Practical tasks

Repo

Instructions:

- install sagemath
- `sage --python -m pip install -r requirements.txt`
- if you want to reload data: `sage --python util.py`
- `sage -n`
- in this jupyter session open provided .ipynb

An html export of the notebook is also provided.

Theorem 1 (Triangle Inequality)

Statement

For any connected graph $G = \langle V, E \rangle$:

$\forall x, y, z \in V : dist(x, y) + dist(y, z) \geq dist(x, z)$

Definitions

We define $dist(x, y)$ as $\min(|x \rightsquigarrow y|)$

Proof

Since G is connected, $\forall x, y, z \in V : \exists x \rightsquigarrow y, y \rightsquigarrow z, x \rightsquigarrow z$

Let's say that $dist(x, y) = |P_1|, dist(y, z) = |P_2|, dist(x, z) = |P_3|$, where P_1, P_2, P_3 are paths $x \rightsquigarrow y, y \rightsquigarrow z, x \rightsquigarrow z$ respectively.

Then we have a concatenation walk $W = P_1 + P_2$

$|W| = |P_1| + |P_2|$

If W is a path (has no repeating vertices), then by our definition of distance $|W| \geq dist(x, z)$, and thus $dist(x, y) + dist(y, z) \geq dist(x, z)$

Otherwise, we remove all edges in W between pairs of repeating vertices recursively until it's a path, which we will call W'

Since W' was obtained from W by removing edges, $|W| \geq |W'|$

At the same time, by the same logic as shown above, $|W'| \geq dist(x, z)$, and thus, once again $dist(x, y) + dist(y, z) \geq dist(x, z)$

Theorem 2

Statement

For any connected graph G: $rad(G) \leq diam(G) \leq 2rad(G)$

Definitions:

$\varepsilon(v) = \max\limits_{u \in V} dist(v, u)$

$rad(G) = \min\limits_{v \in V} \varepsilon(v)$ $diam(G) = \max\limits_{v \in V} \varepsilon(v)$

Proof:

$rad(G) \leq diam(G)$ by definition.

Let's choose $v, u, w \in V$ such that $\varepsilon(v) = rad(G)$, $dist(u, w) = diam(G)$.

Per theorem 1: $dist(u, w) \leq dist(u, v) + dist(v, w)$.

Since $\varepsilon(v) = rad(G)$, $dist(u, v) \leq rad(G)$ and $dist(v, w) \leq rad(G)$.

Thus $diam(G) = dist(u, w) \leq dist(u, v) + dist(v, w) \leq 2rad(G)$.

Theorem 3

Statement

A connected graph $G = \langle V, E \rangle$ is a tree *iff* $|E| = |V| - 1$

Lemma 1

Statement

For any connected graph $G = \langle V, E \rangle$: $|E| \geq |V| - 1$

Proof

Let's start with graph $G' = \langle V, \emptyset \rangle$, and add edges from E that reduce the number of connected components one by one. G' has $|V|$ connected components, and each added edge reduces that number 1, so we will need to add $|V| - 1$ edges with this process to get a connected graph. Since G is connected, $|E| \geq |V| - 1$.

Proof

First, let's assume $|E| = |V| - 1$ and prove that G is a tree.

Suppose G contains a cycle C.

Per lemma 1, if we remove any one edge from G, it will cease to be connected.

Let's remove an edge $e \in C$, calling the resulting graph G'

Since G is connected, $\forall v, u \in V : \exists P = v \rightsquigarrow u$ in G. We can replace all occurrences of e in these paths with $C \setminus \{e\}$, getting equally valid paths.

But since the only difference between G and G' is the edge e, these new paths exist in G' as well, which means G' is connected. The contradiction means our assumption was incorrect, and G contains no cycles, making it a tree by definition.

Now, let's prove that for any tree $T\langle V, E\rangle : |E| = |V| - 1$

Let's build a minimal connected subgraph G' as in lemma 1. If we add another edge v, u from T, we create a cycle, as G' was already connected, thus having a path between v and u, and our newly added edge is another. But since T is a tree it by definition can't contain any cycles, which means there is no such edge, $T = G'$ and thus $|E| = |V| - 1$.

Theorem 4

Statement

Given a connected graph $G = \langle V, E\rangle$ with n vertices, if $\delta(G) \geq \lfloor n/2 \rfloor$, then $\lambda(G) = \delta(G)$.

Proof

Let $k = \lfloor n/2 \rfloor$

$\lambda(G) \leq \delta(G)$, since we can remove all edges incident to a vertex with minimum degree to detach it from the rest of the graph.

Let's look at a minimum edge cut, and consider the smallest connected component produced by such a cut, calling it $H\langle V', E'\rangle$.

Let $m = |V'|, l = |E'|$.

Note that $m \geq 1$

Since there have to be at least 2 components, $m <= k$.

Each vertex in V' can have at most $m-1$ incident edges in H, while in G it must have had at least $\delta(G)$ incident edges.

This means at least $m(\delta(G) - (m-1))$ edges must have been removed.

This is a quadratic function with a peak at $\frac{\delta(G)+1}{2}$, so its minimum on our interval $1 \leq m \leq k$ will lie at either $m = 1$ or $m = k$ depending on which is further from the peak.

Since $\delta(G) \geq k$, $\frac{\delta(G)+1}{2} - 1 \geq k - \frac{\delta(G)+1}{2}$, so we can use the value at $m = 1$, which is $\delta(G)$.

Thus we have $\delta(G) \geq \lambda(G) \geq \delta(G)$, and so $\lambda(G) = \delta(G)$.

Theorem 5

Statement

Every block of a block graph is a clique.

Proof

We'll be looking at a graph G, its block graph H and a block J of H.

Suppose J is not a clique of H, meaning $\exists v, u \in V(J)$ that are not adjacent.

Since J is a block, v and u lie on a cycle, but since they are not adjacent, this cycle must be at least of length 4. Let's call the shortest such cycle C.

Since H is the block graph of G, each vertex in $V(C)$ corresponds to a block in G, and each edge in $E(C)$ corresponds to a cut vertex in G.

For each $z \in V(C)$ we can find a path in the corresponding block of G connecting the cut vertices of G corresponding to edges incident to z in C.

Note that each pair of paths corresponding to adjacent vertices in $V(C)$ share an endpoint - the cut vertex corresponding to the edge connecting them.

Moreover, the only vertex intersections any pair of these paths can have are cut vertices, since each lies in a separate block of G.

But there are no intersections other than those already discussed, since otherwise 2 vertices in $V(C)$ are connected by an edge in $E(J)$ that is not in $E(C)$, which would mean we can shorten C by replacing several edges with that one.

All of this means we can chain these paths to produce a cycle in G that passes through several blocks.

This is a contradiction, since any 2 vertices on an cycle must lie in the same block, which means our assumption was incorrect and every block of H is a clique.