ME572: Homework #4

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Jedediah Frey

Problem 1

Device 1:

$$
\begin{aligned}
P_x &= L_3\left(\cos(\theta_2)\sin(\theta_1)\sin(\theta_3) + \cos(\theta_3)\sin(\theta_1)\sin(\theta_2)\right) + d1\cos(\theta_1) + L_2\sin(\theta_1)\sin(\theta_2) && (1)\\
&= d_1\cos(\theta_1) + L_3\sin(\theta_1)\sin(\theta_2+\theta_3) + L_2\sin(\theta_1)\sin(\theta_2) && (2)\\
&= d_1\cos(\theta_1) + \sin(\theta_1)(L_3\sin(\theta_2+\theta_3) + L_2\sin(\theta_2)) && (3)\\
P_y &= d_1\sin(\theta_1) - L_3(\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) + \cos(\theta_1)\cos(\theta_3)\sin(\theta_2)) - L_2\cos(\theta_1)\sin(\theta_2) && (4)\\
&= d_1\sin(\theta_1) - L_3\cos(\theta_1)\sin(\theta_2+\theta_3) - L_2\cos(\theta_1)\sin(\theta_2) && (5)\\
&= d_1\sin(\theta_1) - \cos(\theta_1)(L_3\sin(\theta_2+\theta_3) + L_2\sin(\theta_2)) && (6)\\
P_z &= L_1 + L_3\cos(\theta_2+\theta_3) + L_2\cos(\theta_2) && (7)\\
T_w &= \begin{bmatrix} \cos(\theta_1) & -\cos(\theta_2+\theta_3)\sin(\theta_1) & -\sin(\theta_2+\theta_3)\sin(\theta_1) & P_x \\ \sin(\theta_1) & \cos(\theta_2+\theta_3)\cos(\theta_1) & \sin(\theta_2+\theta_3)\sin(\theta_1) & P_y \\ 0 & \sin(\theta_2+\theta_3) & \cos(\theta_2+\theta_3) & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} && (8)
\end{aligned}
$$

Device 2:

$$
\begin{aligned}
P_x &= d_1\cos(\theta_1) + S_3\sin(\theta_1)\sin(\theta_2) && (9)\\
P_y &= d_1\sin(\theta_1) - S_3\cos(\theta_1)\sin(\theta_2) && (10)\\
P_z &= L_1 + S_3\cos(\theta_2) && (11)\\
T_w &= \begin{bmatrix} \cos(\theta_1) & -\cos(\theta_2)\sin(\theta_1) & \sin(\theta_1)\sin(\theta_2) & P_x \\ \sin(\theta_1) & \cos(\theta_1)\cos(\theta_2) & -\cos(\theta_1)\sin(\theta_2) & P_y \\ 0 & \sin(\theta_2) & \cos(\theta_2) & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} && (12)
\end{aligned}
$$

Problem 2

Device 2:

[Completed first since it was simpler]

Solve for S_3 by squaring and adding all of the terms. Subtract L_1 from the P_z equation first.

$$P_x^2 + P_y^2 + (P_z - L_1)^2 = S_3^2 + d_1^2 \tag{13}$$
$$S_3^2 = P_x^2 + P_y^2 + (P_z - L_1)^2 - d_1^2 \tag{14}$$
$$S_3 = \pm\sqrt{P_x^2 + P_y^2 + (P_z - L_1)^2 - d_1^2} \tag{15}$$

Given the world transform looks like:

$$T_w = \begin{bmatrix} N_x & O_x & A_x & P_x \\ N_y & O_y & A_y & P_y \\ N_z & O_z & A_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

The desired variables can be isolated by manipulating the transforms. (Easier to me than trig identities).

$$\phi_2 T_2 \phi_3 T_3 = (\phi_1 T_1)^{-1} Tw \tag{17}$$
$$P_x \cos(\theta_1) + P_y \sin(\theta_1) = d_1 \tag{18}$$

Basic definitions to substitute into the above equation.

$$R = \sqrt{P_x^2 + P_y^2} \tag{19}$$
$$\phi = tan^{-1}\left(\frac{P_y}{P_x}\right) \tag{20}$$
$$P_x = R\cos\phi \tag{21}$$
$$P_y = R\sin\phi \tag{22}$$

Values substituted in and reduced to simplest form.

$$R\cos(\phi)\cos(\theta_1) + R\sin(\phi)\sin(\theta_1) = d_1 \tag{23}$$
$$R\cos(\phi - \theta_1) = d_1 \tag{24}$$

Simplify the angle into α for now.

$$\phi - \theta_1 = \alpha \tag{25}$$
$$R\cos(\alpha) = d_1 \tag{26}$$

Multiply both sides by sin until we end up with tan on one side.

$$R\cos(\alpha)\sin(\alpha) = d_1\sin(\alpha) \tag{27}$$
$$\frac{R\cos(\alpha)\sin(\alpha)}{d_1} = \sin(\alpha) \tag{28}$$
$$\frac{R\sin(\alpha)}{d_1} = \frac{\sin(\alpha)}{\cos(\alpha)} \tag{29}$$
$$\frac{R\sin(\alpha)}{d_1} = \tan\alpha \tag{30}$$

Problem 2 continued on next page...

More trig identities so we can substitute in what we already know.

$$\sin\alpha^2 + \cos\alpha^2 = 1 \tag{31}$$
$$\sin\alpha = \sqrt{1-\cos\alpha^2} \tag{32}$$
$$= \sqrt{1-\frac{d_1^2}{R^2}} \tag{33}$$

Solve for the angle, replace with our definition for α. Final symbolic equation is in Eqn. 36.

$$\alpha = \tan^{-1}\left(\frac{\pm\sqrt{\left(1-\frac{d_1^2}{P_x^2+P_y^2}\right)\left(P_x^2+P_y^2\right)}}{d_1}\right) \tag{34}$$
$$\phi - \theta_1 = \tag{35}$$
$$\theta_1 = \tan^{-1}\left(\frac{P_y}{P_x}\right) - \tan^{-1}\left(\frac{\pm\sqrt{\left(1-\frac{d_1^2}{P_x^2+P_y^2}\right)\left(P_x^2+P_y^2\right)}}{d_1}\right) \tag{36}$$

From the same transform that was done to T_w above, the P_x and P_y equations become:

$$P_y\cos(\theta_1) - Px\sin(\theta_1) = -S_3\sin(\theta_2) \tag{37}$$
$$P_z = L_1 + S_3\cos(\theta_2) \tag{38}$$
$$\sin(\theta_2) = \frac{Px\sin(\theta_1) - P_y\cos(\theta_1)}{S_3} \tag{39}$$
$$\cos(\theta_2) = \frac{P_z - L_1}{S_3} \tag{40}$$
$$\tan(\theta_2) = \frac{\frac{Px\sin(\theta_1)-P_y\cos(\theta_1)}{S_3}}{\frac{P_z-L_1}{S_3}} \tag{41}$$

One thing to note on the above equation, S_3 can not be canceled since it puts the solution in the right quadrant when using atan2 (the only real tan^{-1} there is).
Device 1: θ_1 is solved in the same manner as above. The equation below is the same so the solutions for θ_1 will be too.

$$P_x\cos(\theta_1) + P_y\sin(\theta_1) = d_1 \tag{42}$$
$$\theta_1 = \tan^{-1}\left(\frac{P_y}{P_x}\right) - \tan^{-1}\left(\frac{\pm\sqrt{\left(1-\frac{d_1^2}{P_x^2+P_y^2}\right)\left(P_x^2+P_y^2\right)}}{d_1}\right) \tag{43}$$

Problem 3

Device 1:
Device 2: There are 2 unique solutions for both S_3 and θ_1 and thus 4 solutions to the problem.
$S_3 = +\sqrt{43}$ $\theta_1 = 254.2776^o$ $\theta_3 = 319.6844^o$
$S_3 = -\sqrt{43}$ $\theta_1 = 254.2776^o$ $\theta_3 = 139.6844^o$
$S_3 = +\sqrt{43}$ $\theta_1 = 347.6500^o$ $\theta_3 = 40.3156^o$
$S_3 = -\sqrt{43}$ $\theta_1 = 347.6500^o$ $\theta_3 = 220.3156^o$