1 Buildings

Definition 1 *A Polyhedral complex is a certain finite dimensional CW-complex. Each n-cell of the polyhedral complex is*

Definition 2 *Suppose P is a simple convex polytope in X^n. Let F_i be the codimension-one faces of P. Suppose that, for any two faces Fi and Fj, if their intersection is non-empty, then the dihedral angle between the faces is pi/mij, for some mij in $2, 3, 4, \ldots$. Now set $mii = 1, mij = inf$ if Fi, Fj empty intersection. Let s_i be the reflection of X^n across Fi, and let W be the group generated by the set of si's. Then W is the Coxeter group with generators s_i, and Coxeter matrix (mij). Furthermore, W is a discrete subgroup of Isom(X^n), P is a strict fundamental domain for the W action, and P tiles X^n.*

Definition 3 *Let (W, S) be a Coxeter group generated by a simple convex polytope P. A building of type (W, S) is a polyhedral complex, which is a union of subcomplexs, called apartments. An apartment is isometric to the tiling of X^n derived from P, and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy*

1. *Given any two chambers, there exists an aprtment containing both of them.*

2. *Given any two apartments A and B, there exists an isometry from A to B which fixes $A \cap B$ pointwise.*

Example 1 *Let us consider a single copy of X^n. We can tile this copy by P, and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.*

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P, they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. *If they share a common edge, then reflection along this edge gives us our isometry.*

2. *If they only share a common point*

Example 2 *Now we can consider a spherical building. Take the Coxeter group*

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4.