Learning Tractable Graphical Models: Latent Trees

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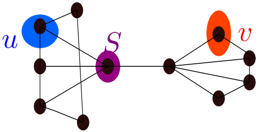
Joint work with Anima Anandkumar and U.N. Niranjan.

High-Dimensional Graphical Modeling

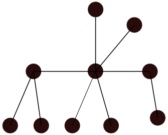
Modeling Conditional Independencies through Graphs

- $X_u \perp\!\!\!\perp X_v | X_S$.
- Learning and inference are NP-hard.



Tractable Models: Tree Models

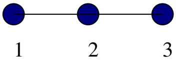
- Efficient inference using belief propagation



Walk-up: Learning Tree Models

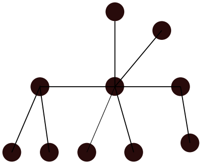
Data processing inequality for Markov chains

$$I(X_1; X_3) \le I(X_1; X_2), I(X_2; X_3).$$



Tree Structure Estimation (Chow and Liu '68)

- MLE: Max-weight tree with estimated mutual information weights



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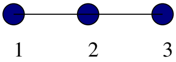

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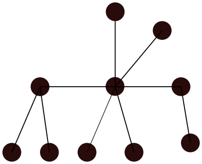
- Pairwise statistics suffice



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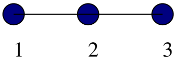

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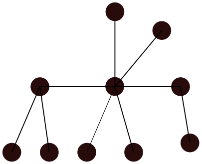
- Pairwise statistics suffice
- n samples and p nodes

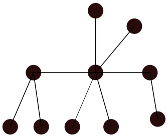


Sample complexity: $\dfrac{\log p}{n} = O(1).$

Learning Tractable Graphical Models

Tractable Models: Tree Models

- Efficient inference using belief propagation
- MLE is easy to compute.
- Tree models are highly restrictive.



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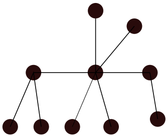

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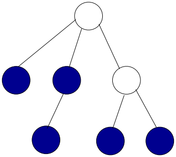
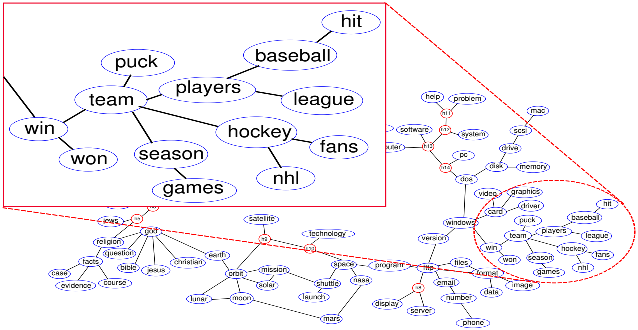
Latent tree graphical models



- Tree models with hidden variables.
- Number and location of hidden variables unknown.

Application: Hierarchical Topic Modeling

- Data: Word co-occurrences.
- Graph: Topic-word structure.



Application of Latent Trees: Object Recognition

- Challenge: Succinct representation of large-scale data
  - Input: ~ 100 object categories, ~ 4000 training images
  - Goal: learn $\sim 2^{100}$ co-occurrence probabilities
- Solution: Latent tree graphical models

"Context Models and Out-of-context Objects," M. J. Choi, A. Torralba, and A. S. Willsky, Pattern Recognition Letters, 2012.

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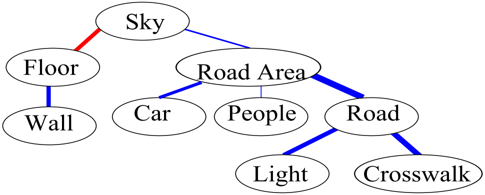

In this talk: learning latent tree models and tree mixtures.

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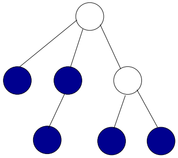
Summary of Results

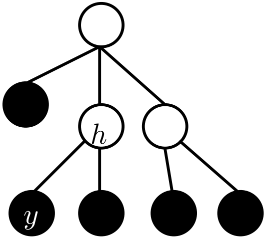
Latent Tree Models

- Number of hidden variables and location unknown
- Integrated structure and parameter estimation.
- Local learning with global consistency guarantees.



Outline

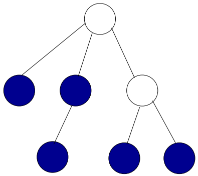
Learning Latent Tree Graphical Models



Linear Multivariate Models

- Conditional independence w.r.t tree
- Categorical k-state hidden variables.
- Multivariate d-dimensional observed variables. $k \leq d$.
- When y is nbr. of h, $\mathbb{E}[y|h] = Ah$.
- Includes discrete, Poisson and Gaussian models, Gaussian mixtures etc.

Additive Tree Distances



Information Distances $[d_{i,j}]$ for Tree Models

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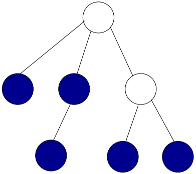

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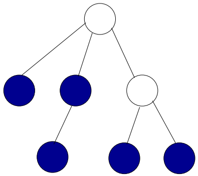

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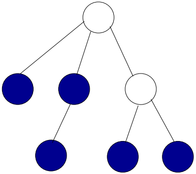

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$[d_{i,j}]$ is an additive tree metric: $\boxed{d_{k,l} = \sum_{(i,j) \in \mathrm{Path}(k,l;E)} d_{i,j}.}$

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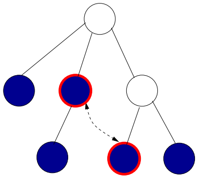

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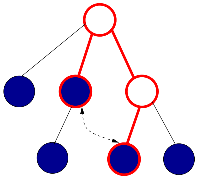

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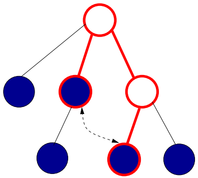

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Learning latent tree using $[\hat{d}_{i,j}]$

Siblings Test Based on Information Distances

Exact Statistics: Distances $[d_{i,j}]$

Let $\Phi_{ijk} := d_{i,k} - d_{j,k}$.

- $-d_{i,j} < \Phi_{ijk} = \Phi_{ijk'} < d_{i,j}\ \forall\, k, k' \neq i, j,\ \iff\ i, j$ leaves with common parent
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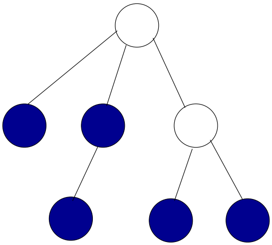

Sample Statistics: ML Estimates $[\hat{d}_{i,j}]$

Use only short distances: $d_{i,k}, d_{j,k} < \tau$, Relax equality relationships

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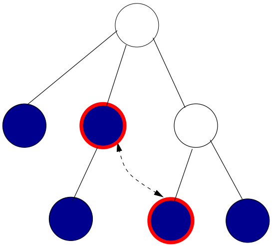

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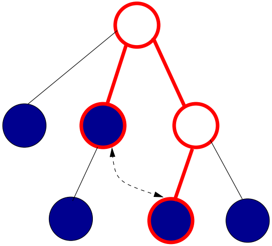

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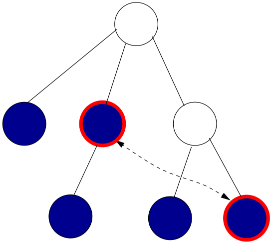

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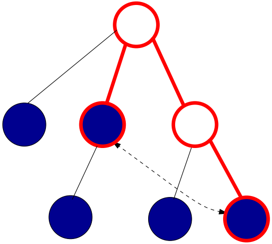

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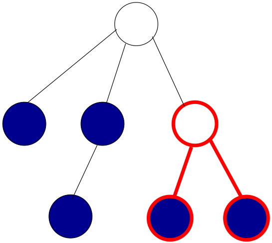

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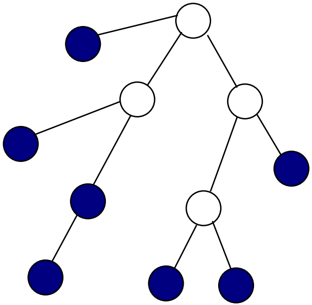
Recursive Grouping

Recursive Grouping Algorithm (Choi, Tan, Anandkumar, Willsky)

- Sibling test and remove leaves
- Build tree from bottom up



- Consistent structure estimation.
- Serial method, high computational complexity.

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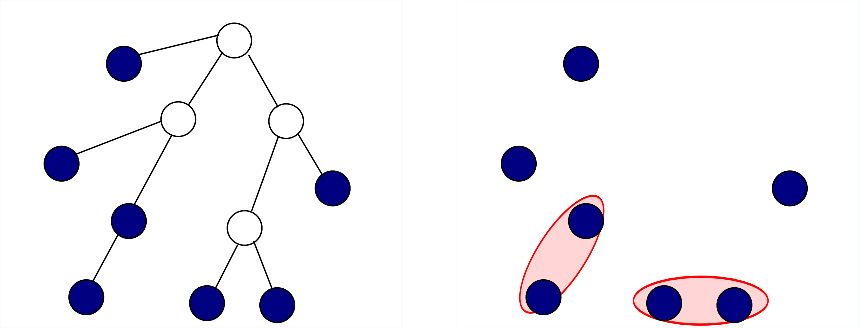

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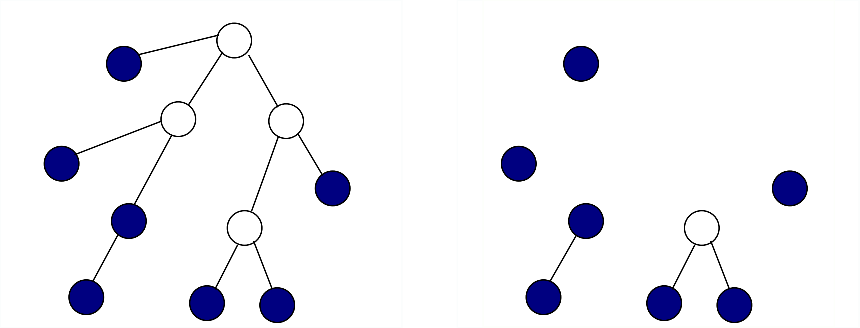

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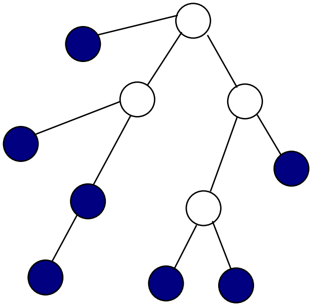

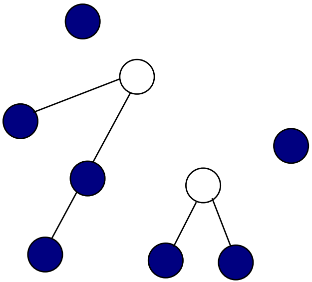

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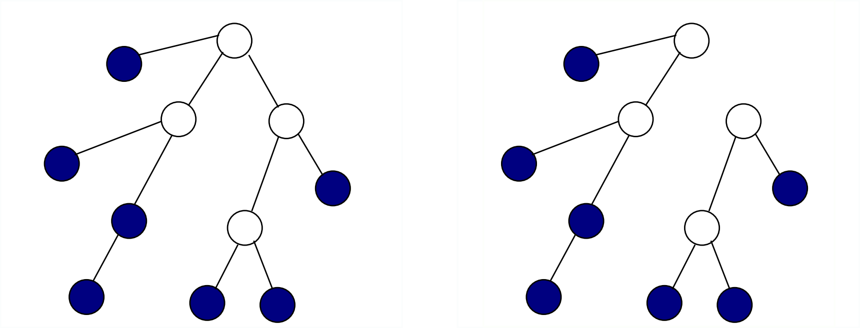

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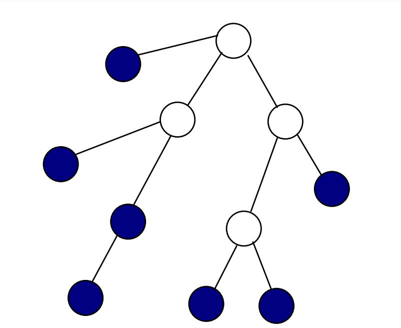

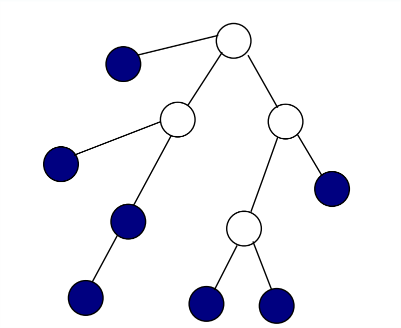

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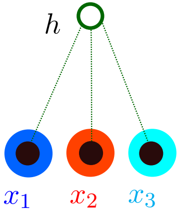
Outline

Overview of Proposed Parameter Learning Method

Toy Model: 3-star

- Linear multivariate model.
- $A^r_{x_i|h} := \mathbb{E}(x_i|h = e_r)$. and $\lambda_r := \mathbb{P}[h = e_r]$.



$$\boxed{\mathbb{E}(x_1 \otimes x_2 \otimes x_3) = \textstyle\sum_{r=1}^{k} \lambda_r A^r_{x_1|h} \otimes A^r_{x_2|h} \otimes A^r_{x_3|h}.}$$

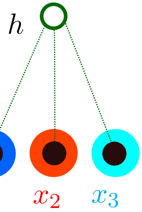
Guaranteed Recovery through Tensor Decomposition

- Transition matrices $A_{x_i|h}$ have full column rank.
- Linear algebraic operations: SVD and tensor power iterations.

"Tensor Decompositions for Learning Latent Variable Models" by A. Anandkumar, R. Ge, D. Hsu, S.M. Kakade and M. Telgarsky. Preprint, October 2012.

Overview of Tensor Decomposition Technique

- Let $a_r = \mathbb{E}(x_i|h = e_r)$ for all i and $\lambda_r := \mathbb{P}[h = e_r]$.
- $M_3 = \mathbb{E}[x_1 \otimes x_2 \otimes x_3] = \sum_{i=1}^{k} \lambda_i a_i^{\otimes 3}$.



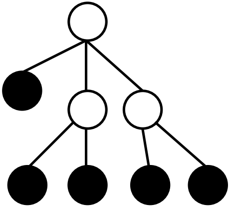
Intuition: if a_i are orthogonal

- $M_3(I, a_1, a_1) := \sum_i \lambda_i \langle a_i, a_1 \rangle^2 a_i = \lambda_1 a_1$.
- a_i are eigenvectors of the tensor M_3.

Convert to an orthogonal tensor using pairwise moments

- $M_2 := \mathbb{E}[x_1 \otimes x_2] = \sum_i \lambda_i a_i^{\otimes 2}$.
- Whitening matrix: $W^\top M_2 W = I$.
- Consider tensor $M_3(W, W, W) := \sum_i \lambda_i (W^\top a_i)^{\otimes 3}$. It is an orthogonal tensor.

Parameter Learning in Latent Trees



Learning through Hierarchical Tensor Decomposition

- Assume known tree structure.
- Decompose different triplets: hidden variable is join point on tree.

Alignment issue

- Tensor decomposition is an unsupervised method.
- Hidden labels permuted across different triplets.
- Solution: Align using common node in triplets.

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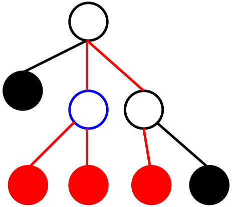

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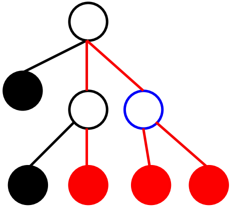
- Solution: Align using common node in triplets.

Parameter Learning in Latent Trees



Learning through Hierarchical Tensor Decomposition

- Assume known tree structure.
- Decompose different triplets: hidden variable is join point on tree.

Alignment issue

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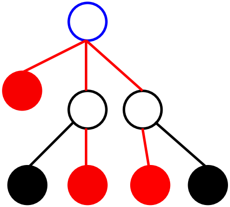

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Outline

Integrated Learning

So far..

- Consistent structure learning through sibling tests on distances.
- Parameter learning through tensor decomposition on triplets.

Challenges

- How to integrate structure and parameter learning?
- Can we save on computations through integration?
- Can we learn parameters as we learn the structure?
- Can we parallelize learning for scalability?

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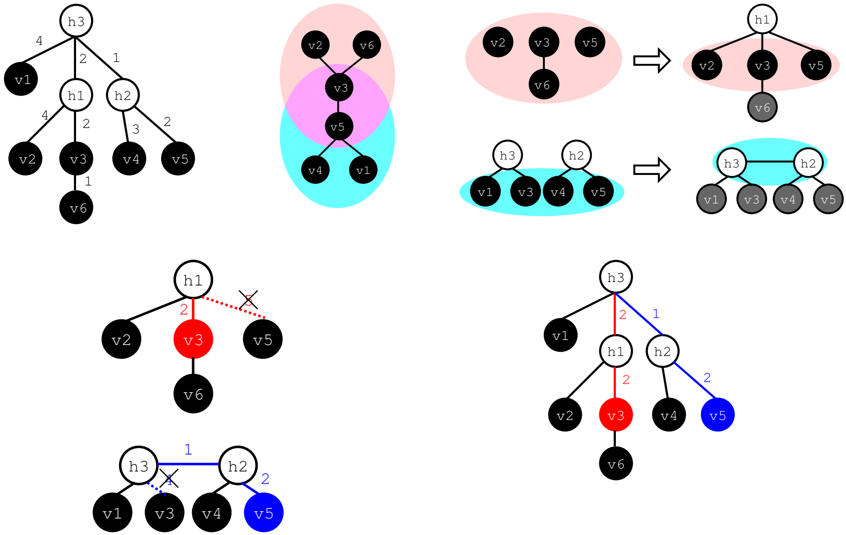
Key Ideas

- Divide and conquer: find (overlapping) groups of observed variables.
- Learn local subtrees (and parameters) over the groups independently.
- Merge subtrees and tweak parameters to obtain global latent tree model.

Parallel Chow-Liu Based Grouping Algorithm

Minimum spanning tree using information distance $[\hat{d}_{i,j}]$.

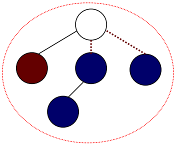


Alignment of Parameters

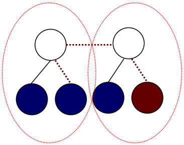
Alignment Correction

- In-group
- Across-group
- Across-neighborhood

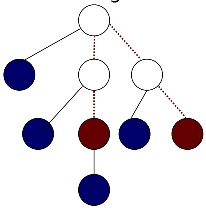
In-group



Across-group



Across-neighborhood



Consistency Guarantees

Theorem

The proposed method consistently recovers the structure with $O(\log p)$ samples and parameters with $\mathrm{poly}(p)$ samples.

Extent of parallelism

- Size of groups $\boxed{\Gamma \leq \Delta^{1+\frac{u}{l}\delta}.}$
- Effective depth $\delta := \max_i\{\min_j\{\textsf{path}(v_i, v_j; \mathcal{T})\}$.
- Maximum degree in latent tree: Δ.
- Upper and lower bound on distances between neighbors in the latent tree: u and l.

Implications

- For homogeneous HMM, constant sized groups.
- Worst case: star graphs.

Computational Complexity

- N samples, d dimensional observed variables, k state hidden variables.
- p number of observed variables. z non-zero entries per sample.
- Γ sized groups.

Algorithm Steps	Time/worker	Degree of parallelism
Information Distance Estimation	$O(Nz + d + k^3)$	$O(p^2)$
Structure: Minimum Spanning Tree	$O(\log p)$	$O(p^2)$
Structure: Local Recursive Grouping	$O(\Gamma^3)$	$O(p/\Gamma)$
Parameter: Tensor Decomposition	$O(\Gamma k^3 + \Gamma d k^2)$	$O(p/\Gamma)$
Merging and Alignment Correction	$O(dk^2)$	$O(p/\Gamma)$

"Integrated Structure and Parameter Learning in Latent Tree Graphical Models" by F. Huang, U. N. Niranjan, A. Anandkumar. Preprint, June 2014.

Proof Ideas

Relating Chow-Liu Tree with Latent Tree

- Surrogate $\mathrm{Sg}(i)$ for node i: observed node with strongest correlation

$$\boxed{\mathrm{Sg}(i) := \operatorname*{argmin}_{j \in V} d_{i,j}}$$

- Neighborhood preservation

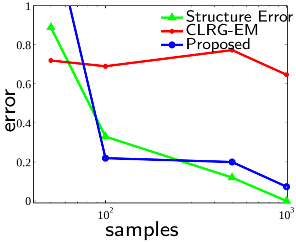
$$\boxed{(i,j) \in T \Rightarrow (\mathrm{Sg}(i), \mathrm{Sg}(j)) \in T_{\mathrm{ML}}.}$$

Chow-Liu grouping reverses edge contractions

Proof by induction

Experiments

- $d = k = 2$ dimensions, $p = 9$ number of variables.



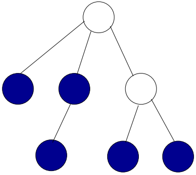
d	p	N	Struct Error	Param Error	Running Time(s)
10	9	50K	0	0.0104	3.8
100	9	50K	0	0.0967	4.4
1000	9	50K	0	0.1014	5.1
10,000	9	50K	0	0.0917	29.9
100,000	9	50k	0	0.0812	56.5
100	9	50K	0	0.0967	10.9
100	81	50K	0.06	0.1814	323.7
100	729	50K	0.16	0.1913	4220.1

Outline

Summary and Outlook

Learning Latent Tree Models

- Integrated Structure and Parameter Learning
- High level of parallelism without losing consistency.



Learning Graphical Model Mixtures

- Tree mixture approximations
- Combinatorial search + spectral decomposition
- Computational and sample guarantees