen's inequality.

$$\log\prod_{\tau=1}^{t}\mathfrak{Q}(\tau)t\log\left[1-\tfrac{1}{n}+\tfrac{0.5}{n^2}-\tfrac{3}{n}t^{-\beta}+\tfrac{t^{-\beta}}{2}\left(1-t^{-\beta}\right)^2\right].$$

That means, $\log\prod_{\tau=1}^{t}\mathfrak{Q}(\tau)\leq\left(t\log\left(1-\frac{1}{n}\right)\right)$ holds under that condition that $t>\left(\frac{n}{2}\right)^{\frac{1}{\beta}}$.

So,

$$\mathbb{E}[\Phi(X(2t))]\leq(1-\frac{1}{n})^t\mathbb{E}[\Phi(X(0)].$$

holds at least under the condition that $\frac{1}{2}\leq\beta\leq 1$ and $t>\left(\frac{n}{2}\right)^{\frac{1}{\beta}}$.

Additionally, the fact that $\beta\in[0.5,1]$ leads to system convergence implies that it's subset $\beta(a)\in[0.5,1]$ leads to system convergence as well. In a nutshell, $t^{-1}\leq\min_a\gamma_t(a)\leq\max_a\gamma_t(a)\leq t^{-0.5}$ is a sufficient condition for the potential function shrinking with time. $\square$

VI. Proof for Lemma 3

Proof: Denote Λ as the event of large perturbation, i.e.,

$$\Lambda=\bigcup_{ij}\left|p_{ij}\left(X_i(2k),X_j(2k+1)\right)-p_{ij}\left(X_i(2k),X_j(2k)\right)\right|>\delta,$$

and the event of small perturbation is

$$\Lambda^C=\bigcap_{ij}\left|p_{ij}\left(X_i(2k),X_j(2k+1)\right)-p_{ij}\left(X_i(2k),X_j(2k)\right)\right|\leq\delta.$$

According to basic theory of probability, the following formula holds:

$$\begin{aligned}&\mathbb{E}\left[\widehat{\Phi}(X(2k+2))|X(2k)\right]\\&=\mathbb{E}\left[\widehat{\Phi}(X(2k+2))|X(2k),\Lambda\right]P(\Lambda)\\&\quad+\mathbb{E}\left[\widehat{\Phi}(X(2k+2))|X(2k),\Lambda^C\right]P(\Lambda^C)\end{aligned}.$$

In the case of small perturbation Λ^C,

$$\begin{aligned}&\left|\mathbb{E}\left[\widehat{\Phi}(X(2k+2))|X(2k)=x\right]-\mathbb{E}\left[\Phi(X(2k+2))|X(2k)=x\right]\right|\\&+\mathbb{E}\left[\Phi(X(2k+2))|X(2k)=x\right]\\&\le\left|\left(2\gamma_{2k}\tfrac{n^3}{n-1}-\gamma_{2k}\tfrac{n^2}{n-1}\right)\overline{x}\delta+n^2\delta+n^2\delta^2\right|\\&\quad+\mathbb{E}\left[\Phi(X(2k+2))|X(2k)\right]\end{aligned}\tag{5}$$

The dominant term in equation (5) is still $\mathbb{E}\left[\Phi(X(2k+2))\right.$ $\left.|X(2k)\right]$ under the condition that $\delta\le\sqrt{\frac{\gamma_{2k}m^2}{n^3}}$.

In other words,

$$\begin{aligned}&\left|\mathbb{E}(\widehat{\Phi}(X(2k+2)))-\mathbb{E}(\Phi(X(2k+2)))\right|+\mathbb{E}[\Phi(X(2k+2))]\\&\le\left(\tfrac{\gamma_{2k}}{n-1}\right)^2\left[\left(\tfrac{n-1}{\gamma_{2k}}-1\right)^2+\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)^2n^2\right.\\&\quad\left.-4\left(\tfrac{n-1}{\gamma_{2k}}-1\right)\left(1-\tfrac{n\gamma_{2k}}{n-1}\right)\right]\Phi(X(2k))\end{aligned}$$

still holds under the condition that $\delta\le\sqrt{\frac{\gamma_{2k}m^2}{n^3}}$.

According to lemma 1 and 2, an upper bound of expected convergence time is achieved. The concentration bound also holds as well:

$$\mathbb{E}[T]\le n\log\frac{4n^2}{\epsilon^2}+\left(\frac{n}{2}\right)^{\frac{1}{\beta}}\tag{6}$$

$$\Pr\left\{\left|\Phi\left(X(T)\right)-\epsilon^2\frac{m^2}{4n^2}\right|\ge m\theta\right\}\le e^{-2\theta^2}\tag{7}$$

in the sufficient region of $\beta\in[\frac{1}{2},1]$ and $m>n^2$.

In case of large perturbation Λ, we need δ to be higher in order to reduce the probability of bad event $P(\Lambda)$. When $\delta=\sqrt{\frac{\gamma_{2k}m^2}{n^3}}$,

$$\mathbb{E}\left[\widehat{\Phi}\left(X(2k+2)\right)|X(2k)=x,\Lambda\right]P(\Lambda)\le\tfrac{\epsilon^2m^2}{4n^2}$$

is satisfied if

$$t\le\left(\frac{n^3}{m^3}\log\frac{4n^4}{\epsilon^2}\right)^{\frac{1}{\beta}}.\tag{8}$$

□

VII. Proof of Proposition 5

Proof: Thanks to Chernoff-Hoeffding bound, we know that

$$\Pr\left(\frac{|X_j(2k)-\mathbb{E}[X_j(2k)]|}{M(2k)}>\epsilon_1\right)\le e^{-2M(2k)\epsilon_1^2}\tag{9}$$

This means that, $\mathbb{E}[X_j(2k)]$ can accurately represent real $X_j(2k)$ with a high probability.

Also according to Equation (9) ,

$$\begin{aligned}&\Pr\left\{\left|p_{ij}^F\left(X_i^F(2k),X_j^F(2k+1)\right)\right.\right.\\&\left.\left.-\mathbb{E}\left[p_{ij}^F\left(X_i^F(2k),X_j^F(2k+1)\right)\right]\right|>\delta\right\}\\&=\Pr\left\{\tfrac{|X_j(2k)-\mathbb{E}[X_j(2k)]|}{M(2k)}>\delta\tfrac{X_i^F(2k)n}{M(2k)}\right\}\\&\le e^{-2M(2k)\delta^2}.\end{aligned}\tag{10}$$

Note that in equation (10), after 1 round of exploration and backtracking, there won't be sparsity in any of the server loads, thus the equation is achieved.

If concentration holds, the following result holds as well:

$$\begin{aligned}&\Pr\left\{\left|p_{ij}^F\left(X_i^F(2k),X_j^F(2k+1)\right)-p_{ij}\left(X_i(2k),X_j(2k+1)\right)\right|>\delta\right\}\\&\le\Pr\left\{\left|p_{ij}-\mathbb{E}[p_{ij}^F]\right|>\delta\right\}+\Pr\left\{\left|\mathbb{E}[p_{ij}^F]-p_{ij}^F\right|>\delta\right\}\\&\le\Pr\left\{\left|p_{ij}-\mathbb{E}[p_{ij}^F]\right|>\delta\right\}+e^{-2M(2k)\delta^2}\end{aligned}.\tag{11}$$

Furthermore,

$$\begin{aligned}&\Pr\left\{\left|p_{ij}-\mathbb{E}[p_{ij}^F]\right|>\delta\right\}\\&\overset{(a)}{=}\Pr\left\{\frac{1}{n}\left|\frac{X_j(2k+1)}{X_i(2k)}-\right.\right.\\&\left.\left.\frac{X_j(2k+1)+Z_j(2k)(1-\gamma_{2k})+(M(t)-Z_j(2k))\frac{\gamma_{2k}}{n-1}}{X_i(2k)+Z_i(2k)}\right|>\delta\right\}\\&\overset{(b)}{=}\Pr\left\{\frac{1}{n}\left|\frac{X_j(2k+1)+Z_j(2k)+\frac{m\gamma_{2k}}{n-1}}{X_i(2k)+Z_i(2k)}-\frac{X_j(2k+1)}{X_i(2k)}\right|>\delta\right\}\\&\le\Pr\left\{\left|\frac{Z_j(2k)+\frac{m\gamma_{2k}}{n-1}}{X_i(2k)}\right|>n\delta\right\}.\end{aligned}\tag{12}$$

(a) holds because of the system model analysis; (b) is achieved by utilizing the asymptotic unbiased estimation of $\widehat{M}(2k)=m+nZ_j(2k)$. Equation (11) and (12) lead to the union bound of transition probability perturbation.

$$\begin{aligned}&\Pr\left\{\bigcup_{ij}\left|p_{ij}^F(X_i^F(2k),X_j^F(2k+1))-p_{ij}(X_i(2k),X_j(2k+1))\right|>\delta\right\}\\&\le\Pr\left\{\bigcup_{ij}\left|\frac{Z_j(2k)+\frac{m\gamma_2k}{n-1}}{X_i(2k)}\right|>n\delta\right\}+n^2e^{-2M(2k)\delta^2}\end{aligned}$$

The proposition is proved. □

VIII. Proof of Lemma 4

Proof: Denote Λ as the event of large perturbation, i.e.,

$$\begin{aligned}\Lambda=&\\&\left\{\bigcup_{ij}|p_{ij}^F\left(X_i(2k),X_j(2k+1)\right)-p_{ij}\left(X_i(2k),X_j(2k+1)\right)|>\delta\right\}\\&\bigcap\left\{M(t)\ge\frac{\epsilon m}{n}\right\},\end{aligned}$$

and the event of small perturbation is

$$\Lambda^C=$$

$$\left\{\bigcap_{ij}|p_{ij}^F\left(X_i(2k),X_j(2k+1)\right)-p_{ij}\left(X_i(2k),X_j(2k+1)\right)|\le\delta\right\}$$
$$\bigcup\left\{M(t)<\frac{\epsilon m}{n}\right\}.$$

According to basic theory of probability, the following formula holds:

$$\begin{aligned}&\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k)\right]\\&=\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k),\Lambda\right]P(\Lambda)\\&+\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k),\Lambda^C\right]P(\Lambda^C).\end{aligned}$$

In case of small perturbation Λ^C, it is obvious that the dominant term in $\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k)\right]$ is still $\mathbb{E}\left[\Phi(X(2k+2))|X(2k)\right]$ as long as $\delta<\sqrt{\frac{\gamma_{2k}m^2}{n^3}}$. Thus lemma 1 and 2 are satisfied.

In case of large perturbation Λ, we need δ to be larger in order to reduce the probability of bad event $P(\Lambda)$. When $\delta=\sqrt{\frac{\gamma_{2k}m^2}{n^3}}$,

$$\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k)=x,\Lambda\right]P(\Lambda)\le\frac{\epsilon^2\max\left\{m,M(2k)\right\}^2}{4n^2}$$

is required for ϵ-Nash equilibrium.

It is known that $\mathbb{E}\left[\widehat{\Phi}(X^F(2k+2))|X(2k)=x,\Lambda\right]P(\Lambda)\le M(2k)^2P(\Lambda)$. As a result, if we could restrict on $Z_i(2k),i\in\{1,2,\dots,n\}$ such that

$$\begin{aligned}P(\Lambda)&\le\Pr\left\{\bigcup_{ij}\left|\frac{Z_j(2k)+\frac{m\gamma_{2k}}{n-1}}{X_i(2k)}\right|>n\delta\right\}+n^2e^{-2M(2k)\delta^2}\\&\le\Pr\left\{\bigcup_{ij}\left|\frac{Z_j(2k)+\frac{m\gamma_{2k}}{n-1}}{X_i(2k)}\right|>n\delta\right\}+n^2e^{-2\epsilon\frac{m}{n}\delta^2}\\&\le\frac{\epsilon^2}{4n^2},\end{aligned}$$

then ϵ-Nash equilibrium is achieved.

So, under the conditions that

$$\Pr\left\{\left|M(2k)-(1-\frac{\gamma_{2k}}{n-1})m\right|>n^{-\frac{1}{2}}m^2\sqrt{\gamma_{2k}}\right\}\le\frac{\epsilon^2}{4n^4},\tag{13}$$

and

$$t\le\left(\frac{n^4}{\epsilon m^3}\log\frac{4n^4}{\epsilon^2}\right)^{\frac{1}{\beta}}\tag{14}$$

an ϵ-Nash equilibrium is obtained.

Equation (13) is due to $\Pr\left\{\bigcup_{ij}\left|\frac{Z_j(2k)+\frac{m\gamma_{2k}}{n-1}}{X_i(2k)}\right|>n\delta\right\}\le\frac{\epsilon^2}{4n^2}$, while equation (14) is achieved by imposing $n^2e^{-2\epsilon\frac{m}{n}\delta^2}\le\frac{\epsilon^2}{4n^2}$.

□

References

[1] T. M. Apostol. *Introduction to Analytic Number Theory*. 2004.