1 Progress on the question

1.1 Statistics on positive folds

We now restrict to looking at Weyl chamber orientations over affine Coxeter complexes. This means that we have a complex Σ, with a boundary $\partial\Sigma$, and that our orientations are induced by a boundary chamber orientation. Here, we can get a partial answer to our main question of calculating the shadow of a given gallery. To do this, we define a ϕ-valuation map on our set of alcoves. We can then prove a recursive algorithm for calculating the shadow of a gallery.

First, given a gallery, we want to calculate the number of positive folds of this gallery that we can make. A proof of this proposition can be found in [1].

Proposition 1.1. Consider the largest element w_0 in W_0. Given an $x \in W$, and a ϕ-postive (multi)folding γ of γ_x, we have

$$l_R(xy^{-1}) \leq |F(\gamma)| \leq l(w_0),$$

where $y := \tau(\mathrm{ft}(\gamma))$.

Definition 1.1. Let $\mathcal{H}(\Sigma)$ be the set of all hyperplanes contained in our Coxeter complex. For an alcove c of Σ, let $\mathcal{H}(c)$ be the subset of $\mathcal{H}(\Sigma)$ which separates c and the fixed identity alcove 1. Now $\mathcal{H}(c) = \mathcal{H}^+_\phi(c) \sqcup \mathcal{H}^-_\phi(c)$.

Definition 1.2. Let $Ch(\Sigma)$ denote the set of all alcoves in Σ. The ϕ-valuation map is the map $\mathrm{v}_\phi : \mathrm{Ch}(\Sigma) \longrightarrow \mathbb{Z}$, with

$$c \mapsto \mathrm{v}_\phi(c) := |\mathcal{H}^+_\phi(c)| - |\mathcal{H}^-_\phi(c)|.$$

Definition 1.3. Let $p_\phi : \mathrm{Ch}(\Sigma) \times \mathcal{H} \longrightarrow \{0, 1\}$ be the function

$$p_\phi(c, H) := \begin{cases} 1 & \text{if } c \text{ is on a } \phi\text{-positive side of } H, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.1.

$$\mathrm{v}_\phi(c) = \sum_{H \in \mathcal{H}(\Sigma)} (p_\phi(c, H) - p_\phi(1, H)).$$

Proof. □

Lemma 1.2.

$$l(x) \geq \mathrm{v}_\phi(c_x).$$

Proof. □

Definition 1.4. We call an alcove c *dominant* with respect to ϕ if $\mathrm{v}_\phi(c) = l(c)$.

Lemma 1.3.

$$l(x) = \max_{a \in W_0} \mathrm{v}_{\tilde{\phi}_a}(c_x).$$

Proof. □

Lemma 1.4. Let $\phi \in$Dir(W), $r \in W$ be a reflection across the hyperplane H_r and $x \in W$. Then $\mathrm{v}_\phi(x) >\mathrm{v}_\phi(rx)$ if and only if x lies in the ϕ-positive side of H_r.

Proof. □

References

[1] Elizabeth Milićević, Petra Schwer, and Anne Thomas. Dimensions of affine deligne-lusztig varieties: a new approach via labeled folded alcove walks and root operators. 2015.