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1 Buildings

Definition 1. A Polyhedral complex is a certain finite dimensional CW-complex. Each n -cell of the polyhedral complex is

Definition 2. Suppose P is a simple convex polytope in X^n . Let F_i be the codimension-one faces of P . Suppose that, for any two faces F_i and F_j , if their intersection is non-empty, then the dihedral angle between the faces is π/m_{ij} , for some m_{ij} in $2, 3, 4, \dots$. Now set $m_{ii} = 1, m_{ij} = \infty$ if F_i, F_j empty intersection. Let s_i be the reflection of X^n across F_i , and let W be the group generated by the set of s_i 's. Then W is the Coxeter group with generators s_i , and Coxeter matrix (m_{ij}) . Furthermore, W is a discrete subgroup of $\text{Isom}(X^n)$, P is a strict fundamental domain for the W action, and P tiles X^n .

Definition 3. Let (W, S) be a Coxeter group generated by a simple convex polytope P . A building of type (W, S) is a polyhedral complex, which is a union of subcomplexes, called apartments. An apartment is isometric to the tiling of X^n derived from P , and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy

1. Given any two chambers, there exists an apartment containing both of them.
2. Given any two apartments A and B , there exists an isometry from A to B which fixes $A \cap B$ pointwise.

Example 1. Let us consider a single copy of X^n . We can tile this copy by P , and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P , they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. If they share a common edge, then reflection along this edge gives us our isometry.
2. If they only share a common point

Example 2. Now we can consider a spherical building. Take the Coxeter group

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4 .

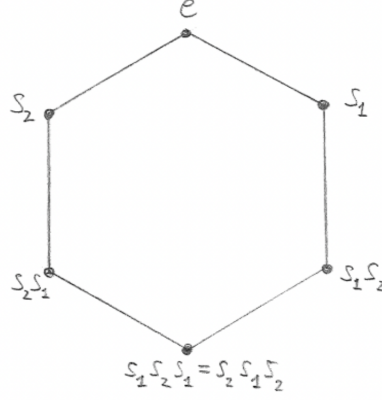
2 Cayley graphs

Definition 4. The Cayley graph $\text{Cay}(G, S)$ of a group G with respect to a generating set S , $1 \notin S$, is the graph (V, E) , with $V = G$, and directed edges

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If $s \in S$ is an involution, we only put a single undirected edge between g and gs , and label the edge s .

Example 3. The Cayley graph of D_6 , with generating set $\{s_1, s_2\}$ is



3 Reflection systems

Definition 5. Let G be a group. A pre-reflection system for G is a pair (X, R) . X is a connected simplicial graph which is acted upon by G , and R is a subset of G . This must satisfy

1. every element of R is an involution;
2. R is closed under conjugation;
3. R generates G ;
4. given an edge of X , there is a unique element of R which flips the edge;
and
5. for every element r of R , there is at least one edge of X which is flipped by r .

Example 4. Let (W, S) be any Coxeter system. Let X be the Cayley graph of (W, S) , and let

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.

Definition 6. Consider a pre-reflection system (X, R) . For each element r of R , the wall H_r is the set of midpoints of all the edges flipped by r .

Definition 7. Consider a pre-reflection system (X, R) . If, additionally, it satisfies

6. for every element r in R , $X \setminus H_r$ has exactly two components,

then (X, R) is a reflection system.

Theorem 1. Suppose we have a group W , generated by a set S of distinct involutions. Then the following are equivalent:

1. (W, S) is a Coxeter system;
2. (X, R) is a reflection system, where $X = \text{Cay}(W, S)$ and

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

3. (W, S) satisfies the Deletion Condition; and
4. (W, S) satisfies the Exchange Condition.

4 Tits' solution to the word problem

Definition 8. Let W be generated by a set S of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let m_{st} be the order of st in W . If m_{st} is finite, consider a word in S with the subword $sts...$ with m_{st} letters. A braid move on the word replaces the subword $sts...$ with $tst...$, again with m_{st} letters.

Theorem 2. (Tits) Suppose we have a group W , generated by a set S of distinct involutions. Also suppose that the Exchange Condition holds. Then

1. A word $s_1s_2...s_k$ is reduced iff we cannot shorten it by a sequence of
 - deleting an instance of ss from the word, or
 - applying a braid move to the word.
2. Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.

5 Tits' Representation Theorem

Theorem 3. (*Tits' Representation Theorem*) Let (W, S) be a Coxeter system. Then there is a map

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

which is a faithful representation, with $N = |S|$, such that

1. $\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and
2. If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order m_{ij} .

Definition 9. The representation ρ above is called the Tits representation, or sometimes the standard geometric representation.

5.1 Corollaries

6 Construction of a geometric realisation

6.1 Simplicial complexes

Definition 10. Let V be a, possibly infinite, set, called the vertex set. Let X be a collection of finite subsets of V such that

1. $\{v\} \in X$ for all elements $v \in V$; and
2. if $\Delta \in X$ and Δ' is a subset of Δ , then Δ' is in X .

Then (V, X) is an abstract simplicial complex.

Definition 11. A simplex is any element of X . A simplex Δ has dimension

$$\dim \Delta = |\Delta| - 1.$$

A simplex of dimension k is called a k -simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension k is the k -skeleton $X^{(k)}$.

Lemma 1. The k -skeleton is also a simplicial complex.

Definition 12. *The dimension of X is*

$$\dim X = \max\{\dim(\Delta) \mid \Delta \in X\}.$$

If all the maximal elements of X have the same dimension, then the simplicial complex is pure.

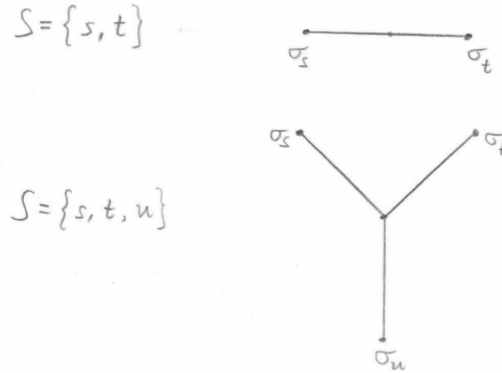
Definition 13. *The standard n -simplex Δ^n is the convex hull of the $(n+1)$ points $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ in \mathbb{R}^{n+1} .*

Given an n -simplex Δ in X , we can identify Δ with a copy of Δ^n .

6.2 The basic construction

Definition 14. *Let X be a connected Hausdorff topological space. Let (W, S) be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of X . Then $(X_s)_{s \in X}$ is a mirror structure on X over S , and X_s is called the s -mirror.*

Example 5. *Consider the cone with $|S|$ vertices. This is the graph with a node in the centre, and a branch for each element in $|S|$. Label the vertices $\{\sigma_s \mid s \in S\}$. Then we can set $X_s = \sigma_s$. This means that we take, for each element of S , the closed set of a single point as the s -mirror.*



Example 6. *Consider the n -simplex, with $n = |S| - 1$. Let $\{\Sigma_s \mid s \in S\}$ be the codimension-one faces. Now let $X_s = \Sigma_s$. This means that we take, for every element of S , a codimension-one element of the simplex as the s -mirror.*

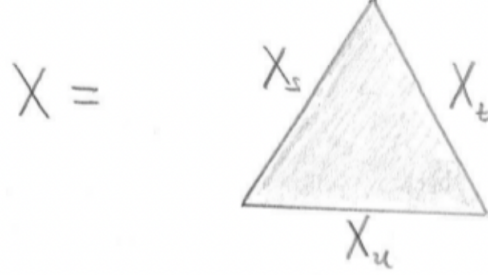


Figure 4.3: X is a 2-simplex, with codimension-one faces $\{\Delta_s \mid s \in S\}$ where $S = \{s, t, u\}$.

Definition 15. For each $x \in X$, define the set

$$S(x) := \{s \in S \mid x \in X_s\}.$$

Example 7. From example 5, we have

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s \mid s \in S\}, \\ \{s\}, & \text{if } x = \sigma_s. \end{cases}$$

Example 8. From example 6, we have

We now want to define an equivalence relation on $W \times X$.

Definition 16. (w, x) is equivalent to $(w'x')$, i.e $(w, x) \sim (w'x')$, if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.

Now we want to equip our group W with the discrete topology, and then $W \times X$ with the product topology. Then we define

$$\mathcal{U}(W, X) = W \times X / \sim .$$

Now we will denote by $[w, x]$ the equivalence class of (w, x) , and we will write wX for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. Now this must be well-defined, as $x \mapsto [w, x]$ is an embedding. We call each wX a chamber.

Example 9. Let W be the $(3, 3, 3)$ -triangle group, i.e

$$W = \langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$

Now we let our topological space be $X = \text{Cone}\{\sigma_s, \sigma_t, \sigma_u\}$. So we have

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\}, \\ \{s\}, \{t\}, \{u\} & \text{if } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$

So $W_{S(x)}$ is one of $1, \{1, s\}, \{1, t\}$, or $\{1, u\}$.

Definition 17. Let (W, S) be a Coxeter system. Let X be a simplex with codimension-one faces $\{\Sigma_s | s \in S\}$. Then $\mathcal{U}(W, X)$ is called the Coxeter complex.

6.3 Properties of $\mathcal{U}(W, X)$

Lemma 2. As a topological space, $\mathcal{U}(W, X)$ is connected.

Definition 18. We define $\mathcal{U}(W, X)$ as locally finite if, given $[w, x] \in \mathcal{U}(W, X)$, we can find an open neighbourhood of $[w, x]$ which meets only a finite number of chambers.

Lemma 3. The following are equivalent:

1. $\mathcal{U}(W, X)$ is locally finite;
2. given any $x \in X$, $W_{S(x)}$ is finite;
3. Given any $T \subset S$ such that its special subgroup W_T is infinite, we have $\bigcap_{x \in T} X_t = \emptyset$.

Example 10. Let $W = \langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$. The Coxeter complex of W is not locally finite.

6.4 Action of W on $\mathcal{U}(W, X)$

We note that W acts naturally on $W \times X$ by $w' \cdot (w, x) = (w'w, x)$.

Lemma 4. W acts on $\mathcal{U}(W, X)$ by $w' \cdot [w, x] = [w'w, x]$.

We note that this action also induces an action on the set of chambers. On the set of chambers, this action is transitive, and is free if there is a point $x \in X$ which is not contained in any mirror.

Now for the point $[w, x] \in \mathcal{U}(W, X)$, its stabiliser is $wW_{S(x)}w^{-1}$.

Definition 19. *Let G be a discrete group, and let Y be a Hausdorff space. An action by homeomorphisms of G on Y is properly discontinuous if*

1. Y/G is Hausdorff;
2. for any $y \in Y$, $G_y = \text{stab}_G(Y)$ is finite;
3. for any $y \in Y$, we can find an open neighbourhood U_y of y such that U_y is stabilised by G_y , and $gU_y \cap U_y = \emptyset$ for all $g \in G \setminus G_y$.

Lemma 5. *The action of W on $\mathcal{U}(W, X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for all $x \in X$.*

6.5 Universal property

We now claim that $\mathcal{U}(W, X)$ satisfies the following universal property.

Theorem 4. (Vinberg) *Let (W, S) be any Coxeter system. Let W act by homeomorphisms on a connected Hausdorff space Y . Assume that for any $s \in S$, the fixed point set Y^s is non-empty. Assume that X is a connected Hausdorff space, and has a mirror structure $(X_S)_{s \in S}$. Let $f; X \rightarrow Y$ be a continuous map with $f(X_s) \subset Y^s$ for all $s \in S$. Then there is a unique extension of f to a W -equivariant map $\tilde{f} : \mathcal{U}(W, X) \rightarrow Y$. This map is given by $\tilde{f}([w, x]) = w \cdot f(x)$.*

7 Geometric Reflection groups and the Davis Complex

Theorem 5. *Let X be the simple convex polytope, (W, S) the Coxeter group etc. Let \bar{s}_i be the reflection in F_i , and let \bar{W} be the group generated by the reflections. Then*

1. the map $\phi : W \rightarrow \bar{W}$, induced by $s_i \mapsto \bar{s}_i$, is an isomorphism;
2. the induced map $\mathcal{U}(W, P^n)$ is a homeomorphism;

3. the Coxeter group W acts properly discontinuously on \mathbb{X}^n , with strict fundamental domain P^n . Therefore, W is a discrete subgroup of $\text{Isom}(\mathbb{X}^n)$ and \mathbb{X}^n is tiled by copies of P^n .

To prove this theorem, the idea is that we first show that $s_i \mapsto \bar{s}_i$. Then we show that the inclusion map $f : P \longrightarrow \mathbb{X}^n$ induces a W -equivariant map which is a homeomorphism. We do this by defining a \mathbb{X}^n -structure.

7.1 The Davis complex

Definition 20. Let (W, S) be a Coxeter group. The Davis complex $\Sigma = \Sigma(W, S)$ of (W, S) is $\mathcal{U}(W, K)$, where the chamber K has mirror structure $(K_s)_{s \in S}$ such that $\forall x \in K$, $W_{S(x)}$ is finite