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# HOMEWORK 2 - PART A

1. Probability of selecting a letter from a word in the sentence $= \dfrac{\text{no. of letters in the word}}{\text{total no. of letters in the sentence}}$

By law of the lazy statician, $E(Y)$ can be calculated as the sum of weighted probabilities.

$$\therefore\ E(Y) = \left[\frac{3}{42}\times 3 + \frac{8}{42}\times 8 + \frac{8}{42}\times 8 + \frac{7}{42}\times 7 + \frac{3}{42}\times 3 + \frac{6}{42}\times 6 + \frac{2}{42}\times 2 + \frac{1}{42}\times 1 + \frac{4}{42}\times 4\right]$$

$$E(Y) = \frac{1}{42}\left(3^2 + 8^2 + 8^2 + 7^2 + 3^2 + 6^2 + 2^2 + 1^2 + 4^2\right)$$

$$E(Y) = \frac{252}{42} = 6$$

$$\therefore\ E(Y) = 6$$

2. $f(x,y) = 12y^2$ for $0 \le y \le x \le 1$, $E(XY) = ?$

Firstly, checking for the pdf ~~of~~ $f(x,y)$.

$$F(x,y) = \int_0^1\int_0^x 12y^2\,dy\,dx = \int_0^1\left[\frac{12}{3}y^3\right]_0^x dx$$

$$F(x,y) = \int_0^1 4x^3\,dx = \left.\frac{4x^4}{4}\right]_0^1 = 1$$

$$E(XY) = \int_0^1\int_0^x xy\,f(x)f(y)\,dy\,dx \qquad \because f(x,y) = f(x)f(y)$$

$$= \int_0^1\int_0^x xy\,12y^2\,dy\,dx = \int_0^1\int_0^x 12y^3x\,dy\,dx$$

$$= \int_0^1 x\left[\frac{12}{4}y^4\right]_0^x dx = \int_0^1 3x^5\,dx = \left[\frac{3}{6}x^6\right]_0^1 = \frac{1}{2}$$

3. Find $E[(X_1 - 2X_2 + X_3)^2]$

since $X_1$, $X_2$ and $X_3$ are random variables from a random sample from the interval $[0, 1]$, their expected values are :

$$E(X_1) = E(X_2) = E(X_3) = 0.5$$

$$E[(X_1 - 2X_2 + X_3)^2] = [E(X_1) - 2E(X_2) + E(X_3)]^2$$

$$= [0.5 - 1 + 0.5]^2 = 0$$

$$\therefore E[(X_1 - 2X_2 + X_3)^2] = 0$$

4. $f(x) = e^{-x}$ , $x > 0$

$Y = e^{\frac{3X}{4}}$ . $E(Y) = 0$

$$E(Y) = \int_0^\infty e^{3x/4} \cdot e^{-x}\, dx = \int_0^\infty e^{-x/4}\, dx$$

Let $-\frac{x}{4} = u \Rightarrow x = -4u$

$dx = (-4)\, du$

$$\int_{0/-4}^{\infty/-4} e^u (-4) du$$

$$E(Y) = \int_{-\infty}^{0} e^u (-4)\, du = 4[e^u]_{-\infty}^{0} = 4[e^0 - e^{-\infty}]$$

$$E(Y) = 4$$

5. $Y = g(X) = 2X^2 + 1$

$E(Y) = ?$

probability of the outcome of rolling the fair die :

$P(x) = \frac{1}{n}$ for an n-sided die .

$$\therefore E(Y) = \sum_{x=1}^{n} (2x^2 + 1) \frac{1}{n}$$

Assuming the die has 6 sides, $n=6$

$$\Rightarrow E(Y) = \sum_{x=1}^{6}(2x^2+1)\frac{1}{6} = \frac{1}{6}*\left(\sum_{x=1}^{6}2x^2\right)+1$$

$$E(Y) = \frac{2}{6}\left(\sum_{x=1}^{6}x^2\right)+1 = \frac{\cancel{2}\cdot 3n(n+1)(2n+1)}{6}+1$$

$$E(Y) = \frac{\cancel{6}\cdot(7\times 13)}{2}+1 = \frac{93}{2} = 46.5$$

6. $f(x) = 2(1-x)$, $0<x<1$, $Y = (2X+1)$, $E(Y^2)=?$

checking for pdf: $F(x) = \int_0^1 2(1-x)\,dx = \left[2x-\frac{2x^2}{2}\right]_0^1 = 2-1=1$

$$E(Y^2) = \int_0^1 2(2x+1)^2(1-x)\,dx$$

$$= 2\int_0^1(4x^2+4x+1)(1-x)\,dx = 2\int_0^1(\cancel{4x^2}+4x+1-4x^3-\cancel{4x^2}-x)\,dx$$

$$E(Y^2) = 2\int_0^1(-4x^3+3x+1)\,dx = 2\left[-x^4+\frac{3}{2}x^2+x\right]_0^1$$

$$E(Y^2) = 2\left[-1+\frac{3}{2}+1\right] = 3$$

7. $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for $n \in Z^+$

show that $E[(ax+b)^n] = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i E(X^{n-i})$

$$E[(ax+b)^n] = E\left[\sum_{K=0}^{n} \binom{n}{K} (ax)^{n-K} b^K\right]$$

$$E[(ax+b)^n] = \sum_{K=0}^{n} \binom{n}{K} a^{n-K} \cdot b^K E(X^{n-k})$$ since a, b are constants

8. proportion of defective parts $= p$

$E(X-Y) = ?$ since $X$ = # of defective parts in the random sample of n parts

~~E(X) = np~~ ~~E(Y)~~

$X = np$.

$\therefore E(X) = np$.

$\downarrow$

$\Rightarrow Y = (n-X)$

$\therefore E(X-Y) = E[X-(n-X)]$ ~~E(X)~~

~~E(X)~~ $E(X-Y) = E[2X-n]$

If $n = 20$, $p = 0.05$

$E(X-Y) = E(2 \times 20 \times 0.05 - 20) = \cancel{E}\ 2 - 20 = -18$

No. of defective parts in the random sample $= 20 \times 0.05 = 2$

~~In a ran random sample~~

To summarize, it is expected that a random sample of 20 will have 1 defective part for every 19 good parts.