1 Progress on the question

1.1 Statistics on positive folds

We now restrict to looking at Weyl chamber orientations over affine Coxeter complexes. This means that we have a complex Σ, with a boundary $\partial\Sigma$, and that our orientations are induced by a boundary chamber orientation. Here, we can get a partial answer to our main question of calculating the shadow of a given gallery. To do this, we define a ϕ-valuation map on our set of alcoves. We can then prove a recursive algorithm for calculating the shadow of a gallery.

First, given a gallery, we want to calculate the number of positive folds of this gallery that we can make. A proof of this proposition can be found in [2].

Proposition 1.1. Consider the largest element w_0 in W_0. Given an $x \in W$, and a ϕ-postive (multi)folding γ of γ_x, we have

$$l_R(xy^{-1}) \leq |F(\gamma)| \leq l(w_0),$$

where $y := \tau(\mathrm{ft}(\gamma))$.

Definition 1.1. Let $\mathcal{H}(\Sigma)$ be the set of all hyperplanes contained in our Coxeter complex. For an alcove c of Σ, let $\mathcal{H}(c)$ be the subset of $\mathcal{H}(\Sigma)$ which separates c and the fixed identity alcove 1. Now $\mathcal{H}(c) = \mathcal{H}^+_\phi(c) \sqcup \mathcal{H}^-_\phi(c)$.

Definition 1.2. Let $Ch(\Sigma)$ denote the set of all alcoves in Σ. The ϕ-valuation map is the map $\mathrm{v}_\phi : \mathrm{Ch}(\Sigma) \longrightarrow \mathbb{Z}$, with

$$c \mapsto \mathrm{v}_\phi(c) := |\mathcal{H}^+_\phi(c)| - |\mathcal{H}^-_\phi(c)|.$$

Definition 1.3. Let $p_\phi : \mathrm{Ch}(\Sigma) \times \mathcal{H} \longrightarrow \{0, 1\}$ be the function

$$p_\phi(c, H) := \begin{cases} 1 & \text{if } c \text{ is on a } \phi\text{-positive side of } H, \\ 0 & \text{otherwise.} \end{cases}$$

We now want to relate this function to our ϕ-valuation map.

Lemma 1.1.

$$\mathrm{v}_\phi(c) = \sum_{H \in \mathcal{H}(\Sigma)} (p_\phi(c, H) - p_\phi(1, H)).$$

Proof. We are assuming that our oritentation ϕ is a chamber orientation. So, in particular, this orientation is locally non-trivial. Therefore, every hyperplane H has a positive and negative side. First consider when 1 and c lie on the same side of H. Then H is not an element of $\mathcal{H}(c)$. But in this case, $p_\phi(1, H) = p_\phi(c, H)$ and so this hyperplane does not contribute to the above sum. Now consider when 1 and c lie on opposite sides of H. In this case, $H \in \mathcal{H}(c)$. If c lies on the positive side of H, then $H \in \mathcal{H}^+_\phi(c)$ and $p_\phi(c, H) = 1$ and

$p_\phi(1,H)=0$, and so H contributes $+1$ to the sum above. Similarly, if c lies on the negative side of H, then $H \in \mathcal{H}^-_\phi(c)$ and $p_\phi(c,H)=0$ and $p_\phi(1,H)=1$, and so H contributes -1 to the sum above. Therefore, we are just counting the size of $\mathcal{H}^+_\phi(c)$ minus the size of $\mathcal{H}^-_\phi(c)$, which is exactly $\mathrm{v}_\phi(c)$. $\square$

The next lemma comes from the trivial observation that

$$|\mathcal{H}^+_\phi(c)| + |\mathcal{H}^+_\phi(c)| \geq |\mathcal{H}^+_\phi(c)| - |\mathcal{H}^+_\phi(c)|.$$

Lemma 1.2.

$$l(x) \geq \mathrm{v}_\phi(c_x).$$

Definition 1.4. We call an alcove c *dominant* with respect to ϕ if $\mathrm{v}_\phi(c) = l(c)$.

Lemma 1.3.

$$l(x) = \max_{a\in W_0} \mathrm{v}_{\tilde{\phi}_a}(c_x).$$

Proof. $\square$

Lemma 1.4. Let $\phi\in$Dir(W), $r \in W$ be a reflection across the hyperplane H_r and $x \in W$. Then $\mathrm{v}_\phi(x) >\mathrm{v}_\phi(rx)$ if and only if x lies in the ϕ-positive side of H_r.

Proof. $\square$

1.2 Computation of regular shadows

We now want to see how we can use this new valuation map to define a recursive definition of a shadow. To do this, we need the next important theorem. A proof of this theorem can be found in [1, pp.142-143].

Let Dir(W) represent the set of chambers in the boundary complex $\partial\Sigma$. We call elements of Dir(W) *directions in W.*

Theorem 1.1. *Let $\phi \in$Dir(W), $x \in W$ and $s \in S$. Then*

(i) If s is in the right descent set $D_R(x)$ of x, then we have

$$\mathrm{Sh}_\phi(x) = \mathrm{Sh}_\phi(xs)\cdot s \cup \{z \in \mathrm{Sh}_\phi(xs) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\}.$$

(ii) If s is in the left descent set $D_R(x)$ of x, then we have

$$\mathrm{Sh}_\phi(x) = \begin{cases} s\cdot \mathrm{Sh}_\phi(sx) \cup \mathrm{Sh}_\phi(sx) & if\ \mathrm{v}_\phi(s) < 0, \\ s\cdot \mathrm{Sh}_\phi(sx) & if\ \mathrm{v}_\phi(s) > 0. \end{cases}$$

Now we can use this theorem to show that the next two lemmas both give us recursive defintions for the shadow of a gallery.

Lemma 1.5. (Algorithm L) Let $\phi \in \mathrm{Dir}(W)$ and $x \in W$. Let $w = (s_1, ..., s_n)$ be a reduced word for x. Let $A_0 = \{1\}$ and let

$$A_i := A_{i-1} \cdot s_i \cup \{z \in A_{i-1} | v_\phi(zs) < v_\phi(z)\}.$$

Then $A_n = \mathrm{Sh}_\phi(x)$.

Proof. Using the theorem above, we can show by induction that $A_i = \mathrm{Sh}_\phi(s_1 ... s_i)$ for $i = 0, ..., n$. Firstly, for $i = 0$ it is trivial, as $\mathrm{Sh}(1) = \{1\}$. Then assume that $A_i = \mathrm{Sh}_\phi(s_1 ... s_i)$ for $i < j$. By part (i) of the theorem,

$$\begin{aligned}
\mathrm{Sh}(s_1 ... s_j) &= \mathrm{Sh}(s_1 ... s_j s_j) \cdot s_j \cup \{z \in \mathrm{Sh}(s_1 ... s_j s_j) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= \mathrm{Sh}(s_1 ... s_{j-1}) \cdot s_j \cup \{z \in \mathrm{Sh}(s_1 ... s_{j-1}) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= A_{j-1} \cdot s_j \cup \{z \in A_{j-1} : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= A_j.
\end{aligned}$$

□

Lemma 1.6. (Algorithm R) Let $\phi \in \mathrm{Dir}(W)$ and $x \in W$, with $(s_n, ..., s_1)$ a reduced expression for x. Let $B_0^\phi := \{1\}$ and define

$$B_i^\phi = \begin{cases} s_i B_{i-1}^{s_i\phi} \cup B_{i-1}^\phi & \text{if } \mathrm{v}_\phi(s_i) < 0, \\ s_i B_{i-1}^{s_i\phi} & \text{if } \mathrm{v}_\phi(s_i) > 0. \end{cases}$$

Then $B_n^\phi = \mathrm{Sh}_\phi(x)$ for all $\phi \in \mathrm{Dir}(W)$.

Proof. Again, we can use the theorem above to prove by induction that $B_i^\phi = \mathrm{Sh}_\phi(s_i ... s_1)$ for all $i = 0, ..., n$. For $i = 0$ it is trivial as $\mathrm{Sh}_\phi(1) = \{1\}$. Now assume that $B_i^\phi = \mathrm{Sh}_\phi(s_i ... s_1)$ for all $i < j$. By part (ii) of the theorem, if $v(s_j) < 0$, then

$$\begin{aligned}
Sh_\phi(s_j ... s_1) &= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_j s_j ... s_1) \cup \mathrm{Sh}_\phi(s_j s_j ... s_1) \\
&= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_{j-1} ... s_1) \cup \mathrm{Sh}_\phi(s_{j-1} ... s_1) \\
&= s_j \cdot B_{j-1}^{s_j\phi} \cup B_{j-1}^\phi \\
&= B_j^\phi.
\end{aligned}$$

Similarly, if $v(s_j) < 0$, then

$$\begin{aligned}
Sh_\phi(s_j ... s_1) &= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_j s_j ... s_1) \\
&= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_{j-1} ... s_1) \\
&= s_j \cdot B_{j-1}^{s_j\phi} \\
&= B_j^{s_j\phi}.
\end{aligned}$$

□

References

[1] Marius Graeber and Petra Schwer. Shadows in coxeter groups. *Annals of Combinatorics*, 24(1):119–147, 2020.

[2] Elizabeth Milićević, Petra Schwer, and Anne Thomas. Dimensions of affine deligne-lusztig varieties: a new approach via labeled folded alcove walks and root operators. 2015.