Buildings

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1 Buildings

Definition 1.1. A Polyhedral complex is a certain finite dimensional CW-complex. Each n-cell of the polyhedral complex is

Definition 1.2. Suppose P is a simple convex polytope in X^n. Let F_i be the codimension-one faces of P. Suppose that, for any two faces Fi and Fj, if their intersection is non-empty, then the dihedral angle between the faces is pi/mij, for some mij in $2, 3, 4, ...$. Now set $mii = 1, mij = inf$ if Fi, Fj empty intersection. Let s_i be the reflection of X^n across Fi, and let W be the group generated by the set of si's. Then W is the Coxeter group with generators s_i, and Coxeter matrix (mij). Furthermore, W is a discrete subgroup of Isom(X^n), P is a strict fundamental domain for the W action, and P tiles X^n.

Definition 1.3. Let (W, S) be a Coxeter group generated by a simple convex polytope P. A building of type (W, S) is a polyhedral complex, which is a union of subcomplexs, called apartments. An apartment is isometric to the tiling of X^n derived from P, and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy

1. Given any two chambers, there exists an aprtment containing both of them.

2. Given any two apartments A and B, there exists an isometry from A to B which fixes $A \cap B$ pointwise.

Example 1.1. Let us consider a single copy of X^n. We can tile this copy by P, and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P, they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. If they share a common edge, then reflection along this edge gives us our isometry.

2. If they only share a common point

Example 1.2. Now we can consider a spherical building. Take the Coxeter group

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4.

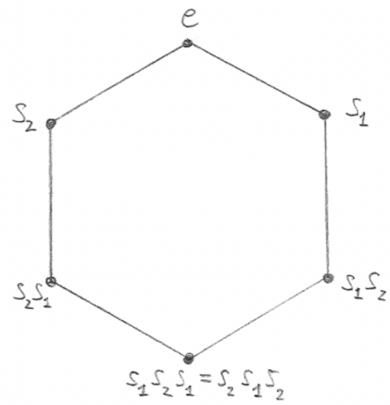
2 Cayley graphs

Definition 2.1. The Cayley graph Cay(G,S) of a group G with respect to a generating set S, $1 \notin S$, is the graph (V, E), with $V = G$, and directed edges

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If $s \in S$ is an involution, we only put a single undirected edge between g and gs, and label the edge s.

Example 2.1. The Cayley graph of D_6, with generating set $\{s_1, s_2\}$ is



3 Reflection systems

Definition 3.1. Let G be a group. A pre-reflection system for G is a pair (X, R). X is a connected simplicial graph which is acted upon by G, and R is a subset of G. This must satisfy

1. every element of R is an involution;
2. R is closed under conjugation;
3. R generates G;
4. given an edge of X, there is a unique element of R which flips the edge; and
5. for every element r of R, there is at least one edge of X which is flipped by r.

Example 3.1. Let (W, S) be any Coxeter system. Let X be the Cayley graph of (W, S), and let

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.

Definition 3.2. Consider a pre-reflection system (X, R). For each element r of R, the wall H_r is the set of midpoints of all the edges flipped by r.

Definition 3.3. Consider a pre-reflection system (X, R). If, additionally, it satisfies

6. for every element r in R, $X \backslash H_r$ has exactly two components,

then (X, R) is a reflection system.

Theorem 3.1. *Suppose we have a group W, generated by a set S of distinct involutions. Then the following are equivalent:*

1. *(W, S) is a Coxeter system;*

2. *(X, R) is a reflection system, where $X = Cay(W, S)$ and*

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

3. *(W, S) satisfies the Deletion Condition; and*

4. *(W, S) satisfies the Exchange Condition.*

4 Tits' solution to the word problem

Definition 4.1. Let W be generated by a set S of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let m_{st} be the order of st in W. If m_{st} is finite, consider a word in S with the subword $sts...$ with m_{st} letters. A braid move on the word replaces the subword $sts...$ with $tst...$, again with m_{st} letters.

Theorem 4.1. *(Tits) Suppose we have a group W, generated by a set S of distinct involutions. Also suppose that the Exchange Condition holds. Then*

1. *A word $s_1 s_2 ... s_k$ is reduced iff we cannot shorten it by a sequence of*
   - *deleting an instance of ss from the word, or*
   - *applying a braid move to the word.*

2. *Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.*

5 Tits' Representation Theorem

Theorem 5.1. *(Tits' Representation Theorem) Let (W, S) be a Coxeter system. Then there is a map*

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

which is a faithful representation, with $N = |S|$, such that

1. *$\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and*

2. *If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order m_{ij}.*

Definition 5.1. The representation ρ above is called the Tits representation, or sometimes the standard geometric representation.

5.1 Corollaries

6 Construction of a geometric realisation

6.1 Simplicial complexes

Definition 6.1. Let V be a, possibly infinite, set, called the vertex set. Let X be a collection of finite subsets of V such that

1. $\{v\} \in X$ for all elements $v \in V$; and
2. if $\Delta \in X$ and Δ' is a subset of Δ, then Δ' is in X.

Then (V, X) is an abstract simplicial complex.

Definition 6.2. A simplex is any element of X. A simplex Δ has dimension

$$\dim \Delta = |\Delta| - 1.$$

A simplex of dimension k is called a k-simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension k is the k-skeleton $X^{(k)}$.

Lemma 6.1. The k-skeleton is also a simplicial complex.

Definition 6.3. The dimension of X is

$$\dim X = max\{\dim(\Delta) Delta \in X\}.$$

If all the maximal elements of X have the same dimension, then the simplicial complex is pure.

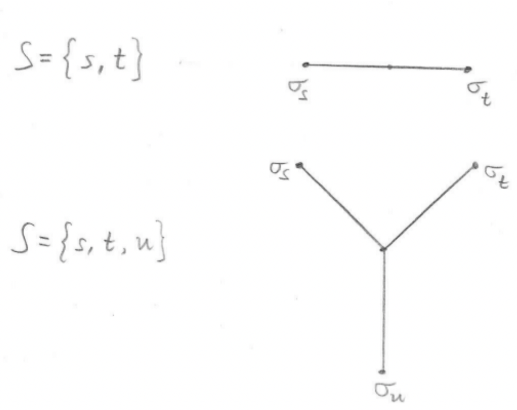
Definition 6.4. The standard n-simplex Δ^n is the convex hull of the $(n+1)$ points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$.

Given an n-simplex Δ in X, we can identify Δ with a copy of Δ^n.

6.2 The basic construction

Definition 6.5. Let X be a connected Hausdorff topological space. Let (W, S) be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of X. Then $(X_s)_{s \in X}$ is a mirror structure on X over S, and X_s is called the s-mirror.

Example 6.1. Consider the cone with $|S|$ vertices. This is the graph with a node in the centre, and a branch for each element in $|S|$. Label the vertices $\{\sigma_s | s \in S\}$. Then we can set $X_s = \sigma_s$. This means that we take, for each element of S, the closed set of a single point as the s-mirror.



Example 6.2. Consider the n-simplex, with $n = |S| - 1$. Let $\{\Sigma_s | s \in S\}$ be the codimension-one faces. Now let $X_s = \Sigma_s$. This means that we take, for every element of S, a codimension-one element of the simplex as the s-mirror.

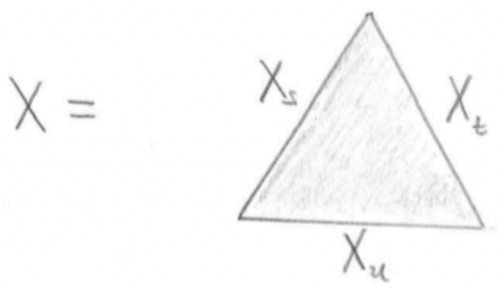


Figure 4.3: X is a 2-simplex, with codimension-one faces $\{\Delta_s \mid s \in S\}$ where $S = \{s, t, u\}$.

Definition 6.6. For each $x \in X$, define the set

$$S(x) := \{s \in S | x \in X_s\}.$$

Example 6.3. From example 5, we have

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s | s \in S\}, \\ \{s\}, & \text{if } x = \sigma_s. \end{cases}$$

Example 6.4. From example 6, we have

We now want to define an equivalence relation on $W \times X$.

Definition 6.7. (w, x) is equivalent to $(w'x')$, i.e $(w, x) \sim (w'x')$, if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.

Now we want to equip our group W with the discrete topology, and then $W \times X$ with the product topology. Then we define

$$\mathcal{U}(W, X) = W \times X / \sim .$$

Now we will denote by $[w, x]$ the equivalence class of (w, x), and we will write wX for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. Now this must be well-defined, as $x \mapsto [w, x]$ is an embedding. We call each wX a chamber.

Example 6.5. Let W be the $(3, 3, 3)$-triangle group, i.e

$$W = \langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$

Now we let our topological space be $X = Cone\{\sigma_s, \sigma_t, \sigma_u\}$. So we have

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\}, \\ \{s\}, \{t\}, \{u\} & \text{if } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$

So $W_{S(x)}$ is one of $1, \{1, s\}, \{1, t\}$, or $\{1, u\}$.

Definition 6.8. Let (W, S) be a Coxeter system. Let X be a simplex with codimension-one faces $\{\Delta_s | s \in S\}$. Then $\mathcal{U}(W, X)$ is called the Coxeter complex.

6.3 Properties of $\mathcal{U}(W, X)$

Lemma 6.2. As a topological space ,$\mathcal{U}(W, X)$ is connected.

Definition 6.9. We define $\mathcal{U}(W, X)$ as locally finite if, given $[w, x] \in \mathcal{U}(W, X)$, we can find an open neighbourhood of $[w, x]$ which meets only a finite number of chambers.

Lemma 6.3. The following are equivalent:

1. $\mathcal{U}(W, X)$ is locally finite;
2. given any $x \in X$, $W_{S(x)}$ is finite;
3. Given any $T \subset S$ such that its special subgroup W_T is infinite, we have $\cap_{x \in T} X_t = \emptyset$.

Example 6.6. Let $W = \langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle$. The Coxeter complex of W is not locally finite.

6.4 Action of W on $\mathcal{U}(W, X)$

We note that W acts naturally on $W \times X$ by $w' \cdot (w, x) = (w'w, x)$.

Lemma 6.4. W acts on $\mathcal{U}(W, X)$ by $w' \cdot [w, x] = [w'w, x]$.

We note that this action also induces an action on the set of chambers. On the set of chambers, this action is transitive, and is free if there is a point $x \in X$ which is not contained in any mirror.

Now for the point $[w, x] \in \mathcal{U}(W, X)$, its stabiliser is $wW_{S(x)}w^{-1}$.

Definition 6.10. Let G be a discrete group, and let Y be a Hausdorff space. An action by homeomorphisms of G on Y is properly discontinuous if

1. Y/G is Hausdorff;
2. for any $y \in Y$, $G_y = \mathrm{stab}_G(Y)$ is finite;
3. for any $y \in Y$, we can find an open nieghbourhood U_y of y such that U_y is stabilised by G_y, and $gU_y \cap U_y = \emptyset$ for all $g \in G \backslash G_y$.

Lemma 6.5. THe action of W on $\mathcal{U}(W, X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for all $x \in X$.

6.5 Universal property

We now claim that $\mathcal{U}(W, X)$ satisfies the following universal property.

Theorem 6.1. *(Vinberg) Let (W, S) be any Coxeter system. Let W act by homeomorphisms on a connected Hausdorff space Y. Assume that for any $s \in S$, the fixed point set Y^s is non-empty. Assume that X is a connected Hausdorff space, and has a mirror structure $(X_S)_{s \in S}$. Let $f; X \longrightarrow Y$ be a continuous map with $f(X_s) \subset Y^s$ for all $s \in S$. Then there is a unique extension of f to a W-equivariant map $\tilde{f} : \mathcal{U}(W, X) \longrightarrow Y$. This map is given by $\tilde{f}([w, x]) = w \cdot f(x)$.*

7 Geometric Reflection groups and the Davis Complex

Theorem 7.1. *Let X be the simple convex polytope, (W, S) the Coxeter group etc. Let $\bar{s}_i$ be the reflection in F_i, and let $\bar{W}$ be the group generated by the reflections. Then*

1. *the map $\phi : W \longrightarrow \bar{W}$, induced by $s_i \mapsto \bar{s}_i$, is an isomorphism;*
2. *the induced map $\mathcal{U}(W, P^n)$ is a homeomorphism;*
3. *the Coxeter group W acts properly discontinuously on $\mathbb{X}^n$, with strict fundamental domain P^n. Therefore, W is a discrete subgroup of Isom($\mathbb{X}^n$) and $\mathbb{X}^n$ is tiled by copies of P^n.*

To prove this theorem, the idea is that we first show that $s_i \mapsto \bar{s}_i$. Then we show that the inclusion map $f : P \longrightarrow \mathbb{X}^n$ induces a W-equivariant map which is a homeomorphism. We do this by defining a $\mathbb{X}^n$-structure.

7.1 The Davis complex

Definition 7.1. Let (W, S) be a Coxeter group. The davis complex $\Sigma = \Sigma(W, S)$ of (W, S) is $\mathcal{U}(W, K)$, where the chamber K has mirror structure $(K_s)_{s \in S}$ such that $\forall x \in K$, $W_{S(x)}$ is finite.

Definition 7.2. We say that a subset $T \subseteq S$ is spherical if $W_T = \langle T \rangle$ is finite. In this case, we call W_T a spherical special subgroup.

How do we construct K? Consider the set

$$L = \{T \subseteq S | T \neq \emptyset, T \text{ is spherical}\}.$$

Let us note that this set itself forms an abstract simplicial complex. Also note that $\{s\} \in L$ for all $s \in S$.

Definition 7.3. This set $L = L(W, S)$ is called the nerve of (W, S).

The set L has vertex set S, and a simplex σ_T spanning each non-empty spherical T.

Example 7.1. Consider a finite Coxeter group (W, S). Then obviously all spherical subgroups of W are also finite. Therefore, the nerve of W is the full simplex on S.

Example 7.2.

Definition 7.4. A flag complex is a simplicial complex L such that each finite, non-empty set of vertices T spans a simplex in L if and only if any two elements of T span an edge in L.

Lemma 7.1. Consider a right-angled Coxeter system (W, S). Then $L(W, S)$ is a flag complex.

8 Topology of the Davis complex

9 Geometry of the Davis complex

10 Boundaries of Coxeter groups

11 Buildings as apartment systems

Definition 11.1. (Tits 1950s) Let (W, S) be a Coxeter group. A building of type (W, S) is a simplicial complex, which is a union of subcomplexes, called apartments. An apartment is a copy of the Coxeter complex for (W, S). The maximal simplicies in the simplicial complex are called chambers. Now the apartments and chambers must satisfy

1. Given any two chambers, there exists an aprtment containing both of them.
2. Given any two apartments A and B, there exists an isomorphism from A to B which fixes $A \cap B$ pointwise.

We have two descriptions of the Coxeter complex

1. It is given by the basic construction $\mathcal{U}(W, X)$, where X is the simplex with codimension-one faces $\{\Delta_s | s \in S\}$ and mirrors $X_s = \Delta_s$. So concretely, $\mathcal{U}(W, X)$ is W-many copies of X, with the s-mirrors wX and wsX glued together.
2. It is the geometric realisation of the poset $\{wW_T | T \subseteq S, w \in W\}$, where we order by inclusion.

We also can expand the above definition to let the complex be a poyhedral complex, and allow the apartments to be other geometric realisations of (W, S).

Specifically, if (W, S) is a geometric reflection group on $\mathbb{X}^n$, then the apartments can be copies of $\mathbb{X}^n$ tiled by copies of P - which are the chambers.

Example 11.1. Let us consider a single copy of X^n. We can tile this copy by P, and we get a thin building. This means that we only have one apartment.

Definition 11.2. If the apartment is a Coxeter complex, or more generally a tiling of $\mathbb{X}^n$, a panel is a codimension-one face of a chamber. If the apartment is a Davis complex, a panel is a copy of a mirror.

Definition 11.3. A building is thick if every panel is contained in at least three chambers.