

# 1 Buildings

**Definition 1** A Polyhedral complex is a certain finite dimensional CW-complex. Each  $n$ -cell of the polyhedral complex is

**Definition 2** Suppose  $P$  is a simple convex polytope in  $X^n$ . Let  $F_i$  be the codimension-one faces of  $P$ . Suppose that, for any two faces  $F_i$  and  $F_j$ , if their intersection is non-empty, then the dihedral angle between the faces is  $\pi/m_{ij}$ , for some  $m_{ij}$  in  $2, 3, 4, \dots$ . Now set  $m_{ii} = 1, m_{ij} = \infty$  if  $F_i, F_j$  empty intersection. Let  $s_i$  be the reflection of  $X^n$  across  $F_i$ , and let  $W$  be the group generated by the set of  $s_i$ 's. Then  $W$  is the Coxeter group with generators  $s_i$ , and Coxeter matrix  $(m_{ij})$ . Furthermore,  $W$  is a discrete subgroup of  $\text{Isom}(X^n)$ ,  $P$  is a strict fundamental domain for the  $W$  action, and  $P$  tiles  $X^n$ .

**Definition 3** Let  $(W, S)$  be a Coxeter group generated by a simple convex polytope  $P$ . A building of type  $(W, S)$  is a polyhedral complex, which is a union of subcomplexes, called apartments. An apartment is isometric to the tiling of  $X^n$  derived from  $P$ , and each copy of  $P$  in the tiling is called a chamber. Now the apartments and chambers must satisfy

1. Given any two chambers, there exists an apartment containing both of them.
2. Given any two apartments  $A$  and  $B$ , there exists an isometry from  $A$  to  $B$  which fixes  $A \cap B$  pointwise.

**Example 1** Let us consider a single copy of  $X^n$ . We can tile this copy by  $P$ , and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of  $P$ , they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. If they share a common edge, then reflection along this edge gives us our isometry.
2. If they only share a common point

**Example 2** Now we can consider a spherical building. Take the Coxeter group

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to  $D_4$ .

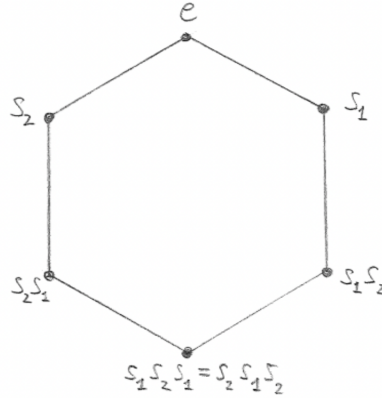
## 2 Cayley graphs

**Definition 4** The Cayley graph  $\text{Cay}(G, S)$  of a group  $G$  with respect to a generating set  $S$ ,  $1 \notin S$ , is the graph  $(V, E)$ , with  $V = G$ , and directed edges

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If  $s \in S$  is an involution, we only put a single undirected edge between  $g$  and  $gs$ , and label the edge  $s$ .

**Example 3** The Cayley graph of  $D_6$ , with generating set  $\{s_1, s_2\}$  is



## 3 Reflection systems

**Definition 5** Let  $G$  be a group. A pre-reflection system for  $G$  is a pair  $(X, R)$ .  $X$  is a connected simplicial graph which is acted upon by  $G$ , and  $R$  is a subset of  $G$ . This must satisfy

1. every element of  $R$  is an involution;
2.  $R$  is closed under conjugation;

3.  $R$  generates  $G$ ;
4. given an edge of  $X$ , there is a unique element of  $R$  which flips the edge;  
and
5. for every element  $r$  of  $R$ , there is at least one edge of  $X$  which is flipped by  $r$ .

**Example 4** Let  $(W, S)$  be any Coxeter system. Let  $X$  be the Cayley graph of  $(W, S)$ , and let

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then  $(X, R)$  is a pre-reflection system.

**Definition 6** Consider a pre-reflection system  $(X, R)$ . For each element  $r$  of  $R$ , the wall  $H_r$  is the set of midpoints of all the edges flipped by  $r$ .

**Definition 7** Consider a pre-reflection system  $(X, R)$ . If, additionally, it satisfies

6. for every element  $r$  in  $R$ ,  $X \setminus H_r$  has exactly two components,

then  $(X, R)$  is a reflection system.

**Theorem 1** Suppose we have a group  $W$ , generated by a set  $S$  of distinct involutions. Then the following are equivalent:

1.  $(W, S)$  is a Coxeter system;
2.  $(X, R)$  is a reflection system, where  $X = \text{Cay}(W, S)$  and

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

3.  $(W, S)$  satisfies the Deletion Condition; and
4.  $(W, S)$  satisfies the Exchange Condition.

## 4 Tits' solution to the word problem

**Definition 8** *Let  $W$  be generated by a set  $S$  of distinct involutions. Let  $s, t \in S$ , with  $s \neq t$ , and let  $m_{st}$  be the order of  $st$  in  $W$ . If  $m_{st}$  is finite, consider a word in  $S$  with the subword  $sts\dots$  with  $m_{st}$  letters. A braid move on the word replaces the subword  $sts\dots$  with  $tst\dots$ , again with  $m_{st}$  letters.*

**Theorem 2 (Tits)** *Suppose we have a group  $W$ , generated by a set  $S$  of distinct involutions. Also suppose that the Exchange Condition holds. Then*

1. *A word  $s_1s_2\dots s_k$  is reduced iff we cannot shorten it by a sequence of*
  - *deleting an instance of  $ss$  from the word, or*
  - *applying a braid move to the word.*
2. *Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.*