1 Buildings

Definition 1 *A Polyhedral complex is a certain finite dimensional CW-complex. Each n-cell of the polyhedral complex is*

Definition 2 *Suppose P is a simple convex polytope in X^n. Let F_i be the codimension-one faces of P. Suppose that, for any two faces Fi and Fj, if their intersection is non-empty, then the dihedral angle between the faces is pi/mij, for some mij in $2, 3, 4, ...$. Now set $mii = 1, mij = inf$ if Fi, Fj empty intersection. Let s_i be the reflection of X^n across Fi, and let W be the group generated by the set of si's. Then W is the Coxeter group with generators s_i, and Coxeter matrix (mij). Furthermore, W is a discrete subgroup of Isom(X^n), P is a strict fundamental domain for the W action, and P tiles X^n.*

Definition 3 *Let (W, S) be a Coxeter group generated by a simple convex polytope P. A building of type (W, S) is a polyhedral complex, which is a union of subcomplexs, called apartments. An apartment is isometric to the tiling of X^n derived from P, and each copy of P in the tiling is called a chamber. Now the apartments and chambers must satisfy*

1. *Given any two chambers, there exists an aprtment containing both of them.*

2. *Given any two apartments A and B, there exists an isometry from A to B which fixes $A \cap B$ pointwise.*

Example 1 *Let us consider a single copy of X^n. We can tile this copy by P, and we get a thin building. This means that we only have one apartment. Clearly this satisfies the first condition - any two chambers immediately lie in the only apartment.*

Now let us look at the second condition. If the two chambers have no intersection, then, as each chamber is a copy of P, they are clearly isometric, and we are done. Now if the two chambers have a non-empty intersection, we have two cases:

1. *If they share a common edge, then reflection along this edge gives us our isometry.*

2. *If they only share a common point*

Example 2 *Now we can consider a spherical building. Take the Coxeter group*

$$W = \langle s_1, s_2 | s_i^2 = 1, (s_1 s_2)^2 = 1 \rangle.$$

This Coxeter group is isomorphic to D_4.

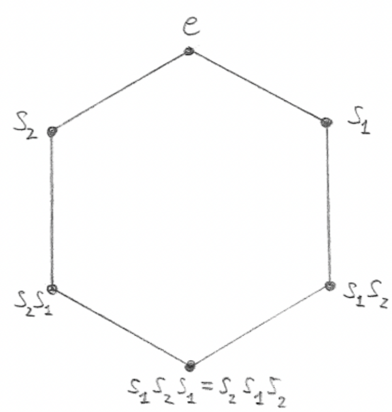
2 Cayley graphs

Definition 4 *The Cayley graph Cay(G,S) of a group* G *with respect to a generating set* S, $1 \notin S$, *is the graph* (V, E), *with* $V = G$, *and directed edges*

$$E = \{(g, gs) | g \in G, s \in S\}.$$

If $s \in S$ *is an involution, we only put a single undirected edge between* g *and* gs, *and label the edge* s.

Example 3 *The Cayley graph of* D_6, *with generating set* $\{s_1, s_2\}$ *is*



3 Reflection systems

Definition 5 *Let* G *be a group. A pre-reflection system for* G *is a pair* (X, R). X *is a connected simplicial graph which is acted upon by* G, *and* R *is a subset of* G. *This must satisfy*

1. *every element of* R *is an involution;*

2. R *is closed under conjugation;*

3. *R generates G;*

4. *given an edge of X, there is a unique element of R which flips the edge; and*

5. *for every element r of R, there is at least one edge of X which is flipped by r.*

Example 4 *Let (W,S) be any Coxeter system. Let X be the Cayley graph of (W,S), and let*

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

Then (X,R) is a pre-reflection system.

Definition 6 *Consider a pre-reflection system (X,R). For each element r of R, the wall H_r is the set of midpoints of all the edges flipped by r.*

Definition 7 *Consider a pre-reflection system (X,R). If, additionally, it satisfies*

6. *for every element r in R, $X \backslash H_r$ has exactly two components,*

then (X,R) is a reflection system.

Theorem 1 *Suppose we have a group W, generated by a set S of distinct involutions. Then the following are equivalent:*

1. *(W,S) is a Coxeter system;*

2. *(X,R) is a reflection system, where $X = Cay(W,S)$ and*

$$R = \{wsw^{-1} | w \in W, s \in S\}.$$

3. *(W,S) satisfies the Deletion Condition; and*

4. *(W,S) satisfies the Exchange Condition.*

4 Tits' solution to the word problem

Definition 8 *Let W be generated by a set S of distinct involutions. Let $s, t \in S$, with $s \neq t$, and let m_{st} be the order of st in W. If m_{st} is finite, consider a word in S with the subword $sts...$ with m_{st} letters. A braid move on the word replaces the subword $sts...$ with $tst...$, again with m_{st} letters.*

Theorem 2 *(Tits) Suppose we have a group W, generated by a set S of distinct involutions. Also suppose that the Exchange Condition holds. Then*

1. *A word $s_1 s_2 ... s_k$ is reduced iff we cannot shorten it by a sequence of*
   - *deleting an instance of ss from the word, or*
   - *applying a braid move to the word.*
2. *Two reduced words will represent the same group element iff a sequence of braid moves sends one to the other.*

5 Tits' Representation Theorem

Theorem 3 *(Tits' Representation Theorem) Let (W, S) be a Coxeter system. Then there is a map*

$$\rho : W \longrightarrow GL(N, \mathbb{R})$$

which is a faithful representation, with $N = |S|$, such that

1. *$\sigma_i = \rho(s_i)$ is a linear involution, whose fixed set is a hyperplane; and*
2. *If $s_i, s_j \in S$ are distinct, then $\sigma_i \sigma_j$ has order m_{ij}.*

Definition 9 *The representation ρ above is called the Tits representation, or sometimes the standard geometric representation.*

5.1 Corollaries

6 Construction of a geometric realisation

6.1 Simplicial complexes

Definition 10 *Let V be a, possibly infinite, set, called the vertex set. Let X be a collection of finite subsets of V such that*

1. *$\{v\} \in X$ for all elements $v \in V$; and*
2. *if $\Delta \in X$ and Δ' is a subset of Δ, then Δ' is in X.*

Then (V, X) is an abstract simplicial complex.

Definition 11 *A simplex is any element of X. A simplex Δ has dimension*

$$\dim \Delta = |\Delta| - 1.$$

A simplex of dimension k is called a k-simplex. A 0-simplex is called a vertex, and a 1-simplex is called an edge, intuitively. The set of all simplices of dimension k is the k-skeleton $X^{(k)}$.

Lemma 1 *The k-skeleton is also a simplicial complex.*

Definition 12 *The dimension of X is*

$$\dim X = max\{\dim(\Delta) Delta \in X\}.$$

If all the maximal elements of X have the same dimension, then the simplicial complex is pure.

6.2 The basic construction

Definition 13 *Let X be a connected Hausdorff topological space. Let (W, S) be a Coxeter system. Let $(X_s)_{s \in X}$ be a collection of non-empty, closed subspaces of X. Then $(X_s)_{s \in X}$ is a mirror structure on X over S, and X_s is called the s-mirror.*

Example 5