

1 Buildings

Definition 1 A Polyhedral complex is a certain finite dimensional CW-complex. Each n -cell of the polyhedral complex is

Definition 2 Suppose P is a simple convex polytope in X^n . Let F_i be the codimension-one faces of P . Suppose that, for any two faces F_i and F_j , if their intersection is non-empty, then the dihedral angle between the faces is π/m_{ij} , for some m_{ij} in $2, 3, 4, \dots$. Now set $m_{ii} = 1$, $m_{ij} = \infty$ if F_i, F_j empty intersection. Let s_i be the reflection of X^n across F_i , and let W be the group generated by the set of s_i 's. Then W is the Coxeter group with generators s_i , and Coxeter matrix (m_{ij}) . Furthermore, W is a discrete subgroup of $\text{Isom}(X^n)$, P is a strict fundamental domain for the W action, and P tiles X^n .

Definition 3 Let (W, S) be a Coxeter group generated by a simple convex polytope P . A building of type (W, S) is a polyhedral complex, which is a union of subcomplexes, called apartments. An apartment is isometric to the tiling of X^n derived from P , and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy

1. Given any two chambers, there exists an apartment containing both of them.
2. Given any two apartments A and B , there exists an isometry from A to B which fixes $A \cap B$ pointwise.

Example 1 Let us consider a single copy of X^n . We can tile this copy by P , and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P , they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. If they share a common edge, then reflection along this edge gives us our isometry.
2. If they only share a common point

Example 2 Now we can consider a spherical building. Take the Coxeter group

$$W = \langle s_1, s_2 \mid s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4 .

2 Reflection systems

Definition 4 Let G be a group. A pre-reflection system for G is a pair (X, R) . X is a connected simplicial graph which is acted upon by G , and R is a subset of G . This must satisfy

1. every element of R is an involution;
2. R is closed under conjugation;
3. R generates G ;
4. given an edge of X , there is a unique element of R which flips the edge;
and
5. for every element r of R , there is at least one edge of X which is flipped by r .

Example 3 Let (W, S) be any Coxeter system. Let X be the Cayley graph of (W, S) , and let

$$R = \{wsw^{-1} \mid w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.