Lectures on Buildings

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Contents

1 Chamber systems

Definition 1.1. A set C is called a *chamber system* over a set I if each $i \in I$ is an equivalence relation on the elements of C. Each i partitions our set C. We say two elements $x, y \in C$ are *i-adjacent*, and we write $x \sim_i y$, if they lie in the same part of the partition, i.e they are equivalent with respect to the equivalence relation corresponding to i. The elements of C are called *chambers*. The *rank* of a chamber system is the size of I.

Other texts enforce that for each i there are at least two elements which are i-adjacent.

Example 1.1. Given a group G, a subgroup B, and an indekxing set I, let there be a subgroup $B < P_i < G$ for all $i \in I$. Then we take as our chamber set C the left cosets of B, and we define an equivalence relation

$$gB \sim_i hB \text{ if and only if } gP_i = hP_i.$$

Definition 1.2. A finite sequence $(c_0, ..., c_k)$ such that c_i is adjacent to c_{i+1} is called a *gallery*. Its *type* is a word $i_1, ..., i_k$ in I such that c_{i-1} is i-adjacent to c_i. We assume that no two consecutive chambers are equal.

Definition 1.3. We call C *connected* if we can connect any two chambers with a gallery. A *J-residue* is a J-connected component. We call $\{i\}$-residues *i-panels*.

Definition 1.4. Let C and D be two chamber systems over the same indexing set I. A *morphism* between C and D is a map $C \longrightarrow D$ which preserves i-adjacency.

1.1 The geometric realisation

Definition 1.5. Let R be a J-residue and S be a K-residue. Then S is a *face* of R if $R \subset S$ and $J \subset K$. The *cotype* of J is the set $I - J$.

Observe that if R is a residue of cotype J, we have

1. for $K \subset J$, there is a unique face of R which has cotype K.
2. Let S_1, S_2 be faces of R with cotypes K_1 and K_2. Then S_1 and S_2 have a shared face of cotype $K_1 \cap K_2$.

With these observations, we can form a *cell complex* of our chamber system. To do this, we form a vertex for each residue of corank 1. Then, we can associate to each residue of cotype $\{i, j\}$ an edge. From the observation above, this has as its boundary the residues of cotype $\{i\}$ and of cotype $\{j\}$. Then this can be continued inductively.

Definition 1.6. Let σ be a simplex of our cell complex. The set *star* $St(\sigma)$ is the corresponding residue.

1.2 $A_n(k)$ Buildings

Let us consider an $n+1$ dimensional vector space V over a field k. We define the chambers of our chamber system to be the maximal sequences

$$V_1 \subset V_2 \subset ... \subset V_n$$

of subspaces of V, where V_i has dimension i. We can then define adjancency by saying that two sequences $V_1 \subset V_2 \subset ... \subset V_n$ and $V_1' \subset V_2' \subset ... \subset V_n'$ are i-adjacent if and only if $V_j = v_j'$ for all $j \neq i$. Then the residues of type i correspond to 1 spaces in the 2 space V_{i+1}/V_{i-1}.

We then get a geometric realisation of this chmaber system. Here, a residue of cotype $J = \{j_1, ..., j_r\}$ corresponds to a sequence

$$V_{j_1} \subset V_{j_2} \subset ... \subset V_{j_r}.$$

This residue has chambers which are maximal flags $V_1' \subset V_2' \subset ... \subset V_n'$ such that $V_j' = v_j$ if $j \in J$.

In particular, residues of cotype $\{i\}$ correspond to the subspaces of V.

1.3 $C_n(k)$ Buildings

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2 Coxeter Complexes

Given a Coxeter group W, take as chambers the elements of W, and define an i-adjancency by $w \sim_i wr_i$, where $\{s_1, ..., s_n\}$ are the set of generators of the Coxeter group. If the Coxeter group has Coxeter matrix M, we call this building a *Coxeter complex of type M.*

Diagram $\widetilde{A}_2$. 

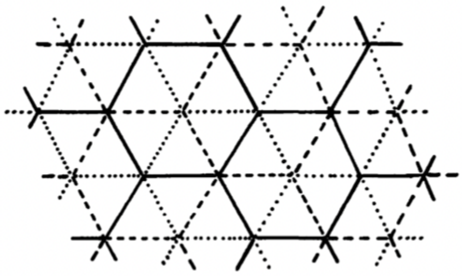

Figure 2.1

Diagram $\widetilde{C}_2$. 

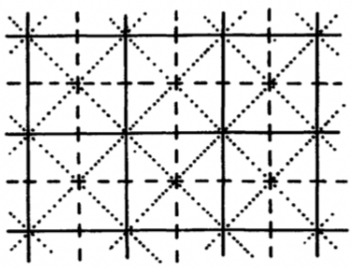

Figure 2.2

Lemma 2.1. The automorphism group of the Coxeter complex is isomorphic to the Coxeter group, and this acts simple-transitively on the set of chambers.

Definition 2.1. A *reflection* r of W is a conjugate of the generators of W. The wall M_r of a reflection r is the set of simplicies in the Coxeter complex which is fied by r when r acts on the complex by left multiplication. Then M_r is a subcomplex of codimension 1.

Theorem 2.1. *There is a bijection between the set of reflections of a Coxeter group, and the set of walls in the corresponding Coxeter complex.*

Definition 2.2. A gallery $(c_0, ..., c_k)$ *crosses* M_r if there is an i such that M_r interchanges c_{i-1} and c_i.

Lemma 2.2. 1. Any minimal gallery does not cross a wall twice.

2. Every gallery from two alcoves x and y have the same parity of crossings of any wall.

Definition 2.3. Each hyperplane splits an apartment into two half-apartments called *roots*. If α is one root, we denote the other corresponding root by $-\alpha$.

Definition 2.4. A set of alcoves is called *convex* if any minimal gallery between two alcoves of the set lies entirely within the set.

Proposition 2.1. 1. Roots are convex.

2. Let α be a root, and let x and y be adjacent chambers with $x \in \alpha$ and $y \in -\alpha$. Then

$$\alpha = \{c | d(x, c) < d(y, c)\}.$$