ME572: Homework #7

Due on March 21, 2012

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A a planar two link manipulator shown has equal link lengths of 20 and is to move from (6, 0.01) to (-4,0.01) in 0.5 seconds.

Generate values for θ_1, θ_2, $\dot{\theta}_1$, $\dot{\theta}_2$ at every 0.025 seconds first using JIM then using CIM. In JIM the joints should reach constant speed in 0.05 s. In CIM the end point should reach constant speed in 0.05 s. Deceleration of each of the parameters should also occur in 0.05 s and use a linear change in speed during both acceleration and deceleration

First generate the forward kinematic equations for the manipulator.

$$\phi_1 = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$T_1 = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 20.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \tag{2}$$

$$\phi_2 = \begin{bmatrix} \cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ \sin(\theta_2) & \cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$T_2 = \begin{bmatrix} 1.00 & 0.00 & 0.00 & 20.00 \\ 0.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & 1.00 \end{bmatrix} \tag{4}$$

$$T_w = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & 20\cdot\cos(\theta_1+\theta_2)+20\cdot\cos(\theta_1) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & 20\cdot\sin(\theta_1+\theta_2)+20\cdot\sin(\theta_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Since we only care about the end position and not orientation we only need the equations for x and y coordinates:

$$x = \left[20\cdot\cos(\theta_1+\theta_2)+20\cdot\cos(\theta_1)\right] \tag{6}$$

$$y = \left[20\cdot\sin(\theta_1+\theta_2)+20\cdot\sin(\theta_1)\right] \tag{7}$$

The inverse kinematics for θ_1 can be found by using atan2 to find the angle of the point (x, y) from the world coordinates. ($\hat{\theta}$ in the sketch below). Then use the law of cosines to find the interior angle of the triangle. As drawn the interior angle is always additive to $\hat{\theta}$ to get θ_1, but it can also be oriented in the other direction. θ_2 can be found using atan2 from the end of link 1 to the end of link 2 then subtracting off θ_1 since θ_2 is measured from coordinate frame rotating with the first linkage.

$$\theta_1 = \tan\left(\frac{y}{x}\right)^{-1} + \cos\left(\frac{l_1^2 - l_2^2 + x^2 + y^2}{2l_1\sqrt{x^2+y^2}}\right)^{-1} \tag{8}$$

$$\theta_2 = \tan\left(\frac{y - l_1\sin\theta_1}{x - l_1\cos\theta_1}\right)^{-1} - \theta_1 \tag{9}$$

Where $l_1 = l_2 = 20$.

For the the initial and final points the following joint angles were found:

X	Y	θ_1	θ_2
6	0.01	81.4686^o	197.2539^o
-4	0.01	264.1176^o	191.4784^o

For each of the types of motion the velocity profile must ramp up and then down the area under the velocity curve will be the distance traveled by either the θs for JIM or the tip (x, y) for CIM.

Divide the constant velocity time steps up into areas of size A*, with magnitude $\dot{Q}_{max}$ and duration Δ.

$$0.25A^* + 0.75A^* + 16A^* + 0.75A^* + 0.25A^* = Q_{final} - Q_{initial} \tag{10}$$

For all variables $A^* = \frac{\Delta Q}{18}$. The δ of each variable for each step is given by $0.25A^*$ & $0.75A^*$ during acceleration and reverse during deceleration. At all points in between it is equal to A^*.

Step	**Time**	$\Delta\theta_1$	θ_1	$\Delta\theta_2$	θ_2	**X**	**Y**	$\dot{\theta}_1$	$\dot{\theta}_2$	$\dot{X}$	$\dot{Y}$
0	0.000	0.000	81.469	0.000	197.254	6.000	0.010				
1	0.025	2.537	84.005	-0.080	197.174	5.947	0.534	101.472	-3.209	-2.660	26.176
2	0.050	7.610	91.616	-0.241	196.933	5.848	1.048	304.415	-9.626	-4.928	25.715
3	0.075	10.147	101.763	-0.321	196.612	5.468	2.005	405.887	-12.834	-19.006	47.852
4	0.100	10.147	111.910	-0.321	196.291	4.937	2.863	405.887	-12.834	-26.578	42.910
5	0.125	10.147	122.057	-0.321	195.970	4.276	3.599	405.887	-12.834	-33.026	36.820
6	0.150	10.147	132.204	-0.321	195.650	3.512	4.196	405.887	-12.834	-38.185	29.818
7	0.175	10.147	142.352	-0.321	195.329	2.674	4.639	405.887	-12.834	-41.937	22.164
8	0.200	10.147	152.499	-0.321	195.008	1.789	4.922	405.887	-12.834	-44.218	14.130
9	0.225	10.147	162.646	-0.321	194.687	0.889	5.041	405.887	-12.834	-45.010	5.988
10	0.250	10.147	172.793	-0.321	194.366	0.002	5.002	405.887	-12.834	-44.348	-1.992
11	0.275	10.147	182.940	-0.321	194.045	-0.844	4.810	405.887	-12.834	-42.313	-9.560
12	0.300	10.147	193.087	-0.321	193.724	-1.625	4.481	405.887	-12.834	-39.027	-16.484
13	0.325	10.147	203.235	-0.321	193.404	-2.318	4.029	405.887	-12.834	-34.652	-22.567
14	0.350	10.147	213.382	-0.321	193.083	-2.905	3.476	405.887	-12.834	-29.376	-27.645
15	0.375	10.147	223.529	-0.321	192.762	-3.374	2.845	405.887	-12.834	-23.415	-31.595
16	0.400	10.147	233.676	-0.321	192.441	-3.713	2.158	405.887	-12.834	-16.995	-34.337
17	0.425	10.147	243.823	-0.321	192.120	-3.920	1.441	405.887	-12.834	-10.351	-35.835
18	0.450	10.147	253.970	-0.321	191.799	-3.995	0.719	405.887	-12.834	-3.715	-36.098
19	0.475	7.610	261.581	-0.241	191.559	-4.013	0.363	304.415	-9.626	-0.915	-17.788
20	0.500	2.537	264.118	-0.080	191.478	-4.000	0.010	101.472	-3.209	0.654	-17.670

Table 1: Joint interpolated motion

Step	**Time**	θ_1	θ_2	ΔX	**X**	ΔY	**Y**	$\dot{\theta}_1$	$\dot{\theta}_2$	$\dot{X}$	$\dot{Y}$
0.0	0.0	81.4686	197.2539	0.0	6.0000	0.0	0.01				
1.0000	0.0250	81.8754	196.4494	-0.1389	5.8611	0.0	0.01	16.2746	-32.1783	-5.5556	0.0
2.0000	0.0500	82.2823	195.6458	-0.4167	5.4444	0.0	0.01	16.2772	-32.1457	-16.6667	0.0
3.0000	0.0750	83.0968	194.0408	-0.5556	4.8889	0.0	0.01	32.5785	-64.2002	-22.2222	0.0
4.0000	0.1000	83.9130	192.4385	-0.5556	4.3333	0.0	0.01	32.6458	-64.0896	-22.2222	0.0
5.0000	0.1250	84.7323	190.8387	-0.5556	3.7778	0.0	0.01	32.7738	-63.9921	-22.2222	0.0
6.0000	0.1500	85.5573	189.2410	-0.5556	3.2222	0.0	0.01	32.9996	-63.9074	-22.2222	0.0
7.0000	0.1750	86.3923	187.6452	-0.5556	2.6667	0.0	0.01	33.3994	-63.8353	-22.2222	0.0
8.0000	0.2000	87.2460	186.0508	-0.5556	2.1111	0.0	0.01	34.1495	-63.7758	-22.2222	0.0
9.0000	0.2250	88.1395	184.4576	-0.5556	1.5556	0.0	0.01	35.7413	-63.7285	-22.2222	0.0
10.0	0.2500	89.1403	182.8652	-0.5556	1.0000	0.0	0.01	40.0310	-63.6930	-22.2222	0.0
11.0000	0.2750	90.6521	181.2736	-0.5556	0.4444	0.0	0.01	60.4728	-63.6657	-22.2222	0.0
12.0000	0.3000	264.6974	180.3196	-0.5556	-0.1111	0.0	0.01	6961.8117	-38.1596	-22.2222	0.0
13.0000	0.3250	268.1855	181.9102	-0.5556	-0.6667	0.0	0.01	139.5244	63.6226	-22.2222	0.0
14.0000	0.3500	267.7802	183.5021	-0.5556	-1.2222	0.0	0.01	-16.2142	63.6763	-22.2222	0.0
15.0000	0.3750	267.1304	185.0947	-0.5556	-1.7778	0.0	0.01	-25.9934	63.7058	-22.2222	0.0
16.0000	0.4000	266.4103	186.6884	-0.5556	-2.3333	0.0	0.01	-28.8036	63.7460	-22.2222	0.0
17.0000	0.4250	265.6600	188.2833	-0.5556	-2.8889	0.0	0.01	-30.0102	63.7981	-22.2222	0.0
18.0000	0.4500	264.8937	189.8799	-0.5556	-3.4444	0.0	0.01	-30.6518	63.8626	-22.2222	0.0
19.0000	0.4750	264.5066	190.6789	-0.4167	-3.8611	0.0	0.01	-15.4832	31.9594	-16.6667	0.0
20.0	0.5000	264.1176	191.4784	-0.1389	-4.0000	0.0	0.01	-15.5626	31.9803	-5.5556	0.0

Table 2: Cartesian interpolated motion

For both of the tables the velocities of the end effector and joint angles are the average velocity during that step. For example for CIM the average x velocity during step 1 is -5.5556/s, since speed is ramping linearly (constant acceleration) the velocity at the beginning of step 1 is 0/s and at the end is -11.1111/s, for step 2 it starts at -11.1111/s and ends at -22.2222/s, averaging -16.6667/s, and reaching a constant speed by the end of step 2 (0.05s).

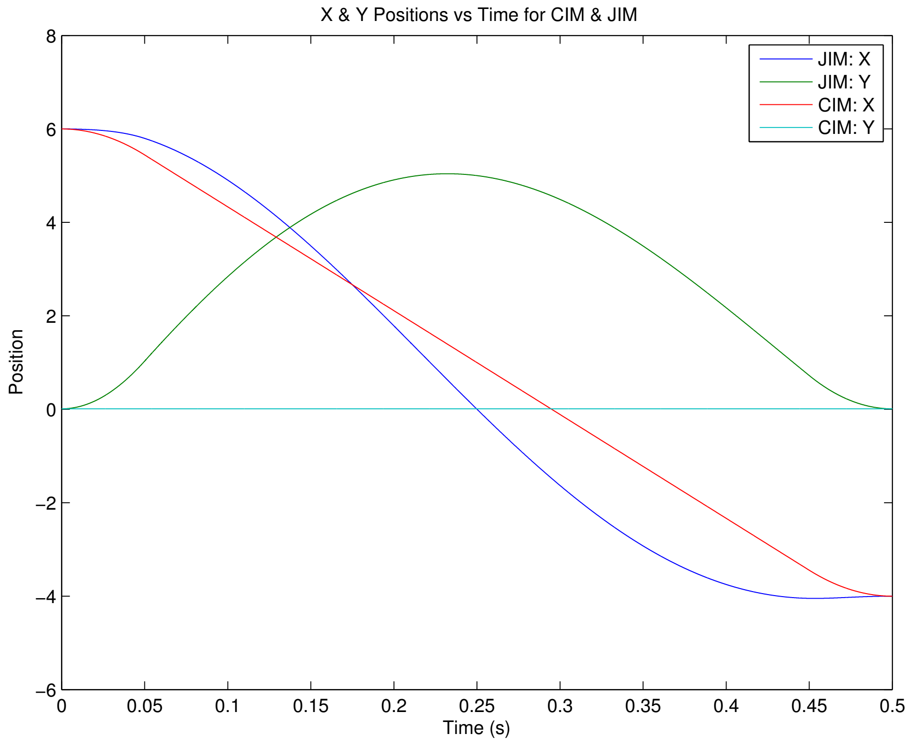

Figure 1: End Position X & Y coordinates vs time for both types of motion

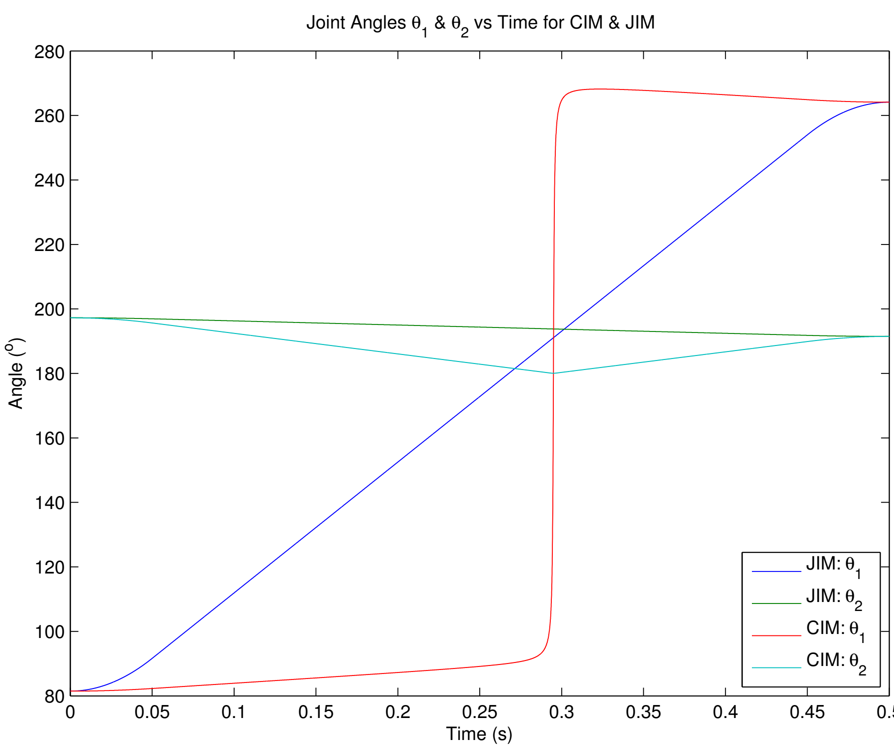

Figure 2: Joint angles θ_1 & θ_2 vs time for both types of motion

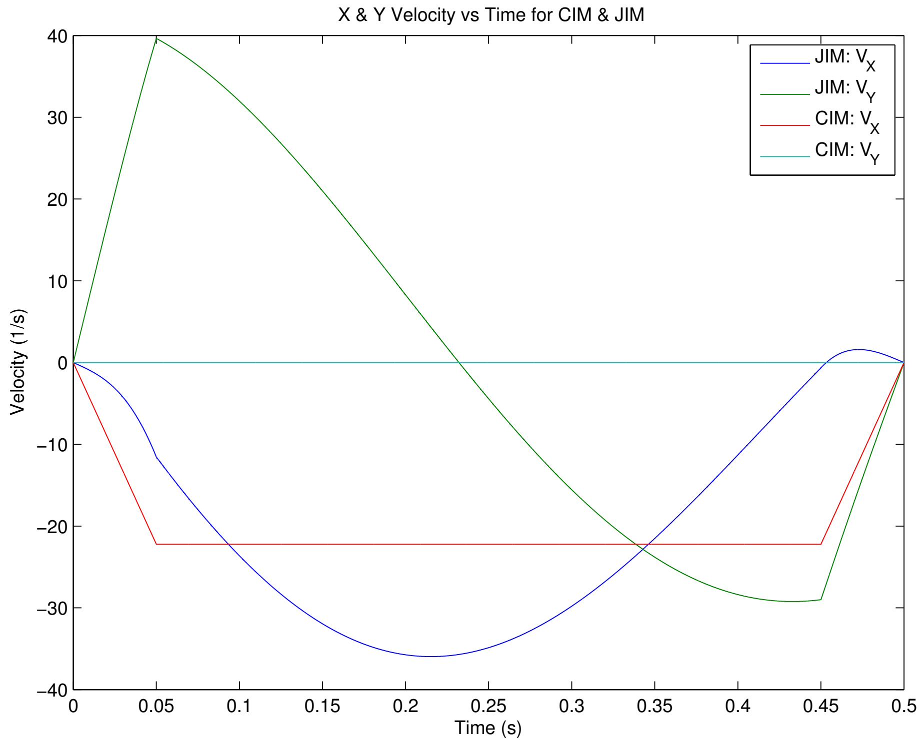

Figure 3: Velocity of X & Y position vs time for both types of motion

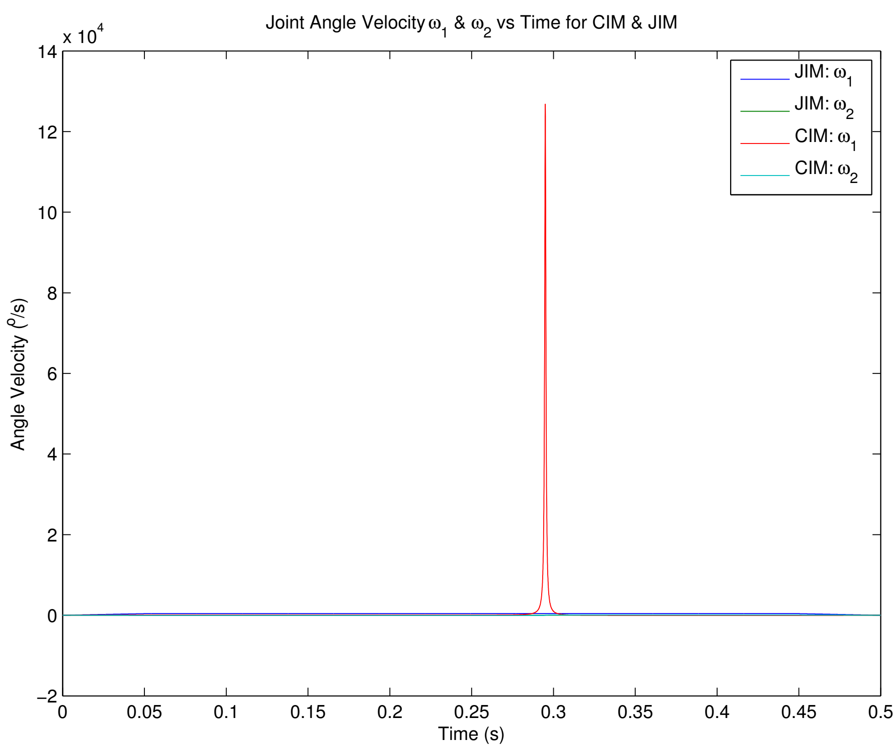

Figure 4: Angular velocity of θ_1 & θ_2 position vs time for both types of motion

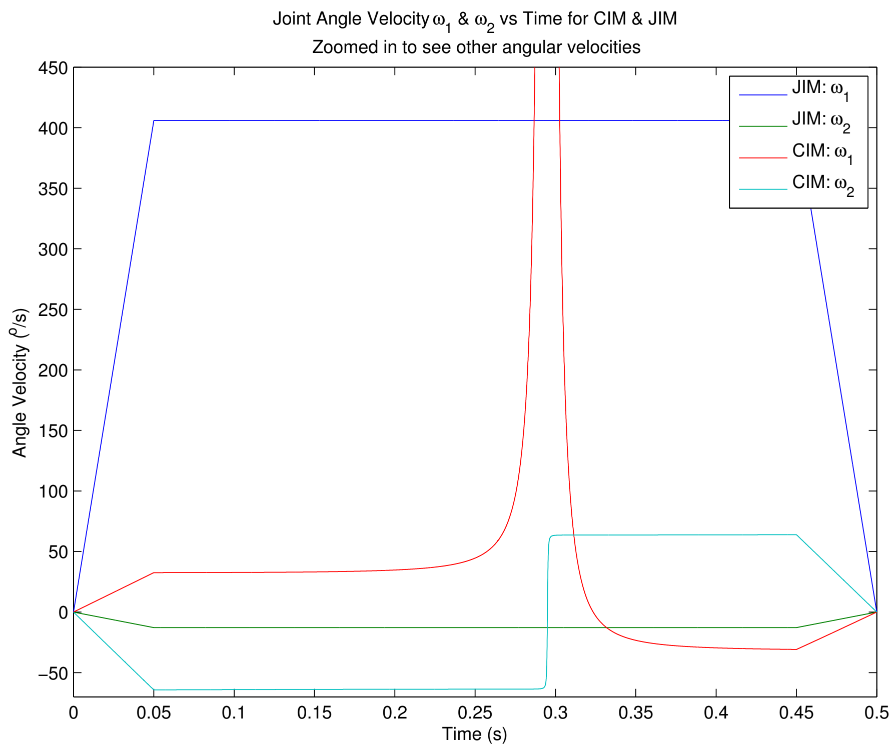

Figure 5: Angular velocity plot zoomed in to see the detail of the other angular velocities

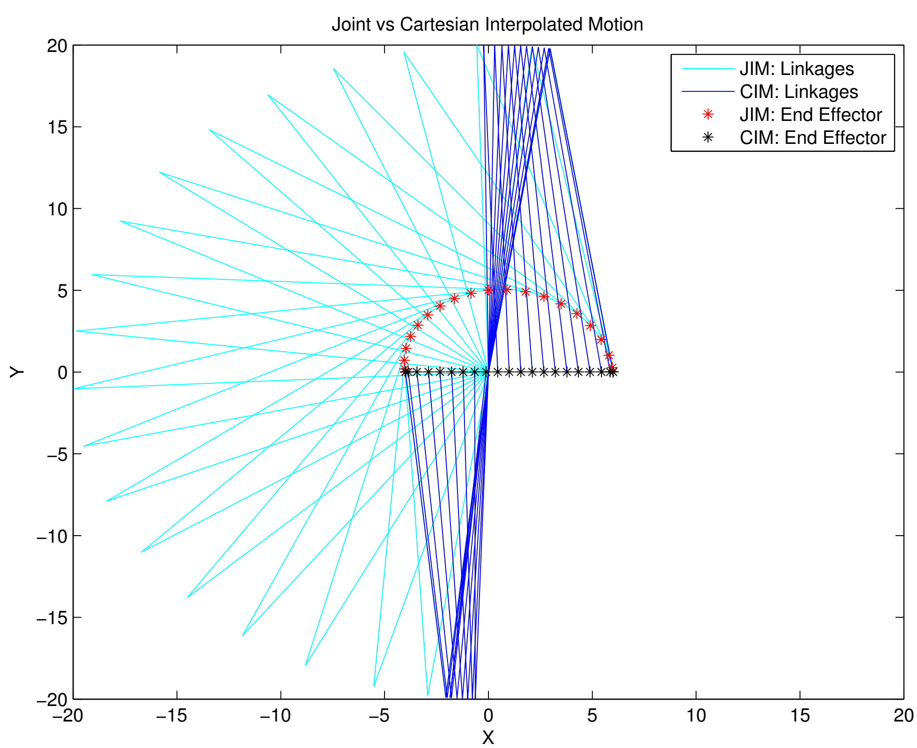

Figure 6: Manipulator motion in for both types of motion

Comment on the results obtained

Joint interpolated motion kept the angular velocities of each of the joints reasonable however it didn't follow a straight line path from start to finish. The cartesian interpolated motion followed a straight line from start to finish however since it passed near the singularity the angular velocity of θ_1 spiked as it swung around from quadrant II to III. Dexterity ellipsoid.

The Jacobian matrix is in the form of:

$$\begin{Bmatrix} \dot{P}_x \\ \dot{P}_y \\ \dot{P}_z \\ \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = \begin{bmatrix} J \\ 6 \times 2 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} \tag{11}$$

The Jacobian matrix for this problem is:

$$J = \begin{Bmatrix} \vec{a}_0 \times \left(\vec{P}_E - \vec{P}_0\right) & \vec{a}_1 \times \left(\vec{P}_E - \vec{P}_1\right) \\ \vec{a}_0 & \vec{a}_1 \end{Bmatrix} \tag{12}$$

Where:

$$P_e = \begin{bmatrix} 20 \cdot \cos(\theta_1 + \theta_2) + 20 \cdot \cos(\theta_1) \\ 20 \cdot \sin(\theta_1 + \theta_2) + 20 \cdot \sin(\theta_1) \\ 0 \end{bmatrix} \tag{13}$$

$$a_0 = \begin{bmatrix} 0.00 \\ 0.00 \\ 1.00 \end{bmatrix} \tag{14}$$

$$P_0 = \begin{bmatrix} 20 \cdot \cos(\theta_1 + \theta_2) + 20 \cdot \cos(\theta_1) \\ 20 \cdot \sin(\theta_1 + \theta_2) + 20 \cdot \sin(\theta_1) \\ 0 \end{bmatrix} \tag{15}$$

$$a_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{16}$$

$$P_1 = \begin{bmatrix} 20 \cdot \cos(\theta_1 + \theta_2) \\ 20 \cdot \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \tag{17}$$

$$J = \begin{bmatrix} -20 \cdot \sin(\theta_1 + \theta_2) - 20 \cdot \sin(\theta_1) & -20 \cdot \sin(\theta_1 + \theta_2) \\ 20 \cdot \cos(\theta_1 + \theta_2) + 20 \cdot \cos(\theta_1) & 20 \cdot \cos(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \tag{18}$$

The manipulability of linear velocity is given by $\dot{\vec{x}} = J_{top}\dot{q}$ where J_{top} is square. For the manipulator:

$$J_{top} = \begin{bmatrix} -20 \cdot \sin(\theta_1+\theta_2) - 20 \cdot \sin(\theta_1) & -20 \cdot \sin(\theta_1+\theta_2) \\ 20 \cdot \cos(\theta_1+\theta_2) + 20 \cdot \cos(\theta_1) & 20 \cdot \cos(\theta_1+\theta_2) \end{bmatrix} \tag{19}$$

$$B = J_{top}J_{top}^T \tag{20}$$

$$B_{(1,1)} = \left[400 \cdot \sin(\theta_1+\theta_2)^2 + 400 \cdot (\sin(\theta_1+\theta_2) + \sin(\theta_1))^2\right] \tag{21}$$

$$B_{(1,2)} = \left[-400 \cdot \sin(2 \cdot \theta_1 + \theta_2) - 200 \cdot \sin(2 \cdot \theta_1) - 400 \cdot \sin(2 \cdot \theta_1 + 2 \cdot \theta_2)\right] \tag{22}$$

$$B_{(2,1)} = \left[-400 \cdot \sin(2 \cdot \theta_1 + \theta_2) - 200 \cdot \sin(2 \cdot \theta_1) - 400 \cdot \sin(2 \cdot \theta_1 + 2 \cdot \theta_2)\right] \tag{23}$$

$$B_{(2,2)} = \left[(20 \cdot \cos(\theta_1+\theta_2) + 20 \cdot \cos(\theta_1))^2 + 400 \cdot \cos(\theta_1+\theta_2)^2\right] \tag{24}$$

The ellipsoid can then be determined from the eigenvalues and eigenvectors where

$$a_x = \frac{1}{\sqrt{\lambda_1}} \tag{25}$$

$$b_y = \frac{1}{\sqrt{\lambda_2}} \tag{26}$$

$$\tag{27}$$

For the point near the singularity where: $\theta_1 = 264.6974$ & $\theta_2 = 180.3196$:
$\beta = 175.0170$, $a_x = 0.0500$, $b_y = 8.9637$.
For the point at the start where: $\theta_1 = 81.469$ & $\theta_2 = 197.254$:
$\beta = 170.4411$, $a_x = 0.0499$, $b_y = 0.1688$.
The shape becomes very cylindrical near the singularity because b_y is very large compared to a_x.