Shadows in Coxeter complexes

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Contents

1 Introduction

In this project, we will be looking at the geometric object of buildings, and in particular Coxeter complexes. Buildings can be defined in many ways. We can look at buildings as geometric objects, combinatorial objects, or as representations of groups. These viewpoints give us unique information of the building and group. We will look at galleries in buildings, which are walks around the alcoves of a building. We then look at foldings of these galleries, with respect to orientations of our building. An important combinatorial question of foldings is which alcoves of the building can be reached by folding a certain gallery. We call this set the *shadow* of a gallery. We shall see some progress in answering the question of calculating the shadow, and we will discuss tools which could be used to improve these answers.

2 Chamber systems

We first start our exploration of buildings with an abstract chamber system - a set with some equivalence relations on it.

Definition 2.1. A set C is called a *chamber system* over a set I if each $i \in I$ is an equivalence relation on the elements of C. Each i partitions our set C. We say two elements $x, y \in C$ are *i-adjacent*, and we write $x \sim_i y$, if they lie in the same part of the partition, i.e they are equivalent with respect to the equivalence relation corresponding to i. The elements of C are called *chambers*. The *rank* of a chamber system is the size of I.

A very important example is obtained by looking at a group G, and a subgroup B, and defining the following equivalence relations:

Example 2.1. Given a group G, a subgroup B, and an indexing set I, let there be a subgroup $B < P_i < G$ for all $i \in I$. Then we take our chamber system C to be the left cosets of B, and we define an equivalence relation

$$gB \sim_i hB \text{ if and only if } gP_i = hP_i.$$

We now look at galleries of a chamber system. These are walks around the chambers, where we only move from one chamber to an adjacent chamber.

Definition 2.2. A finite sequence $(c_0, ..., c_k)$ such that c_i is adjacent to c_{i+1} is called a *gallery*. Its *type* is a word $i_1, ..., i_k$ in I such that c_{i-1} is i-adjacent to c_i.

Definition 2.3. We call C *connected* if there is a gallery between any two chambers. Given a subset $J \subset I$, a *residue of type* J is a J-connected component, i.e. there is a gallery between any two elements of a J-connected component whose type is a word in J. The *cotype* of J is $I - J$.

2.1 The geometric realisation

We now want to construct a geometric realisation of this chamber system. We construct a simplicial complex, where each simplex represents a residue of our chamber system.

Definition 2.4. Let R be a J-residue and S be a K-residue. Then S is a *face* of R if $R \subset S$ and $J \subset K$.

Observe that if R is a residue of cotype J, we have

1. for $K \subset J$, there is a unique face of R which has cotype K.
2. Let S_1, S_2 be faces of R with cotypes K_1 and K_2. Then S_1 and S_2 have a shared face of cotype $K_1 \cap K_2$.

With these observations, we can form a *cell complex* of our chamber system. To do this, we form a vertex for each residue of corank 1. Then, we can associate to each residue of cotype $\{i, j\}$ an edge. From the observation above, this has, as its boundary, the residues of cotype $\{i\}$ and of cotype $\{j\}$.

Then this can be continued inductively. So given a residue R which has cotype of size r, we associate a dimension $r - 1$ simplex. Now the faces of this simplex are exactly the faces of R, as defined above.

2.2 $A_n(k)$ Buildings

A key example of a chamber system is formed by considering the subspaces of an $n + 1$ dimensional vector space V over a field k. We define the chambers of our chamber system to be the maximal sequences

$$V_1 \subset V_2 \subset ... \subset V_n$$

of subspaces of V, where V_i has dimension i. We can then define adjacency by saying that two sequences $V_1 \subset V_2 \subset ... \subset V_n$ and $V_1' \subset V_2' \subset ... \subset V_n'$ are i-adjacent if and only if $V_j = V_j'$ for all $j \neq i$. Then the residues of type i correspond to 1-spaces in the 2-space V_{i+1}/V_{i-1}.

We then get a geometric realisation of this chamber system. Here, a residue of cotype $J = \{j_1, ..., j_r\}$ corresponds to a sequence

$$V_{j_1} \subset V_{j_2} \subset ... \subset V_{j_r}.$$

This residue has chambers which are maximal flags $V_1' \subset V_2' \subset ... \subset V_n'$ such that $V_j' = V_j$ if $j \in J$.

In particular, residues of cotype $\{i\}$ correspond to the subspaces of V.

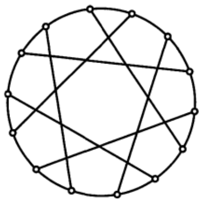


Figure 1.1

I will create my own picture with coloured dots.

Example 2.2. Here we have the geometric realisation of $A_2(\mathbb{F}_2)$. There are 7 one-dimensional subspaces, represented here as white, and there are 7 two-dimensional subspaces, represented as black.

3 Coxeter complexes

We can now construct a specific type of chamber system which arises from a given Coxeter group.

Given a Coxeter group (W, S), take as chambers the elements of W, and define an i-adjacency by $w \sim_i ws_i$, where $S = \{s_1, ..., s_n\}$ are the set of generators of the Coxeter group. If the Coxeter group has Coxeter matrix M, we call this building a *Coxeter complex of type M*.

Now we can look at the geometric realisation of this chamber system.

Definition 3.1. Let (W, S) be a Coxeter system. Let $S' \subset S$. We define the *standard parabolic subgroup* $W_{s'}$ of W to be the subgroup generated by the subset S'. Then $(W_{s'}, S')$ is also a Coxeter group.

This idea corresponds to our more general idea of residues of chamber systems. For Coxeter complexes, we have an explicit definition of the residues as the parabolic subgroups.

Definition 3.2. The maximal simplicies in the simplicial complex are called *alcoves*, and the codimension-one faces are called *panels*.

Definition 3.3. If a panel p corresponds to the element $xW_{S\setminus\{s\}}$, we say that p has *type* s, and write $\tau(p) = s$.

Diagram $\tilde{A}_2$. 

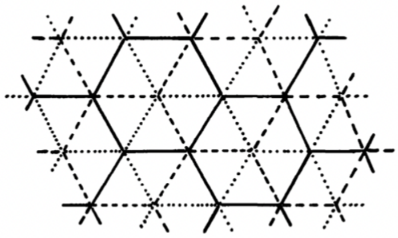


Figure 2.1

Example 3.1. Here we are looking at the Coxeter group

$$\tilde{A}_2 = \langle s_0, s_1, s_2 | s_i = 1, (s_1 s_2)^3 = (s_2 s_3)^3 = (s_3 s_1)^3 = 1 \rangle.$$

To form our geometric realisation, we note that our indexing set is $I = \{0, 1, 2\}$. Then we create a vertex for every maximal parabolic subgroup. This is the set of elements of the form $xW_{\{s_i, s_j\}}$, with $x \in W$ and $i, j \in \{0, 1, 2\}$. Now we join two vertices

Now we can make a very similar definition of a gallery for Coxeter complexes.

Definition 3.4. Given a Coxeter complex Σ, a *combinatorial gallery* is a sequence

$$\gamma = (c_0, p_1, c_1, p_2, ..., p_n, c_n),$$

where the c_i are alcoves and the p_i are panels of Σ, such that p_i is contained in c_{i-1} and c_i for all $i - 1, ..., n$. The length of a combinatorial gallery γ is $n + 1$ - this counts how many alcoves there are in the sequence. Then γ is *minimal* if there does not exist a shorter gallery starting at c_0 and ending at c_n.

So a gallery is a path between c_0 and c_n through alcoves, such that adjacent alcoves in the path share a commmon panel. This is very similar to how we defined a gallery for a general chamber system.

Lemma 3.1. A Coxeter complex is connected.

Proof. Given any two elements $x = s_{i_1} \dots s_{i_n}$ and $y = s_{j_1} \dots s_{j_m}$, we have a gallery

$$s_{i_1} \dots s_{i_n} \sim_{i_n} s_{i_1} \dots s_{i_{n-1}} \sim_{i_{n-2}} \dots \sim_{i_n} 1 \sim_{j_1} s_{j_1} \sim_{j_2} s_{j_1} s_{j_2} \sim_{j_3} \dots \sim_{j_m} s_{j_1} \dots s_{j_m}.$$

So any two chambers are connected by a gallery. □

Lemma 3.2. The automorphism group of the Coxeter complex is isomorphic to the Coxeter group, and this acts simple-transitively on the set of chambers.

Proof. W acts on itself by left multiplication. This action clearly preserves the i-adjacency defined above. Now consider an automorphism which has a fixed chamber. Then all adjacent chambers must be fixed. This is because each rank 1 residue must have exactly two chambers, by definition. So, as Coxeter complexes are connected, we must fix the whole of the complex. So this action is simple-transitive. □

3.1 Reflections and walls

Definition 3.5. A *reflection* r of W is a conjugate of the generators of W. The wall M_r of a reflection r is the set of simplicies in the Coxeter complex which is fixed by r when r acts on the complex by left multiplication. Then M_r is a subcomplex of codimension 1.

By this definition, a panel lies in a wall M_r if and only if it is fixed by the reflection r.

Example 3.2. In the case that the Coxeter complex is infinite and affine, the walls correspond to hyperplanes of the geometric realisation.

Theorem 3.1. *There is a bijection between the set of reflections of a Coxeter group, and the set of walls in the corresponding Coxeter complex.*

Proof. Let p be an i-panel of a chamber x. Then the unique reflection which fixes p and interchanges x and xs_i is the map $r = xs_ix^{-1}$. So the map from reflections to walls is a bijection. □

3.2 Roots

We now want to consider the halfplanes defined by our walls.

Definition 3.6. A gallery $(c_0, ..., c_k)$ *crosses* M_r if there is an i such that M_r interchanges c_{i-1} and c_i.

Lemma 3.3.
1. Any minimal gallery does not cross a wall twice.
2. Every gallery from two alcoves x and y have the same parity of crossings of any wall.

Definition 3.7. Each hyperplane splits an apartment into two half-apartments called *roots*. If α is one root, we denote the other corresponding root by $-\alpha$.

Definition 3.8. A set of alcoves is called *convex* if any minimal gallery between two alcoves of the set lies entirely within the set.

Proposition 3.1.
1. Roots are convex.
2. Let α be a root, and let x and y be adjacent chambers with $x \in \alpha$ and $y \in -\alpha$. Then

$$\alpha = \{c | d(x, c) < d(y, c)\}.$$

3. There are bijections between the set of all reflections of a Coxeter group, the set of walls, and the set of pairs of opposite roots.

Definition 3.9. A *folding of W onto α* is the map which fixes α and sends $-\alpha$ to α by relfecting across the the defining wall of α.

Proposition 3.2. Consider any two chambers x and y. Let $(x, p_1, x_1, ..., x_{k-1,} p_k, y)$ be a minimal gallery from x to y. Define β_i to be the root which contains x_{i-1} and which does not contain x_i. Then the β_i are all distinct, and this set is all the roots which contain x but do not contain y. So in particular, $d(x, y) = k$ is the size of the set of roots containing x but not containing y.

Proposition 3.3. Given two chambers x and y, a third chamber z lies on a minimal gallery from x to y if and only if it is contained within every root which also contains x and y.

Let R be a residue. Now we can define a map, called $\mathrm{proj}_R w$, which maps w to the unique chamber of R closest to w.

Proposition 3.4. Given a residue R and a chamber $x \in R$, for any chamber w there is a minimal gallery from x to w which passes through $\mathrm{proj}_R w$.

Lemma 3.4. Residues are convex.

Theorem 3.2. *Given a gallery γ of type f, γ is minimal if and only if f is reduced.*

3.3 Finite Coxeter complexes

Now we assume that our group W is finite, and so our Coxeter complex is also finite.

Definition 3.10. The *diameter*, $\mathrm{diam}(W)$, of W is the maximum distance between two chambers of the Coxeter complex. Two chambers are said to be *opposite* if the distance between them is $\mathrm{diam}(W)$.

Theorem 3.3. *1. $diam(W) = 1/2 * |\{roots\ of\ W\}|$.*

2. Two chambers are contained in no common root if and only if they are opposite.

3. For any given chamber, there is a unique opposite chamber.

4. Any chamber lies on a minimal gallery between two opposite chambers.

4 Buildings

Definition 4.1. Let (W, S) be a Coxeter group with Coxeter matrix M, and let I be an indexing set for the generators S of W. A *building of type M* is a chamber system Δ over

I, such that each panel lies on at least two chambers, i.e. every $\{i\}$-residue contains at least two elements. We also require a W-distance function

$$\delta : \Delta \times \Delta \to W,$$

such that if f is a reduced word in S, then we have that $\delta(x, y) = s_f$ if and only if there is a gallery of type f between x and y. We denote a building by $\Sigma = \Sigma(W, S)$.

Example 4.1. Taking our W-distance function to be $\delta(x, y) = x^{-1}y$, Coxeter complexes are buildings.

Some key properties of buildings are as follows:

1. Δ is connected.
2. δ is surjective.
3. $\delta(x, y) = \delta(y, x)^{-1}$.
4. $\delta(x, y) = s_i$ if and only if $x \neq y$ and $x \sim_i y$.
5. For $i \neq j$, i- and j-adjacency are mutually exclusive.
6. For chambers x and y, if there is a gallery from x to y of type f, and f is homotopic to g, then there is a gallery from x to y of type g.
7. A gallery if minimal if and only if its type is reduced.
8. If there is a gallery of type f from x to y, and f is reduced, then this gallery is unique.

4.1 Properties of buildings

Theorem 4.1. *Any J-residue is a building of type M_J.*

Theorem 4.2. *Any isometry from a subset of W into Δ can be extended to an isometry of W into Δ.*

Corollary 4.1. Any two chambers lie in a common apartment.

Theorem 4.3. *Apartments are convex.*

5 Retractions of buildings

Definition 5.1. Let A be a chosen apartment in a building X, and let $c \in A$ be an alcove. We define the *retraction from X to A based at c* as the map $r_{A,c} : X \longrightarrow A$ where we send any alcove d to its image under the isomorphism from the apartment containing both c and d to A.

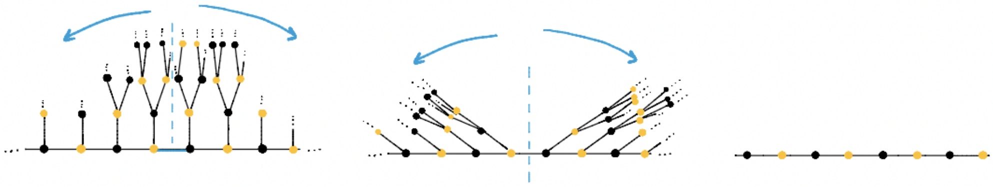


FIGURE 13. The retraction based at the fundamental alcove flattens the tree outwards.

Definition 5.2. Let A be a chosen apartment in an affine building X, and let $C \in \partial A$ be a chamber at infinity of the apartment. We define the *retraction from X to A based at C* as the map $\rho_{A,C} : X \longrightarrow A$, which sends an alcove d to its image under the isomorphism from the apartment containing both d and a Weyl chamber representing C to A.

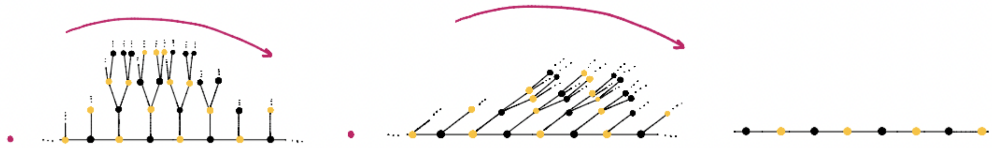


FIGURE 14. The retraction of a tree based at a Weyl chamber at infinity, indicated by the pink dot to the left of the horizontal line, flattens the tree away from that direction at infinity.

6 Orientations of buildings

We want to consider an orientation on the panels of our buildings. Ultimately, we want to define foldings of galleries, and we only want to consider postive foldings with respect to some orientation.

Definition 6.1. An *orientation* ϕ of a building Σ is a map from the set of pairs (p, c), where p is a panel and c is an alcove containing p, to the set $\{+1, -1\}$. If $\phi(p, c) = +1$, then we say that c is on a *ϕ-positive side* of p, otherwise we say that c is on a *ϕ-negative side.*

Example 6.1. The trivial positive orientation is the map which sends all pairs to $+1$. Similarly, the trivial negative orientation is the map which sends all pairs to -1.

Often, we do not want to have orientations which locally behave like trivial orientations. Hence, we define the following concepts:

Definition 6.2. Given an orientation ϕ of Σ, we have

1. ϕ is *locally non-negative* if, for each panel, there is at least one alcove which is on the ϕ-positive side.
2. ϕ is *locally non-trivial* if, for each panel, there is exactly one alcove which is on the ϕ-positive side.

There is a natural action of W on the set of all possible orientations of Σ, induced by the action of W on on the alcoves and panels. It is defined as

$$(x \cdot \phi)(p, c) := \phi(x^{-1}p, x^{-1}c).$$

Definition 6.3. Given an orientation ϕ of Σ, we say that ϕ is *wall consistent* if, given any wall H, for all pairs c, d of alcoves which lie in the same halfspace of H, with panels p and q lying in H, we have that $\phi(p, c) = \phi(q, d)$. If our orientation is wall consistent, we can then define the *positive side* H^ϵ of H as the half-space such that all alcoves c in H^ϵ have $\phi(p, c) = +1$ for all panels of c. Then the *negative side* is defined similarly.

We want to look at several natural ways to orient a Coxeter complex. First, we will look at an orientation which is derived from either a choice of alcove, or a choice of panel. This orientation works for any Coxeter group.

Definition 6.4. Choose a fixed alcove c in Σ. Now given any alcove d, and panel p, we define their orientation as $\phi(p, d) = +1$ if and only if c and d lie in the same side of the wall which is spanned by p. We call this orientation the *alcove orientation towards* c.

Definition 6.5. Choose a fixed simplex b in Σ. Now given any alcove c, and panel p in c, we define their orientation as $\phi(q, c) = +1$ if and only if either c and b lie in the same side of the wall H containing p, or if b lies inside H. We call this orientation the *simplex orientation towards* b.

Example 6.2. Here we see two simplex orientations of an A_2 Coxeter complex. In this complex, the alcoves are edges, and the panels are vertices.

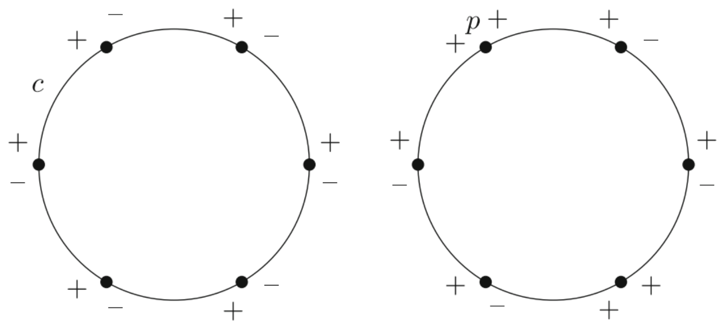


FIGURE 2. An alcove (left) and panel orientation (right) on the type A_2 Coxeter complex

Lemma 6.1. Consider a Coxeter group (W, S) with Coxeter complex Σ. We have the following:

(i) If ϕ is a simplex orientation of Σ, then ϕ is wall consistent and locally non-negative.

(ii) If ϕ is an alcove orientation of Σ, then ϕ is wall consistent and locally non-trivial.

Proof. (i) Let b be the simplex defining the orientation, and consider a wall H in our Coxeter complex. First let us consider the case in which b lies inside H. Then, by definition of the simplex orientation, both sides of the wall are defined to be positive. So any two alcoves, and any respective panels, lying in the same halfplane of H will have the same orientation. So this wall satisfies the conditions of wall consistency, and both sides are defined as positive so it is locally non-negative.

Now assume that b does not lie in H, so b lies in exactly one halfplane of H. Then this side of the halfplane is the positive side, and any two alcoves, and any respective panels, in this halfplane are given a positive orientation. Similarly, any two alcoves, and any respective panels, in the other halfplane are given a negative orientation. So again, this wall satisfies the conditions of wall-consistency, and both sides are defined as positive so it is locally non-negative.

(ii) An alcove orientation is a type of simplex orientation, so part (i) implies that ϕ is wall consistent. Now considering the cases from part (i), we can never be in the first case. This is because an alcove has one higher dimension whan a wall, and so an alcove can never fully lie within a wall. So therefore we are always in case two, and so by the same argument as part (i), we conclude that ϕ is locally non-trivial. □

6.1 The affine case

Now we want to consider when our Coxeter complex Σ is affine. To define an orientation on Σ, we choose a chamber at infinity.

If ϕ is a wall consistent orientation, then, given two chambers c, d which share a common panel p, c and d are given the same orientation if they lie in the same half-space of the hyperplane spanned by p. This amounts to picking a positive side of the hyperplane.

However, we did not have to pick these positive sides in any consistent way.

Definition 6.6. Let ϕ be a wall consistent orientation of an affine Coxeter complex. We say that ϕ is *periodic* if, given two parallel hyperplanes H_1, H_2 and corresponding half-spaces $H_1^\epsilon, H_2^\epsilon$, if $H_1^\epsilon \subset H_2^\epsilon$, then H_1^ϵ is positive if and only if H_2^ϵ is positive.

Example 6.3. If ϕ is a trivial orientation on an affine Coxeter complex, then ϕ is periodic.

Example 6.4. Simplex orientations are not periodic, as, for every set of parallel hyperplanes, we can find pairs of representatives which have the simplex on different sides.

If ϕ is a periodic orientation, then we have a natural orientation induced on the boundary. Similarly, if we have an orinetation defined on the boundary of a Coxeter complex, then we have a periodic orientation on the Coxeter complex which induces this orientation.

Lemma 6.2. [2, p.125] Given a periodic orientation ϕ on an affine Coxeter complex Σ, there is an induced wall-consistent orientation $\partial\phi$ on the boundary complex $\partial\Sigma$. Now if ϕ is locally non-negative or non-trivial, so is $\partial\phi$.

Proof. Consider a wall M in the boundary $\partial\Sigma$. This corresponds to a set of parallel walls in Σ. Consider a chamber $a \in \partial\Sigma$, which has a panel p lying in M. Now we can find a Weyl chamber C_a of Σ which represents a. This has a bounding wall H_M in the set of parallel walls corresponding to M. Let c be the alcove at the tip of C_a. So c has a panel q which lies in H_M. We now define the orientation of the boundary by

$$\partial\phi(a, p) = \phi(c, q).$$

This is well-defined as ϕ is periodic, so the choice of C_a does not affect the orientation. Also, as ϕ is periodic, this orientation is wall-consistent. If ϕ is locally non-negative, then given a panel q of Σ, we can find an alcove d such that $\phi(d, q) = +1$. Then under the projection map from Σ to the boundary $\partial\Sigma$, we get a chamber a and panel p such that $\partial\Sigma(a, p) = +1$. So $\partial\phi$ is locally non-negative. The same argument can be made to show that if ϕ is locally non-trivial, then $\partial\phi$ is also locally non-trivial. □

Lemma 6.3. Given a wall-consistent orientation ϕ of the boundary complex $\partial\Sigma$, there exists a unique periodic orientation $\tilde{\phi}$ of Σwhich induces the orientation ϕ.

Proof. Let H be a wall in Σ. Given a halfplane H^ϵ of H, we define H^ϵ to be the positive side of H if the corresponding halfplane ∂H^ϵ is the positive side of the wall ∂H with respect to the orientation ϕ. Otherwise we define H^ϵ to be the negative side of H. This is well-defined as ϕ is wall-consistent. This definition also uniquely defines the orientation $\tilde{\phi}$. □

Definition 6.7. Let σ be a chamber of the boundary Δ of a Coxeter complex Σ. Then we form an orientation ϕ_σ on the boundary Δ. The *Weyl chamber orientation* on Σ is the oritentation on Σ which induces ϕ_σ.

7 Folded galleries

7.1 Definitions

Definition 7.1. Given a gallery γ of Σ, we say that γ is *folded* (or *stammering*) if, within γ, we can find an index i such that $c_i = c_{i-1}$. Then we say that γ has a *fold* at panel p_i. Otherwise, we say that γ is *unfolded* (or *non-stammering*).

Definition 7.2. Given a gallery γ, define the set $\mathrm{F}(\gamma)$ to be the subset of $\{1, ..., n\}$ such that $i \in \mathrm{F}(\gamma)$ if and only if γ has a fold at panel p_i.

To represent a gallery, we draw a path which passes through every chamber and panel in the gallery of the geometric representation of our Coxeter complex. We draw an arrow towards the end alcove of our gallery.

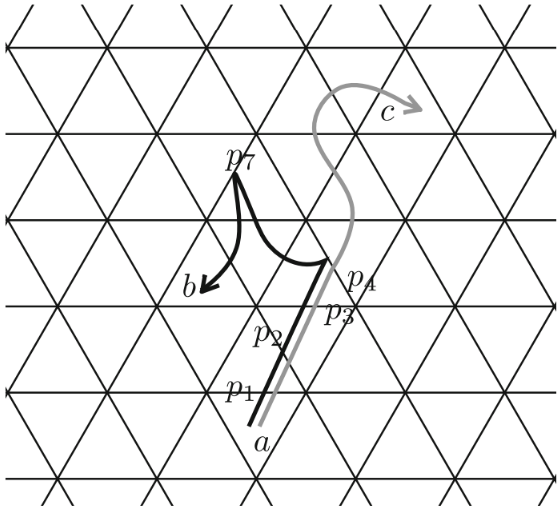


FIGURE 3. This figure shows galleries in type $\tilde{A}_2$ with two folds (black) and no folds (gray)

Definition 7.3. Given a gallery γ in Σ, and an orientation ϕ, we say that γ is *positively folded* with respect to ϕ if, whenever γ is folded at position i, $\phi(p_i, c_i) = +1$. We can similarly define *negatively folded*.

We note that, as W has a natural left action on Σ, W also acts on the set of galleries in Σ. For instance, $x \in W$ sends $\gamma = (c_0, p_1, c_1, ..., p_n, c_n)$ to the gallery $\gamma = (xc_0, xp_1, xc_1, ..., xp_n, xc_n)$.

Lemma 7.1. Consider an affine Coxeter system (W, S) with a Coxeter complex Σ. Let a be a chamber in the boundary complex $\partial\Sigma$. Now a gallery γ is ϕ_a-positively folded if and only if $x \cdot \gamma$ is ϕ_a-positively folded. So the action of W on $\partial\Sigma$ preserves the condition of being 'ϕ_a-positively folded'.

7.2 Galleries and words

We can now define the type and decorated type of a gallery. Note that here we use tilde notation to denote a decorated word, but in other texts, such as [2], they use hat notation to denote the decorated word.

Definition 7.4. Consider a gallery $\gamma = (c_0, p_1, c_1, ..., p_n, c_n)$. Let panel p_i of γ have type $s_{j_i} \in S$. We define its *type* $\tau(\gamma)$ as the word

$$\tau(\gamma) := s_{j_1} ... s_{j_n}.$$

We denote by $\Gamma^+_\phi(w)$ the set of all ϕ-positively folded galleries which have type w.

Definition 7.5. The *decorated type* $\tilde{\tau}(\gamma)$ of a gallery $\gamma = (c_0, p_1, c_1, ..., p_n, c_n)$ is the decorated word

$$\tilde{\tau}(\gamma) := s_{j_1}...\tilde{s_{j_i}}...s_{j_n},$$

where we place a tilde on the elements s_{j_i} of the word which correspond to a fold $c_{i-1} = c_i$ of the gallery. We denote by $\Gamma^+_\phi(\tilde{w})$ the set of all ϕ-positively folded galleries which have decorated type $\tilde{w}$.

Lemma 7.2. [2, p.128] Let c_0 be a fixed alcove in our Coxeter complex Σ.

(i) There is a bijection between words in S and unfolded galleries starting at c_0.

(ii) There is a bijection between decorated words in S and gallleries starting at c_0.

Proof. Given a word $s_1 \dots s_n$ in S, we can define an unfolded gallery starting at c_0 by multiplying c_0 by s_1, and in general define c_i by multiplying c_{i-1} by s_i, and set p_i to be the unique panel contained in both c_{i-1} and c_i. This gives our bijection for part (i). Now given a decorated word in S, we can define a general gallery by multiplying c_{i-1} by s_i, as above, if s_i is not decorated. If s_i is decorated, then let $c_i = c_{i-1}$, and let the panel p_i be the unique panel of c_i which has type s_i. This gives the bijection for part (ii). □

The next lemma gives some easy results from the definitions of type and decorated type.

Lemma 7.3. [2, p.128] Let γ be a gallery. Then

1. $F(\gamma) = \emptyset$ if and only if $\tau(\gamma) = \tilde{\tau}(\gamma)$.
2. γ is minimal if and only if $F(\gamma) = \emptyset$ and $\tau(\gamma)$ is reduced.

We want to be able to characterise the last alcove in a gallery. We do this by constructing another gallary which removes any folds from our original gallery. This leads to an unfolded gallery which has shorter length than the original gallery.

Definition 7.6. Consider a gallery $\gamma = (c_0, p_1, c_1, ..., p_n, c_n)$ in Σ. We create a new gallery, called the *footprint* $\mathrm{ft}(\gamma)$ *of* γ, by deleting all pairs p_i, c_i such that the letter s_i has a hat in $\hat{\tau}(\gamma)$.

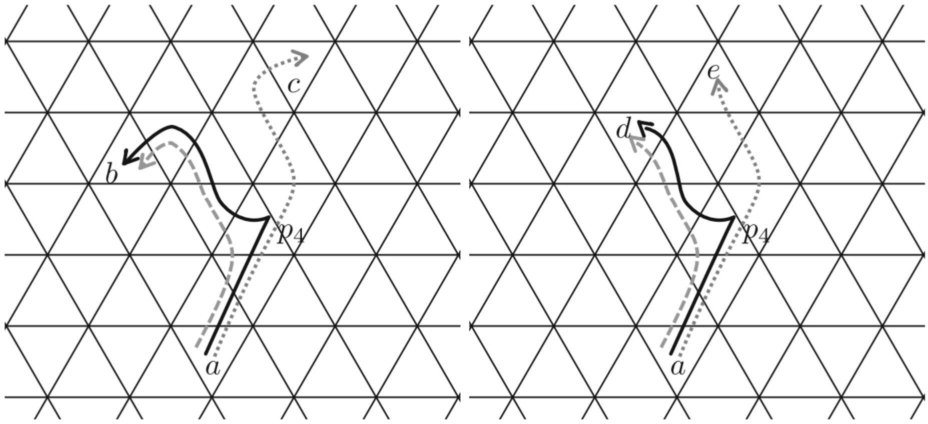


FIGURE 4. This figure shows galleries (black), their unfolded images (dotted gray), and footprints (dashed gray)

Lemma 7.4. We can calculate the final alcove of a gallery as the element $c_n = c_0 \cdot w$, where $w = \tau(\mathrm{ft}(\gamma))$.

Proof. The footprint of a gallery is exactly the gallery achieved by deleting repeated alcoves and the corresponding panels. So we are left with a gallery $\mathrm{ft}(\gamma) = (c_o = d_0, q_1, d_1, ..., q_m, d_m = c_n)$. Now d_i is exactly defined as the alcove obtained by multiplying d_{i-1} by s_i, where s_i is the type of the panel q_i. So, by induction, d_m, which equals c_n, can be calculated by multiplying $d_0 = c_0$ by the type of $\mathrm{ft}(\gamma)$. $\square$

7.3 Folding and unfolding galleries

Now we have defined galleries, and in particular folded galleries, we want to be able to create folded galleries ourselves from unfolded galleries. We do this in a natural way, where folding along a panel leads to a reflection of the rest of the gallery with respect to that panel. For instance, this figure shows foldings in the 4th and 7th panel of the given gallery, and also illustrates that foldings are commutative - a fact that we will formally prove.

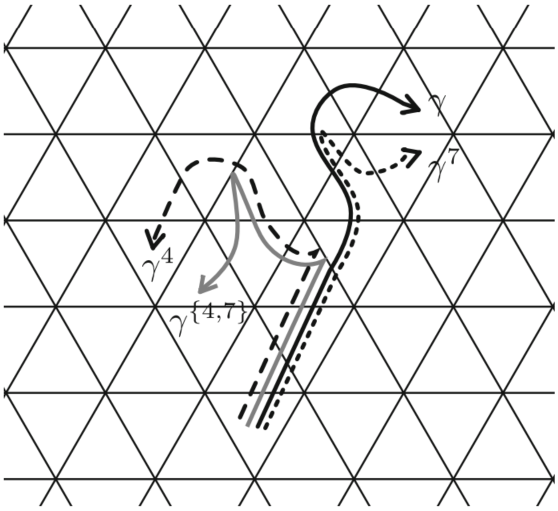


FIGURE 5. This figure shows commuting folds at panels 4 and 7 of the black gallery γ

Definition 7.7. Consider a gallery $\gamma = (c_0, p_1, c_1, ..., p_n, c_n)$. Let H_i be the wall containing the panel p_i, and let r_i be the reflection across H_i. For $i = 1, ..., n$, let

$$\gamma^i := (c_o, p_1, ..., p_i, r_i c_i, r_i p_{i+1}, r_i c_{i+1}, ..., r_i p_n, r_i c_n).$$

If γ was folded at panel p_i, we call γ^i a *unfolding of* γ at p_i. Otherwise, we call it a *folding*.

Lemma 7.5. For all $i = 1, ..., n$, $\tau(\gamma) = \tau(\gamma^i)$. So folding and unfolding does not change the gallery type. Also, $(\gamma^i)^i = \gamma$.

Proof. This is just a result of the definition of the type of a panel, which is invariant under reflections along walls. Also, we note that $r_i r_i = 1$, as reflections are self-inverse, so applying a fold twice at panel p_i will first achieve $(c_o, p_1, \ldots, p_i, r_i c_i, r_i p_{i+1}, r_i c_{i+1}, ..., r_i p_n, r_i c_n)$, and will then achieve $(c_o, p_1, ..., p_i, r_i r_i c_i, r_i r_i p_{i+1}, r_i r_i c_{i+1}, ..., r_i r_i p_n, r_i r_i c_n) = (c_0, p_1, ..., p_n, c_n)$. Hence, $(\gamma^i)^i = \gamma$. $\square$

Lemma 7.6. For all $i, j = 1, \ldots, n$, $(\gamma^i)^j = (\gamma^j)^i$, i.e. foldings are commutative.

Proof. As we have already dealt with the case that $i = j$, we can assume that $i < j$. Let r be the reflection along the wall containing p_i, and let t be the reflection along the wall containing p_j. Then we have, by the definition of (un)-folding,

$$(\gamma^j)^i = (c_0, p_1, c_1, \ldots, c_{i-1}, p_i, rc_i, ..., rp_j, rtc_j, ..., rtc_n).$$

Also, if we let u be the reflection along the wall containing rp_j, we have

$$(\gamma^i)^j = (c_0, p_1, c_1, ..., c_{i-1}, p_i, rc_i, ..., rp_j, urc_j, ..., urc_n).$$

Let us calculate the maps r, t and u. Given a panel p of an alcove c, the reflection along the wall containing p is given by the multiplication map $c\tau(p)c^{-1}$. Hence,

$$r = c_{i-1}\tau(p_i)c_{i-1}^{-1}, t = c_{j-1}\tau(p_j)c_{j-1}^{-1}, \text{ and } u = (rc_{j-1})\tau(rp_j)(rc_{j-1})^{-1}.$$

Now by a previous lemma, reflections preserve type, so we have that $\tau(rp_j) = \tau(p_j)$. Then

$$\begin{aligned} ur &= (rc_{j-1})\tau(rp_j)(rc_{j-1})^{-1}c_{i-1}\tau(p_i)c_{i-1}^{-1} \\ &= r(c_{j-1}\tau(p_j)c_{j-1}^{-1})(c_{i-1}\tau(p_i)c_{i-1}^{-1})(c_{i-1}\tau(p_i)c_{i-1}^{-1}) \\ &= r(c_{j-1}\tau(p_j)c_{j-1}^{-1}) \\ &= rt. \end{aligned}$$

Therefore $ur = rt$, and hence $(\gamma^j)^i = (\gamma^i)^j$. □

Because of this property, we are able to define a *multifolding* with respect to a subset I of $\{1, ..., n\}$ as the (un-)foldings γ^I. Now multifoldng does not affect the type. Then the set of folds of γ^I will be the symmetric difference of the folds of γ and I. In particular, if I and J are subsets of $\{1, ..., n\}$, $(\gamma^I)^J = \gamma^{I\Delta J}$. The following corollary now follows.

Corollary 7.1. Given any gallery γ, there is a subset $I \subset \{1, ..., n\}$ such that γ^I is unfolded, and γ and γ^I have the same type.

Now we fix an alcove of our Coxeter complex, and call this 1. Then, for any word w with elements in S, we let γ_w be the unique unfolded gallery which has type w and starts at 1. Now we write

1. $\gamma \rightharpoonup \eta$ if γ and η are galleries such that $\eta = \gamma^I$ for some index set I, i.e. there is a folding of γ which gives η.

2. $w \rightharpoonup u$ if w and u are words in S such that there is a folding of γ_w which has footprint u.

3. $w \rightharpoonup x$ if x is an element of W such that there is a folding of γ_w which has end alcove c_x.

We denote by $A \overset{\phi}{\rightharpoonup} B$ if the respective gallery is ϕ-positively folded.

8 Braid invariant orientations

Two reduced words in S represent the same element in the Coxeter group if and only if they differ by a squence of braid moves. A braid move replaces a subword $s_is_j\cdots$ of length m_{ij} with the string $s_js_i\cdots$, again of length m_{ij}. We want to define the concept of braid invariant orientations, so we can later conclude that, if we have a braid invariant orientation, our shadows of a gallery do not depend on the chosen word of S representing the end alcove.

Definition 8.1. Consider a Coxeter system (W, S) and a corresponding Coxeter complex Σ. Let ϕ be an orientation on Σ. Then we say that ϕ is *braid invariant* if, given any two braid equivalent words w, w' in S and any $x \in W$, $w \overset{\phi}{\rightharpoonup} x$ if and only if $w' \overset{\phi}{\rightharpoonup} x$. Then if $y\phi$ is braid invariant for all $y \in W$, ϕ is called *strongly braid invariant.*

For instance, trivial orientations are strongly braid invariant, but these orientations are not very interesting. The following proposition gives us a large family of orientations which are braid invariant. A proof of this proposition can be found in [2, pp.135-138].

Proposition 8.1. Weyl chamber orientations are braid invariant.

Happy to prove this proposition, but the proof is very long so wanted to ask before I wrote it.

9 Shadows

Definition 9.1. Consider a Coxeter system W and an orientation ϕ on the Coxeter complex $\Sigma(W, S)$. Let w be a word is S. The *shadow* of w with respect to ϕ is the set

$$\mathrm{Sh}_\phi(w) := \{u \in W | w \overset{\phi}{\rightharpoonup} u\}.$$

If ϕ is braid invariant, we can define $\mathrm{Sh}_\phi(x)$ $=\mathrm{Sh}_\phi(w)$, where w is any reduced expression of $x \in W$. If we have the Weyl chamber orientation ϕ_a with $a \in W_0$, the *regular shadow* of x with respect to a is $\mathrm{Sh}_a(w)$ $:=\mathrm{Sh}_{\phi_a}(w)$. The *full shadow* of x is the union of regular shadows.

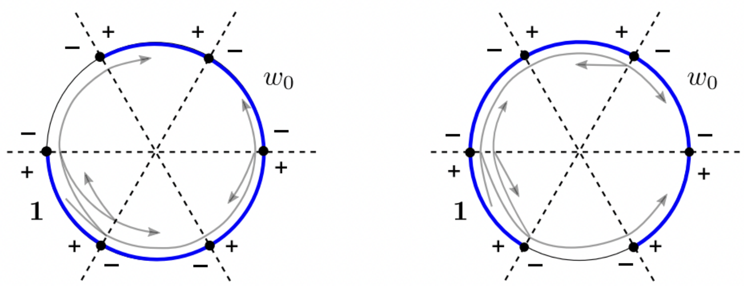
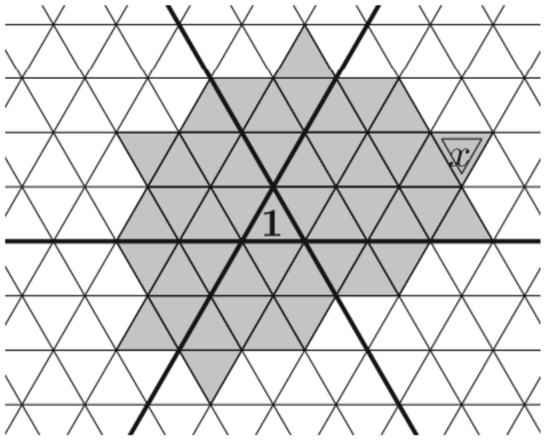


FIGURE 6. The picture shows a non-braid-invariant orientation which hence produces different shadows (shown fat blue) for the two minimal galleries from **1** to w_0. See Example 6.2 for details

Definition 9.2. Let $x = s_1 ... s_n$ be a reduced expression for $x \in W$. Let $y \in W$. We say that $y \leq x$ if there exists a reduced expression for y of the form $s_{i_1} ... s_{i_k}$ with $1 \leq i_1 \leq ... \leq i_j \leq n$. This ordering is called the *Bruhat order.*

Proposition 9.1. Consider the trivial positive orientation ϕ_+, and the alcove orientation ϕ_1 towards 1. For $x, y \in W$, $x \geq y$ if and only if $x \xrightarrow{\phi_+} y$, if and only if $x \xrightarrow{\phi_1} y$.

Now this proposition is telling us that actually both the trivial positive orientation and the alcove orientation towards 1 are not interesting orientations to study in terms of the shadow.



Example 9.1. The picture above shows the shadow for an alcove of a Coxeter complex of type $\tilde{A}_2$, with respect to the trivial positive orientation. By the proposition above, this is also the shadow with respect to the alcove orientation towards 1. Furthermore, the alcoves in this shadow are all the elements y of the Coxeter group such that $y \leq x$ with respect to the Bruhat ordering, and so it is the Bruhat interval $[1, x]$.

10 Progress on calculating shadows

So now our main question becomes whether we can calculate the shadow of a given gallery. This can only be partially answered, and only with specific orientations.

10.1 Statistics on positive folds

We now restrict to looking at Weyl chamber orientations over affine Coxeter complexes. This means that we have a complex Σ, with a boundary $\partial\Sigma$, and that our orientations are induced by a boundary chamber orientation. Here, we can get a partial answer to our main question of calculating the shadow of a given gallery. To do this, we define a ϕ-valuation map on our set of alcoves. We can then prove a recursive algorithm for calculating the shadow of a gallery.

First, given a gallery, we want to calculate the number of positive folds of this gallery that we can make. A proof of this proposition can be found in [4].

Proposition 10.1. Consider the largest element w_0 in W_0. Given an $x \in W$, and a ϕ-postive (multi)folding γ of γ_x, we have

$$l_R(xy^{-1}) \leq |F(\gamma)| \leq l(w_0),$$

where $y := \tau(\mathrm{ft}(\gamma))$.

Definition 10.1. Let $\mathcal{H}(\Sigma)$ be the set of all walls contained in our Coxeter complex. For an alcove c of Σ, let $\mathcal{H}(c)$ be the subset of $\mathcal{H}(\Sigma)$ which separates c and the fixed identity alcove 1. Now $\mathcal{H}(c) = \mathcal{H}^+_\phi(c) \sqcup \mathcal{H}^-_\phi(c)$, where $\mathcal{H}^+_\phi(c)$ is the subset of $\mathcal{H}(c)$ such that c lies on the positive side of the walls, and similarly for $\mathcal{H}^-_\phi(c)$.

Definition 10.2. Let $\mathrm{Ch}(\Sigma)$ denote the set of all alcoves in Σ. The ϕ-valuation map is the map $\mathrm{v}_\phi : \mathrm{Ch}(\Sigma) \longrightarrow \mathbb{Z}$, with

$$c \mapsto \mathrm{v}_\phi(c) := |\mathcal{H}^+_\phi(c)| - |\mathcal{H}^-_\phi(c)|.$$

Definition 10.3. Let $p_\phi : \mathrm{Ch}(\Sigma) \times \mathcal{H} \longrightarrow \{0, 1\}$ be the function

$$p_\phi(c, H) := \begin{cases} 1 & \text{if } c \text{ is on a } \phi\text{-positive side of } H, \\ 0 & \text{otherwise.} \end{cases}$$

We now want to relate this function to our ϕ-valuation map.

Lemma 10.1.

$$\mathrm{v}_\phi(c) = \sum_{H \in \mathcal{H}(\Sigma)} (p_\phi(c, H) - p_\phi(1, H)).$$

Proof. We are assuming that our orientation ϕ is a chamber orientation. So, in particular, this orientation is locally non-trivial and wall consistent. Therefore, every hyperplane H has a positive and negative side. First consider when 1 and c lie on the same side of H. Then H is not an element of $\mathcal{H}(c)$. But, in this case, $p_\phi(1, H) = p_\phi(c, H)$ and so this hyperplane does not contribute to the above sum. Now consider when 1 and c lie on opposite sides of H. In this case, $H \in \mathcal{H}(c)$. If c lies on the positive side of H, then $H \in \mathcal{H}^+_\phi(c)$, $p_\phi(c, H) = 1$ and $p_\phi(1, H) = 0$, and so H contributes $+1$ to the sum above. Similarly, if c lies on the negative side of H, then $H \in \mathcal{H}^-_\phi(c)$, $p_\phi(c, H) = 0$ and $p_\phi(1, H) = 1$, and so H contributes -1 to the sum above. Therefore, we are just counting the size of $\mathcal{H}^+_\phi(c)$ minus the size of $\mathcal{H}^-_\phi(c)$, which is exactly $\mathrm{v}_\phi(c)$. □

The next lemma comes from the trivial observation that

$$|\mathcal{H}^+_\phi(c)| + |\mathcal{H}^+_\phi(c)| \geq |\mathcal{H}^+_\phi(c)| - |\mathcal{H}^+_\phi(c)|.$$

Lemma 10.2.

$$l(x) \geq \mathrm{v}_\phi(c_x).$$

Definition 10.4. We call an alcove c *dominant* with respect to ϕ if $\mathrm{v}_\phi(c) = l(c)$.

The next two lemmas tell us more information about the ϕ-valuation map. Proofs can be found in [2].

Lemma 10.3.

$$l(x) = \max_{a \in W_0} \mathrm{v}_{\tilde{\phi}_a}(c_x).$$

Lemma 10.4. Let $\phi \in$Dir(W), $r \in W$ be a reflection across the hyperplane H_r and $x \in W$. Then $\mathrm{v}_\phi(x) >\mathrm{v}_\phi(rx)$ if and only if x lies in the ϕ-positive side of H_r.

10.2 Computation of regular shadows

We now want to see how we can use this new valuation map to define a recursive definition of a shadow. To do this, we need the next important theorem. A proof of this theorem can be found in [2, pp.142-143].

Let Dir(W) represent the set of chambers in the boundary complex $\partial\Sigma$. We call elements of Dir(W) *directions in W*.

Theorem 10.1. *Let $\phi \in$Dir(W), $x \in W$ and $s \in S$. Then*

(i) If s is in the right descent set $D_R(x)$ of x, then we have

$$\mathrm{Sh}_\phi(x) = \mathrm{Sh}_\phi(xs) \cdot s \cup \{z \in \mathrm{Sh}_\phi(xs) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\}.$$

(ii) If s is in the left descent set $D_R(x)$ of x, then we have

$$\mathrm{Sh}_\phi(x) = \begin{cases} s \cdot \mathrm{Sh}_{s\phi}(sx) \cup \mathrm{Sh}_\phi(sx) & if\ \mathrm{v}_\phi(s) < 0, \\ s \cdot \mathrm{Sh}_{s\phi}(sx) & if\ \mathrm{v}_\phi(s) > 0. \end{cases}$$

Now we can use this theorem to show that the next two lemmas both give us recursive defintions for the shadow of a gallery.

Lemma 10.5. (Algorithm L) Let $\phi \in$ Dir(W) and $x \in W$. Let $w = s_1...s_n$ be a reduced word for x. Let $A_0 = \{1\}$ and let

$$A_i := A_{i-1} \cdot s_i \cup \{z \in A_{i-1} | v_\phi(zs) < v_\phi(z)\}.$$

Then $A_n = \mathrm{Sh}_\phi(x)$.

Proof. Using the theorem above, we can show by induction that A_i =Sh$_\phi(s_1...s_i)$ for $i = 0, ..., n$. Firstly, for $i = 0$ it is trivial, as Sh$(1) = \{1\}$. Then assume that A_i =Sh$_\phi(s_1...s_i)$ for $i < j$. By part (i) of the theorem,

$$\begin{aligned}
\mathrm{Sh}(s_1...s_j) &= \mathrm{Sh}(s_1...s_js_j) \cdot s_j \cup \{z \in \mathrm{Sh}(s_1...s_js_j) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= \mathrm{Sh}(s_1...s_{j-1}) \cdot s_j \cup \{z \in \mathrm{Sh}(s_1...s_{j-1}) : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= A_{j-1} \cdot s_j \cup \{z \in A_{j-1} : \mathrm{v}_\phi(zs) < \mathrm{v}_\phi(z)\} \\
&= A_j.
\end{aligned}$$

□

Lemma 10.6. (Algorithm R) Let $\phi \in \mathrm{Dir}(W)$ and $x \in W$, with $s_n...s_1$ a reduced expression for x. Let $B_0^\phi := \{1\}$ and define

$$B_i^\phi = \begin{cases} s_i B_{i-1}^{s_i\phi} \cup B_{i-1}^\phi & \text{if } \mathrm{v}_\phi(s_i) < 0, \\ s_i B_{i-1}^{s_i\phi} & \text{if } \mathrm{v}_\phi(s_i) > 0. \end{cases}$$

Then B_n^ϕ =Sh$_\phi(x)$ for all $\phi \in \mathrm{Dir}(W)$.

Proof. Again, we can use the theorem above to prove by induction that B_i^ϕ =Sh$_\phi(s_i...s_1)$ for all $i = 0, ..., n$. For $i = 0$ it is trivial, as Sh$_\phi(1) = \{1\}$. Now assume that B_i^ϕ =Sh$_\phi(s_i...s_1)$ for all $i < j$. By part (ii) of the theorem, if $v(s_j) < 0$, then

$$\begin{aligned}
Sh_\phi(s_j...s_1) &= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_js_j...s_1) \cup \mathrm{Sh}_\phi(s_js_j...s_1) \\
&= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_{j-1}...s_1) \cup \mathrm{Sh}_\phi(s_{j-1}...s_1) \\
&= s_j \cdot B_{j-1}^{s_j\phi} \cup B_{j-1}^\phi \\
&= B_j^\phi.
\end{aligned}$$

Similarly, if $v(s_j) > 0$, then

$$\begin{aligned}
Sh_\phi(s_j...s_1) &= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_js_j...s_1) \\
&= s_j \cdot \mathrm{Sh}_{s_j\phi}(s_{j-1}...s_1) \\
&= s_j \cdot B_{j-1}^{s_j\phi} \\
&= B_j^\phi.
\end{aligned}$$

□

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