MA-677 - HOMEWORK - ESTIMATION

1. The MLE, $\hat{p}$, for p is given by:

$$\hat{p} = \frac{12}{70} = \frac{6}{35}$$

2. The likelihood for θ, where $X_1, \ldots, X_n \sim \text{Bern}(\theta)$ and $0<\theta<1$, is given by:

$$L(\theta) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}$$

$$L(\theta) = (1-\theta)^n \quad \text{if } x_i = 0$$

and

$$= \theta^n \quad \text{if } x_i = 1$$

Taking the log likelihood:

$$\ell(\theta) = n \ln(1-\theta) \quad (\text{if every observed value is } 0)$$

Taking the first derivative to get the MLE of θ:

$$\frac{d}{d\theta}[\ell(\theta)] = \frac{-n}{1-\theta} = 0$$

from the above equation, we cannot solve for $\hat{\theta}$.

Similarly, if every observation is 1,

$$\frac{d}{d\theta}[\ell(\theta)] = \frac{n}{\hat{\theta}} = 0$$

$\therefore$ when every observed value is either 0 or 1, the MLE of θ does not exist.

3. $X_1, \ldots, X_n \sim \text{Poisson}(\lambda)$, $\lambda > 0$

$L(\lambda)$, Likelihood $\rightarrow$ $f(x_1, \ldots, x_n \mid \lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

log likelihood : $\ell(\lambda) = \sum_{i=1}^{n} \left(x_i \ln(\lambda) - n\lambda - \ln(x_i!) \right)$

$= \ln(\lambda) \sum_{i=1}^{n} x_i - n\lambda - \sum_{i=1}^{n} \ln(x_i!)$

taking the first derivative :

$$\frac{d}{d\lambda}[\ell(\lambda)] = \frac{1}{\lambda} \sum_{i=1}^{n} x_i - n = 0$$

$$\therefore \hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$$

If every observed value is 0, then we cannot find MLE $\hat{\lambda}$ of λ from $\frac{1}{\lambda} \sum_{i=1}^{n} x_i - n = 0$ as $\sum_{i=1}^{n} x_i - n < 0$ for all $x_i = 0$; and we cannot find only one value of λ to satisfy the inequality.

4. $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, σ^2 is unknown and μ is known.

log likelihood : $\ell(\mu, \sigma^2, x_1, \ldots, x_n) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{j=1}^{n} (x_j - \mu)^2$

taking the 1^{st} derivative :

$$\frac{d}{d\sigma^2}[\ell(\mu, \sigma^2, x_1, \ldots, x_n)] = \frac{-n(2\sigma)}{2\sigma^2} - \frac{(-2)}{2\sigma^3} \sum_{j=1}^{n} (x_j - \mu)^2$$

$$= -\frac{n}{\sigma} + \frac{1}{\sigma^3} \left(\sum_{j=1}^{n} (x_j - \mu)^2 \right) = 0$$

$$\Rightarrow \frac{\sum_{j=1}^{n} (x_j - \mu)^2}{\sigma^3} = \frac{n}{\sigma}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{j=1}^{n} (x_j - \mu)^2}{n}$$