Class_Notes_MA677

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Notes on the next page.

→ X and Y have joint prob. distr.

$f(-1,0) = 0, \quad f(-1,1) = 1/4$

$f(0,0) = 1/6, \quad f(0,1) = 0$

$f(1,0) = 1/12, \quad f(1,1) = 1/2$

Are X and Y independent? What is cov (X, Y)?

		X: -1	0	1	
Y	0	0	1/6	1/12	3/12
	1	1/4	0	1/2	3/4 = 9/12
		1/4	1/6	7/12	1

$E(Y) = \frac{9}{12} = \frac{3}{4}$

$E(X) = \frac{1}{3} \quad = -1 \times \frac{1}{4} + 0 + \frac{7}{12}$

~~E(XY)~~ =

X	Y	XY	P(X,Y)	E(X,Y)
-1	0	0		
-1	1	-1 ✓ →	1/4	-1/4
0	0	0		
0	1	0		
1	0	0		
1	1	1 ✓ →	1/2	1/2

∴ $E(XY) = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

✓ multiply and should get the joints

$Cov(X,Y) = \frac{1}{4} - \frac{1}{3} \times \frac{3}{4} = 0$

$E(XY) - E(X)E(Y)$

↓

if not, then they are not independent.

Inside - joint prob.

Borders - marginal prob. of X and Y.

→ pdf of X is $f(x) = 6x(1-x)$ for $0 < x < 1$

Find the pdf of $Y = X^3$.

$F(x) = \int_0^1 6x(1-x)\,dx = \int_0^1 (6x - 6x^2)\,dx$

$\times \{ \cancel{F(x) = 6[1] - \frac{6}{2}[x^3]_0^1 = 6 - 3 \times 1} \} \times$

$F_Y(y) = P(Y \le y) = P(X^3 \le y) = P(X \le y^{1/3})$

$F_Y(y) = \int_0^{y^{1/3}} f(x)\,dx = \int_0^{y^{1/3}} (6x - 6x^2)\,dx$

$= \frac{6}{2}[y^{2/3}] - 2[y]$

$\frac{d}{dx}[3y^{2/3} - 2y]$ ✓

$\to 2y^{-1/3} - 2$

$2(y^{-1/3} - 1)$

$\frac{d}{dy} F_Y(y) = \frac{d}{dy}[6y^{1/3}(1 - y^{1/3})] = \frac{d}{dy}[6y^{1/3} - 6y^{2/3}]$

$= 2y^{-2/3} - 4y^{-1/3} = 2y^{-1/3}(y^{-1/3} - 2)$ ✗

→ The joint pdf of X and Y is

$f(x,y) = \frac{1}{3}(x+y)$ for $0 < x < 1$, $0 < y < 2$

Find variance of $W = 3X + 4Y - 5$.

$\sigma_x^2 = E(x^2) - \mu_x^2 = E[(x-\mu)^2]$

$Var(\sum a_i x_i) = \sum a_i^2 Var(x_i) + 2 \sum\sum_{i<j} a_i a_j Cov(x_i, x_j)$

$Var(W) = 9 Var(X) + 16 Var(Y) + \underset{\cdot 3 \cdot 4}{2} Cov(X,Y)$

$9[E(X)^2 - [E(X)]^2] + 16[E(Y)^2 - [E(Y)]^2]$

$+ 24[E(XY) - E(X)E(Y)]$

$\therefore$ need to find $E(X)$, $E(X)^2$, $E(Y)$, $E(Y)^2$ and $E(XY)$ to solve the problem

→ $\psi_x(t) = E_x(e^{tx}) = \int_x e^{tx} f(x)\,dx$

$\psi_x'(t) \rightarrow$ $\psi_x'(0) = \mu$ → 1^{st} moment

$\psi_x''(0) = \mu_x^2$ → 2^{nd} moment.

→ The pdf of X is $f(x) = \binom{3}{x}\frac{1}{8}$, for $x = 0, 1, 2, 3$. Calc. μ_x and σ_x directly. Then calculate the mgf of X and use the mgf to verify the mean and the variance.

$\sum_{x=0}^{3} f(x) = 1\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}\right) + 1\left(\frac{1}{8}\right) = 1$

$\Downarrow$

$f(x)$ is the pdf.

$E(X) = \frac{3}{2}$ $\sigma_x^2 = 3 - \left(\frac{3}{2}\right)^2 =$

$E(X^2) = 3$

$M_x(t) = E(e^{tx}) = \frac{1}{8}[e + 3e^t + 3e^{2t} + e^{3t}] = \frac{1}{8}[1+e^t]^3$

$M'_x(0) = \frac{3}{8}\left[1+e^t\right]^2\left[e^t\right]\Big|_{t=0} \Rightarrow \frac{3}{8}\times 4\times 1 = \frac{3}{2}$

$M''_x(0) = \frac{3}{8}\left[2(1+e^t).e^t.e^t + (1+e^t)^2 e^t\right] = \frac{3}{8}\left[2(1+e^t)e^{2t} + (1+e^t)^2 . e^t\right]_{t=0}$

$M''_x(0) = \frac{3}{8}\left[2.2+4\right] = 3$

→ The decay of a radioactive element is exponentially distributed. The time for the nucleus to emit the first α particle is x seconds with pdf $f(x) = \lambda e^{-\lambda x}$ for $\lambda > 0$ and $x > 0$. The process has no memory. Assume that $\lambda = 5$. Find the prob. that the time for a sample of the material to emit 2 particles is no more than 3 seconds.

Use the fact that for X distributed as a Gamma distr $f(s) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, the mgf is $\left(1-\frac{1}{\beta}\right)^{-\alpha}$ for $t < \beta$.

→ mgf of Gamma is $\left(1-\frac{t}{\beta}\right)^{-\alpha}$. So, exponential is Gamma with $\alpha = 1$ and $\beta = \lambda$.

mgf of exponential is $\left(1-\frac{t}{\lambda}\right)^{-1}$