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1 Buildings

Definition 1 *A Polyhedral complex is a certain finite dimensional CW-complex. Each n -cell of the polyhedral complex is*

Definition 2 *Suppose P is a simple convex polytope in X^n . Let F_i be the codimension-one faces of P . Suppose that, for any two faces F_i and F_j , if their intersection is non-empty, then the dihedral angle between the faces is π/m_{ij} , for some m_{ij} in $2, 3, 4, \dots$. Now set $m_{ii} = 1, m_{ij} = \infty$ if F_i, F_j empty intersection. Let s_i be the reflection of X^n across F_i , and let W be the group generated by the set of s_i 's. Then W is the Coxeter group with generators s_i , and Coxeter matrix (m_{ij}) . Furthermore, W is a discrete subgroup of $\text{Isom}(X^n)$, P is a strict fundamental domain for the W action, and P tiles X^n .*

Definition 3 *Let (W, S) be a Coxeter group generated by a simple convex polytope P . A building of type (W, S) is a polyhedral complex, which is a union of subcomplexes, called apartments. An apartment is isometric to the tiling of X^n derived from P , and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy*

1. *Given any two chambers, there exists an apartment containing both of them.*

2. Given any two apartments A and B , there exists an isometry from A to B which fixes $A \cap B$ pointwise.

Example 1 Let us consider a single copy of X^n . We can tile this copy by P , and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P , they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. If they share a common edge, then reflection along this edge gives us our isometry.
2. If they only share a common point

Example 2 Now we can consider a spherical building. Take the Coxeter group

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4 .

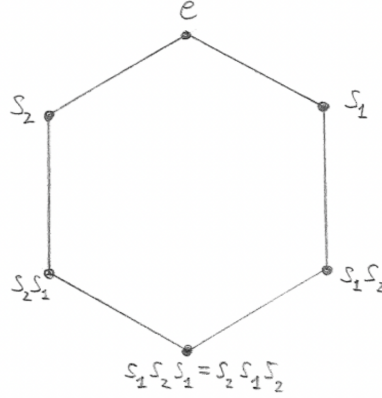
2 Cayley graphs

Definition 4 The Cayley graph $\text{Cay}(G, S)$ of a group G with respect to a generating set S , $1 \notin S$, is the graph (V, E) , with $V = G$, and directed edges

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If $s \in S$ is an involution, we only put a single undirected edge between g and gs , and label the edge s .

Example 3 The Cayley graph of D_6 , with generating set $\{s_1, s_2\}$ is



3 Reflection systems

Definition 5 Let G be a group. A pre-reflection system for G is a pair (X, R) . X is a connected simplicial graph which is acted upon by G , and R is a subset of G . This must satisfy

1. every element of R is an involution;
2. R is closed under conjugation;
3. R generates G ;
4. given an edge of X , there is a unique element of R which flips the edge;
and
5. for every element r of R , there is at least one edge of X which is flipped by r .

Example 4 Let (W, S) be any Coxeter system. Let X be the Cayley graph of (W, S) , and let

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.

Definition 6 Consider a pre-reflection system (X, R) . For each element r of R , the wall H_r is the set of midpoints of all the edges flipped by r .

Definition 7 Consider a pre-reflection system (X, R) . If, additionally, it satisfies

6. for every element r in R , $X \setminus H_r$ has exactly two components,

then (X, R) is a reflection system.

Theorem 1 Suppose we have a group W , generated by a set S of distinct involutions. Then the following are equivalent:

1. (W, S) is a Coxeter system;
2. (X, R) is a reflection system, where $X = \text{Cay}(W, S)$ and

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

3. (W, S) satisfies the Deletion Condition; and
4. (W, S) satisfies the Exchange Condition.

4 Tits' solution to the word problem

Definition 8 Let W be generated by a set S of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let m_{st} be the order of st in W . If m_{st} is finite, consider a word in S with the subword $sts\dots$ with m_{st} letters. A braid move on the word replaces the subword $sts\dots$ with $tst\dots$, again with m_{st} letters.

Theorem 2 (Tits) Suppose we have a group W , generated by a set S of distinct involutions. Also suppose that the Exchange Condition holds. Then

1. A word $s_1s_2\dots s_k$ is reduced iff we cannot shorten it by a sequence of
 - deleting an instance of ss from the word, or
 - applying a braid move to the word.
2. Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.

5 Tits' Representation Theorem

Theorem 3 (*Tits' Representation Theorem*) Let (W, S) be a Coxeter system. Then there is a map

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

which is a faithful representation, with $N = |S|$, such that

1. $\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and
2. If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order m_{ij} .

Definition 9 The representation ρ above is called the Tits representation, or sometimes the standard geometric representation.

5.1 Corollaries

6 Construction of a geometric realisation

6.1 Simplicial complexes

Definition 10 Let V be a, possibly infinite, set, called the vertex set. Let X be a collection of finite subsets of V such that

1. $\{v\} \in X$ for all elements $v \in V$; and
2. if $\Delta \in X$ and Δ' is a subset of Δ , then Δ' is in X .

Then (V, X) is an abstract simplicial complex.

Definition 11 A simplex is any element of X . A simplex Δ has dimension

$$\dim \Delta = |\Delta| - 1.$$

A simplex of dimension k is called a k -simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension k is the k -skeleton $X^{(k)}$.

Lemma 1 The k -skeleton is also a simplicial complex.

Definition 12 *The dimension of X is*

$$\dim X = \max\{\dim(\Delta) \mid \Delta \in X\}.$$

If all the maximal elements of X have the same dimension, then the simplicial complex is pure.

Definition 13 *The standard n -simplex Δ^n is the convex hull of the $(n + 1)$ points $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ in \mathbb{R}^{n+1} .*

Given an n -simplex Δ in X , we can identify Δ with a copy of Δ^n .

6.2 The basic construction

Definition 14 *Let X be a connected Hausdorff topological space. Let (W, S) be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of X . Then $(X_s)_{s \in X}$ is a mirror structure on X over S , and X_s is called the s -mirror.*

Example 5