Contents

1 Buildings

Definition 1. *A Polyhedral complex is a certain finite dimensional CW-complex. Each n-cell of the polyhedral complex is*

Definition 2. *Suppose P is a simple convex polytope in X^n. Let F_i be the codimension-one faces of P. Suppose that, for any two faces Fi and Fj, if their intersection is non-empty, then the dihedral angle between the faces is* *pi/mij, for some mij in* $2, 3, 4, \ldots$. *Now set* $mii = 1, mij = inf$ *if Fi, Fj empty intersection. Let* s_i *be the reflection of* X^n *across Fi, and let W be the group generated by the set of si's. Then W is the Coxeter group with generators* s_i*, and Coxeter matrix* (mij). *Furthermore, W is a discrete subgroup of Isom(*X^n*), P is a strict fundamental domain for the W action, and P tiles* X^n.

Definition 3. *Let* (W, S) *be a Coxeter group generated by a simple convex polytope P. A building of type* (W, S) *is a polyhedral complex, which is a union of subcomplexs, called apartments. An apartment is isometric to the tiling of* X^n *derived from P, and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy*

1. *Given any two chambers, there exists an aprtment containing both of them.*

2. *Given any two apartments A and B, there exists an isometry from A to B which fixes* $A \cap B$ *pointwise.*

Example 1. *Let us consider a single copy of* X^n. *We can tile this copy by P, and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.*

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P, they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. *If they share a common edge, then reflection along this edge gives us our isometry.*

2. *If they only share a common point*

Example 2. *Now we can consider a spherical building. Take the Coxeter group*

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4.

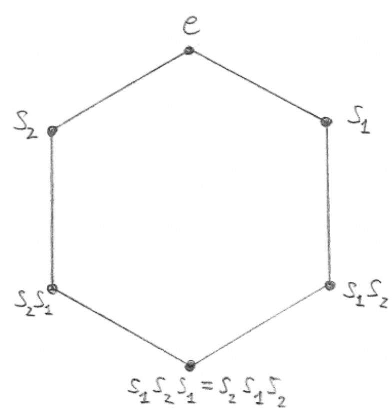
2 Cayley graphs

Definition 4. *The Cayley graph Cay(G,S) of a group G with respect to a generating set S, $1 \notin S$, is the graph (V, E), with $V = G$, and directed edges*

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If $s \in S$ is an involution, we only put a single undirected edge between g and gs, and label the edge s.

Example 3. *The Cayley graph of D_6, with generating set $\{s_1, s_2\}$ is*



3 Reflection systems

Definition 5. *Let G be a group. A pre-reflection system for G is a pair (X, R). X is a connected simplicial graph which is acted upon by G, and R is a subset of G. This must satisfy*

1. *every element of R is an involution;*
2. *R is closed under conjugation;*
3. *R generates G;*
4. *given an edge of X, there is a unique element of R which flips the edge; and*
5. *for every element r of R, there is at least one edge of X which is flipped by r.*

Example 4. *Let (W, S) be any Coxeter system. Let X be the Cayley graph of (W, S), and let*

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then (X, R) is a pre-reflection system.

Definition 6. *Consider a pre-reflection system (X, R). For each element r of R, the wall H_r is the set of midpoints of all the edges flipped by r.*

Definition 7. *Consider a pre-reflection system (X, R). If, additionally, it satisfies*

6. for every element r in R, $X \backslash H_r$ has exactly two components,

then (X, R) is a reflection system.

Theorem 1. *Suppose we have a group W, generated by a set S of distinct involutions. Then the following are equivalent:*

1. *(W, S) is a Coxeter system;*
2. *(X, R) is a reflection system, where $X = Cay(W, S)$ and*
$$R = \{wsw^{-1} | w \in W, s \in S\}.$$
3. *(W, S) satisfies the Deletion Condition; and*
4. *(W, S) satisfies the Exchange Condition.*

4 Tits' solution to the word problem

Definition 8. *Let W be generated by a set S of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let m_{st} be the order of st in W. If m_{st} is finite, consider a word in S with the subword $sts...$ with m_{st} letters. A braid move on the word replaces the subword $sts...$ with $tst...$, again with m_{st} letters.*

Theorem 2. *(Tits) Suppose we have a group W, generated by a set S of distinct involutions. Also suppose that the Exchange Condition holds. Then*

1. *A word $s_1 s_2 ... s_k$ is reduced iff we cannot shorten it by a sequence of*
   - *deleting an instance of ss from the word, or*
   - *applying a braid move to the word.*
2. *Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.*

5 Tits' Representation Theorem

Theorem 3. *(Tits' Representation Theorem) Let (W, S) be a Coxeter system. Then there is a map*

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

which is a faithful representation, with $N = |S|$, such that

1. *$\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and*
2. *If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order m_{ij}.*

Definition 9. *The representation ρ above is called the Tits representation, or sometimes the standard geometric representation.*

5.1 Corollaries

6 Construction of a geometric realisation

6.1 Simplicial complexes

Definition 10. *Let V be a, possibly infinite, set, called the vertex set. Let X be a collection of finite subsets of V such that*

1. *$\{v\} \in X$ for all elements $v \in V$; and*
2. *if $\Delta \in X$ and Δ' is a subset of Δ, then Δ' is in X.*

Then (V, X) is an abstract simplicial complex.

Definition 11. *A simplex is any element of X. A simplex Δ has dimension*

$$\dim \Delta = |\Delta| - 1.$$

A simplex of dimension k is called a k-simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension k is the k-skeleton $X^{(k)}$.

Lemma 1. *The k-skeleton is also a simplicial complex.*

Definition 12. *The dimension of X is*

$$\dim X = max\{\dim(\Delta) Delta \in X\}.$$

If all the maximal elements of X have the same dimension, then the simplicial complex is pure.

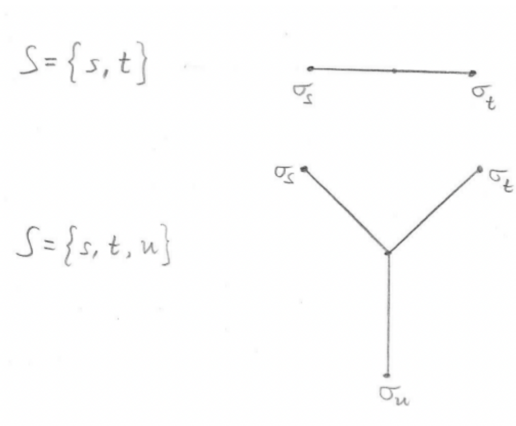
Definition 13. *The standard n-simplex Δ^n is the convex hull of the $(n+1)$ points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$.*

Given an n-simplex Δ in X, we can identify Δ with a copy of Δ^n.

6.2 The basic construction

Definition 14. *Let X be a connected Hausdorff topological space. Let (W, S) be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of X. Then $(X_s)_{s \in X}$ is a mirror structure on X over S, and X_s is called the s-mirror.*

Example 5. *Consider the cone with $|S|$ vertices. This is the graph with a node in the centre, and a branch for each element in $|S|$. Label the vertices $\{\sigma_s | s \in S\}$. Then we can set $X_s = \sigma_s$. This means that we take, for each element of S, the closed set of a single point as the s-mirror.*



Example 6. *Consider the n-simplex, with $n = |S| - 1$. Let $\{\Sigma_s | s \in S\}$ be the codimension-one faces. Now let $X_s = \Sigma_s$. This means that we take, for every element of S, a codimension-one element of the simplex as the s-mirror.*

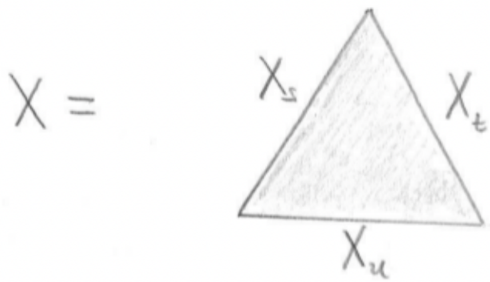


Figure 4.3: X is a 2-simplex, with codimension-one faces $\{\Delta_s \mid s \in S\}$ where $S = \{s, t, u\}$.

Definition 15. *For each $x \in X$, define the set*

$$S(x) := \{s \in S | x \in X_s\}.$$

Example 7. *From example 5, we have*

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s | s \in S\}, \\ \{s\}, & \text{if } x = \sigma_s. \end{cases}$$

Example 8. *From example 6, we have*

We now want to define an equivalence relation on $W \times X$.

Definition 16. *(w, x) is equivalent to $(w'x')$, i.e $(w, x) \sim (w'x')$, if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.*

Now we want to equip our group W with the discrete topology, and then $W \times X$ with the product topology. Then we define

$$\mathcal{U}(W, X) = W \times X / \sim .$$

Now we will denote by $[w, x]$ the equivalence class of (w, x), and we will write wX for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. Now this must be well-defined, as $x \mapsto [w, x]$ is an embedding. We call each wX a chamber.

Example 9. *Let W be the $(3,3,3)$-triangle group, i.e*

$$W = \langle s,t,u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1\rangle.$$

Now we let our topological space be $X = Cone\{\sigma_s, \sigma_t, \sigma_u\}$. So we have

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\}, \\ \{s\}, \{t\}, \{u\} & \text{if } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$

So $W_{S(x)}$ is one of $1, \{1,s\}, \{1,t\}$, or $\{1,u\}$.

Definition 17. *Let (W,S) be a Coxeter system. Let X be a simplex with codimension-one faces $\{\Sigma_s | s \in S\}$. Then $\mathcal{U}(W,X)$ is called the Coxeter complex.*

6.3 Properties of $\mathcal{U}(W,X)$

Lemma 2. *As a topological space ,$\mathcal{U}(W,X)$ is connected.*

Definition 18. *We define $\mathcal{U}(W,X)$ as locally finite if, given $[w,x] \in \mathcal{U}(W,X)$, we can find an open neighbourhood of $[w,x]$ which meets only a finite number of chambers.*

Lemma 3. *The following are equivalent:*

1. *$\mathcal{U}(W,X)$ is locally finite;*
2. *given any $x \in X$, $W_{S(x)}$ is finite;*
3. *Given any $T \subset S$ such that its special subgroup W_T is infinite, we have $\cap_{x \in T} X_t = \emptyset$.*

Example 10. *Let $W = \langle s,t,u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1\rangle$. The Coxeter complex of W is not locally finite.*

6.4 Action of W on $\mathcal{U}(W,X)$

We note that W acts naturally on $W \times X$ by $w' \cdot (w,x) = (w'w, x)$.

Lemma 4. *W acts on $\mathcal{U}(W,X)$ by $w' \cdot [w,x] = [w'w, x]$.*

We note that this action also induces an action on the set of chambers. On the set of chambers, this action is transitive, and is free if there is a point $x \in X$ which is not contained in any mirror.

Now for the point $[w, x] \in \mathcal{U}(W, X)$, its stabiliser is $wW_{S(x)}w^{-1}$.

Definition 19. *Let G be a discrete group, and let Y be a Hausdorff space. An action by homeomorphisms of G on Y is properly discontinuous if*

1. *Y/G is Hausdorff;*
2. *for any $y \in Y$, G_y =stab$_G(Y)$ is finite;*
3. *for any $y \in Y$, we can find an open nieghbourhood U_y of y such that U_y is stabilised by G_y, and $gU_y \cap U_y = \emptyset$ for all $g \in G \backslash G_y$.*

Lemma 5. *THe action of W on $\mathcal{U}(W, X)$ is properly discontinuous if and only if $W_{S(x)}$ is finite for all $x \in X$.*

6.5 Universal property

We now claim that $\mathcal{U}(W, X)$ satisfies the following universal property.

Theorem 4. *(Vinberg) Let (W, S) be any Coxeter system. Let W act by homeomorphisms on a connected Hausdorff space Y. Assume that for any $s \in S$, the fixed point set Y^s is non-empty. Assume that X is a connected Hausdorff space, and has a mirror structure $(X_S)_{s \in S}$. Let $f; X \longrightarrow Y$ be a continuous map with $f(X_s) \subset Y^s$ for all $s \in S$. Then there is a unique extension of f to a W-equivariant map $\tilde{f} : \mathcal{U}(W, X) \longrightarrow Y$. This map is given by $\tilde{f}([w, x]) = w \cdot f(x)$.*

7 Geometric Reflection groups and the Davis Complex

Theorem 5. *Let X be the simple convex polytope, (W, S) the Coxeter group etc. Let $\bar{s}_i$ be the reflection in F_i, and let $\bar{W}$ be the group generated by the reflections. Then*

1. *the map $\phi : W \longrightarrow \bar{W}$, induced by $s_i \mapsto \bar{s}_i$, is an isomorphism;*
2. *the induced map $\mathcal{U}(W, P^n)$ is a homeomorphism;*

3. *the Coxeter group W acts properly discontinuously on $\mathbb{X}^n$, with strict fundamental domain P^n. Therefore, W is a discrete subgroup of Isom($\mathbb{X}^n$) and $\mathbb{X}^n$ is tiled by copies of P^n.*

To prove this theorem, the idea is that we first show that $s_i \mapsto \bar{s}_i$. Then we show that the inclusion map $f : P \longrightarrow \mathbb{X}^n$ induces a W-equivariant map which is a homeomorphism. We do this by defining a $\mathbb{X}^n$-structure.

7.1 The Davis complex

Definition 20. *Let (W, S) be a Coxeter group. The davis complex $\Sigma = \Sigma(W, S)$ of (W, S) is $\mathcal{U}(W, K)$, where the chamber K has mirror structure $(K_s)_{s \in S}$ such that $\forall x \in K$, $W_{S(x)}$ is finite.*

Definition 21. *We say that a subset $T \subseteq S$ is spherical if $W_T = \langle T \rangle$ is finite. In this case, we call W_T a spherical special subgroup.*

How do we construct K? Consider the set

$$L = \{T \subseteq S | T \neq \emptyset, T \text{ is spherical}\}.$$

Let us note that this set itself forms an abstract simplicial complex. Also note that $\{s\} \in L$ for all $s \in S$.

Definition 22. *This set $L = L(W, S)$ is called the nerve of (W, S).*

The set L has vertex set S, and a simplex σ_T spanning each non-empty spherical T.

Example 11. *Consider a finite Coxeter group (W, S). Then obviously all spherical subgroups of W are also finite. Therefore, the nerve of W is the full simplex on S.*

Example 12.

Definition 23. *A flag complex is a simplicial complex L such that each finite, non-empty set of vertices T spans a simplex in L if and only if any two elements of T span an edge in L.*

Lemma 6. *Consider a right-angled Coxeter system (W, S). Then $L(W, S)$ is a flag complex.*

8 Topology of the Davis complex

9 Geometry of the Davis complex

10 Boundaries of Coxeter groups

11 Buildings as apartment systems