# Contents

# 1   Buildings

**Definition 1.** *A Polyhedral complex is a certain finite dimensional CW-complex. Each n-cell of the polyhedral complex is*

**Definition 2.** *Suppose $P$ is a simple convex polytope in $X^n$. Let $F_i$ be the codimension-one faces of $P$. Suppose that, for any two faces $Fi$ and $Fj$, if their intersection is non-empty, then the dihedral angle between the faces is $pi/mij$, for some $mij$ in $2, 3, 4, ....$ Now set $mii = 1, mij = inf$ if $Fi, Fj$ empty intersection. Let $s_i$ be the reflection of $X^n$ across $Fi$, and let $W$ be the group generated by the set of $si$'s. Then $W$ is the Coxeter group with generators $s_i$, and Coxeter matrix $(mij)$. Furthermore, $W$ is a discrete subgroup of $Isom(X^n)$, $P$ is a strict fundamental domain for the $W$ action, and $P$ tiles $X^n$.*

**Definition 3.** *Let $(W, S)$ be a Coxeter group generated by a simple convex polytope $P$. A building of type $(W, S)$ is a polyhedral complex, which is a union of subcomplexes, called apartments. An apartment is isometric to the tiling of $X^n$ derived from $P$, and each copy of $P$ in the tiling is called a chamber. Now the apartments and chambers must satisfy*

    *1. Given any two chambers, there exists an aprtment containing both of them.*

2. *Given any two apartments A and B, there exists an isometry from A to B which fixes $A \cap B$ pointwise.*

**Example 1.** *Let us consider a single copy of $X^n$. We can tile this copy by P, and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.*

*Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P, they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:*

1. *If they share a common edge, then reflection along this edge gives us our isometry.*

2. *If they only share a common point*

**Example 2.** *Now we can consider a spherical building. Take the Coxeter group*
$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$
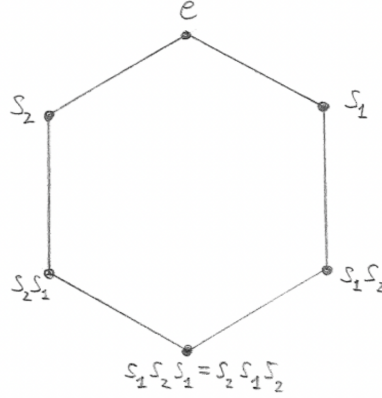*This Coxeter group is isomorphic to $D_4$.*

# 2 Cayley graphs

**Definition 4.** *The Cayley graph Cay(G,S) of a group G with respect to a generating set S, $1 \notin S$, is the graph $(V, E)$, with $V = G$, and directed edges*

$$E = \{(g, gs) | g \in G, s \in S\}.$$

*If $s \in S$ is an involution, we only put a single undirected edge between g and gs, and label the edge s.*

**Example 3.** *The Cayley graph of $D_6$, with generating set $\{s_1, s_2\}$ is*

# 3 Reflection systems

**Definition 5.** *Let $G$ be a group. A pre-reflection system for $G$ is a pair $(X, R)$. $X$ is a connected simplicial graph which is acted upon by $G$, and $R$ is a subset of $G$. This must satisfy*

1. *every element of $R$ is an involution;*

2. *$R$ is closed under conjugation;*

3. *$R$ generates $G$;*

4. *given an edge of $X$, there is a unique element of $R$ which flips the edge; and*

5. *for every element $r$ of $R$, there is at least one edge of $X$ which is flipped by $r$.*

**Example 4.** *Let $(W, S)$ be any Coxeter system. Let $X$ be the Cayley graph of $(W, S)$, and let*
$$R = \{wsw^{-1} | w \in W, s \in S\}.$$
*Then $(X, R)$ is a pre-reflection system.*

**Definition 6.** *Consider a pre-reflection system $(X, R)$. For each element $r$ of $R$, the wall $H_r$ is the set of midpoints of all the edges flipped by $r$.*

**Definition 7.** *Consider a pre-reflection system $(X, R)$. If, additionally, it satisfies*

*6. for every element r in R, $X \backslash H_r$ has exactly two components,*

*then $(X, R)$ is a reflection system.*

**Theorem 1.** *Suppose we have a group $W$, generated by a set $S$ of distinct involutions. Then the following are equivalent:*

1. *$(W, S)$ is a Coxeter system;*

2. *$(X, R)$ is a reflection system, where $X = Cay(W, S)$ and*

$$R = \{wsw^{-1}|w \in W, s \in S\}.$$

3. *$(W, S)$ satisfies the Deletion Condition; and*

4. *$(W, S)$ satisfies the Exchange Condition.*

# 4 Tits' solution to the word problem

**Definition 8.** *Let $W$ be generated by a set $S$ of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let $m_{st}$ be the order of $st$ in $W$. If $m_{st}$ is finite, consider a word in $S$ with the subword sts... with $m_{st}$ letters. A braid move on the word replaces the subword sts... with tst..., again with $m_{st}$ letters.*

**Theorem 2.** *(Tits) Suppose we have a group $W$, generated by a set $S$ of distinct involutions. Also suppose that the Exchange Condition holds. Then*

1. *A word $s_1 s_2 ... s_k$ is reduced iff we cannot shorten it by a sequence of*

   - *deleting an instance of ss from the word, or*
   - *applying a braid move to the word.*

2. *Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.*

# 5  Tits' Representation Theorem

**Theorem 3.** *(Tits' Representation Theorem) Let $(W, S)$ be a Coxeter system. Then there is a map*

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

*which is a faithful representation, with $N = |S|$, such that*

1. *$\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and*

2. *If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order $m_{ij}$.*

**Definition 9.** *The representation $\rho$ above is called the Tits representation, or sometimes the standard geometric representation.*

## 5.1  Corollaries

# 6  Construction of a geometric realisation

## 6.1  Simplicial complexes

**Definition 10.** *Let $V$ be a, possibly infinite, set, called the vertex set. Let $X$ be a collection of finite subsets of $V$ such that*

1. *$\{v\} \in X$ for all elements $v \in V$; and*

2. *if $\Delta \in X$ and $\Delta'$ is a subset of $\Delta$, then $\Delta'$ is in $X$.*

*Then $(V, X)$ is an abstract simplicial complex.*

**Definition 11.** *A simplex is any element of $X$. A simplex $\Delta$ has dimension*

$$\dim \Delta = |\Delta| - 1.$$

*A simplex of dimension $k$ is called a $k$-simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension $k$ is the $k$-skeleton $X^{(k)}$.*

**Lemma 1.** *The $k$-skeleton is also a simplicial complex.*

**Definition 12.** *The dimension of $X$ is*

$$\dim X = max\{\dim(\Delta) Delta \in X\}.$$

*If all the maximal elements of $X$ have the same dimension, then the simplicial complex is pure.*

**Definition 13.** *The standard n-simplex $\Delta^n$ is the convex hull of the $(n+1)$ points $(1, 0, ..., 0), ..., (0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$.*

Given an n-simplex $\Delta$ in $X$, we can identify $\Delta$ with a copy of $\Delta^n$.

## 6.2   The basic construction

**Definition 14.** *Let $X$ be a connected Hausdorff topological space. Let $(W, S)$ be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of $X$. Then $(X_s)_{s \in X}$ is a mirror structure on $X$ over $S$, and $X_s$ is called the s-mirror.*

**Example 5.** *Consider the cone with $|S|$ vertices. This is the graph with a node in the centre, and a branch for each element in $|S|$. Label the vertices $\{\sigma_s | s \in S\}$. Then we can set $X_s = \sigma_s$. This means that we take, for each element of $S$, the closed set of a single point as the s-mirror.*
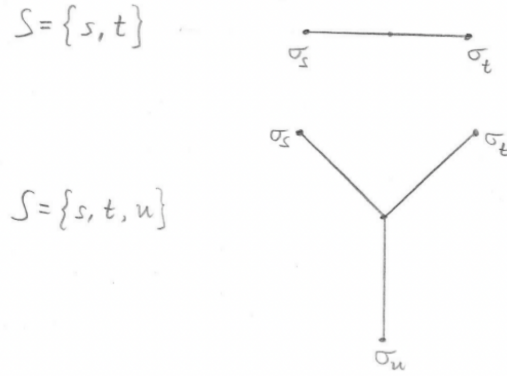


**Example 6.** *Consider the n-simplex, with $n = |S| - 1$. Let $\{\Sigma_s | s \in S\}$ be the codimension-one faces. Now let $X_s = \Sigma_s$. This means that we take, for every element of $S$, a codimension-one element of the simplex as the s-mirror.*
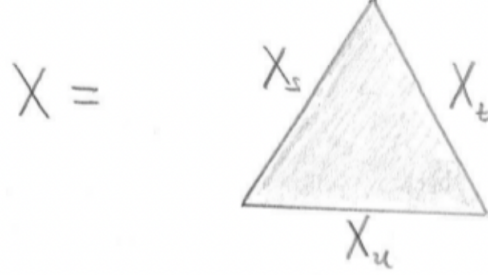
Figure 4.3: $X$ is a 2-simplex, with codimension-one faces $\{\Delta_s \mid s \in S\}$ where $S = \{s, t, u\}$.

**Definition 15.** *For each $x \in X$, define the set*

$$S(x) := \{s \in S \mid x \in X_s\}.$$

**Example 7.** *From example 5, we have*

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s \mid s \in S\}, \\ \{s\}, & \text{if } x = \sigma_s. \end{cases}$$

**Example 8.** *From example 6, we have*

We now want to define an equivalence relation on $W \times X$.

**Definition 16.** *$(w, x)$ is equivalent to $(w'x')$, i.e $(w, x) \sim (w'x')$, if and only if $x = x'$ and $w^{-1}w' \in W_{S(x)}$.*

Now we want to equip our group $W$ with the discrete topology, and then $W \times X$ with the product topology. Then we define

$$\mathcal{U}(W, X) = W \times X / \sim .$$

Now we will denote by $[w, x]$ the equivalence class of $(w, x)$, and we will write $wX$ for the image of $\{w\} \times X$ in $\mathcal{U}(W, X)$. Now this must be well-defined, as $x \mapsto [w, x]$ is an embedding. We call each $wX$ a chamber.

**Example 9.** *Let $W$ be the $(3,3,3)$-triangle group, i.e*

$$W = \langle s, t, u | s^2 = t^2 = u^2 = 1, (st)^3 = (tu)^3 = (us)^3 = 1 \rangle.$$

*Now we let our topological space be $X = Cone\{\sigma_s, \sigma_t, \sigma_u\}$. So we have*

$$S(x) = \begin{cases} \emptyset, & \text{if } x \notin \{\sigma_s, \sigma_t, \sigma_u\} \\ \{s\}, \{t\}, \{u\} & \text{if } x = \sigma_s, \sigma_t, \sigma_u \text{ resp.} \end{cases}$$