Latent Variable Modeling: Tensor and Graphical Approaches

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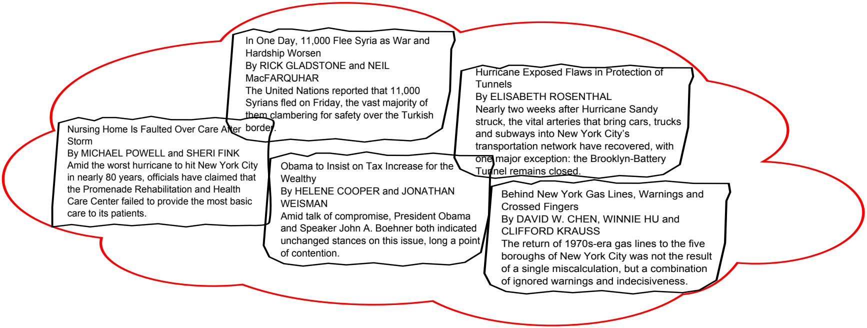
Latent Variable Modeling

Goal: Discover hidden effects from observed measurements

Example: document modeling

- Observations: words. Hidden: topics.



Learning latent variable models: efficient methods and guarantees

Challenges and Approaches

Challenges: High-Dimensional Regime

- Sample and Computational complexities
- Identifiability: when can hidden variables be discovered?

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Method of Moments

- Hidden choice variable and observed samples
- Inverse moment method: solve equations relating hidden variable to observed moments
- Low order tensor form and efficient decomposition methods

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Graphical Modeling

- Qualitative: graph structure. Quantitative: interaction strengths.
- Markov relationships: graphs with long cycles and hidden variables.
- Greedy graph estimation method: efficient tradeoffs.

Results from Two Approches

Learning Mixture Models through Tensor Decomposition

Topic 1	Topic 2	Topic 3
bush	company	show
president	percent	book
government	million	women
official	companies	family
campaign	market	film
political	business	school
law	stock	look
leader	billion	home
george_bush	money	children
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- Top 10 words for three topics from NYTimes data set.

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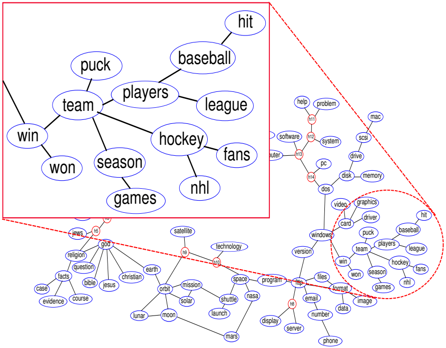
Graph Estimation Through Greedy Methods



- Graph: Topic-Word Relationships.

Other Motivating Applications

Social Network Modeling

- Community detection: Discovering hidden communities
- Dynamic network modeling: Predicting vertex co-presence

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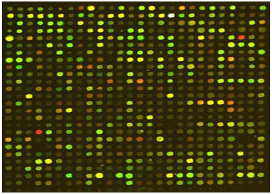

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- Modeling gene associations
- Hidden variables may be regulators that control groups of functionally similar genes

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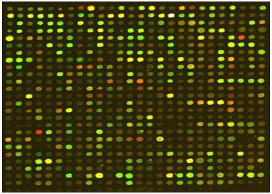

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Computer Vision, Phylogenetics, Financial Modeling . . .

Outline

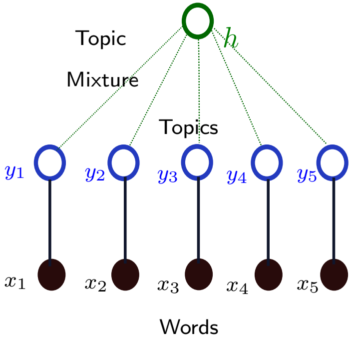
Warmup: Exchangeable Single Topic Models

Exchangeability

- Order of words does not matter
- Sufficient statistics: word counts
- DeFinetti's theorem: latent variable

Exchangeable Topic Models

- l words in a document $x_1, \ldots, x_l$.
- Document: topic mixture (draw of h).
- Word x_i generated from topic y_i.
- Exchangeability: $\boxed{x_1 \perp\!\!\!\perp x_2 \perp\!\!\!\perp \ldots | h}$
- $\boxed{\Phi(i, j) := \mathbb{P}[x_m = i | y_m = j].}$



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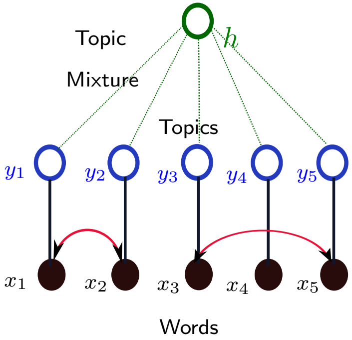

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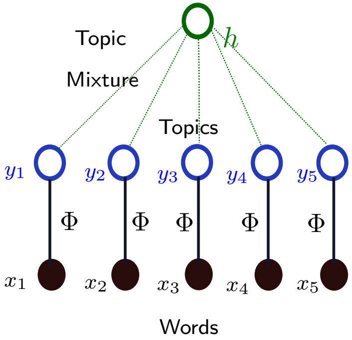

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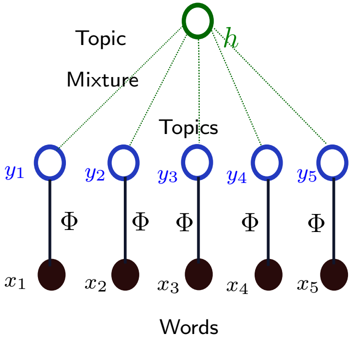

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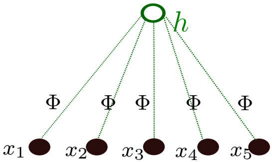
Single topic model

- Each document has only one hidden topic: $y_i = h$.
- h is a discrete variable and let $\boxed{\lambda_i := \mathbb{P}[h = i].}$

Form of Observed Moments

- $\boxed{\vec{\lambda} := [\mathbb{P}[h = i]]_i.}$
  $\boxed{\Phi(i, j) := \mathbb{P}[x_m = i | h = j].}$
- Learning: Loading matrix Φ and Vector $\vec{\lambda}$



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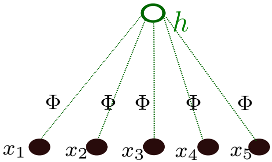

Pairwise Probability Matrix M_2

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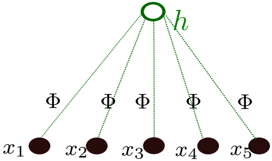

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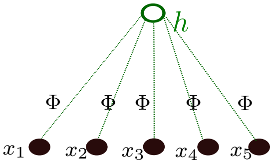

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Matrix and Tensor Forms: $\phi_r := r^{\text{th}}$ column of Φ.

$$\boxed{M_2 = \sum_{r=1}^{k} \lambda_r \phi_r \otimes \phi_r. \qquad M_3 = \sum_{r=1}^{k} \lambda_r \phi_r \otimes \phi_r \otimes \phi_r}$$

Tensor Basics: Multilinear Transformations

- For a tensor M_3, define (for matrices V_i of appropriate dimensions)

$$\boxed{[M_3(V_1, V_2, V_3)]_{i_1,i_2,i_3} := \sum_{j_1,j_2,j_3} (M_3)_{j_1,j_2,j_3} \prod_{m\in[3]} V_1(j_m, i_m)}$$

- For a matrix M_2, $\boxed{M(V_1, V_2) := V_1^\top M_2 V_2}$.

$$\boxed{M_3 = \sum_{r=1}^{k} \lambda_r \phi_r \otimes \phi_r \otimes \phi_r}$$

$$\boxed{\begin{aligned} M_3(W, W, W) &= \sum_{r\in[k]} \lambda_r (W^\top \phi_r)^{\otimes 3} \\ M_3(I, v, v) &= \sum_{r\in[k]} \lambda_r \langle v, \phi_r\rangle^2 \phi_r. \\ M_3(I, I, v) &= \sum_{r\in[k]} \lambda_r \langle v, \phi_r\rangle \phi_r \phi_r^\top. \end{aligned}}$$

Inverse Moment Methods for Learning

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Identifiability Using 2^{nd} and 3^{rd} Order Moments

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Special Case: Orthogonality

- If Φ is an orthogonal matrix $\boxed{M_3(I, \phi_r, \phi_r) = \lambda_r \phi_r}$.
- Loading vectors $\{\phi_r\}$ are eigenvectors of the tensor M_3

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How to obtain an orthogonal tensor form?

Orthogonal Tensor Decomposition

$$\boxed{M_2 = \sum_{r \in [k]} \lambda_r \phi_r \otimes \phi_r, \quad M_3 = \sum_{r \in [k]} \lambda_r \phi_r \otimes \phi_r \otimes \phi_r}$$

- Define $W = UD^{-1}$, where $M_2 = UDU^\top$.
- Let $\widetilde{\phi}_i := \sqrt{\lambda_i}\ W^\top \phi_i$. They are orthonormal.

$$\boxed{M_2(W, W) \ = \ \sum_{i \in [k]} W^\top (\sqrt{\lambda_i} \phi_i)(\sqrt{\lambda_i} \phi_i)^\top W \ = \ \sum_{i \in [k]} \widetilde{\phi}_i \widetilde{\phi}_i^\top \ = \ I,}$$

- Now define $\widetilde{M}_3$, so that

$$\boxed{\widetilde{M}_3 = M_3(W, W, W) = \sum_{i \in [k]} \lambda_i \, (W^\top \phi_i)^{\otimes 3} = \sum_{i \in [k]} \frac{1}{\sqrt{\lambda_i}} \ \widetilde{\phi}_i^{\otimes 3}.}$$

Learning: Tensor Decomposition of $\widetilde{M}_3$

Orthogonal Tensor Eigen Analysis

- Consider orthogonal symmetric tensor $T = \sum_i w_i \mu_i^{\otimes 3}$

$$\boxed{T = \sum_{i=1}^{k} w_i \mu_i^{\otimes 3}. \quad T(I, \mu_i, \mu_i) = w_i \mu_i}$$

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- Challenge: empirical moments

  Solution: robust tensor decomposition methods

Optimization Viewpoint for Tensor Eigen Analysis

Consider Norm Optimization Problem for Tensor T

- $\boxed{\max_u \; T(u,u,u) \qquad s.t. \; u^\top u = I}$
- Constrained stationary fixed points $T(I,u,u) = \lambda u$ and $u^\top u = I$.
- u is a local isolated maximizer if $w^\top (T(I,I,u) - \lambda I) w < 0$ for all w such that $w^\top w = I$ and w is orthogonal to u.

Review for Symmetric Matrices $M = \sum_i w_i \mu_i^{\otimes 2}$

- Constrained stationary points are the eigenvectors
- Only top eigenvector is a maximizer and stable under power iterations

Orthogonal Symmetric Tensors $T = \sum_i w_i \mu_i^{\otimes 3}$

- Stationary points are the eigenvectors (up to scaling)
- All basis vectors $\{\mu_i\}$ are local maximizers and stable under power iterations

Tensor Decomposition: Perturbation Analysis

- Observed tensor $\widetilde{T} = T + E$, where $T = \sum_{i \in k} w_i \mu_i^{\otimes 3}$ is orthogonal tensor and perturbation E, and $\|E\| \leq \epsilon$.
- Recall power iterations $\boxed{u \mapsto \dfrac{\widetilde{T}(I,u,u)}{\|\widetilde{T}(I,u,u)\|}}$

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Perturbation Analysis

After N iterations, eigen pair (w_i, μ_i) is estimated up to $O(\epsilon)$ error, where

$$\boxed{N = O\left(\log k + \log\log \frac{w_{\max}}{\epsilon}\right).}$$

A. Anandkumar, R. Ge, D. Hsu, S.M. Kakade and M. Telgarsky "Tensor Decompositions for Learning Latent Variable Models," Preprint, October 2012.

Robust Tensor Power Method

$$\boxed{\widetilde{T} = \sum_i w_i \mu_i^{\otimes 3} + E}$$

Basic Algorithm

- Pick random initialization vectors
- Run power iterations $\boxed{u \mapsto \dfrac{\widetilde{T}(I,u,u)}{\|\widetilde{T}(I,u,u)\|}}$
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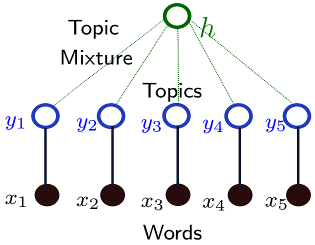
Further Improvements

- Initialization: Use long document vectors for initialization
- Stabilization: $\boxed{u^{(t)} \mapsto \alpha \frac{\widetilde{T}(I,u^{(t-1)},u^{(t-1)})}{\|\widetilde{T}(I,u^{(t-1)},u^{(t-1)})\|} + (1-\alpha)u^{(t-1)}}$

Extensions...

Latent Dirichlet Allocation

- Each document a topic mixture rather than a single topic
- Modified second and third order moments reduce to symmetric tensor.



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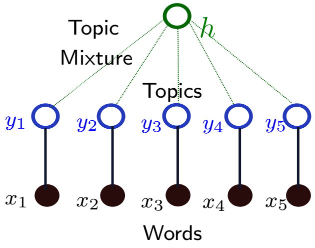

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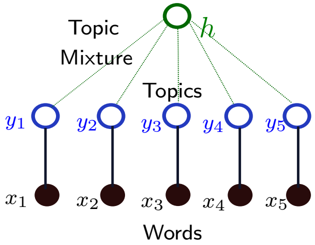

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Community Modeling and Detection in Social Networks

- Mixed membership model (Airoldi et. al): overlapping communities
- Edge counts and 3-star counts: tensor decomposition

A. Anandkumar, R. Ge, D. Hsu, S. Kakade, " Learning Mixed Membership Block Models."

Preliminary Experiments

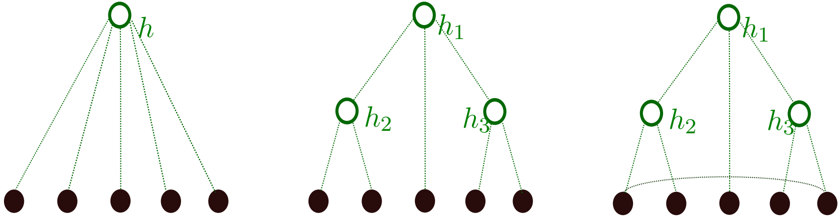
Top 10 words for 5 topics (NYTimes data)

Topic 1	Topic 2	Topic 3	Topic 4	Topic 5
bush	company	show	official	team
president	percent	book	water	game
government	million	women	attack	season
official	companies	family	u_s	player
campaign	market	film	food	play
political	business	school	united_states	games
law	stock	look	afghanistan	point
leader	billion	home	taliban	run
george_bush	money	children	air	win
al_gore	cost	friend	military	won

Outline

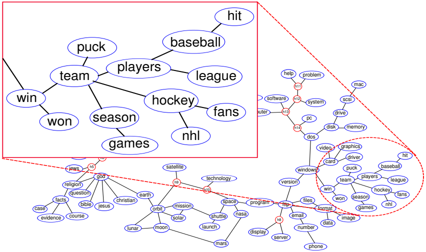
Hierarchical Latent Variable Models

So far... | Latent Tree | Loopy Model

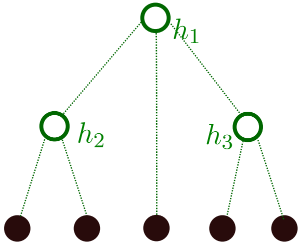


Graph Estimation with Latent Variables

- # and location of hidden variables unknown
- Estimate graph over all variables
- Trees and girth-constrained graphs



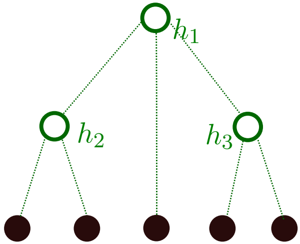
Learning Latent Tree Models



Learning Latent Tree Models

Information Distances $\{d_{ij}\}$

- Gaussian: $d_{ij} := -\log|\rho_{ij}|$.
- Discrete: $d_{ij} := -\log|\operatorname{Det}(P_{i,j})|$.



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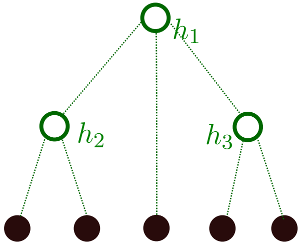

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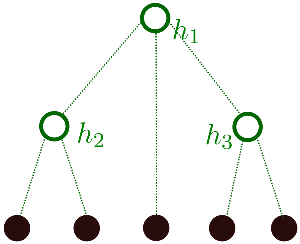


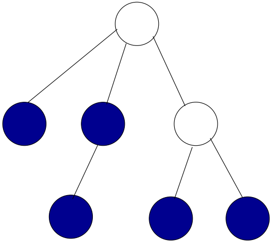
Learning latent tree using $[\hat{d}_{i,j}]$

Siblings Test Based on Information Distances

Exact Statistics: Distances $[d_{i,j}]$

Let $\Phi_{ijk} := d_{i,k} - d_{j,k}$.

- $-d_{i,j} < \Phi_{ijk} = \Phi_{ijk'} < d_{i,j} \; \forall\, k, k' \neq i, j,$ $\iff$ i, j leaves with common parent
- $\Phi_{ijk} = d_{i,j}, \; \forall\, k \neq i, j,$ $\iff$ i is a leaf and j is its parent.



Sample Statistics: ML Estimates $[\hat{d}_{i,j}]$

Use only short distances: $d_{i,k}, d_{j,k} < \tau$, Relax equality relationships

Siblings Test Based on Information Distances

Exact Statistics: Distances $[d_{i,j}]$

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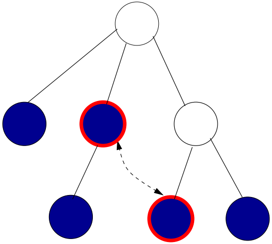

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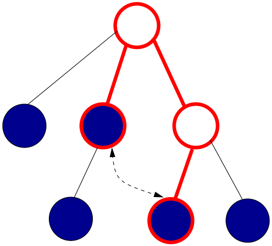

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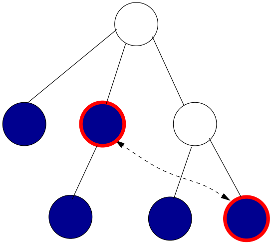

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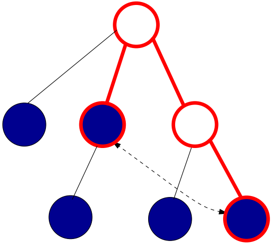

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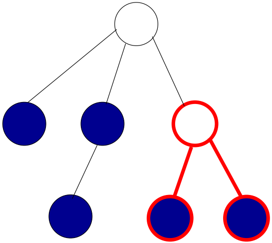

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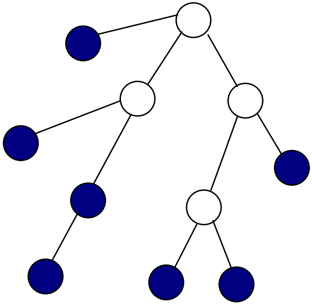
Recursive Grouping Algorithm (Choi, Tan, **A.**, Willsky)

- Sibling test and remove leaves
- Build tree from bottom up



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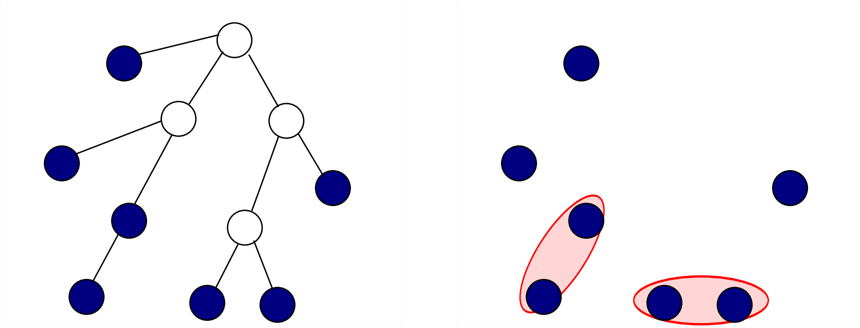

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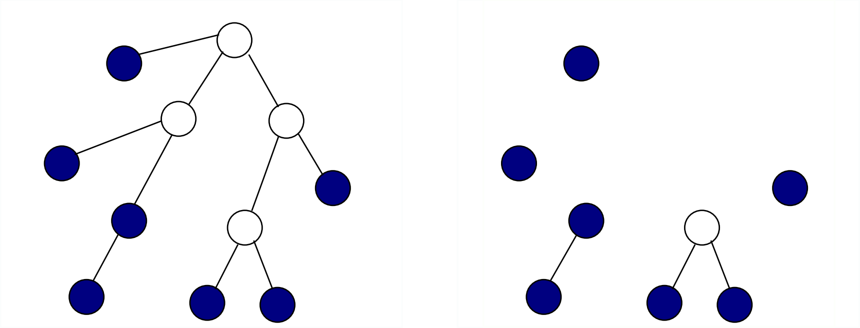

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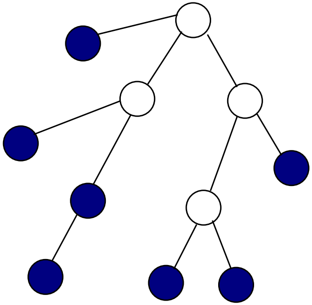

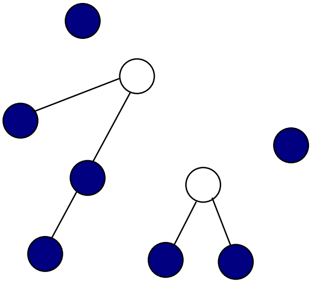

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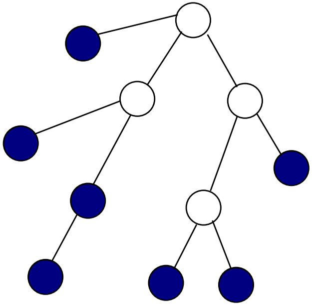

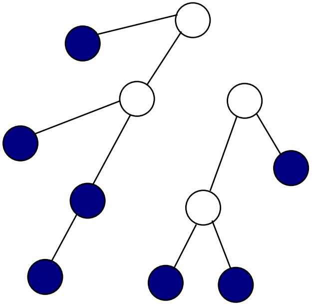

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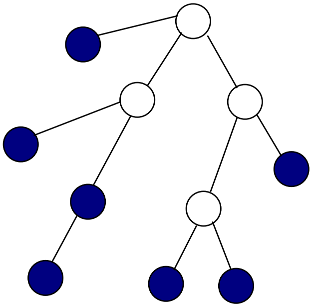

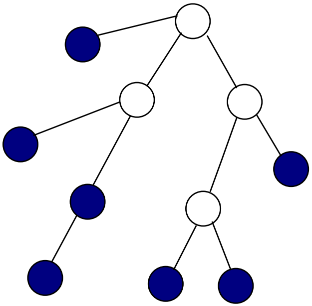

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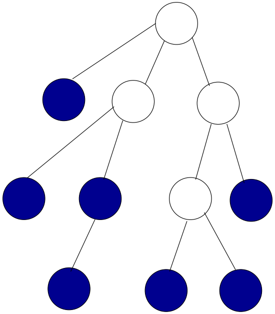
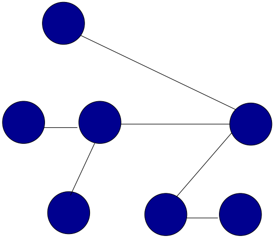
Efficient Initial Tree on Observed Nodes (MST)

Minimum spanning tree using edge weights $[\hat{d}_{i,j}]$.





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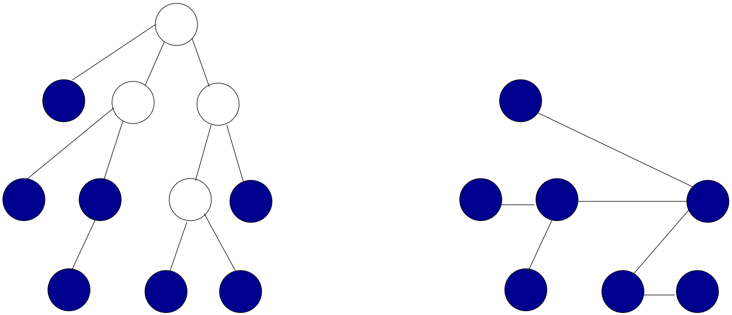

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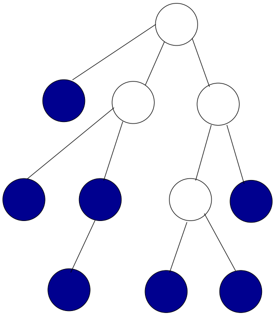

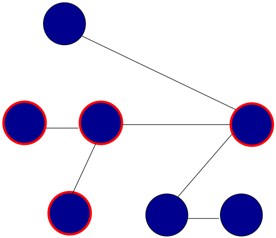

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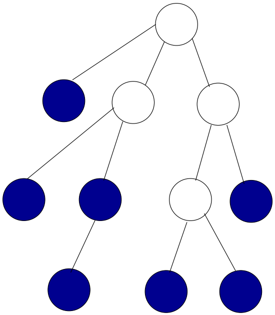

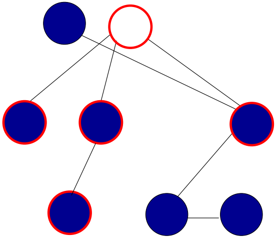

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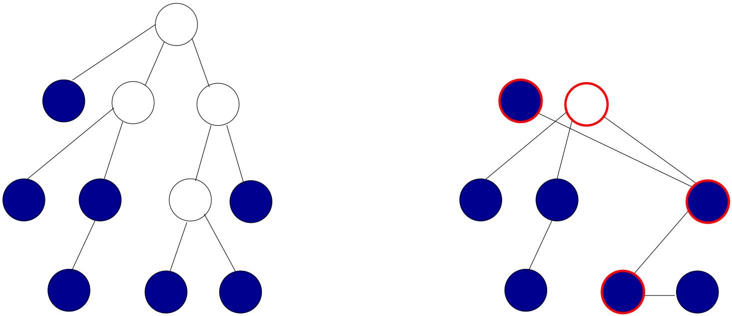

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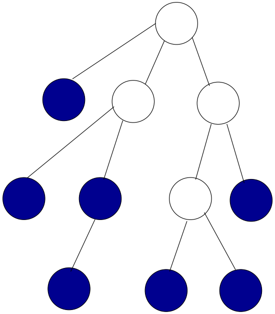

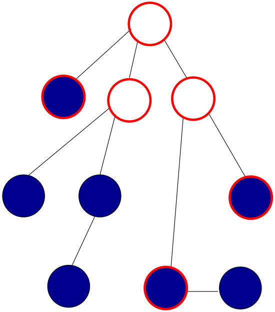

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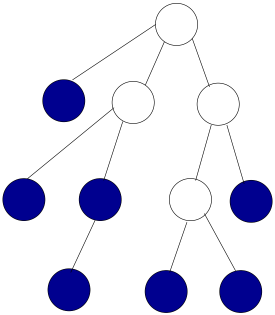

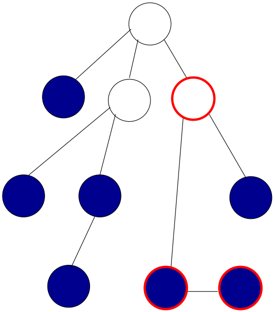

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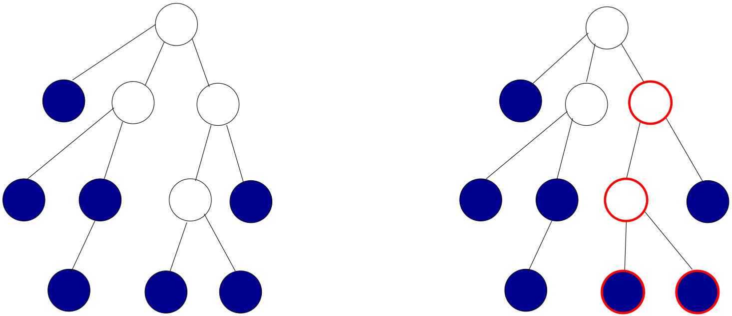


Proof Ideas

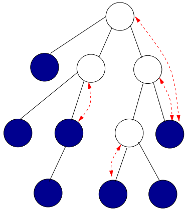
Relating Chow-Liu Tree with Latent Tree

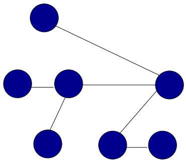
- Surrogate $\mathrm{Sg}(i)$ for node i: observed node with strongest correlation

$$\boxed{\mathrm{Sg}(i) := \operatorname*{argmin}_{j \in V} d_{i,j}}$$

- Neighborhood preservation

$$\boxed{(i,j) \in T \Rightarrow (\mathrm{Sg}(i), \mathrm{Sg}(j)) \in T_{\mathrm{ML}}.}$$





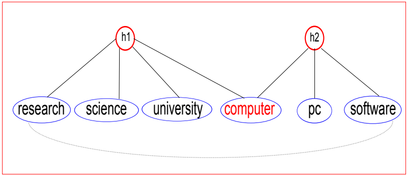
Chow-Liu grouping reverses edge contractions

Proof by induction

Loopy Graphical Models with Latent Nodes

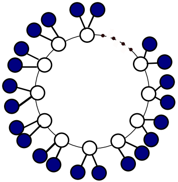
Motivation: Topic Models

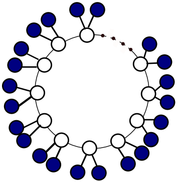
- Common words among topics.
- Latent or hidden nodes.
- Typically long cycles: Locally tree-like.



Overview of Proposed Method

- Consider local neighborhoods for building local MST
- Merge the MSTs to obtain a loopy graph
- Run latent tree routine on different local neighborhoods





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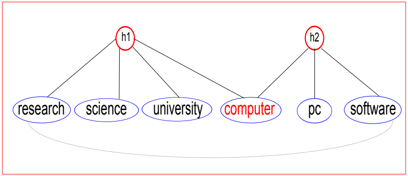

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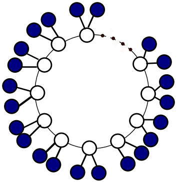

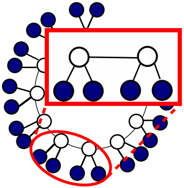

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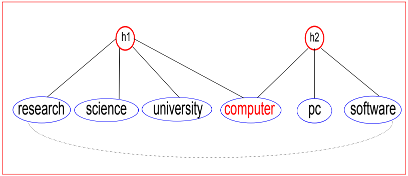

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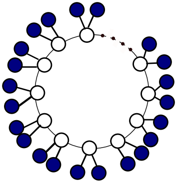

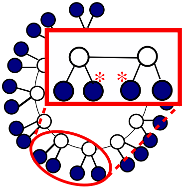

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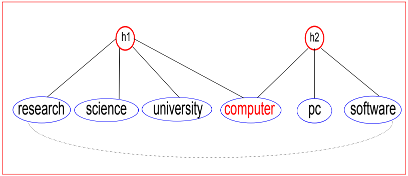

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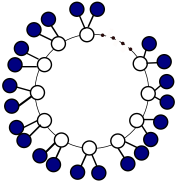

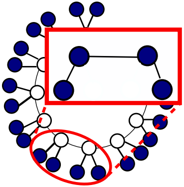

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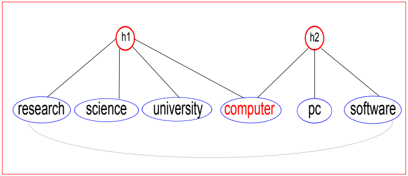

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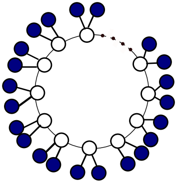

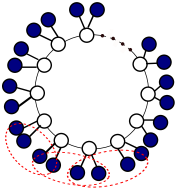

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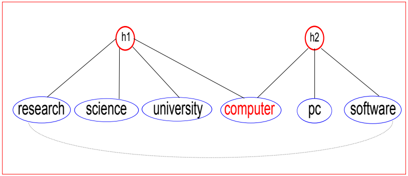

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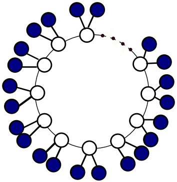

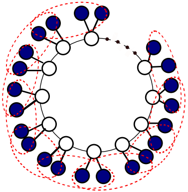

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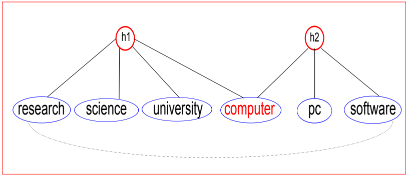

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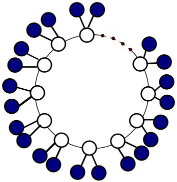

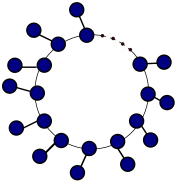

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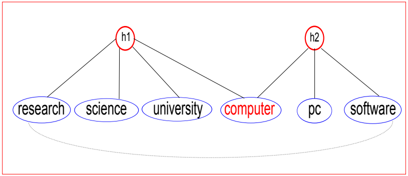

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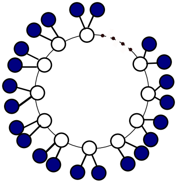
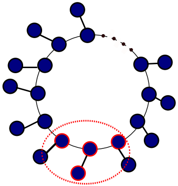
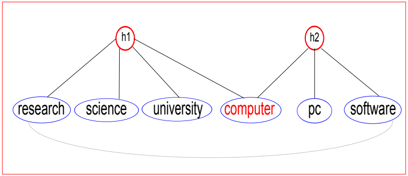




Loopy Graphical Models with Latent Nodes

Motivation: Topic Models

- Common words among topics.
- Latent or hidden nodes.
- Typically long cycles: Locally tree-like.



Overview of Proposed Method

- Consider local neighborhoods for building local MST
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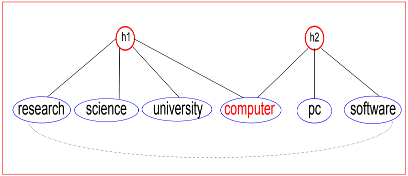

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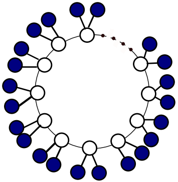

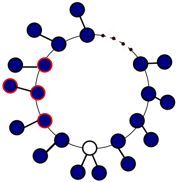

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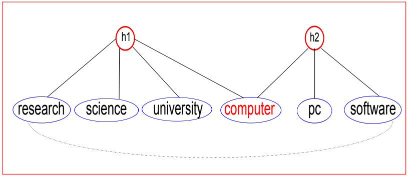

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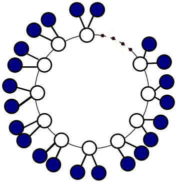

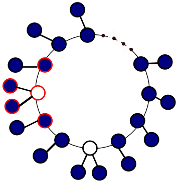

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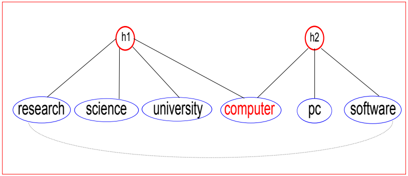

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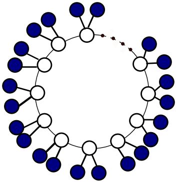

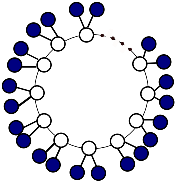

Guarantees for Latent Structure Learning

- Ising model with minimum edge potential $J_{\min}$.

$$\boxed{p(x) \propto \exp\left[\sum_{(i,j)\in G} J_{i,j} x_i x_j + \sum_{i \in V} h_i x_i\right]}$$

- Depth δ: worst-case distance between hidden and observed nodes.
- Parameter β: depends on min. and max. node and edge potentials
  - $\beta = 1$ for homogeneous models.

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Theorem (A. , Valluvan '12)

Proposed method correctly recovers graph structure w.h.p. on p observed nodes and n samples when

$$\boxed{\frac{J_{\min}^{-2\delta\beta(\beta+1)-2} \log p}{n} = O(1).}$$

A. Anandkumar and R. Valluvan "Learning Loopy Graphical Models with Latent Variables: Efficient Methods and Guarantees" Under revision, Annals of Statistics, June 2012.

Insights and Implications

Tradeoff between depth δ and girth g

Roughly require: $\delta < g/4$.

Tradeoff between max. edge strength $J_{\max}$ and degree Δ

Require $J_{\max} < \operatorname{atanh}(\Delta^{-1})$.

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Sample complexity for uniform node sampling

Given ρ fraction of nodes as observed nodes,

$$\boxed{n = \Omega\left(\Delta^2 \rho^{-4} (\log p)^5\right).}$$

Necessary conditions for structure recovery

For any deterministic algorithm, the number of samples n needs to be

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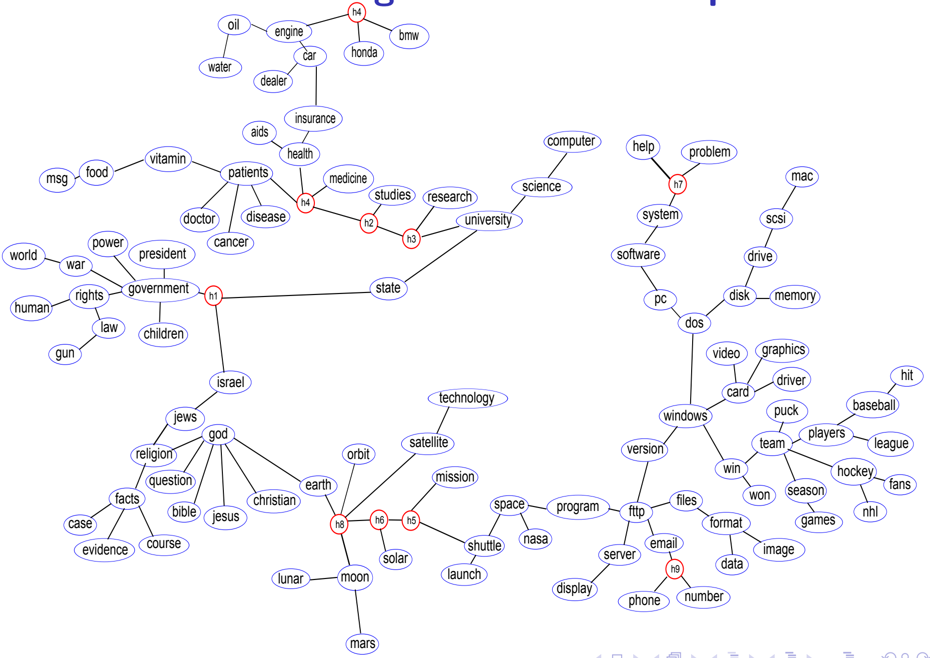
Outline

Discovering Word Relationships



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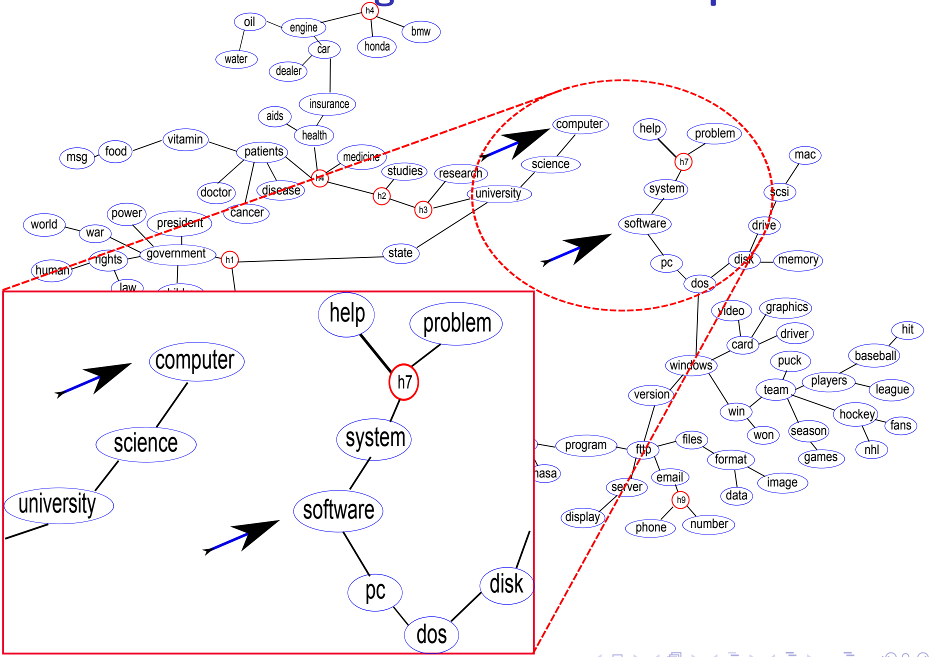

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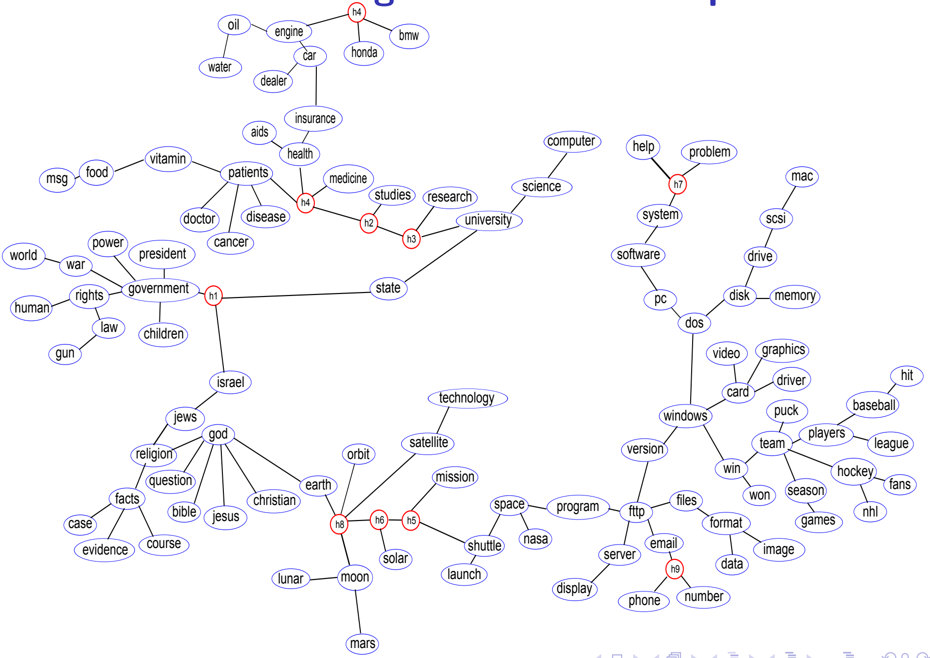

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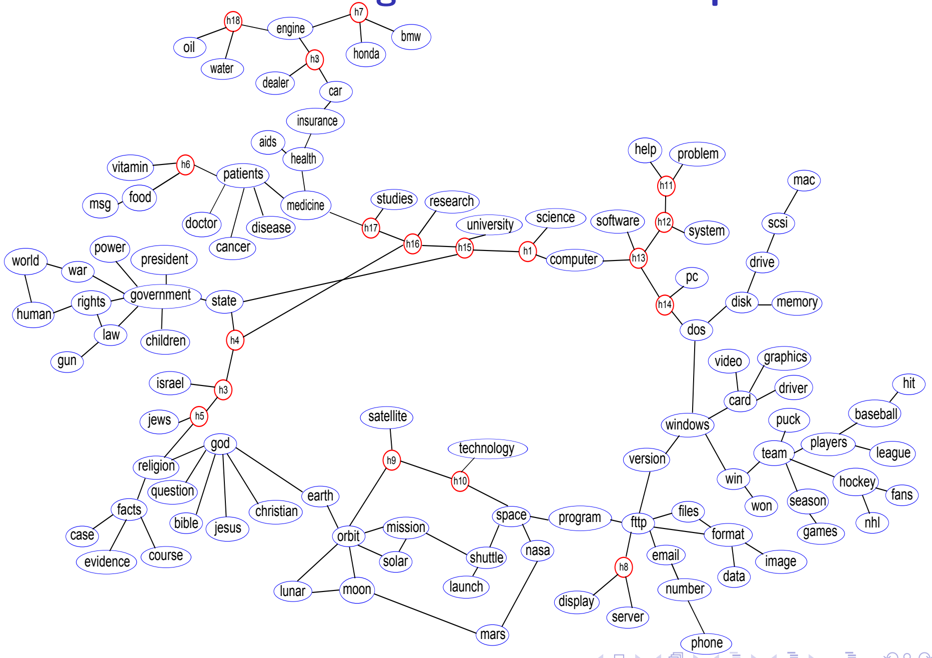

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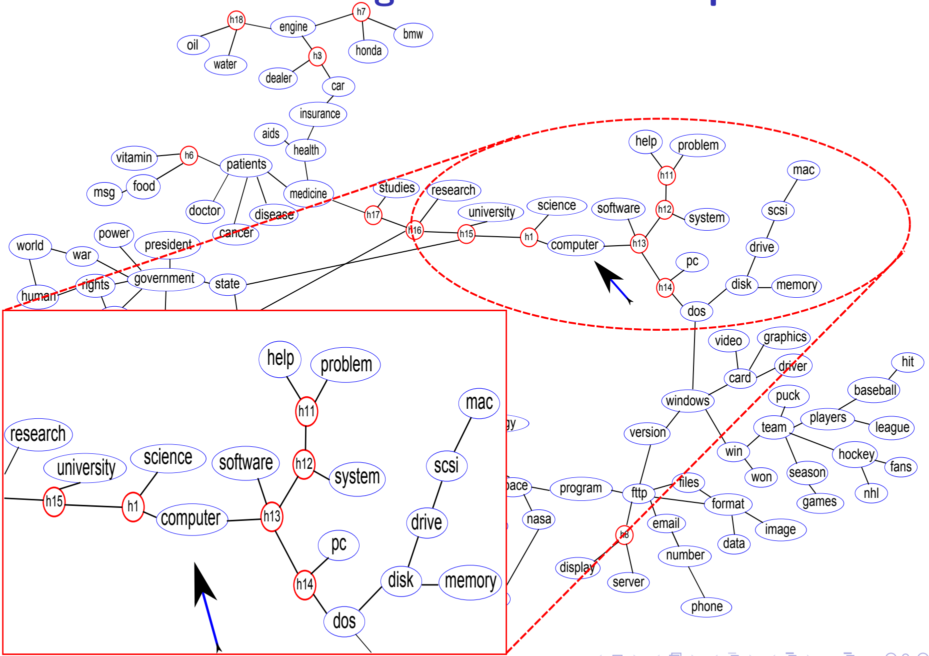

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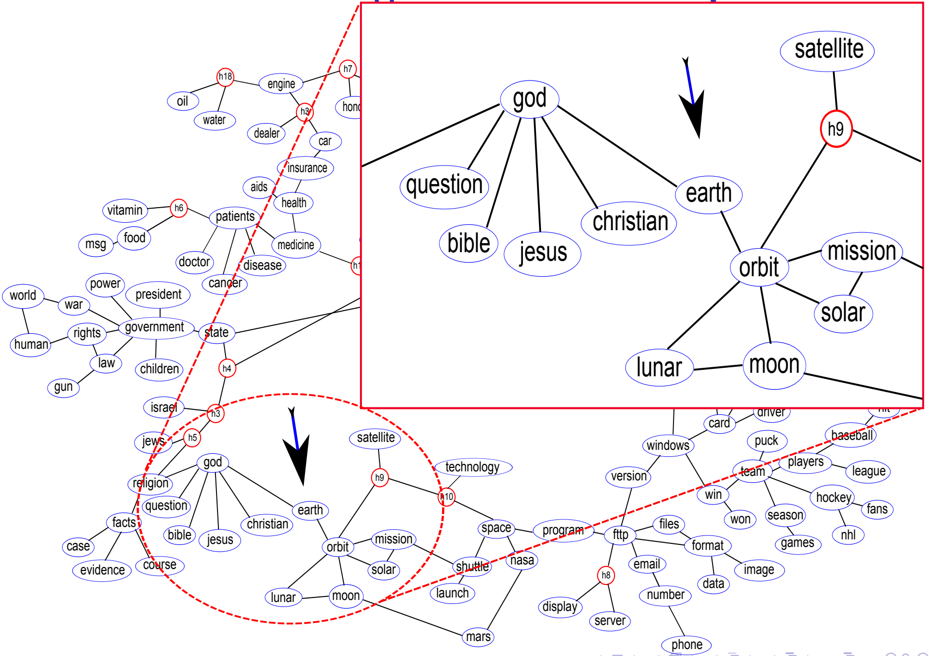

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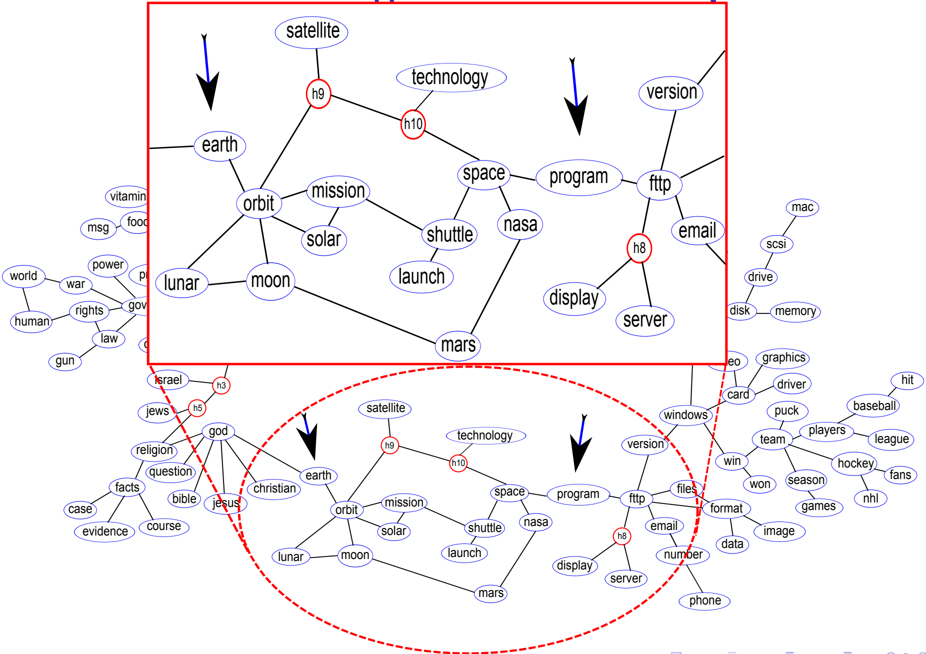

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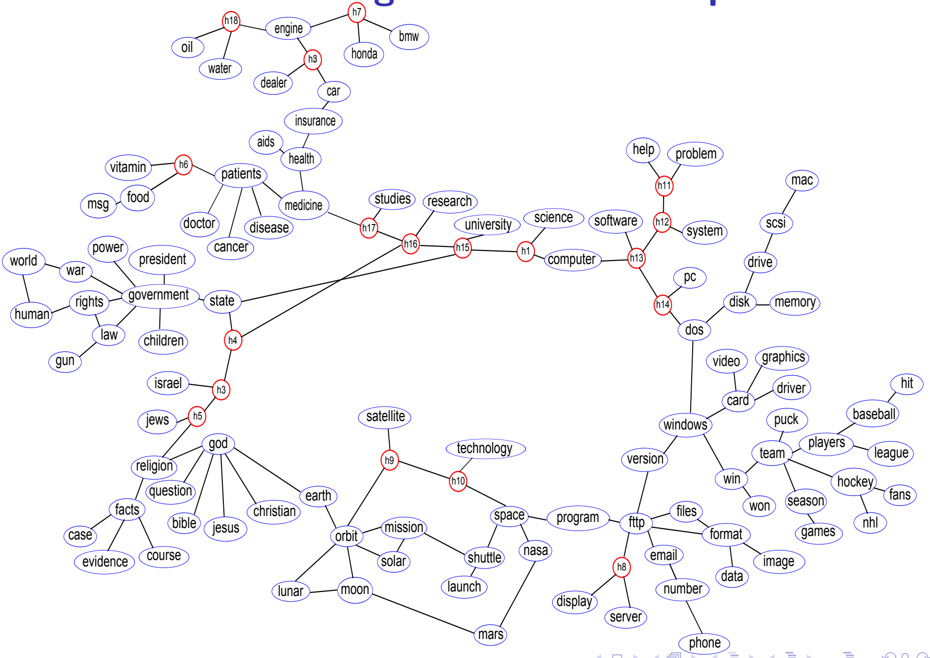
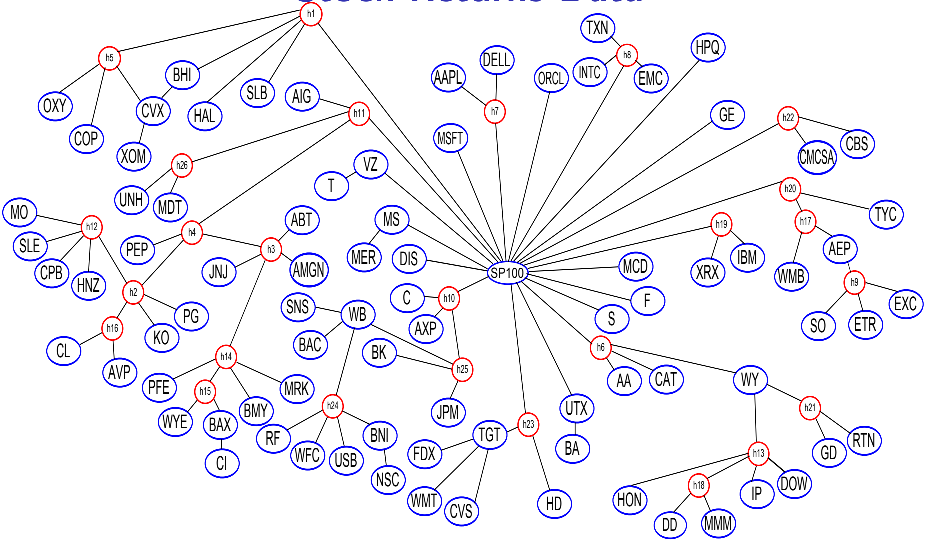


Stock Returns Data



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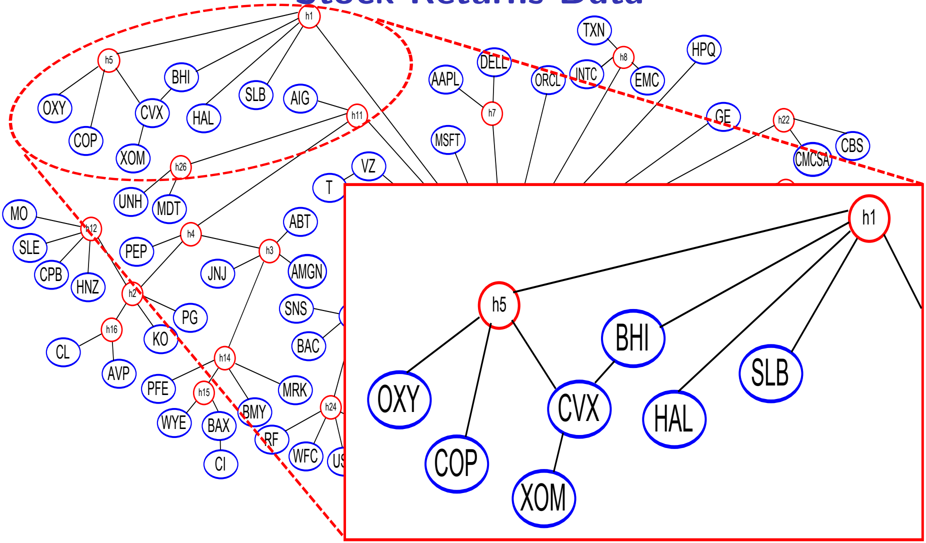

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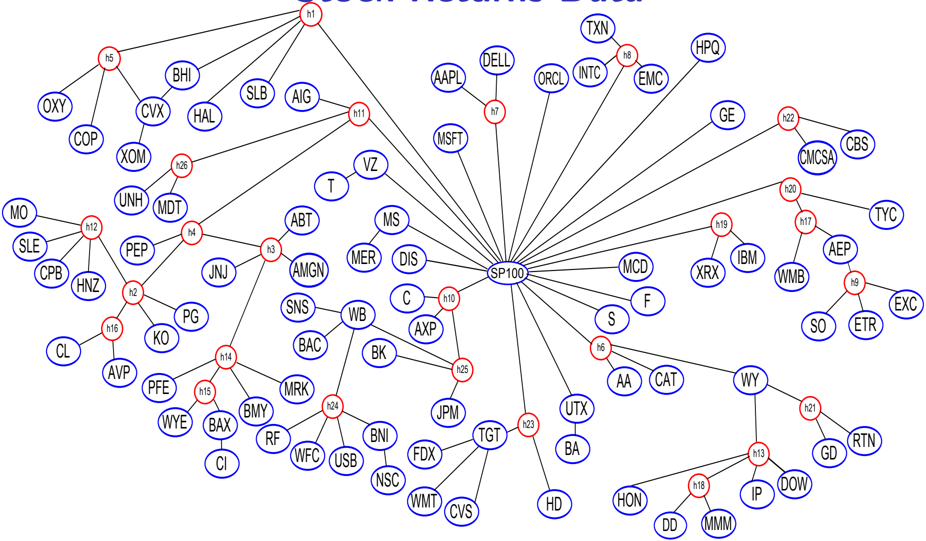

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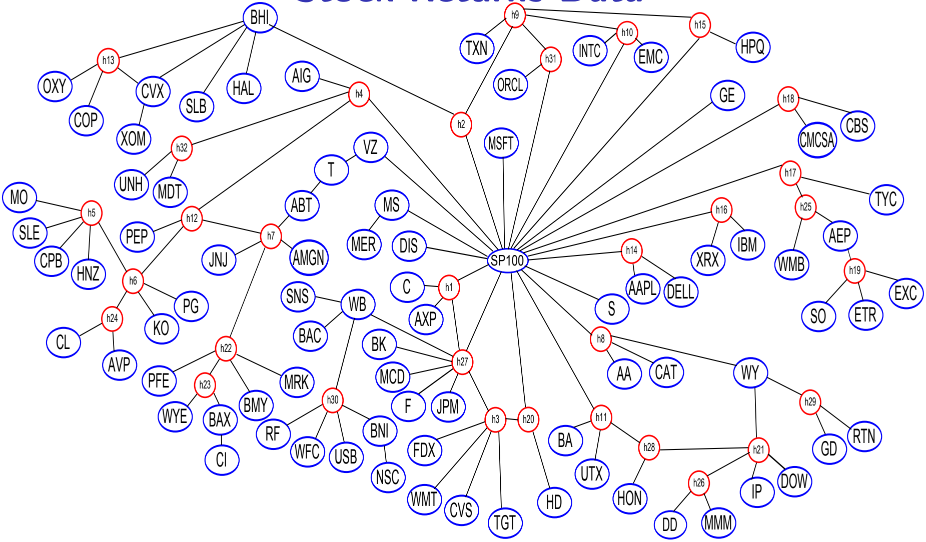

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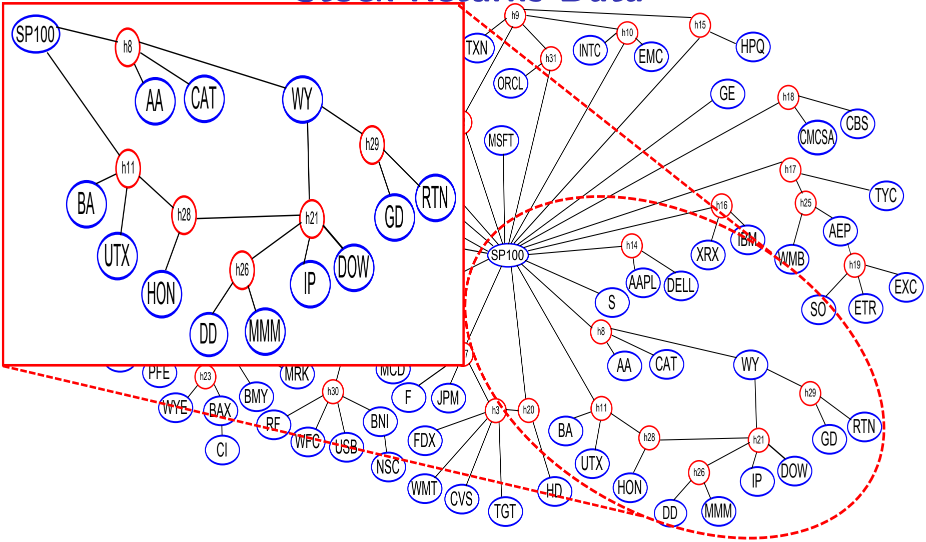

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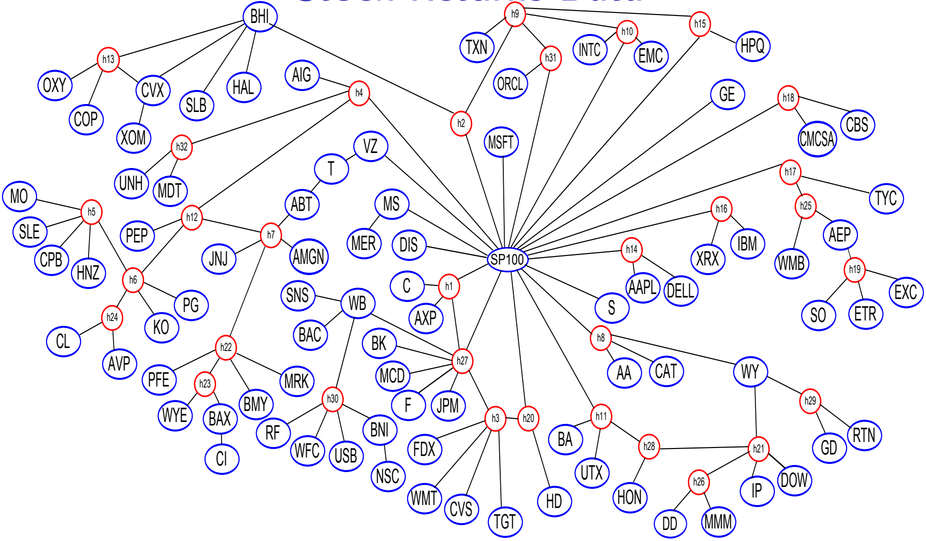

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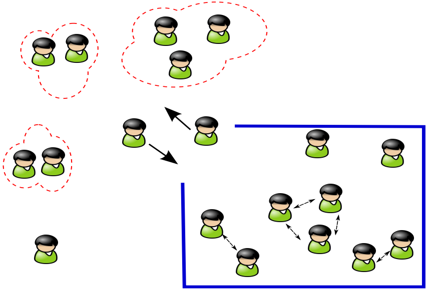
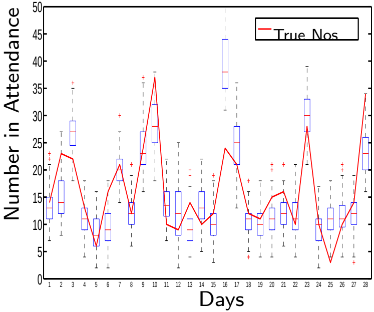
Dynamic Network Modeling

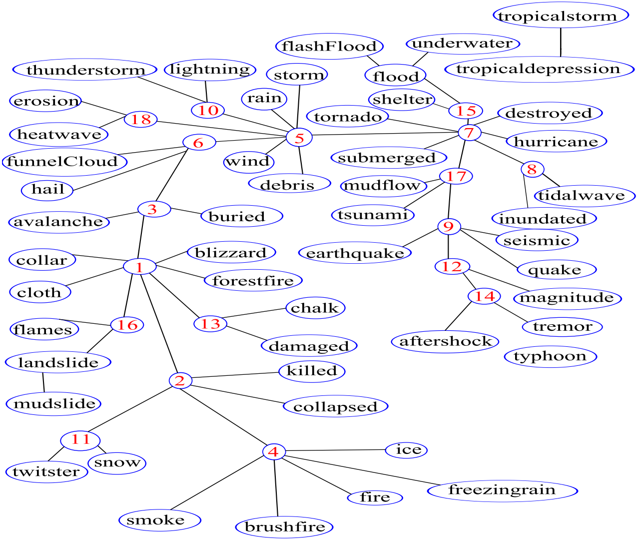
- Observations: series of graph $G_t = (V_t, E_t)$ and covariates
- Modeling vertex participation through latent graphical model
- Logistic regression for edge prediction given vertices
- Data: windsurfer interaction on a beach
- Improvement over baseline: 164% for vertices and 45% for edges.





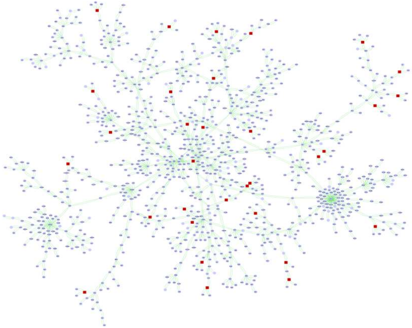
R. Valluvan, Z. W. Almquist, C. T. Butts, and A. Anandkumar. Semi-parametric vertex set prediction for dynamic networks using latent tree models. (Sunbelt 2012).

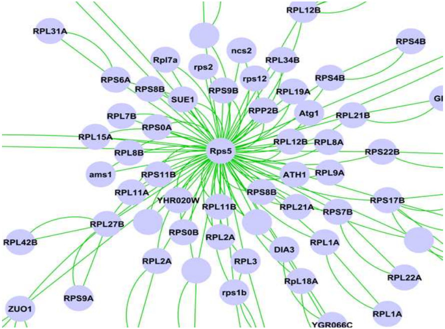
Modeling Hazard-related Tweets



Modeling Gene Associations

- Observed: gene expression levels
- Relationships between genes, e.g. genes that encode ribosomal subunits group together
- Hidden nodes: regulators that control groups of functionally similar genes, e.g. transcription factors





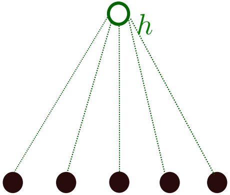
In collaboration with Anthony Gitter (Microsoft) and Ernest Fraenkel (MIT)

Outline

Summary on Learning Latent Variable Models

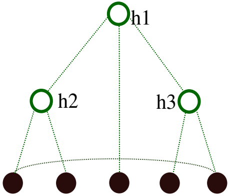
Tensor Methods

- Tensor forms for a range of models
- Efficient decomposition methods
- Perturbation analysis



Graph Estimation

- Latent modeling via graphical approaches
- Efficient methods for graph estimation
- Guarantees on sample and computational complexities



The Big Picture