1 Chamber systems

We first start our exploration of buildings with an abstract chamber system - a set with some equivalence relations on it.

Definition 1.1. [1, ?] A set C is called a *chamber system* over a set I if each $i \in I$ is an equivalence relation on the elements of C. Each i partitions our set C. We say two elements $x, y \in C$ are *i-adjacent*, and we write $x \sim_i y$, if they lie in the same part of the partition, i.e they are equivalent with respect to the equivalence relation corresponding to i. The elements of C are called *chambers*. The *rank* of a chamber system is the size of I.

A very important example is obtained by looking at a group G, and a subgroup B, and defining the following equivalence relations:

Example 1.1. [1, ?] Given a group G, a subgroup B, and an indexing set I, let there be a subgroup $B < P_i < G$ for all $i \in I$. Then we take as our chamber set C the left cosets of B, and we define an equivalence relation

$$gB \sim_i hB \text{ if and only if } gP_i = hP_i.$$

We now look at galleries of a chamber system. These are walks around the chambers, where we only move from one chamber to an adjacent chamber.

Definition 1.2. [1, ?] A finite sequence $(c_0, ..., c_k)$ such that c_i is adjacent to c_{i+1} is called a *gallery*. Its *type* is a word $i_1, ..., i_k$ in I such that c_{i-1} is i-adjacent to c_i. We assume that no two consecutive chambers are equal.

Definition 1.3. [1, ?] We call C *connected* if there is a gallery between any two chambers. Given a subset $J \subset I$, a *residue of type* J is a J-connected component. The *cotype* of J is $I - J$.

1.1 The geometric realisation

We now want to construct a geometric realisation of this chamber system. This will turn out to be an example of a building. We construct a simplicial complex, where each simplex represents a residue of our chamber system.

Definition 1.4. Let R be a J-residue and S be a K-residue. Then S is a *face* of R if $R \subset S$ and $J \subset K$. The *cotype* of J is the set $I - J$.

Observe that if R is a residue of cotype J, we have

1. for $K \subset J$, there is a unique face of R which has cotype K.
2. Let S_1, S_2 be faces of R with cotypes K_1 and K_2. Then S_1 and S_2 have a shared face of cotype $K_1 \cap K_2$.

With these observations, we can form a *cell complex* of our chamber system. To do this, we form a vertex for each residue of corank 1. Then, we can associate to each residue of cotype $\{i, j\}$ an edge. From the observation above, this has as its boundary the residues of cotype $\{i\}$ and of cotype $\{j\}$. Then this can be continued inductively....

1.2 $A_n(k)$ Buildings

A key example of a chamber system is formed by considering the subspaces of an $n+1$ dimensional vector space V over a field k. We define the chambers of our chamber system to be the maximal sequences

$$V_1 \subset V_2 \subset ... \subset V_n$$

of subspaces of V, where V_i has dimension i. We can then define adjancency by saying that two sequences $V_1 \subset V_2 \subset ... \subset V_n$ and $V_1' \subset V_2' \subset ... \subset V_n'$ are i-adjacent if and only if $V_j = v_j'$ for all $j \neq i$. Then the residues of type i correspond to 1 spaces in the 2 space V_{i+1}/V_{i-1}.

We then get a geometric realisation of this chamber system. Here, a residue of cotype $J = \{j_1, ..., j_r\}$ corresponds to a sequence

$$V_{j_1} \subset V_{j_2} \subset ... \subset V_{j_r}.$$

This residue has chambers which are maximal flags $V_1' \subset V_2' \subset ... \subset V_n'$ such that $V_j' = v_j$ if $j \in J$.

In particular, residues of cotype $\{i\}$ correspond to the subspaces of V.

2 Coxeter complexes

Given a Coxeter group W, take as chambers the elements of W, and define an i-adjancency by $w \sim_i wr_i$, where $\{s_1, ..., s_n\}$ are the set of generators of the Coxeter group. If the Coxeter group has Coxeter matrix M, we call this building a *Coxeter complex of type M.*

Diagram $\widetilde{A}_2$. 

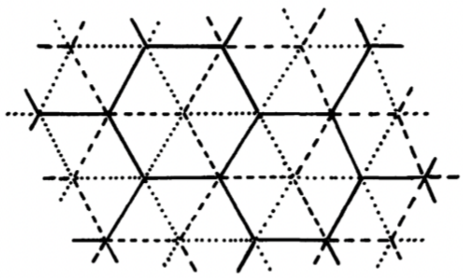
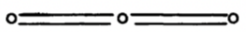


Figure 2.1

Diagram $\widetilde{C}_2$. 

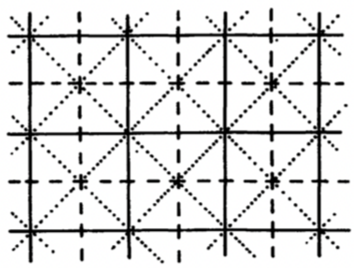


Figure 2.2

Lemma 2.1. The automorphism group of the Coxeter complex is isomorphic to the Coxeter group, and this acts simple-transitively on the set of chambers.

Definition 2.1. A *reflection* r of W is a conjugate of the generators of W. The wall M_r of a reflection r is the set of simplicies in the Coxeter complex which is fied by r when r acts on the complex by left multiplication. Then M_r is a subcomplex of codimension 1.

Theorem 2.1. *There is a bijection between the set of reflections of a Coxeter group, and the set of walls in the corresponding Coxeter complex.*

Now we can make a very similar definition of a gallery for Coxeter complexes.

Definition 2.2. Given a Coxeter complex Σ, a *combinatorial gallery* is a sequence

$$\gamma = (c_0, p_1, c_1, p_2, ..., p_n, c_n),$$

where the c_i are alcoves and the p_i are panels of Σ, such that p_i is contained in c_i and c_{i-1} for all $i - 1, ..., n$. The length of a combinatorial gallery γ is $n + 1$ - this counts how many alcoves there are in the sequence. Then γ is *minimal* if there does not exist a shorter gallery starting at c_0 and ending at c_n.

So a gallery is a path between c_0 and c_n through alcoves, such that adjacent alcoves in the path share a commmon panel.

Definition 2.3. A gallery $(c_0, ..., c_k)$ *crosses* M_r if there is an i such that M_r interchanges c_{i-1} and c_i.

Lemma 2.2.

1. Any minimal gallery does not cross a wall twice.
2. Every gallery from two alcoves x and y have the same parity of crossings of any wall.

Definition 2.4. Each hyperplane splits an apartment into two half-apartments called *roots.* If α is one root, we denote the other corresponding root by $-\alpha$.

Definition 2.5. A set of alcoves is called *convex* if any minimal gallery between two alcoves of the set lies entirely within the set.

Proposition 2.1.

1. Roots are convex.
2. Let α be a root, and let x and y be adjacent chambers with $x \in \alpha$ and $y \in -\alpha$. Then
$$\alpha = \{c | d(x, c) < d(y, c)\}.$$
3. There are bijections between the set of all reflections of a Coxeter group, the set of walls, and the set of pairs of opposite roots.

Definition 2.6. A *folding of W onto α* is the map which fixes α and sends $-\alpha$ to α by relfecting across the the defining wall of α.

Proposition 2.2. Consider any chambers x and y. Let $(x, x_1, ..., x_{k-1}, y)$ be a minimal gallery from x to y. Define β_i to be the root which contains x_{i-1} and which does not contain x_i. Then the β_i are all distinct, and this set is all the roots which contain x but do not contain y. So in particular, $d(x, y) = k$ is the size of the set of roots containing x but not containing y.

Proposition 2.3. Given two chambers x and y, a third chamber z lies on a minimal gallery from x to y if and only if it is contained within every root which also contains x and y.

Let R be a residue. Now we can define a map, called $\text{proj}_R w$, which maps w to the unique chamber of R closest to w.

Proposition 2.4. Given a residue R and a chamber $x \in R$, for any chamber w there is a minimal gallery from x to w which passes through $\text{proj}_R w$.

Lemma 2.3. Residues are convex.

Theorem 2.2. *Given a gallery γ of type f, γ is minimal if and only if f is reduced.*

2.1 Finite Coxeter complexes

Now we assume that our group W is finite, and so our Coxeter complex is also finite.

Definition 2.7. The *diameter*, $\text{diam}(W)$, of W is the maximum distance between two chambers of the Coxeter complex. Two chambers are said to be *opposite* if the distance between them is $\text{diam}(W)$.

Theorem 2.3.

1. *$diam(W) = 1/2 * |\{roots\ of\ W\}|$.*
2. *Two chambers are contained in no common root if and only if they are opposite.*
3. *For any given chamber, there is a unique opposite chamber.*
4. *Any chamber lies on a minimal gallery between two opposite chambers.*

3 Buildings

Definition 3.1. Let (W, S) be a Coxeter group with Coxeter matrix M, and let I be an indexing set for the generators of W. A *building of type M* is a chamber system Δ over I, such that each panel lies on at least two chambers, i.e every $\{i\}$-residue contains at least two elements. We also require a W-distance function

$$\delta : \Delta \times \Delta \to W,$$

such that if f is a reduced word in S, then we have that $\delta(x, y) = s_f$ if and only if there is a gallery of type f between x and y.

Example 3.1. Taking our W-distance function to be $\delta(x, y) = x^{-1}y$, Coxeter complexes are buildings.

Some key properties of bulidings are as follows:

1. Δ is connected.
2. δ is surjective.
3. $\delta(x, y) = \delta(y, x)^{-1}$.
4. $\delta(x, y) = s_i$ if and only if $x \neq y$ and $x \sim_i y$.
5. For $i \neq j$, i- and j-adjacency are mutually exclusive.
6. For chambers x and y, if there is a gallery form x to y of type f, and f is homotopic to g, then there is a gallery from x to y of type g.
7. A gallery if minimal if and only if its type if reduced.
8. If there is a gallery of type f from x to y, and f is reduced, then this gallery is unique.

Theorem 3.1. *Any J-residue is a building of type M_J.*

Theorem 3.2. *Any isometry from a subset of W into Δ can be extended to an isometry of W into Δ.*

Corollary 3.1. Any two chambers lie in a common apartment.

Theorem 3.3. *Apartments are convex.*

References

[1] Mark Ronan. *Lectures on Buildings*, volume 7 of *Perspectives in mathematics*. Academic Press, 1989.