Shadows in the Wild

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Contents

Red notes for Megan

Blue notes for Yusra

1 Roots

In this text, we are considering galleries of the form $\gamma = (\lambda_0, c_0, p_1, c_1, ..., p_n, c_n, \lambda_n)$, i.e galleries which start and end at vertices. This is slightly different to the other convention of starting and ending at alcoves. This changes the concept of minimla galleries slightly.

We also take our orientation to be a fixed Weyl chamber orientation. So we have fixed a chamber at infinity which defines the orientation.

Definition 1.1. Let γ be a gallery in a Coxeter complex Σ. At p_i, the hyperplane containing p_i is said to be *load-bearing* if c_i lies on the positive side of the hyperplane. A hyperplane containing λ_0 is called *load-bearing* if c_0 is on the positive side.

Note that at each panel, we have at most one load-bearing hyperplane, but at the starting vertex we can have multiple. The number of load-bearing hyperplanes at λ_0 is bounded by the number of hyperplanes in the spherical Weyl group, or equivantely the size of a positive root system defining the Weyl group.

Definition 1.2. Given a gallery γ, its *dimension* is the number of pairs (p_i, H), where H is a load-bearing hyperplane at p_i.

Definition 1.3. Given a fixed end-vertex and fixed type τ, a gallery of maximal possible dimension is called a *LS-gallery* of type τ.

Is this related to the ϕ-valuation? This seems very similar to the dominant alcove but 'the other way round'

Proposition 1.1. Let γ be an LS-gallery. Then

1. An application of e_α will increase the dimension of γ by one.
2. An application of f_α will decrease the dimension of γ by one.
3. Applying e_α then f_α gives the original gallery γ. Applying f_α then e_α gives the original gallery γ.
4. The set of LS-galleries is closed under applying e and f operators.
5. Any LS-gallery of the same type as γ can be formed from γ by a finite number of e and f operators.

How can this proposition be true when LS-galleries are defined to be the maximal dimension galleries of a certain type, and I don't think e and f will change the type.

Proposition 1.2. Let λ be a dominant vertex in Σ. Let ϕ be the Weyl chamber orientation which is induced by the anti-dominant Weyl chamber. Then we have

$$\mathrm{Sh}_\phi^\vee(\lambda) = \{\nu \leq \lambda | \nu \text{ is a vertex with the same type as } \lambda\}.$$

Is the ordering here Bruhat ordering?

Definition 1.4. Let A be a chosen apartment in a building X, and let $c \in A$ be an alcove. We define the *retraction from* X *to* A *based at* c as the map $r_{A,c} : X \longrightarrow A$ where we send any alcove d to its image under the isomorphism from the apartment containing both c and d to A.

How do we know that this isomorphism is unique?

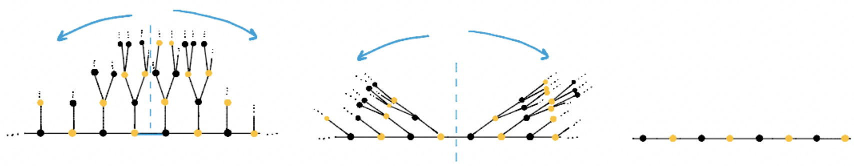

FIGURE 13. The retraction based at the fundamental alcove flattens the tree outwards.

Definition 1.5. Let A be a chosen apartment in an affine building X, and let $C \in \partial A$ be a chamber at infinity of the apartment. We define the *retraction from* X *to* A *based at* C as the map $\rho_{A,C} : X \longrightarrow A$, which sends an alcove d to its image under the isomorphism from the apartment containing both d and a Weyl chamber representing C to A.

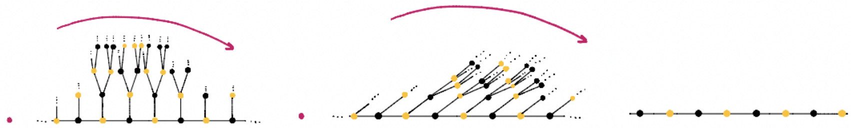

FIGURE 14. The retraction of a tree based at a Weyl chamber at infinity, indicated by the pink dot to the left of the horizontal line, flattens the tree away from that direction at infinity.

2 Kostant Convexity

Theorem 2.1. *We note that we can decompose any reductive group G with assiciated Bruhat-Tits building as*

$$G = UTK,$$

where U is the stabiliser of the parallel class of the anti-dominant Weyl chamber in the base apartment A_0, T is the set of translations of A_0, and K is the stabiliser of the origin on A_0. Then for all $t, t' \in T$, we have

$$Ut'K \cap KtK \neq \emptyset \iff t'K \in \mathrm{conv}(W_0 \cdot tK).$$

Theorem 2.2. *Consider a thick affine building X, with fixed base apartment A_0 with origin λ_0, base alcove c_0 and fundamental Weyl chamber C_0. Choose a special vertex λ in a_0. Then,*

$$\rho_{C_0,A}(r^{-1}_{c_0,A}(W_0 \cdot \lambda)) = \mathrm{conv}(W_0 \cdot \lambda).$$

3 Affine Deligne-Lusztig varieties

Let $F = k((t))$, where k is the algebraic closure of the finite field of order $q = p^m$, where p is prime. Let σ be the Frobenius map of $\mathbb{F}_q$. We denote by $\mathcal{O}$ the ring of integers $k[[t]]$.

Definition 3.1. Let G be a connected reductive group over $\mathbb{F}_q$. The quotient $G(F)/I$ is called the