

1 Shadows in buildings

We want to consider

1.1 Orientations

Definition 1.1. [1, ?] An *orientation* ϕ of Σ is a map from the set of pairs (p, c) , where p is a panel and c is an alcove containing p , to the set $\{+1, -1\}$. If $\phi(p, c) = +1$, then we say that c is on the ϕ -*positive side*, otherwise we say that c is on the ϕ -*negative side*.

Example 1.1. The trivial positive orientation is the map which sends all pairs to $+1$. Similarly, the trivial negative orientation is the map which sends all pairs to -1 .

Often, we do not want to have orientations which locally behave like trivial orientations. Hence, we define the following concept:

Definition 1.2. [1, ?] Given an orientation ϕ of Σ , we have

1. ϕ is *locally non-negative* if, for each panel, there is at least one alcove which is on the ϕ -positive side.
2. ϕ is *locally non-trivial* if, for every panel, there is exactly one alcove which is on the ϕ -positive side.

There is a natural action of W on the set of all possible orientations of Σ , induced by the action of W on the alcoves and panels. It is defined as

$$(x \cdot \phi)(p, c) := (x^{-1}p, x^{-1}c).$$

Definition 1.3. [1, ?] Given an orientation ϕ of Σ , we say that ϕ is *wall consistent* if, given any wall H , for all pairs c, d of alcoves which lie in the same halfspace of H , with panels p and q respectively, we have that $\phi(p, c) = \phi(q, d)$. If our orientation is wall consistent, we can then define the *positive side* H^ϵ of H as the half-space such that all alcoves c in H^ϵ have $\phi(p, c) = +1$ for all panels of c . Then the *negative side* is defined similarly.

We want to look at several natural ways to orient a Coxeter complex. First, we will look at an orientation which is derived from either a choice of alcove, or a choice of panel. This orientation works for any Coxeter group.

Definition 1.4. [1, ?] Choose a fixed alcove c in Σ . Now given any alcove d , and panel p , we define their orientation as $\phi(p, d) = +1$ if and only if c and d lie in the same side of the wall which is spanned by p . We call this orientation the *alcove orientation towards c* .

Definition 1.5. [1, ?] Choose a fixed simplex b in Σ . Now given any alcove c , and panel p in c , we define their orientation as $\phi(p, c) = +1$ if and only if either c and b lie in the same side of the wall H containing p , or if b lies inside H . We call this orientation the *simplex orientation towards b* .

Example 1.2. Here we see two simplex orientations of an A_2 Coxeter complex. In this complex, the alcoves are edges, and the panels are vertices.

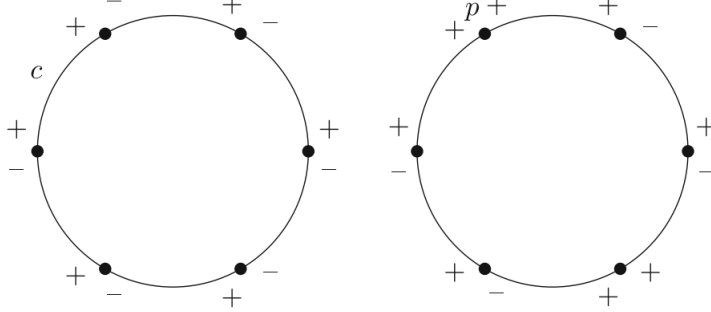


FIGURE 2. An alcove (left) and panel orientation (right) on the type A_2 Coxeter complex

Lemma 1.1. [1, ?] Consider a Coxeter group (W, S) with Coxeter complex Σ . We have the following:

- (i) If ϕ is a simplex orientation of Σ , then ϕ is wall consistent and locally non-negative.
- (ii) If ϕ is an alcove orientation of Σ , then ϕ is wall consistent and locally non-trivial.

Proof. (i) Let b be the simplex defining the orientation, and consider a wall H in our Coxeter complex. First let us consider the case in which b lies inside H . Then, by definition of the simplex orientation, both sides of the wall are defined to be positive. So any two alcoves, and any respective panels, lying in the same halfplane of H will have the same orientation. So this wall satisfies the conditions of wall consistency, and both sides are defined as positive so it is locally non-negative.

Now assume that b does not lie in H , so b lies in exactly one halfplane of H . Then this side of the halfplane is the positive side, and any two alcoves, and any respective panels, in this halfplane are given a positive orientation. Similarly, any two alcoves, and any respective panels, in the other halfplane are given a negative orientation. So again, this wall satisfies the conditions of wall-consistency, and both sides are defined as positive so it is locally non-negative.

(ii) An alcove orientation is a type of simplex orientation, so part (i) implies that ϕ is wall consistent. Now considering the cases from part (i), we can never be in the first case. This is because an alcove has one higher dimension than a wall, and so an alcove can never fully lie within a wall. So therefore we are always in case two, and so by the same argument as part (i), we conclude that ϕ is locally non-trivial. \square

1.2 The affine case

Now we want to consider when our Coxeter complex Σ is affine. To define an orientation on Σ , we choose a chamber at infinity.

If ϕ is a wall consistent orientation, then, given two chambers c, d which share a common panel p , c and d are given the same orientation if they lie in the same half-space of the hyperplane spanned by p . This amounts to picking a positive side of the hyperplane.

However, we did not have to pick these positive sides in any consistent way.

Definition 1.6. Let ϕ be a wall consistent orientation of an affine Coxeter complex. We say that ϕ is *periodic* if, given two parallel hyperplanes H_1, H_2 and corresponding half-spaces $H_1^\epsilon, H_2^\epsilon$, if $H_1^\epsilon \subset H_2^\epsilon$, then H_1^ϵ is positive if and only if H_2^ϵ is positive.

Example 1.3. If ϕ is a trivial orientation on an affine Coxeter complex, then ϕ is periodic.

Example 1.4. Simplex orientations are not periodic, as, for every set of parallel hyperplanes, we can find pairs of representatives which have the simplex on different sides.

If ϕ is a periodic orientation, then we have a natural orientation induced on the boundary. Similarly, if we have an orientation defined on the boundary of a Coxeter complex, then we have a periodic orientation on the Coxeter complex which induces this orientation.

Lemma 1.2. [1, p.125] Given a periodic orientation ϕ on an affine Coxeter complex Σ , there is an induced wall-consistent orientation $\partial\phi$ on the boundary complex $\partial\Sigma$. Now if ϕ is locally non-negative or non-trivial, so is $\partial\phi$.

Proof. Consider a wall M in the boundary $\partial\Sigma$. This corresponds to a set of parallel walls in Σ . Consider a chamber $a \in \partial\Sigma$, which has a panel p lying in M . Now we can find a Weyl chamber C_a of Σ which represents a . This has a bounding wall H_M in the set of parallel walls corresponding to M . Let c be the alcove at the tip of C_a . So c has a panel q which lies in H_M . We now define the orientation of the boundary by

$$\partial\phi(a, p) = \phi(c, q).$$

This is well-defined as ϕ is periodic, so the choice of C_a does not affect the orientation. Also, as ϕ is periodic, this orientation is wall-consistent. If ϕ is locally non-negative, then given a panel p of Σ , we can find an alcove c such that $\phi(c, p) = +1$. Then under the projection map from Σ to the boundary $\partial\Sigma$, we get a chamber d and panel q such that $\partial\phi(d, q) = +1$. So $\partial\phi$ is locally non-negative. The same argument can be made to show that if ϕ is locally non-trivial, then $\partial\phi$ is also locally non-trivial. \square

Lemma 1.3. Given a wall-consistent orientation ϕ of the boundary complex $\partial\Sigma$, there exists a unique periodic orientation $\tilde{\phi}$ of Σ , which induces the orientation ϕ .

Proof. Let H be a wall in Σ . Given a halfplane H^ϵ of H , we define H^ϵ to be the positive side of H if the corresponding halfplane ∂H^ϵ is the positive side of the wall ∂H with respect to the orientation ϕ . Otherwise we define H^ϵ to be the negative side of H . This is well-defined as ϕ is wall-consistent. This definition also uniquely defines the orientation $\tilde{\phi}$. \square

Definition 1.7. Let σ be a chamber of the boundary Δ of a Coxeter complex Σ . Then we form an orientation ϕ_σ on the boundary Δ . The *Weyl chamber orientation* on Σ is the induced orientation by ϕ_σ .

2 Folded galleries

2.1 Definitions

Definition 2.1. Given a Coxeter complex Σ , a *combinatorial gallery* is a sequence

$$\gamma = (c_0, p_1, c_1, p_2, \dots, p_n, c_n),$$

where the c_i are alcoves and the p_i are panels of Σ , such that p_i is contained in c_i and c_{i-1} for all $i = 1, \dots, n$. The length of a combinatorial gallery γ is $n + 1$ - this counts how many alcoves there are in the sequence. Then γ is *minimal* if there does not exist a shorter gallery starting at c_0 and ending at c_n .

So a gallery is a path between c_0 and c_n through alcoves, such that adjacent alcoves in the path share a common panel.

Definition 2.2. Given a gallery γ of Σ , we say that γ is *folded (or stammering)* if, within γ , we can find an index i such that $c_i = c_{i-1}$. Then we say that γ has a *fold* at panel p_i . Otherwise, we say that γ is *unfolded (or non-stammering)*.

Definition 2.3. Given a gallery γ , define the set $F(\gamma)$ to be the subset of $\{1, \dots, n\}$ such that $i \in F(\gamma)$ if and only if γ has a fold at panel p_i .

To represent a gallery, we draw a path which passes through every chamber and panel in the gallery of the Euclidean representation of our Coxeter complex. We draw an arrow towards the sink of our gallery.

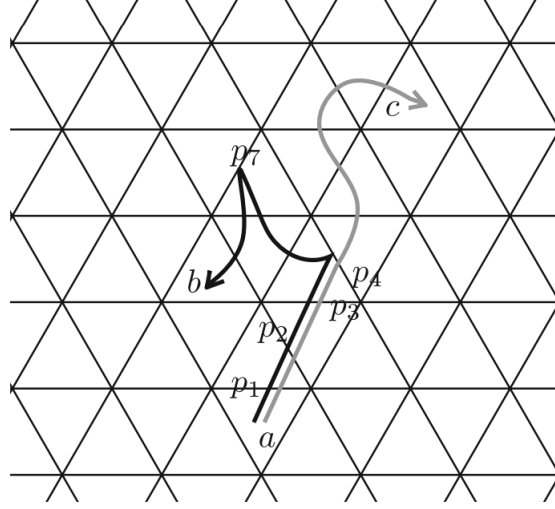


FIGURE 3. This figure shows galleries in type \tilde{A}_2 with two folds (black) and no folds (gray)

Definition 2.4. Given a gallery γ in Σ , and an orientation ϕ , we say that γ is *positively folded* with respect to ϕ if, whenever γ is folded at position i , $\phi(p_i, c_i) = +1$. We can similarly define *negatively folded*.

2.2 Galleries and Words

Definition 2.5. Consider a gallery $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$. Let panel p_i of γ have type $s_{j_i} \in S$. We define its *type* $\tau(\gamma)$ as the word

$$\tau(\gamma) := s_{j_1} \dots s_{j_n}.$$

We denote by $\Gamma_\phi^+(w)$ the set of all ϕ -positively folded galleries which have type w .

We can now define the type and decorated type of a gallery. Note that here we use tilde notation to denote a decorated word, but in other texts, such as [1], they use hat notation to denote the decorated word.

Definition 2.6. The *decorated type* $\hat{\tau}(\gamma)$ of a gallery $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$ is the decorated word

$$\tilde{\tau}(\gamma) := s_{j_1} \dots \tilde{s}_{j_i} \dots s_{j_n},$$

where we place a hat on the elements s_{j_i} of the word which correspond to a fold $c_{i-1} = c_i$ of the gallery. We denote by $\Gamma_\phi^+(\tilde{w})$ the set of all ϕ -positively folded galleries which have decorated type \tilde{w} .

Lemma 2.1. [1, p.128] Let c_0 be a fixed alcove in our Coxeter complex Σ .

- (i) There is a bijection between words in S and unfolded galleries starting at c_0 .
- (ii) There is a bijection between decorated words in S and galleries starting at c_0 .

Proof. Given a word $s_1 \dots s_n$ in S , we can define an unfolded gallery starting at c_0 by multiplying c_0 by s_1 , and in general define c_i by multiplying c_{i-1} by s_i , and set p_i to be the unique panel contained in both c_{i-1} and c_i . This gives our bijection for part (i). Now given a decorated word in S , we can define a general gallery by multiplying c_{i-1} by s_i , as above, if s_i is not decorated. If s_i is decorated, then let $c_i = c_{i-1}$, and let the panel p_i be the unique panel of c_i which has type s_i . This gives the bijection for part (ii). \square

The next lemma gives some easy results from the definitions of type and decorated type.

Lemma 2.2. [1, p.128] Let γ be a gallery. Then

- 1. $F(\gamma) = \emptyset$ if and only if $\tau(\gamma) = \tilde{\tau}(\gamma)$.
- 2. γ is minimal if and only if $F(\gamma) = \emptyset$ and $\tau(\gamma)$ is reduced.

We want to be able to characterise the last alcove in a gallery. We do this by constructing another gallery which removes any folds from our original gallery. This leads to an unfolded gallery which has shorter length than the original gallery.

Definition 2.7. Consider a gallery $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$ in Σ . We create a new gallery, called the *footprint* $ft(\gamma)$ of γ , by deleting all pairs p_i, c_i such that the letter s_i has a hat in $\hat{\tau}(\gamma)$.

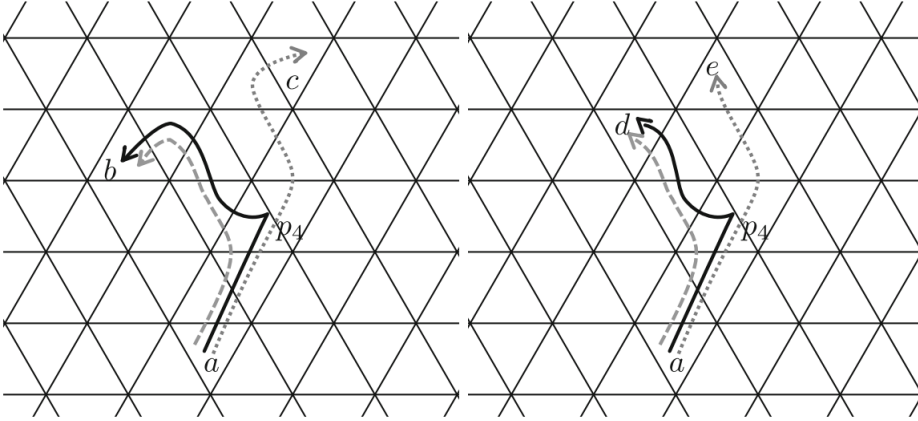


FIGURE 4. This figure shows galleries (black), their unfolded images (dotted gray), and footprints (dashed gray)

Lemma 2.3. We can calculate the final alcove of a gallery as the element $c_n = c_0 \cdot w$, where $w = \tau(ft(\gamma))$.

Proof. The footprint of a gallery is exactly the gallery achieved by deleting repeated alcoves and the corresponding panels. So we are left with a gallery $\text{ft}(\gamma) = (c_0 = d_0, q_1, d_1, \dots, q_m, d_m = c_n)$. Now d_i is exactly defined as the alcove obtained by multiplying d_{i-1} by s_i , where s_i is the type of the panel q_i . So, by induction, d_m , which equals c_n , can be calculated by multiplying $d_0 = c_0$ by the type of $\text{ft}(\gamma)$. \square

2.3 Folding and unfolding galleries

Now we have defined galleries, and in particular folded galleries, we want to be able to create folded galleries ourselves from unfolded galleries. We do this in a natural way, where folding along a panel leads to a reflection of the rest of the gallery with respect to that panel. For instance, this figure shows foldings in the 4th and 7th panel of the given gallery, and also illustrates that foldings are commutative - a fact that we will formally prove.

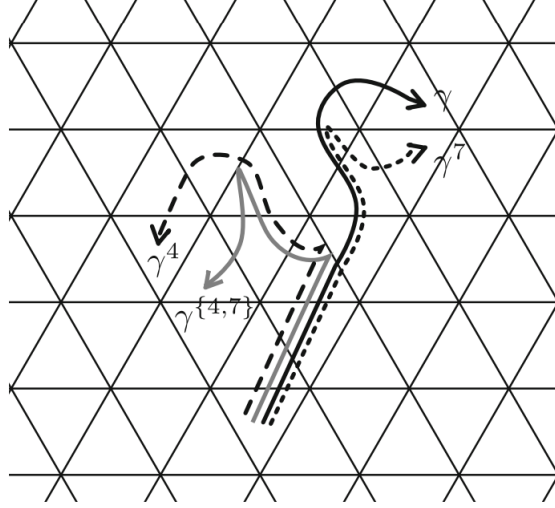


FIGURE 5. This figure shows commuting folds at panels 4 and 7 of the black gallery γ

We note that, as W has a natural left action on Σ , W also acts on the set of galleries in Σ . For instance, $x \in W$ sends $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$ to the gallery $\gamma = (xc_0, xp_1, xc_1, \dots, xp_n, xc_n)$.

Lemma 2.4. MOVE THIS LEMMA Consider an affine Coxeter system (W, S) with a Coxeter complex Σ . Let a be a chamber in the boundary complex $\partial\Sigma$. Now a gallery γ is ϕ_a -positively folded if and only if $x \cdot \gamma$ is ϕ_a -positively folded. So the action of W on $\partial\Sigma$ preserves the condition of being ' ϕ_a -positively folded'.

Proof. \square

Definition 2.8. Consider a gallery $\gamma = (c_0, p_1, c_1, \dots, p_n, c_n)$. Let H_i be the hyperplane

containing the panel p_i , and let r_i be the reflection across H_i . For $i = 1, \dots, n$, let

$$\gamma^i := (c_o, p_1, \dots, p_i, r_i c_i, r_i p_{i+1}, r_i c_{i+1}, \dots, r_i p_n, r_i c_n).$$

If γ was folded at panel p_i , we call γ^i a *unfolding* of γ at p_i . Otherwise, we call it a *folding*.

Lemma 2.5. For all $i = 1, \dots, n$, $\tau(\gamma) = \tau(\gamma^i)$. So folding and unfolding does not change the gallery type. Also, $(\gamma^i)^i = \gamma$.

Proof. This is just a result of the definition of the type of a panel, which is invariant under reflections along walls. Also, we note that $r_i r_i = 1$, as reflections are self-inverse, so applying a fold twice at panel p_i will first achieve $(c_o, p_1, \dots, p_i, r_i c_i, r_i p_{i+1}, r_i c_{i+1}, \dots, r_i p_n, r_i c_n)$, and will then achieve $(c_o, p_1, \dots, p_i, r_i r_i c_i, r_i r_i p_{i+1}, r_i r_i c_{i+1}, \dots, r_i r_i p_n, r_i r_i c_n) = (c_o, p_1, \dots, p_i, c_i, p_{i+1}, c_{i+1}, \dots, p_n, c_n)$. Hence, $(\gamma^i)^i = \gamma$. \square

Lemma 2.6. For all $i, j = 1, \dots, n$, $(\gamma^i)^j = (\gamma^j)^i$, i.e. foldings are commutative.

Proof. As we have already dealt with the case that $i = j$, we can assume that $i < j$. Let r be the reflection along the wall containing p_i , and let t be the reflection along the wall containing p_j . Then we have, by the definition of (un)-folding,

$$(\gamma^j)^i = ((c_o, p_1, c_1, \dots, c_{i-1}, p_i, r c_i, \dots, r p_j, r t c_j, \dots, r t c_n)).$$

Also, if we let u be the reflection along the wall containing $r p_j$, we have

$$(\gamma^i)^j = ((c_o, p_1, c_1, \dots, c_{i-1}, p_i, r c_i, \dots, r p_j, u r c_j, \dots, u r c_n)).$$

Let us calculate the maps r , t and u . Given a panel p of an alcove c , the reflection along the wall containing p is given by the multiplication map $c\tau(p)c^{-1}$. Hence,

$$r = c_{i-1}\tau(p_i)c_{i-1}^{-1}, t = c_{j-1}\tau(p_j)c_{j-1}^{-1}, \text{ and } u = (rc_{j-1})\tau(rp_j)(rc_{j-1})^{-1}.$$

Now by lemma ..., reflections preserve type, so we have that $\tau(rp_j) = \tau(p_j)$. Then

$$\begin{aligned} ur &= (rc_{j-1})\tau(rp_j)(rc_{j-1})^{-1}c_{i-1}\tau(p_i)c_{i-1}^{-1} \\ &= r(c_{j-1}\tau(p_j)c_{j-1}^{-1})(c_{i-1}\tau(p_i)c_{i-1}^{-1})(c_{i-1}\tau(p_i)c_{i-1}^{-1}) \\ &= r(c_{j-1}\tau(p_j)c_{j-1}^{-1}) \\ &= rt. \end{aligned}$$

Therefore $ur = rt$, and hence $(\gamma^j)^i = (\gamma^i)^j$. \square

Because of this property, we are able to define a *multifolding* with respect to a subset I of $\{1, \dots, n\}$ as the (un)-foldings γ^I . Now multifolding does not affect the type. Then the set of folds of γ^I will be the symmetric difference of the folds of γ and I . In particular, if I and J are subsets of $\{1, \dots, n\}$, $(\gamma^I)^J = \gamma^{I \Delta J}$. The following corollary now follows.

Corollary 2.1. Given any gallery γ , there is a subset $I \subset \{1, \dots, n\}$ such that γ^I is unfolded, and γ and γ^I have the same type.

Now we fix an alcove of our Coxeter complex, and call this 1. Then, for any word w with elements in S , we let γ_w be the unique unfolded gallery which has type w and starts at 1. Now we write

1. $\gamma \rightarrow \eta$ if γ and η are galleries such that $\eta = \gamma^I$ for some index set I ,
2. $w \rightarrow u$ if w and u are words in S such that there is a folding of γ_w which has footprint u .
3. $w \rightarrow x$ if x is an element of W such that there is a folding of γ_w which has end alcove c_x .

We denote by $A \xrightarrow{\phi} B$ if the respective gallery is ϕ -positively folded.

3 Braid invariant orientations

Two reduced words in S represent the same element in the Coxeter group if and only if they differ by a sequence of braid moves [CITE]. A braid move replaces a subword $s_i s_j \dots$ of length m_{ij} with the string $s_j s_i \dots$, again of length m_{ij} . We want to define the concept of braid invariant orientations, so we can later conclude that, if we have a braid invariant orientation, our shadows of a gallery do not depend on the chosen word of S representing the end alcove.

Definition 3.1. Consider a Coxeter system (W, S) and a corresponding Coxeter complex Σ . Let ϕ be an orientation on Σ . Then we say that ϕ is *braid invariant* if, given any two braid equivalent words w, w' in S and any $x \in W$, $w \xrightarrow{\phi} x$ if and only if $w' \xrightarrow{\phi} x$. Then if $y\phi$ is braid invariant for all $y \in W$, ϕ is called *strongly braid invariant*.

For instance, trivial orientations are strongly braid invariant, but these orientations are not very interesting. The following proposition gives us a large family of orientations which are braid invariant. A proof of this proposition can be found in [1, pp.135-138].

Proposition 3.1. Weyl chamber orientations are braid invariant.

Happy to prove this proposition, but the proof is quite long so wanted to ask before I wrote it.

4 Shadows

Definition 4.1. Consider a Coxeter system W and an orientation ϕ on the Coxeter complex $\Sigma(W, S)$. Let w be a word in S . The *shadow* of w with respect to ϕ is the set

$$\text{Sh}_\phi(w) := \{u \in W \mid w \xrightarrow{\phi} u\}.$$

If ϕ is braid invariant, we can define $\text{Sh}_\phi(x) = \text{Sh}_\phi(w)$, where w is any reduced expression of $x \in W$. If we have the Weyl chamber orientation ϕ_a with $a \in W_0$, the *regular shadow* of x with respect to a is $\text{Sh}_a(w) := \text{Sh}_{\phi_a}(w)$. The *full shadow* of x is the union of regular shadows.

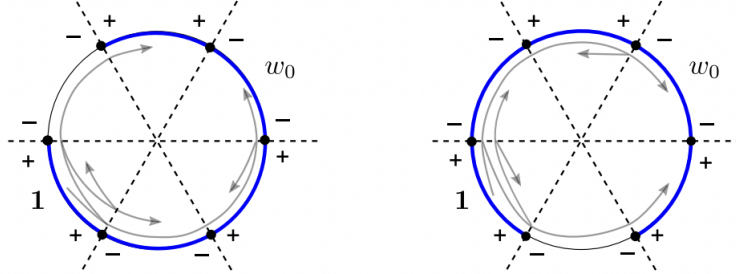


FIGURE 6. The picture shows a non-braid-invariant orientation which hence produces different shadows (shown fat blue) for the two minimal galleries from 1 to w_0 . See Example 6.2 for details

Definition 4.2. Let $x = s_1 \dots s_n$ be a reduced expression for $x \in W$. Let $y \in W$. We say that $y \leq x$ if there exists a reduced expression for y of the form $s_{i_1} \dots s_{i_k}$ with $1 \leq i_1 \leq \dots \leq i_k \leq n$. This ordering is called the *Bruhat order*.

Proposition 4.1. Consider the trivial positive orientation ϕ_+ , and the alcove orientation ϕ_1 towards 1 . For $x, y \in W$, $x \geq y$ if and only if $x \xrightarrow{\phi_+} y$, if and only if $x \xrightarrow{\phi_1} y$.

References

- [1] Marius Graeber and Petra Schwer. Shadows in coxeter groups. *Annals of Combinatorics*, 24(1):119–147, 2020.