MTH30002 - Differential Equations
Assignment 4

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Question 3.L

Given the Legendre's equation

$$y^{''} - \frac{2x}{1-x^2}y^{'} + \frac{n(n+1)}{1-x^2}y = 0 \tag{1}$$

find the Taylor series expansion for $\frac{1}{1-x^2}$ near $x_0 = 0$, and thus show that equation **1** is analytical at $x_0 = 0$.

Answer

A function f is called analytic at x_0 if it is given by a convergent power series.

Consider the Taylor series expansion of $f(x) = \frac{1}{1-x^2}$ around $x_0 = 0$.

$$f(x) = f(x_0) + \frac{f^{'}(x_0)(x-x_0)}{1!} + \frac{f^{''}(x_0)(x-x_0)^2}{2!} + ... + \frac{f^{n}(x_0)(x-x_0)^n}{n!} \tag{2}$$

$$\begin{aligned}
&f(0) = 1 \\
&f^{'}(x) = \frac{2x}{(1-x^2)^2} \therefore f^{'}(x)x = 0 \\
&f^{''}(x) = \frac{8x^2}{(1-x^2)^3} + \frac{2}{(1-x)^2} \therefore \frac{f^{''}(0)x^2}{2!} = x^2 \\
&f^{'''}(x) = \frac{24x}{(1-x^2)^3} + \frac{48x^3}{(1-x^2)^4} \therefore f^{'''}(0)x^3 = 0 \\
&f^{''''}(x) = \frac{288x^2}{(1-x^2)^4} + \frac{24}{(1-x^2)^3} + \frac{384x^4}{(1-x^2)^5} \therefore \frac{f^{''''}(0)x^4}{4!} = x^4
\end{aligned}$$

Therefore, the Taylor series about $x_0 = 0$ is clearly $\sum_{n=0}^{\infty} x^{2n}$.

$$L = \lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| = \lim_{n\to\infty} \left\| \frac{x^{2(n+1)}}{x^{2n}} \right\| = \left| x^2 \right|. \tag{3}$$

$$\begin{aligned}
R = \|x^2\| &< 1 \\
\sqrt{\|x^2\|} &< \sqrt{1}
\end{aligned}$$

$$R = \|x\| < 1$$

The series is by definition convergent.

Continuing...

$$\frac{-2x}{1-x^2} = -2x \cdot \sum_{n=0}^{\infty} x^{2n} = -2 \cdot \sum_{n=0}^{\infty} x^{2n+1} \text{ around } x_0 = 0.$$

$$L = \lim_{n \to \infty} \left\| \frac{a_{n+1}}{a_n} \right\| = \lim_{n \to \infty} \left\| \frac{x^{2(n+1+1)}}{x^{2n+1}} \right\| = \left| x^2 \right|. \tag{4}$$

$$\begin{aligned} R = \|x^2\| &< 1 \\ \sqrt{\|x^2\|} &< \sqrt{1} \end{aligned}$$

$$R = \|x\| < 1$$

The series is also convergent.

$$\frac{n(n+1)}{1-x^2} = n(n+1) \sum_{m=0}^{\infty} x^{2m} \text{ around } x_0 = 0. \tag{5}$$

Clearly, at $x_0 = 0$, the equation **1** is analytic as shown in **3**, **4**, and **5**.

Question 3.M

Using the recurrence equation

$$a_{s+2} = \frac{-(n-s)(n+s+1) \cdot a_s}{(s+2)(s+1)} \Big| s \geq 0 \tag{6}$$

show that the radius of convergence R = 1.

Answer

Given that **6** is true for $s \geq 0$ we let $s = 0$.

$$a_2 = \frac{-n(n+1)a_0}{2!}$$

$$\begin{aligned} L = \lim_{n \to \infty} \left\| \frac{a_{n+1}}{a_n} \right\| = \lim_{n \to \infty} &\left\| \frac{-2 \cdot (n+1)(n+2) \cdot a_0}{-2 \cdot (n+1) \cdot n \cdot a_0} \right\| \\ &\lim_{n \to \infty} \left\| 1 + \frac{2}{n} \right\| \therefore L = 1 \end{aligned}$$

$$R = \frac{1}{L} \therefore R = 1.$$