LAB 9: Numerical Integration

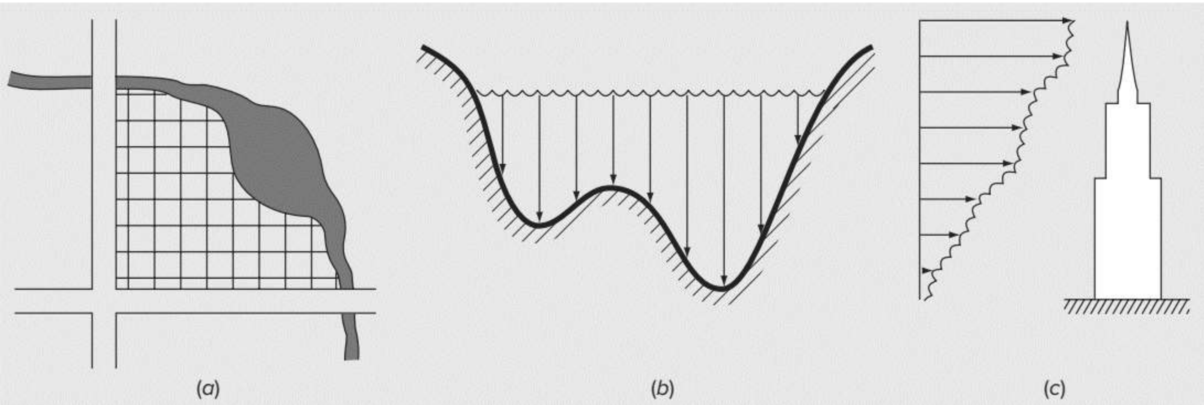
Objective: To numerically obtain the integration of any given function.

Theory: Mathematically, definite integration is represented by

$$I = \int_a^b f(x)\, dx$$

which stands for the integral of the function $f(x)$ with respect to the independent variable x, evaluated between the limits $x = a$ to $x = b$.

The meaning of integration is the total value, or summation, of a f (x) dx over the range x = a to b. In fact, the symbol ∫ is actually a stylized capital S that is intended to signify the close connection between integration and summation.



Examples of how integration is used to evaluate areas in engineering and scientific applications. (*a*) A surveyor might need to know the area of a field bounded by a meandering stream and two roads. (*b*) A hydrologist might need to know the cross-sectional area of a river. (*c*) A structural engineer might need to determine the net force due to a nonuniform wind blowing against the side of a skyscraper.

Method to numerically evaluate integration-

TRAPEZOIDAL RULE –

The *Newton-Cotes formulas* are the most common numerical integration schemes. They are based on the strategy of replacing a complicated function or tabulated data with a polynomial that is easy to integrate:

$$I = \int_a^b f(x)\, dx \cong \int_a^b f_n(x)\, dx$$

where $f_n(x)$ = a polynomial of the form

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

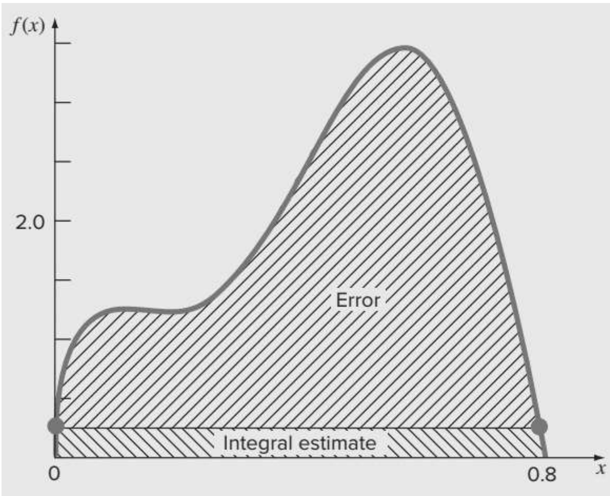
The trapezoidal rule is the first of the Newton-Cotes closed integration formulas.

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a} (x - a) \right] dx$$

The result of the integration is

$$I = (b - a)\frac{f(a) + f(b)}{2} \qquad\qquad I = \text{width} \times \text{average height}$$

which is called the *trapezoidal rule*.

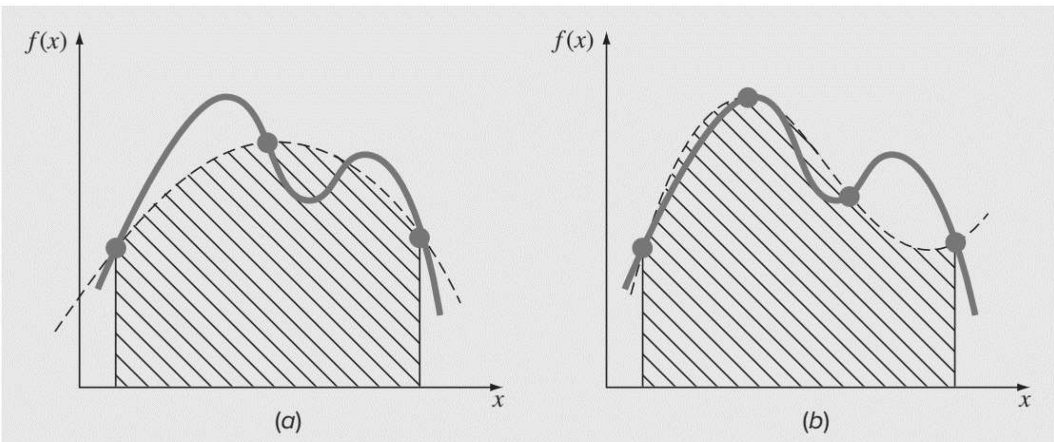


Error estimate

Simpson's 1/3 Rule –

Simpson's 1⁄3 rule corresponds to the case where the polynomial

(*a*) Graphical depiction of Simpson's 1/3 rule: It consists of taking the area under a parabola connecting three points. (*b*) Graphical depiction of Simpson's 3/8 rule: It consists of taking the area under a cubic equation connecting four points.



$$I = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \right.$$

$$\left. + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

$$I = \frac{h}{3} [f(x_0) + 4f(x_1) + f(x_2)]$$

where, for this case, $h = (b - a)/2$. This equation is known as *Simpson's 1/3 rule.*

$$I = (b - a) \frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

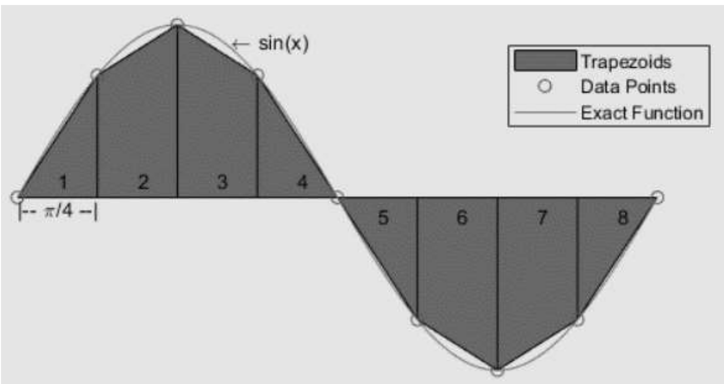
where $a = x_0$, $b = x_2$, and x_1 = the point midway between a and b, which is given by $(a + b)/2$.

Problem Statement. Use Eq. (19.23) to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$ using analytical, Trapezoidal and 1/3 simpson rule.

Segments (n)	Points	Name	Formula	Truncation Error
1	2	Trapezoidal rule	$(b-a)\frac{f(x_0)+f(x_1)}{2}$	$-(1/12)h^3 f''(\xi)$
2	3	Simpson's 1/3 rule	$(b-a)\frac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-(1/90)h^5 f^{(4)}(\xi)$
3	4	Simpson's 3/8 rule	$(b-a)\frac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-(3/80)h^5 f^{(4)}(\xi)$



`trapz` performs numerical integration via the trapezoidal method. This method approximates the integration over an interval by breaking the area down into trapezoids with more easily computable areas. For example, here is a trapezoidal integration of the sine function using eight evenly-spaced trapezoids:

For an integration with `N+1` evenly spaced points, the approximation is

$$\int_a^b f(x)dx \approx \frac{b-a}{2N}\sum_{n=1}^{N}(f(x_n)+f(x_{n+1}))$$

$$=\frac{b-a}{2N}[f(x_1)+2f(x_2)+...+2f(x_N)+f(x_{N+1})] ,$$

where the spacing between each point is equal to the scalar value $\frac{b-a}{N}$. By default MATLAB® uses a spacing of 1.

If the spacing between the `N+1` points is not constant, then the formula generalizes to

$$\int_a^b f(x)dx \approx \frac{1}{2}\sum_{n=1}^{N}(x_{n+1}-x_n)[f(x_n)+f(x_{n+1})] ,$$

where $a = x_1 < x_2 < \ ... \ < x_N < x_{N+1} = b$, and $(x_{n+1} - x_n)$ is the spacing between each consecutive pair of points.

MATLAB code for evaluating area under the sin curve.

```
clc;clear all; close all;
X = 0:pi/100:pi;
Y = sin(X);
Q = trapz(X,Y);
```

Assignment -

One numerical as suggested by Faculty.

Sample numerical.

19.4 Evaluate the following integral:

$$\int_{-2}^{4}(1 - x - 4x^3 + 2x^5)\,dx$$

(a) analytically, **(b)** single application of the trapezoidal rule, **(c)** composite trapezoidal rule with $n = 2$ and 4, **(d)** single application of Simpson's 1/3 rule, **(e)** Simpson's 3/8 rule, and

```
% Define the function
function y = f(x)
  y = (cos(x) - x) / (1 + x);
end

% Set the limits of integration
a = 0;
b = 1;

% Set the number of steps
n = 6;

% Calculate the width of each step
h = (b - a) / n;

% Initialize the sum for the trapezoidal rule
sum = 0;

% Calculate the first and last term (f(a) and f(b))
sum = sum + f(a) + f(b);

% Calculate the sum of the function values at interior points
for i = 1:n-1
  x = a + i*h;
  sum = sum + 2*f(x);
end

% Apply the trapezoidal rule formula
integral = h/2 * sum;

% Print the result
fprintf('The integral using the trapezoidal rule is: %f\n',integral);

% Define the function
function y = g(x)
    y = (cos(x) - x) / (1 + x);
end

% Set the limits of integration
a = 0;
b = 1;

% Set the number of steps
n = 2;

% Calculate the width of each step
h = (b - a) / n;

% Initialize the sum for the trapezoidal rule
```

```
sum = 0;

% Calculate the first and last term (f(a) and f(b))
sum = sum + f(a) + f(b);

% Calculate the sum of the function values at interior points
for i = 1:n-1
    x = a + i * h;
    sum = sum + 2 * f(x);
end

% Apply the trapezoidal rule formula
integral = h / 2 * sum;

% Print the result
fprintf('The integral using the trapezoidal rule with n=2 is: %f\n', integral);

% Define the function
function y = w(x)
    y = (cos(x) - x) / (1 + x);
end

% Set the limits of integration
a = 0;
b = 1;

% Set the number of intervals (Simpson's 1/3 rule needs an even number, here n=2 for one segment)
n = 2;

% Calculate the width of each interval
h = (b - a) / n;

% Initialize the variables for the sum using Simpson's 1/3 rule
fa = f(a);
fb = f(b);
fm = f((a + b) / 2); % Midpoint evaluation

% Apply Simpson's 1/3 rule formula
integral = (h / 3) * (fa + 4*fm + fb);

% Print the result
fprintf('The integral using Simpson''s 1/3 rule is: %f\n', integral);

% Define the function
function y = s(x)
    y = (cos(x) - x) / (1 + x);
end
```

```matlab
% Set the limits of integration
a = 0;
b = 1;

% Set the number of intervals
n = 3;   % Simpson's 3/8 rule uses 3 intervals

% Calculate the width of each interval
h = (b - a) / n;

% Calculate the values at the four points needed for Simpson's 3/8 rule
f0 = f(a);
f1 = f(a + h);
f2 = f(a + 2*h);
f3 = f(b);

% Apply Simpson's 3/8 rule formula
integral = (3 * h / 8) * (f0 + 3*f1 + 3*f2 + f3);

% Print the result
fprintf('The integral using Simpson''s 3/8 rule is: %f\n', integral);
```

Command Window

```
The integral using the trapezoidal rule is: 0.296947
The integral using the trapezoidal rule with n=2 is: 0.318399
The integral using Simpson's 1/3 rule is: 0.296173
The integral using Simpson's 3/8 rule is: 0.295113
>>
```