Contents

1 Coxeter complexes

Definition 1. Let (W, S) be a Coxeter system. Let $S' \subset S$. We define the *standard parabolic subgroup* $W_{s'}$ of W to be the subgroup generated by the subset S'. Then $(W_{s'}, S')$ is also a Coxeter group.

We can now define an abstract simplicial complex Σ by taking all the left cosets $xW_{S'}$ of all the standard parabolic subgroups, and defining a partial order on this set by reverse inclusion.

Why do we choose to order by reverse inclusion?

The vertex set of this simplicial complex corresponds to cosets of the maximal parabolic subgroups. These maximal parabolic subgroups are formed by taking a subset of S with one element removed. So these maximal parabolic subgroups are in bijection with the elements of S.

Definition 2. The maximal simplicies in the simplicial complex are called *alcoves*, and the codimension-one faces are called *panels*.

We notes that there is a correspondence between between panels and vertices. This is because a vertex corresponds to $xW_{S\setminus\{s\}}$, whilst a panel corresponds to an element if the form $xW_{S\setminus\{s\}}$.

Definition 3. If a panel p corresonds to the element $xW_{S\setminus\{s\}}$, we say that p has *type* s, and write $\tau(p) = s$.

The we can consider W acting on this simplicial complex. We want to consider the set of elements of W which exactly fix a hyperplane in the simplicial complex. This subset is

$$R := \bigcup_{x \in W} xSx^{-1},$$

and the elements of this set are called reflections. Given an element $r \in R$, we denoted the hyperplane it fixes by H_r. Then the hyperplane H_r *separates* two alcoves if they are contained in different half-spaces defined by H_r.

Now let us consider a Euclidean (What actually is a Euclidean Coxeter group?) Coxeter system of type $\tilde{X}$. This group can be split into a semi-direct product of a spherical Weyl group W_0 and a translation group T which acts on Σ.

Definition 4. A vertex of Σ whose stabiliser in W is isomorphic to W_0 is called a *special vertex*.

Now when we have an irreducible Euclidean Coxeter system, Σ can be geometrically realised as a tiling of the Euclidean n-space, where $n = |S| - 1$. Now here, the group T is isomorphic to $\mathbb{Z}^n$. This correpsonds to the coroot lattice.

Now let us consider this geometric realisation of Σ, which we also call Σ. Then we fix a special vertex 0, which we call the *origin* of Σ. We want to consider the set $\mathcal{H}_v$ of all hyperplanes through a special vertex v which is in the orbit of 0 under T.

Definition 5. The *Weyl chambers* are the closures of the connected components of $\Sigma \setminus \bigcup_{H \in \mathcal{H}} H$.

Now the set of equivalence classes of parallel rays in Σ form what we call the *boundary sphere*, denoted by $\partial\Sigma$. This sphere inherits a tiling from the oirginal tiling of the Euclidean plane. To do this, we take, as the hyperplanes, the parallel classes of hyperplanes in Σ.

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2 Orientations

For this section, let (W, S)be any Coxeter system, and Σ be its associated Coxeter complex.

Definition 6. An *orientation* ϕ of $\Sigma$is a map from the set of pairs (p, c), where p is a panel and c is an alcove containing p, to the set $\{+1, -1\}$. If $\phi(p, c) = +1$, then we say that c is on the *ϕ-positive side*, otherwise we say that c is on the *ϕ-negative side*.

Example 1. The trivial positive orientation is the map which sends all pairs to $+1$. Similarly, the trivial negative orientation is the map which sends all pairs to -1.

Often, we do not want to have orientations which locally behave like trivial orientations. Hence, we define the following concept:

Definition 7. Given an orientation ϕ of Σ, we have

1. ϕ is *locally non-negative* if, for each panel, there is at least one alcove which is on the ϕ-positive side.

2. ϕ is *locally non-trivial* if, for every