Lectures on Buildings

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Contents

1 Chamber systems

Definition 1.1. A set C is called a *chamber system* over a set I if each $i \in I$ is an equivalence relation on the elements of C. Each i partitions our set C. We say two elements $x, y \in C$ are *i-adjacent*, and we write $x \sim_i y$, if they lie in the same part of the partition, i.e they are equivalent with respect to the equivalence relation corresponding to i. The elements of C are called *chambers*. The *rank* of a chamber system is the size of I.

Example 1.1. Given a group G, a subgroup B, and an indekxing set I, let there be a subgroup $B < P_i < G$ for all $i \in I$. Then we take as our chamber set C the left cosets of B, and we define an equivalence relation

$$gB \sim_i hB \text{ if and only if } gP_i = hP_i.$$

Definition 1.2. A finite sequence $(c_0, ..., c_k)$ such that c_i is adjacent to c_{i+1} is called a *gallery*. Its *type* is a word $i_1, ..., i_k$ in I such that c_{i-1} is i-adjacent to c_i. We assume that no two consecutive chambers are equal.

Definition 1.3. If (W, S) is a Coxeter group, and $f = i_1...i_k$, then s_f represents the word $s_{1_1}...s_{i_k}$.

Definition 1.4. We call C *connected* if we can connect any two chambers with a gallery. A *J-residue* is a J-connected component. We call $\{i\}$-residues *i-panels*.

Definition 1.5. Let C and D be two chamber systems over the same indexing set I. A *morphism* between C and D is a map $C \to D$ which preserves i-adjacency.

1.1 The geometric realisation

Definition 1.6. Let R be a J-residue and S be a K-residue. Then S is a *face* of R if $R \subset S$ and $J \subset K$. The *cotype* of J is the set $I - J$.

Observe that if R is a residue of cotype J, we have

1. for $K \subset J$, there is a unique face of R which has cotype K.
2. Let S_1, S_2 be faces of R with cotypes K_1 and K_2. Then S_1 and S_2 have a shared face of cotype $K_1 \cap K_2$.

With these observations, we can form a *cell complex* of our chamber system. To do this, we form a vertex for each residue of corank 1. Then, we can associate to each residue of cotype $\{i, j\}$ an edge. From the observation above, this has as its boundary the residues of cotype $\{i\}$ and of cotype $\{j\}$. Then this can be continued inductively.

Definition 1.7. Let σ be a simplex of our cell complex. The set *star* $St(\sigma)$ is the corresponding residue.

1.2 $A_n(k)$ Buildings

Let us consider an $n+1$ dimensional vector space V over a field k. We define the chambers of our chamber system to be the maximal sequences

$$V_1 \subset V_2 \subset ... \subset V_n$$

of subspaces of V, where V_i has dimension i. We can then define adjancency by saying that two sequences $V_1 \subset V_2 \subset ... \subset V_n$ and $V_1' \subset V_2' \subset ... \subset V_n'$ are i-adjacent if and only if $V_j = v_j'$ for all $j \neq i$. Then the residues of type i correspond to 1 spaces in the 2 space V_{i+1}/V_{i-1}.

We then get a geometric realisation of this chmaber system. Here, a residue of cotype $J = \{j_1, ..., j_r\}$ corresponds to a sequence

$$V_{j_1} \subset V_{j_2} \subset ... \subset V_{j_r}.$$

This residue has chambers which are maximal flags $V_1' \subset V_2' \subset ... \subset V_n'$ such that $V_j' = v_j$ if $j \in J$.

In particular, residues of cotype $\{i\}$ correspond to the subspaces of V.

1.3 $C_n(k)$ Buildings

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2 Coxeter Complexes

Given a Coxeter group W, take as chambers the elements of W, and define an i-adjancency by $w \sim_i wr_i$, where $\{s_1, ..., s_n\}$ are the set of generators of the Coxeter group. If the Coxeter group has Coxeter matrix M, we call this building a *Coxeter complex of type M*.

Diagram $\tilde{A}_2$. 

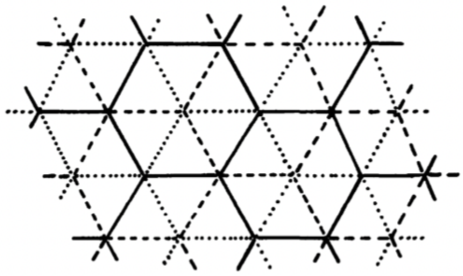
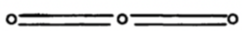


Figure 2.1

Diagram $\tilde{C}_2$. 

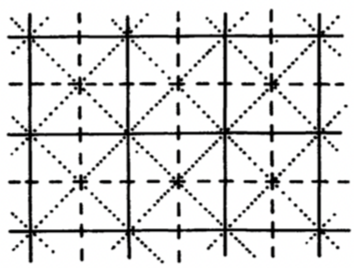


Figure 2.2

Lemma 2.1. The automorphism group of the Coxeter complex is isomorphic to the Coxeter group, and this acts simple-transitively on the set of chambers.

Definition 2.1. A *reflection* r of W is a conjugate of the generators of W. The wall M_r of a reflection r is the set of simplicies in the Coxeter complex which is fied by r when r acts on the complex by left multiplication. Then M_r is a subcomplex of codimension 1.

Theorem 2.1. *There is a bijection between the set of reflections of a Coxeter group, and the set of walls in the corresponding Coxeter complex.*

Definition 2.2. A gallery $(c_0, ..., c_k)$ *crosses* M_r if there is an i such that M_r interchanges c_{i-1} and c_i.

Lemma 2.2.

1. Any minimal gallery does not cross a wall twice.
2. Every gallery from two alcoves x and y have the same parity of crossings of any wall.

Definition 2.3. Each hyperplane splits an apartment into two half-apartments called *roots*. If α is one root, we denote the other corresponding root by $-\alpha$.

Definition 2.4. A set of alcoves is called *convex* if any minimal gallery between two alcoves of the set lies entirely within the set.

Proposition 2.1.

1. Roots are convex.
2. Let α be a root, and let x and y be adjacent chambers with $x \in \alpha$ and $y \in -\alpha$. Then
$$\alpha = \{c|d(x, c) < d(y, c)\}.$$
3. There are bijections between the set of all reflections of a Coxeter group, the set of walls, and the set of pairs of opposite roots.

Definition 2.5. A *folding of W onto α* is the map which fixes α and sends $-\alpha$ to α by relfecting across the the defining wall of α.

Proposition 2.2. Consider any chambers x and y. Let $(x, x_1, ..., x_{k-1}, y)$ be a minimal gallery from x to y. Define β_i to be the root which contains x_{i-1} and which does not contain x_i. Then the β_i are all distinct, and this set is all the roots which contain x but do not contain y. So in particular, $d(x, y) = k$ is the size of the set of roots containing x but not containing y.

Proposition 2.3. Given two chambers x and y, a third chamber z lies on a minimal gallery from x to y if and only if it is contained within every root which also contains x and y.

Let R be a residue. Now we can define a map, called $\text{proj}_R w$, which maps w to the unique chamber of R closest to w.

Proposition 2.4. Given a residue R and a chamber $x \in R$, for any chamber w there is a minimal gallery from x to w which passes through $\text{proj}_R w$.

Lemma 2.3. Residues are convex.

Theorem 2.2. *Given a gallery γ of type f, γ is minimal if and only if f is reduced.*

2.1 Finite Coxeter complexes

Now we assume that our group W is finite, and so our Coxeter complex is also finite.

Definition 2.6. The *diameter*, $\text{diam}(W)$, of W is the maximum distance between two chambers of the Coxeter complex. Two chambers are said to be *opposite* if the distance between them is $\text{diam}(W)$.

Theorem 2.3.

1. *$diam(W) = 1/2 * |\{roots\ of\ W\}|$.*
2. *Two chambers are contained in no common root if and only if they are opposite.*
3. *For any given chamber, there is a unique opposite chamber.*
4. *Any chamber lies on a minimal gallery between two opposite chambers.*

3 Buildings

Definition 3.1. Let (W, S) be a Coxeter group with Coxeter matrix M, and let I be an indexing set for the generators of W. A *building of type M* is a chamber system Δ over I, such that each panel lies on at least two chambers, i.e every $\{i\}$-residue contains at least two elements. We also require a W-distance function

$$\delta : \Delta \times \Delta \to W,$$

such that if f is a reduced word in S, then we have that $\delta(x, y) = s_f$ if and only if there is a gallery of type f between x and y.

Do we actually require this W-distance function, or are we saying that a building is a structure such that we could find a W-distance function satisfying the conditions?

Example 3.1. Taking our W-distance function to be $\delta(x, y) = x^{-1}y$, Coxeter complexes are buildings.

Some key properties of bulidings are as follows:

1. Δ is connected.
2. δ is surjective.
3. $\delta(x, y) = \delta(y, x)^{-1}$.
4. $\delta(x, y) = s_i$ if and only if $x \neq y$ and $x \sim_i y$.
5. For $i \neq j$, i- and j-adjacency are mutually exclusive.
6. For chambers x and y, if there is a gallery form x to y of type f, and f is homotopic to g, then there is a gallery from x to y of type g.

7. A gallery if minimal if and only if its type if reduced.

8. If there is a gallery of type f from x to y, and f is reduced, then this gallery is unique.

Theorem 3.1. *Any J-residue is a building of type M_J.*

Theorem 3.2. *Any isometry from a subset of W into Δ can be extended to an isometry of W into Δ.*

Corollary 3.1. Any two chambers lie in a common apartment.

Theorem 3.3. *Apartments are convex.*

4 BN pairs

Definition 4.1. Let G be a group. A *Tits system* or *BN-pair* of G is a a pair of subgroups B, N such that

1. $G = \langle B, N \rangle$

2. $B \cap N$ is normal inside N and $W = N \backslash B \cap N$ is a Coxeter group with generators $s_1, ..., s_n$.

3. If $w \in W$ and $s = s_i$, then $BsBwB \subset BwB \cup BswB$.

4. For all $s = s_i$, $sBs \neq B$.

Lemma 4.1. 1. If $BwB = Bw'B$, then we have $w = w'$. So G is the disjoint union $\bigcup BwB$. This union is called the Bruhat decomposition.

2. $l(sw) > l(w)$ implies that $BsBwB = BswB$.

Definition 4.2. We say that a group G of automorphisms on a building Δ is *strongly transitive* if

1. G is transitive on the sets of pairs of chambers with the same W-distance, and

2. there exists an apartment Σ whose stabiliser in G is transitive on the apartments of Σ.

Theorem 4.1. *Let G act strongly transitively on a thick building Δ. Let Σ be as above, and take W to be the corresponding Coxeter group. Choose a chamber c in Σ, and take $B =$stab$_G c$, $N =$stab$_G \Sigma$. Then (B, N) is a BN-pair, and $\delta(c, d) = w$ if and only if $d \subset BwB$, where we view d to be a left coset of B.*

Theorem 4.2. *Given a BN-pair, we can define a building. The chambers are left cosets of B, and i-adjacency if defined by*

$$gb \sim_i hB \iff g^{-1}h \in B\langle s_i \rangle B.$$

Then the distance function is defined by

$$\delta(B, gB) = w \iff gB \subset BwB.$$

Now the subgroup N stabilises an apartment, and G acts strongly transitively.

Example 4.1. Let $G = GL_{n+1}(k)$. Let B be the set of upper-triangular matrices, and N be the set of matrices with exactly one non-zero entry in each row and column. Then $B \cap N$ is the set of diagonal matrices, and this group gives $W \cong S_{n+1}$. W has generators $s_1, ..., s_n$, where s_i is the permutation matrix which switches i and $i+1$.

4.1 Parabolic subgroups

Definition 4.3. Given a subset $J \subset I$, we define the subgroup

$$W_J = \langle s_j | j \in J \rangle.$$

Now we define

$$P_J = BW_JB.$$

Then P_J is a subgroup of G by (BN2). Then a *parabolic subgroup* is a conjugate of a P_J. If $J = \emptyset$, then the conjugates of $B := P_\emptyset$ are called *Borel subgroups*.

Theorem 4.3.

1. *The set of subgroups containing B is precisely the set of P_J.*
2. *$P_J \cap P_K = P_{J \cap K}$ and $\langle P_J, P_K \rangle = P_{J \cup K}$.*
3. *The normaliser $N_G(P_J)$ of P_J is P_J. The stabiliser $stab_G(K)$ of the J-residue K containing c is P_J.*
4. *There is a bijection between the double coset spaces $W_J \backslash W / W_K$ and $P_J \backslash G / P_K$, defined by $W_J w W_K \mapsto P_J w P_K$.*

Example 4.2. Consider our previous example of $GL_{n+1}(k)$. Take $J = \{t, t+1, ..., t+m-1\}$. Now P_J has $GL_m(k)$ as its quotient group, and P_J consists of matrices which are upper-triangular, except for the $m \times m$ block starting at (t, t).

5 Spherical buildings and root groups

In this section, we assume that our Coxeter group is finite. Then a building where its associated Coxeter group is finite is called a *spherical building*. We assume in this section that all buildings are thick.

Recall that we call two chambers in an apartment *opposite* if their distance is equal to the diameter of W. We extend this idea to any pair of simplicies of a spherical building.

Definition 5.1. Let A be an apartment of a building. The *opposite involution* is the map $\mathrm{op}_A : A \to A$ which sends each chamber of A to its opposite chamber.

Definition 5.2. Two simplicies of W are *opposite* if they are interchanged by op_W.

Lemma 5.1. Let π and π' be opposite panels in a building. Let $x \in \mathrm{St}(\pi)$ and $y \in \mathrm{St}(\pi')$. Then $d(x, y) = d = \mathrm{Diam}(W)$ unless $x = \mathrm{proj}_\pi(y)$, in which case $d(x, y) = d - 1$.

Recall that we define a *root* of a building as a root of an apartment of the building. These are the half-apartments arisings from the walls of an apartment.

Lemma 5.2. Consider a root α of a building. Let x be a chamber such that there is a panel π of x which lies in $\partial\alpha$. Then there is exactly one root containing both x and $\partial\alpha$. Furthermore, if $x \notin \alpha$ then there is exactly one apartment containing both x and α.

Definition 5.3. Define the set $E_1(c)$ to be the set of chambers which are adjacent to c.

Proposition 5.1. Consider a spherical building, and let c and d be two chambers which are opposite. Let ϕ be an automorphism of the building which fixes b and all elements of $E_1(c)$. Then ϕ is the identity automorphism.

5.1 Root Groups

Definition 5.4. The *root group* U_α of a root α in a spherical building is the set

$$U_\alpha = \{g \in \mathrm{Aut}\Delta |\ g \text{ fixes every chamber having a panel in } \alpha - \partial\alpha\}.$$

This definition is only for when the "diagram" has no isolated nodes. What diagram are they referring to?

Now if this diagram is connected, then we conclude, using the proposition from the previous subsection, that the only element of U_α which fixes an apartment A which contains α is the identity automorphism.

Definition 5.5. A building is *Moufang* if, for every root α, U_α is transitive.

By the observation above, if a building is transitive, then it must be simple-transitive.

Noting that, if g sends one root α to another root β, then $gU_\alpha g^{-1} = u_\beta$, for a building to be Moufang it is enough to require U_α to be transitive for any root α in an apartment.