Multivariate Control Chart

Seminar Paper

Course: Statistical Quality Control

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1 Introduction

1.1 What is Multivariate Quality Control

Multivariate quality control refers to the statistical techniques used to monitor and control processes involving multiple interrelated quality characteristics. These techniques are essential in situations where variables are correlated and cannot be adequately assessed using individual univariate tools.

The foundation of multivariate statistical process control (MSPC) was laid by Hotelling in 1947, who applied multivariate procedures to analyze bombsight data during World War II. Since then, multivariate methods have evolved significantly and are now widely used in advanced manufacturing, medical diagnostics, and service industries.

1.2 Univariate Control Charts and Their Limitations

Traditionally, Statistical Process Control (SPC) has relied on univariate control charts—such as the Shewhart, CUSUM, and EWMA charts—to monitor a single quality characteristic over time. These methods are effective in detecting shifts in the mean or variance of one variable, assuming the variables are independent.

However, in most real-world processes, multiple variables interact and exhibit correlations. Monitoring them individually not only ignores these dependencies but can also lead to misleading conclusions. For example, a univariate chart might not signal any alarm, even though a significant shift has occurred in the joint distribution of the variables.

Univariate methods also increase the probability of false alarms when multiple charts are used simultaneously—known as the multiple testing problem. Additionally, these methods are not equipped to detect changes in the correlation structure or variance-covariance relationship between variables, which can be critical in high-precision environments.

1.3 Multivariate Control Charts: Mean and Covariance Monitoring

Multivariate control charts are designed to overcome the limitations of univariate methods by analyzing the process as a whole. The most widely used chart is **Hotelling's T^2 chart**, which is the multivariate extension of the Shewhart chart. It helps detect changes in the average values (mean vector) of multiple related quality variables.

Besides monitoring the process mean, it is also important to monitor variability and the relationship between variables. This is usually done through control charts for the variance-covariance structure (the Σ matrix). These charts help detect shifts not only in average values but also in the spread or shape of the data cloud, which can indicate process instability.

There are also advanced multivariate charts such as the **Multivariate EWMA (MEWMA)** and **Multivariate CUSUM (MCUSUM)**. These are more sensitive to smaller and gradual shifts compared to traditional Shewhart-style charts. They are especially useful when data are autocorrelated or when early detection of subtle changes is needed.

In high-dimensional settings or when historical data are limited, modern tools such as shrinkage estimators and regularized covariance estimation are used. These improve the reliability of the control limits and help maintain the performance of control charts in complex environments.

Overall, multivariate control charts provide a powerful framework for monitoring correlated process variables together. By doing so, they help reduce false alarms, improve detection of joint shifts, and offer a clearer picture of the process over time.

1.4 Relevance in Modern Industry and Applications

Multivariate control charts like Hotelling's $\mathbf{T^2}$ and MEWMA are used in many industries to track several related variables at once. They are helpful when univariate charts might miss important changes. For example, in radiotherapy, these charts help ensure that treatment plans are accurate and safe by checking multiple risk factors together. Even with high-dimensional data, techniques like PCA make it possible to apply multivariate monitoring in complex settings. Overall, these charts are useful for detecting both mean shifts and variability across several variables, giving a fuller picture of the process.

1.5 Main purpose of this paper

This paper explores the context and motivation behind Statistical Process Control (SPC), focusing on the transition from univariate to multivariate monitoring. It emphasizes the importance of jointly monitoring multiple interrelated quality characteristics, introduces key theoretical tools such as Hotelling's T^2 and MEWMA charts, and demonstrates practical implementation using statistical software. While Hotelling's $\mathbf{T^2}$ chart can also be applied to individual observations (n = 1), this paper focuses on the subgrouped version, which is more common in many manufacturing and quality control settings.

2 Theoretical Background

2.1 The Multivariate Normal Distribution

In **univariate** statistical quality control, a single variable is modeled using the **normal distribution**, with the probability density function (pdf):

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{1}$$

- The exponent term $\left(\frac{x-\mu}{\sigma}\right)^2$ represents the **squared standardized distance** from the mean, showing how far x is from μ, scaled by standard deviation σ.

Note: leaving the negative sign in the exponent, the expression in the exponential function of the multivariate normal pdf can be written as:

$$(\mathbf{x} - \boldsymbol{\mu})^\top \Sigma^{-1} (\mathbf{x} - \boldsymbol{\mu}) \tag{2}$$

Generalizing to the Multivariate Case:

- Let $\mathbf{x}' = [x_1, x_2, \ldots, x_p]$ be a **p-dimensional vector** of variables.
- Let $\boldsymbol{\mu}' = [\mu_1, \mu_2, \ldots, \mu_p]$ be the **mean vector**.
- Let Σ be the **covariance matrix**, which is $p \times p$. Diagonal entries = variances; off-diagonals = covariances.

The Multivariate Normal Density Function

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \cdot e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})} \tag{3}$$

- The term $(\mathbf{x}-\boldsymbol{\mu})^\top \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is the **squared Mahalanobis distance**, which measures how far the vector $\mathbf{x}$ is from the mean vector $\boldsymbol{\mu}$, while accounting for correlations between the variables.
- This ensures the area under the density surface equals 1 for any number of variables p.
- The function describes a probability **surface**, not a curve (as in the univariate case).
- For $p = 2$, the result is a **bivariate normal distribution**, visualized as an **elliptical surface**, with shape influenced by the correlation between variables.

This function gives the **density** or **relative likelihood** of observing a particular combination of values $\mathbf{x}$.

- High $f(\mathbf{x}) \Rightarrow$ values are **likely**.
- Low $f(\mathbf{x}) \Rightarrow$ values are **unlikely**.

The shape of this function, when plotted, forms a **probability surface**. In the bivariate case ($p = 2$), this appears as a **bell-shaped hill**, with:

- The peak located at the mean vector $\boldsymbol{\mu}$.
- The spread and orientation determined by the covariance matrix Σ.

Visualization:
If you graph this function for two variables:

- You get **ellipses of equal density** centered around the mean.
- If the variables are **correlated**, the ellipses are tilted.
- If the variables are **uncorrelated**, the ellipses align with the coordinate axes.

2.2 The Sample Mean Vector and Covariance Matrix

Suppose we collect a random sample of n observations, where each observation is a p-dimensional vector:

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$$

Each vector is of the form:

$$\mathbf{x}_i = [x_{i1}, x_{i2}, \ldots, x_{ip}]$$

Sample Mean Vector:

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i \tag{4}$$

This vector represents the average of all n observations, computed component-wise.

Sample Covariance Matrix:

$$\mathbf{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^\top \tag{5}$$

Where:

- Diagonal entries of $\mathbf{S}$ are the **sample variances**: s_j^2
- Off-diagonal entries are the **sample covariances**: s_{jk}

Each variance:

$$s_j^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2 \tag{6}$$

Each covariance:

$$s_{jk} = \frac{1}{n-1} \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ik} - \bar{x}_k) \tag{7}$$

Summary:

- The multivariate normal distribution is a natural extension of the univariate normal distribution to multiple correlated variables.
- It is fully characterized by a **mean vector $\boldsymbol{\mu}$** and a **covariance matrix** Σ.
- The **generalized squared distance** accounts for correlations and is used in quality control methods like **Hotelling's T^2 statistic**.
- In practice, the parameters of a multivariate distribution are estimated using the **sample mean vector** and the **sample covariance matrix**.

2.3 Hotelling's T^2 Control Chart

2.3.1 Multivariate Visualization Using the Control Ellipse (Hotelling's T^2)

When monitoring two quality characteristics simultaneously (e.g., x_1 for diameter and x_2 for weight), the control ellipse provides a useful geometric representation of the joint control region. Each point on the graph represents the mean of a subgroup, plotted in two dimensions. The central question becomes: *Is the process in control at a particular observation point?*

This is where the ellipse helps. It acts as a boundary defined by the chi-square distribution with $p = 2$ degrees of freedom. If a point falls inside the ellipse, the process is considered in control; if it falls outside, the process is out of control.

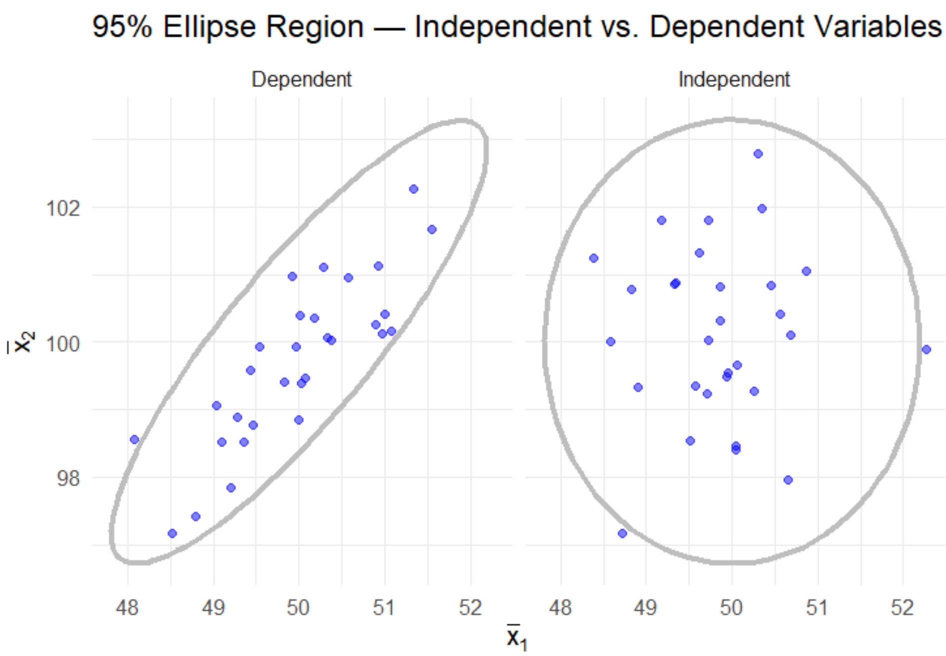


Figure 1: 95% Control Ellipses for Independent vs. Dependent Variables

What is the Control Ellipse?

The control ellipse is a contour of constant probability under the bivariate normal distribution. The center of the ellipse corresponds to the in-control mean vector (μ_1, μ_2). Its shape and orientation are determined by the variances of each variable and the covariance between them. Specifically:

- When $s_{12} = 0$ (independent variables), the ellipse aligns with the coordinate axes.
- When $s_{12} \neq 0$ (correlated variables), the ellipse tilts in the direction of the correlation.

How Is This Related to Hotelling's T^2?

The ellipse is a visual boundary defined by the Hotelling's T^2 statistic:

$$T^2 = n(\bar{\mathbf{x}} - \boldsymbol{\mu})^\top \Sigma^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) \tag{8}$$

If the computed T^2 for a point satisfies:

$$T^2 \leq \chi^2_{\alpha,2} \quad \text{(inside ellipse} \rightarrow \text{in control)}$$

$$T^2 > \chi^2_{\alpha,2} \quad \text{(outside ellipse} \rightarrow \text{out of control)}$$

then the point is considered either in or out of control.

Interpretation Using an Example

In Figure 1 , the left panel shows the ellipse for two correlated variables (dependent case). The ellipse is elongated and rotated, reflecting positive correlation. In contrast, the right panel displays the ellipse for independent variables — here, the ellipse is circular and aligned with the axes.

This visualization emphasizes a key idea: even if individual charts for x_1 and x_2 show in-control signals, a joint observation may fall outside the ellipse, signaling a multivariate anomaly.

Limitations of Ellipse Visualization

Although intuitive and powerful in two dimensions, this approach has limitations:

Limitation	**Why It Matters**
Only works for 2 variables	Cannot visualize higher-dimensional ellipses (e.g., $p > 2$) easily.
No time tracking	Time-ordering is lost; cannot tell when the issue occurred unless manually labeled.
No scalar summary	Difficult to use for continuous process monitoring over time.

Table 1: Challenges of using control ellipses for process monitoring

Conclusion and Motivation for T^2 Control Chart

The control ellipse offers valuable insight when dealing with two variables. However, for practical multivariate process monitoring—especially when $p > 2$ or when tracking shifts over time—a scalar statistic like Hotelling's T^2 is more suitable. The T^2 control chart transforms the multidimensional problem into a univariate monitoring framework, retaining the joint structure of the data while enabling real-time tracking and interpretation.

2.4 Chi-Square Control Chart (Hotelling's T^2 Chart)

The Hotelling's T^2 control chart, often referred to as the chi-square control chart, is a widely used method for monitoring multivariate processes. Instead of visualizing variables simultaneously in an elliptical confidence region, the T^2 chart simplifies the monitoring task by reducing the multivariate data into a single statistic, plotted over time. Each plotted point represents the multivariate distance of a subgroup from the process center, taking into account the correlation among quality variables.

The main strengths of this chart lie in its ability to monitor multiple, potentially correlated variables in a unified and scalable way. It effectively captures time-based shifts in the multivariate mean vector, offering a compact view of process behavior. Additionally, the use of a single statistic (T^2) simplifies interpretation and decision-making.

For a sample of size n measuring p quality characteristics, the Hotelling's T^2 statistic is defined as:

$$T^2 = n(\bar{\mathbf{x}} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\bar{\mathbf{x}} - \boldsymbol{\mu}) \tag{9}$$

where $\bar{\mathbf{x}}$ is the vector of sample means ($p \times 1$), $\boldsymbol{\mu}$ is the true mean vector of the process, and $\boldsymbol{\Sigma}$ is the covariance matrix ($p \times p$) that reflects the process variation and correlation structure.

Since $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are rarely known in real applications, they are estimated from a set of preliminary samples, known as Phase I data. Phase I refers to the initial use of control charts to assess whether the process was in control when the m preliminary subgroups were drawn. The objective of Phase I is to compute reliable estimates of the process mean vector $\bar{\bar{\mathbf{x}}}$ and covariance matrix $\mathbf{S}$, ensuring they are derived from an in-control dataset. These estimates are then used to establish control limits for Phase II.

Phase II refers to the ongoing use of the chart for monitoring future production. Here, the control limits established during Phase I are applied to new data in order to detect any shifts in the process mean vector.

Using Phase I estimates, the Hotelling's T^2 statistic becomes:

$$T^2 = n(\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}})' \mathbf{S}^{-1} (\bar{\mathbf{x}} - \bar{\bar{\mathbf{x}}}) \tag{10}$$

The control limits for the T^2 chart differ between Phase I and Phase II. In Phase I, the upper control limit is given by:

$$\mathrm{UCL}_{\text{Phase I}} = \frac{p(m-1)(n-1)}{mn - m - p + 1} F_{\alpha, p, mn-m-p+1}, \quad \mathrm{LCL} = 0 \tag{11}$$

For Phase II monitoring, the control limit is modified as follows:

$$\mathrm{UCL}_{\text{Phase II}} = \frac{p(m+1)(n-1)}{mn - m - p + 1} F_{\alpha, p, mn-m-p+1}, \quad \mathrm{LCL} = 0$$

When the number of Phase I samples m is large, and the sample estimates are considered reliable, the T^2 statistic approximately follows a chi-square distribution with p degrees of freedom. In such cases, a simplified control limit can be used:

$$\mathrm{UCL} = \chi^2_{\alpha, p}$$

This simplification allows for ease of implementation in Phase II without requiring the F-distribution, provided the assumptions hold.

Montgomery and other authorities emphasize that Phase I analysis must be conducted with care to ensure that the initial dataset is truly representative of an in-control process. A recommended number of Phase I subgroups is often $m \geq 200$ to improve the reliability of estimated parameters. Once control limits are established, Phase II monitoring can provide effective real-time surveillance of multivariate quality characteristics.

Interpretation of Out-of-Control Signals

While the Hotelling's T^2 chart is effective in detecting multivariate shifts, its primary limitation lies in its diagnostic power. Specifically, when the chart signals an out-of-control condition, it indicates that the overall process mean vector has changed—but it does not reveal which individual variable(s) are responsible for the signal. This lack of specificity can make root cause analysis challenging, particularly in high-dimensional settings.

The core difficulty arises from the structure of the T^2 statistic: although it aggregates information across all p variables, it produces a single summary value. Consequently, identifying which specific variable (or combination thereof) caused the signal requires additional analysis.

To address this issue, several diagnostic techniques have been proposed in the literature:

1. Bonferroni-Adjusted Univariate Control Charts Alt (1985) proposed a method based on applying individual $\bar{X}$ charts to each variable, with modified control limits to account for multiple comparisons. Specifically, a Bonferroni correction is applied to maintain the overall Type I error rate. Instead of using the conventional control limit based on $Z_{\alpha/2}$, the adjusted control limit becomes:

$$Z_{\alpha/2p}$$

where p is the number of variables. This adjustment reduces the likelihood of false alarms that may occur when evaluating multiple variables independently.

2. Simultaneous Confidence Intervals Hayter and Tsui (1994) suggested constructing simultaneous confidence intervals for all variables in the vector $\bar{\mathbf{x}}$. This method allows for the identification of variables that significantly deviate from the estimated mean vector and does not rely on the assumption of multivariate normality. By inspecting these joint intervals, practitioners can assess which individual variables contributed to the out-of-control condition.

3. Principal Component Control Charts An alternative approach, introduced by Jackson (1980), involves transforming the original correlated variables into a set of uncorrelated linear combinations known as principal components. Control charts are then constructed for the principal component scores. Since these components are orthogonal, this method can help isolate the source of variation more effectively than working with correlated variables.

However, a notable drawback of principal component analysis is interpretability. Because each principal component is a weighted combination of original variables, it can be difficult to map a signal in component space back to specific, meaningful quality characteristics in the original measurement space.

These diagnostic strategies serve as valuable complements to the T^2 chart, enabling practitioners not only to detect shifts in the process but also to investigate their underlying causes. Incorporating these tools into the multivariate monitoring framework enhances both the sensitivity and actionability of quality control systems.

Decomposition of T^2: Identifying Contributing Variables

To address the challenge of interpreting out-of-control signals from a Hotelling's T^2 chart, Runger, Alt, and Montgomery (1996) proposed a diagnostic method that decomposes the overall T^2 statistic into contributions from individual variables. This approach provides valuable insight into which variables are most responsible for the deviation from the multivariate mean.

The method is based on the idea of leave-one-out analysis. Specifically, the contribution of the i-th variable is defined as:

$$d_i = T^2 - T^2_{(-i)} \tag{12}$$

Where:

- T^2 is the full Hotelling's statistic computed using all p variables.
- $T^2_{(-i)}$ is the Hotelling's statistic recalculated after excluding the i-th variable.
- d_i measures how much variable i contributes to the overall out-of-control signal.

Interpretation: A larger value of d_i indicates that the corresponding variable i has a greater influence on the T^2 signal and is more likely to be the root cause of the observed shift. This decomposition technique is particularly useful when the T^2 chart signals a fault, but the specific dimension responsible is unclear.

Illustrative Example: Suppose we are monitoring three standardized quality characteristics: x_1, x_2, and x_3, with the following covariance matrix:

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & 0.9 & 0.9 \\ 0.9 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{bmatrix}$$

An observed data point is:

$$\mathbf{y}' = (2, 0, 0)$$

The full Hotelling's statistic is computed as:

$$T^2 = 27.14$$

The contributions from each variable (obtained by leaving that variable out in turn) are:

$$d_1 = T^2 - T^2_{(-1)} = 27.14 - 0 = 27.14$$
$$d_2 = T^2 - T^2_{(-2)} = 27.14 - 21.05 = 6.09$$
$$d_3 = T^2 - T^2_{(-3)} = 27.14 - 21.05 = 6.09$$

This clearly shows that x_1 contributed the most to the out-of-control signal.

Rule of Thumb for Significance: To determine whether a variable's contribution is statistically significant, a common threshold is the 99^{th} percentile of the chi-square distribution with 1 degree of freedom:

$$d_i > \chi^2_{0.01,1} = 6.63$$

Any variable with a d_i exceeding this threshold can be flagged as significantly deviating from the expected mean.

This decomposition technique enhances the interpretability of multivariate control charts and supports more targeted quality improvement actions by identifying the variable(s) most likely to be out of control.

2.5 Multivariate EWMA (MEWMA) Control Chart

The Multivariate Exponentially Weighted Moving Average (MEWMA) chart extends the univariate EWMA to multivariate settings. It is particularly effective in detecting small and moderate shifts in the process mean vector. MEWMA is typically used in **Phase II monitoring**, where the process has already been brought under statistical control and the goal is to detect future deviations from this baseline.

In contrast to Hotelling's T^2 chart, which only uses the current sample (a so-called "Type I" chart), MEWMA incorporates past information by using a weighted average of current and previous observations. This increases sensitivity to subtle process shifts.

Motivation: Traditional T^2 charts rely solely on current sample information, which limits their effectiveness for small shift detection. MEWMA addresses this limitation by using past data with exponentially decreasing weights, making it more responsive to gradual process changes.

Definition

The MEWMA statistic is recursively defined as:

$$\mathbf{Z}_i = \lambda \mathbf{X}_i + (1-\lambda)\mathbf{Z}_{i-1}, \quad \text{with } \mathbf{Z}_0 = 0 \tag{13}$$

Here, $\mathbf{Z}_i$ is the EWMA vector at time i, $\mathbf{X}_i$ is the current observation vector, and λ: smoothing parameter ($0 < \lambda \leq 1$), controls how much weight recent data gets.

The test statistic plotted on the chart is:

$$T_i = \mathbf{Z}_i' \mathbf{\Sigma}_{\mathbf{Z}_i}^{-1} \mathbf{Z}_i \tag{14}$$

Where the covariance matrix of $\mathbf{Z}_i$ is:

$$\mathbf{\Sigma}_{\mathbf{Z}_i} = \frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]\mathbf{\Sigma} \tag{15}$$

In steady-state (as $i \to \infty$), this converges to:

$$\mathbf{\Sigma}_Z = \frac{\lambda}{2-\lambda}\mathbf{\Sigma} \tag{16}$$

Control Limits

The MEWMA control chart uses an upper control limit (UCL), typically denoted by H. The value of H depends on the number of quality characteristics p, the smoothing parameter λ, and the desired in-control average run length (ARL), such as 200 or 500. A lower control limit is usually not necessary, as MEWMA is designed to detect increases in the statistic due to process shifts.

Sensitivity to Process Shifts

The MEWMA chart is directionally invariant, meaning it is sensitive to shifts in any direction of the multivariate space. Its detection ability is quantified using the noncentrality parameter:

$$\delta = \left[(\mu_1 - \mu_0)' \Sigma^{-1} (\mu_1 - \mu_0)\right]^{1/2} \tag{17}$$

where μ_0 is the in-control mean vector and μ_1 is the shifted mean vector. Larger values of δ correspond to greater shifts and result in faster detection by the MEWMA chart.

ARL Performance

The Average Run Length (ARL) is a key metric in evaluating control chart performance. ARL_0 denotes the average number of samples before a false alarm when the process is in control, while ARL_1 refers to the average number of samples required to detect a real shift.

Smaller values of λ, such as $\lambda = 0.05$, have been shown to increase ARL_0 (fewer false alarms) and decrease ARL_1 (faster detection of small shifts).

Example ARL Table (adapted from Prabhu and Runger)

Table 2: Zero-State Average Run Lengths (ARLs) for MEWMA Chart with $p = 2$, $\lambda = 0.2$, $H = 9.65$

Shift Size δ	Average Run Length (ARL)
0.0 (In-control)	200.00
0.5	35.17
1.0	10.20
1.5	5.49
2.0	3.78
3.0	2.42

This table illustrates MEWMA's sensitivity: a small shift of $\delta = 0.5$ is detected after only 35 samples on average, compared to 200 under in-control conditions.

2.6 Regression Adjustment in Multivariate Control

The Hotelling T^2 (and related chi-square) control chart is fundamentally rooted in hypothesis testing. It tests whether the mean vector $\boldsymbol{\mu}$ of a multivariate normal distribution equals a specified constant vector, against the alternative that it does not. Under multivariate normality, T^2 is the optimal statistic for detecting arbitrary shifts in the process mean vector.

However, while Hotelling's T^2 performs well in a strict hypothesis-testing context, its performance in practical process monitoring is more nuanced. The T^2 chart often fails to detect small or structured shifts efficiently — particularly when only a subset of variables deviate from their in-control states. In contrast, control charts like the MEWMA (Multivariate Exponentially Weighted Moving Average) are often better suited for detecting gradual and moderate shifts due to their memory properties and lower Average Run Lengths (ARL_1).

Moreover, because T^2 and MEWMA are both based on quadratic forms, their sensitivity extends beyond shifts in location — they also respond to changes in process variability. This dual sensitivity can be problematic in cases where the primary interest lies in monitoring the mean, as variance changes can obscure detection of actual mean shifts.

To address these limitations, alternative strategies have been proposed that move beyond the T^2 structure. One such method is regression adjustment, introduced by Hawkins (1991). This technique is particularly useful for individual observation scenarios, which are common in modern manufacturing and service processes. Rather than relying on multivariate statistics, the regression-adjusted method constructs a set of univariate control charts using residuals from linear regressions.

In this approach, each variable is regressed against the remaining $p-1$ variables using ordinary least squares (OLS), and the residuals from these regressions are monitored over time. Because the residuals represent the variation in a variable that is unexplained by the others, this method isolates shifts that are not simply due to correlation structure, enhancing detection power.

How It Works: Suppose a multivariate process consists of p quality characteristics, and let y_1 be one of the variables of interest. We regress y_1 on the remaining variables:

$$y_1 = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_9 x_9 + \varepsilon \tag{18}$$

The residual is computed as the difference between the observed and predicted value:

$$\text{residual} = y_1 - \hat{y}_1$$

Control charts (typically individuals and moving range charts) are then constructed based on these residuals, rather than on the raw data. This removes predictable variation due to known relationships between variables, allowing the chart to focus on unexpected shifts.

Hawkins demonstrated that the ARL performance of regression-adjusted residual charts is comparable to or better than traditional multivariate charts in many cases. However, the effectiveness of this method depends on the type of univariate chart applied to the residuals and the structure of the underlying correlations.

A particularly valuable application of regression adjustment arises in *cascade processes* — systems in which upstream (input) variables influence downstream (output) variables. In such settings, regression adjustment serves to isolate the behavior of output variables by removing variation attributable to input effects, enabling more accurate monitoring of true process shifts.

3 Control Charts for Monitoring Variability

3.1 Control Charts for Monitoring Variability

In multivariate statistical process control (MSPC), it is crucial to monitor not only the mean vector ($\boldsymbol{\mu}$ or $\bar{\mathbf{x}}$) but also the covariance matrix $\boldsymbol{\Sigma}$. While charts like Hotelling's T^2 and MEWMA focus on detecting shifts in the process mean, this section addresses methods for identifying changes in the process variability and correlation structure.

3.1.1 Motivation for Monitoring Covariance

The covariance matrix $\boldsymbol{\Sigma}$ encapsulates both the individual variances of the variables (diagonal elements) and the covariances (off-diagonal elements) that describe how the variables vary together. In practice, changes in either the variances or covariances can indicate an out-of-control process. Detecting such changes is essential because multivariate processes typically exhibit complex interdependencies between variables.

3.2 Alt's W Statistic: A Generalization of the s^2 Chart

One of the primary methods for detecting changes in process variability is Alt's W statistic, which extends the univariate s^2 control chart to the multivariate setting. The method tests whether the sample covariance matrix $\mathbf{S}_i$ from the i-th subgroup significantly deviates from the known or estimated in-control covariance matrix $\mathbf{\Sigma}$.

The test statistic is defined as:

$$W_i = -pn + pn\ln(n) - n\ln\left(\frac{|\mathbf{\Sigma}|}{|\mathbf{A}_i|}\right) + \mathrm{tr}(\mathbf{\Sigma}^{-1}\mathbf{A}_i) \tag{19}$$

Here, p is the number of process variables, n is the subgroup sample size, and $\mathbf{S}_i$ is the sample covariance matrix. The matrix $\mathbf{A}_i = (n-1)\mathbf{S}_i$ is a scaled version of $\mathbf{S}_i$ to simplify computations. The operator $\mathrm{tr}(\cdot)$ denotes the trace, or sum of the diagonal elements of a matrix.

Control Chart Interpretation

The computed W_i statistic is plotted on a control chart and compared to an upper control limit (UCL) derived from the chi-square distribution:

$$\mathrm{UCL} = \chi^2_{\alpha,\, p(p+1)/2} \tag{20}$$

If W_i exceeds the UCL, this indicates that the variability or correlation structure of the process has changed, suggesting an out-of-control condition.

Illustrative Example

Consider a bivariate process ($p = 2$) with a subgroup size of $n = 5$. The in-control covariance matrix is:

$$\mathbf{\Sigma} = \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix}$$

and the sample covariance matrix from a subgroup is:

$$\mathbf{S}_i = \begin{bmatrix} 5 & 1.5 \\ 1.5 & 2.8 \end{bmatrix}$$

Step 1: Compute $\mathbf{A}_i = 4 \cdot \mathbf{S}_i$:

$$\mathbf{A}_i = \begin{bmatrix} 20 & 6 \\ 6 & 11.2 \end{bmatrix}$$

Step 2: Compute determinants:

$$|\mathbf{A}_i| = 20 \cdot 11.2 - 6 \cdot 6 = 188, \quad |\mathbf{\Sigma}| = 4 \cdot 3 - 2 \cdot 2 = 8$$

Step 3: Invert the in-control matrix:

$$\mathbf{\Sigma}^{-1} = \frac{1}{8}\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$$

Step 4: Compute the trace of $\mathbf{\Sigma}^{-1}\mathbf{A}_i$:

$$\mathbf{\Sigma}^{-1}\mathbf{A}_i \approx \frac{1}{8}\begin{bmatrix} 48 & -4.4 \\ -16 & 32.8 \end{bmatrix} \Rightarrow \mathrm{tr} = \frac{48 + 32.8}{8} = 10.1$$

Step 5: Calculate W_i:

$$W_i = -10 + 10\ln(5) - 5\ln\left(\frac{188}{8}\right) + 10.1 \approx 0.519$$

Assuming a significance level of $\alpha = 0.01$, the control limit is $\chi^2_{0.01,3} \approx 11.34$. Since $W_i <$ UCL, the process is assessed as in control.

Relevance in Practice

Alt's W statistic provides a robust framework for monitoring structural changes in the covariance matrix, capturing shifts in variances and correlations that are not evident from traditional mean-focused charts. As such, it is an essential complement to Hotelling's T^2 and MEWMA charts within a comprehensive multivariate monitoring strategy.

3.3 Monitoring Generalized Variance: The $|S|$ Chart

An alternative approach to monitoring multivariate process variability is through the use of generalized variance — the determinant of the sample covariance matrix. Rather than evaluating individual variances or covariances, this method summarizes the total variability of a multivariate process in a single scalar measure, denoted as:

$$|S| = \det(\mathbf{S}) \tag{21}$$

This value represents the "volume" of the multivariate distribution in p-dimensional space. Significant changes in this generalized variance indicate a shift in overall process dispersion or correlation structure.

Under normal operating conditions, Montgomery (2009) provides expressions for the mean and variance of $|S|$:

$$\mathbb{E}(|S|) = b_1|\mathbf{\Sigma}|, \quad \mathrm{Var}(|S|) = b_2|\mathbf{\Sigma}|^2 \tag{22}$$

Where $\mathbf{\Sigma}$ is the in-control covariance matrix, and b_1, b_2 are constants depending on sample size n and the number of variables p:

$$b_1 = \frac{1}{(n-1)^p}\prod_{i=1}^{p}(n-i) \tag{23}$$

$$b_2 = \frac{1}{(n-1)^{2p}}\left[\prod_{i=1}^{p}(n-i)^2 \cdot \left(\prod_{j=1}^{p}(n-j+2) - \prod_{j=1}^{p}(n-j)\right)\right] \tag{24}$$

These expressions provide the basis for constructing a control chart for $|S|$, using the following limits:

$$\mathrm{UCL} = |\mathbf{\Sigma}|(b_1 + 3\sqrt{b_2}), \tag{25}$$

$$\mathrm{CL} = b_1|\mathbf{\Sigma}|, \tag{26}$$

$$\mathrm{LCL} = \max\left\{0, |\mathbf{\Sigma}|(b_1 - 3\sqrt{b_2})\right\} \tag{27}$$

Illustrative Example

Suppose a bivariate process ($p = 2$) is monitored with subgroups of size $n = 10$. Assume the in-control covariance matrix has determinant $|\mathbf{\Sigma}| = 5$, and for one sample, the computed determinant of the sample covariance matrix is $|S| = 4.5$.

First, compute b_1:

$$b_1 = \frac{1}{(10-1)^2} \cdot (10-1)(10-2) = \frac{1}{81} \cdot (9 \cdot 8) = \frac{72}{81} = 0.8889$$

Next, compute b_2 by evaluating the required products:

$$\prod_{i=1}^{2} (10-i)^2 = (9)^2 \cdot (8)^2 = 81 \cdot 64 = 5184$$

$$\prod_{j=1}^{2} (10-j+2) = 11 \cdot 10 = 110, \quad \prod_{j=1}^{2} (10-j) = 9 \cdot 8 = 72$$

$$b_2 = \frac{1}{(10-1)^4} \cdot 5184 \cdot (110-72) = \frac{1}{6561} \cdot 5184 \cdot 38 \approx 30.03$$

Now compute the control limits:

$$\text{CL} = 0.8889 \cdot 5 = 4.4445$$
$$\text{UCL} = 5 \cdot (0.8889 + 3 \cdot \sqrt{30.03}) \approx 5 \cdot (0.8889 + 3 \cdot 5.48) = 5 \cdot 17.327 = 86.64$$
$$\text{LCL} = 5 \cdot (0.8889 - 3 \cdot 5.48) < 0 \quad \Rightarrow \quad \text{LCL} = 0$$

Since $|S| = 4.5$ lies between 0 and 86.64, the process is considered in control with respect to generalized variance.

Interpretation

The generalized variance chart is particularly useful for detecting structural changes in process variability that affect the overall data spread. Because it collapses the multivariate variance-covariance structure into a single determinant value, it provides a concise and intuitive signal when overall variability increases or decreases. This complements the Alt's W chart, which accounts for matrix shape and structure, by offering a scalar summary of volume-related dispersion.

4 Practical Implementation

4.1 Phase I Analysis of Multivariate Process

This section demonstrates the application of multivariate control charts using R, focusing on Phase I analysis of the `Ryan92` dataset. This dataset is available in the `IAcsSPCR` package and contains 80 observations (20 subgroups with 4 observations each) of two quality characteristics, x_1 and x_2, which are monitored for shifts in mean and covariance.

Packages Used

- `qcc`: The main package used for quality control charting in R. It provides functions for univariate ($\bar{X}$, R, S) and multivariate (T^2) charts. The function `qcc()` generates Shewhart-type charts, while `mqcc()` supports Hotelling's T^2 charts.
- `IAcsSPCR`: Provides access to specialized datasets used in applied SPC research, including the `Ryan92` data.
- `MASS`: Supports matrix operations required by multivariate charting, such as covariance matrix computation and multivariate normal operations.

The Hotelling's T^2 statistic, as defined in the theoretical section,

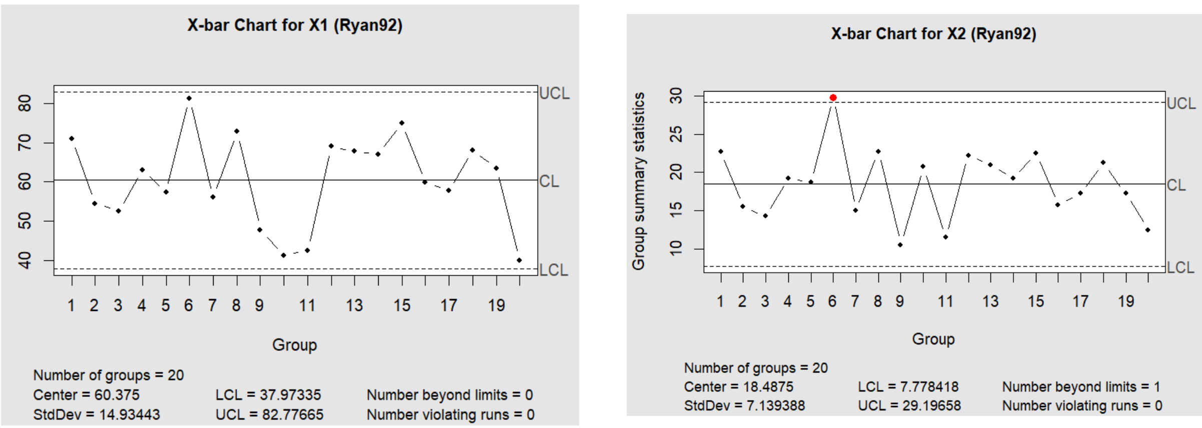
$$T^2 = n(\bar{\mathbf{x}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{x}} - \boldsymbol{\mu})$$

is automatically computed and plotted using the `mqcc()` function in the `qcc` package. The associated confidence ellipse shown by `ellipseChart()` visualizes this statistic by forming a multivariate confidence region, aligned with the Mahalanobis distance.

Phase I analysis aims to identify and remove any out-of-control subgroups in order to reliably estimate the in-control process parameters. Once anomalies (e.g., subgroups 10 and 20) are removed, the process is re-evaluated.

Step 1: Individual X-bar Charts for Each Variable

As an initial screening step, separate $\bar{X}$-charts were constructed for both x_1 and x_2 to identify potential univariate signals.



(a) $\bar{X}$-Chart for x_1

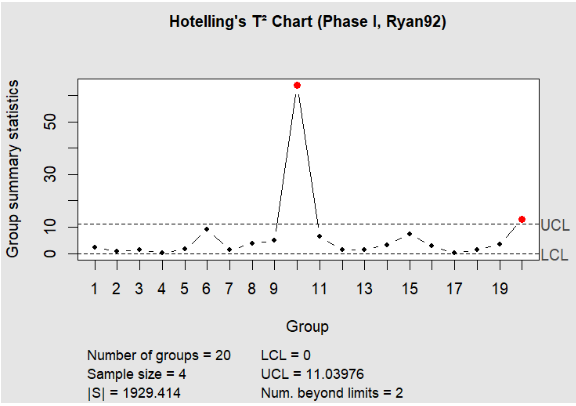
(b) $\bar{X}$-Chart for x_2

Figure 2: Univariate $\bar{X}$-charts for the two variables. Subgroup 6 is flagged as out-of-control in x_2.

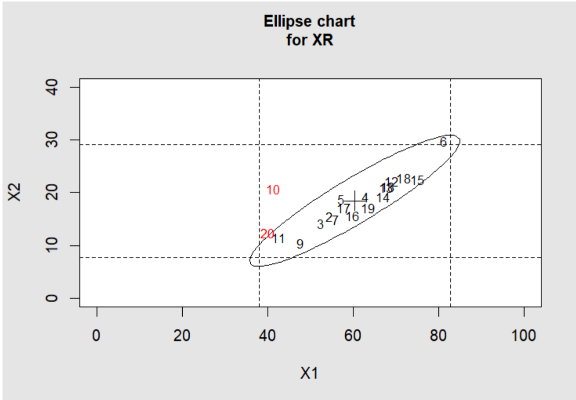
In this case, the X-bar chart for x_1 exhibited no out-of-control signals. However, the chart for x_2 flagged subgroup 6 as exceeding control limits. This highlights a limitation of univariate charts — they assess each variable in isolation and may fail to detect joint shifts across variables.

Initial Hotelling's T^2 and Ellipse Chart (Before Cleaning)

To address this limitation, a Phase I Hotelling's T^2 control chart was applied to jointly monitor both variables.



(a) Initial Hotelling's T^2 Chart



(b) Confidence Ellipse (Initial)

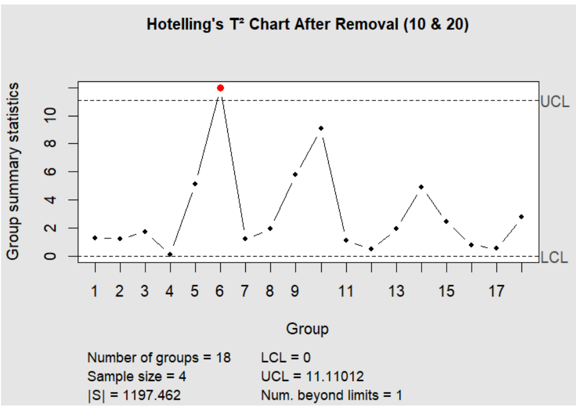
Figure 3: Initial multivariate monitoring: Subgroups 10 and 20 fall outside control limits.

Unlike the univariate charts, the T^2 chart identifies subgroups 10 and 20 as multivariate outliers — neither of which were flagged by both univariate charts. This demonstrates the enhanced sensitivity of multivariate control charts for detecting subtle shifts across variable combinations.

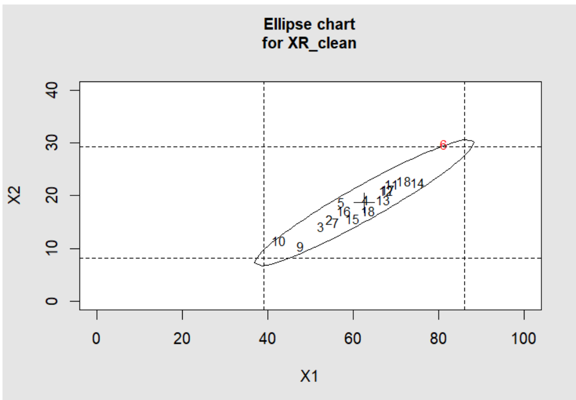
To visualize the relationship between the two variables and locate outliers in multivariate space, a confidence ellipse was generated. The plot clearly places subgroups 10 and 20 outside the 99% confidence ellipse, confirming their atypical behavior.

Cleaned Multivariate Monitoring (After Removal of Points 10 and 20)

Assuming subgroups 10 and 20 reflect assignable causes, they were removed from the dataset. A new Hotelling's T^2 chart was constructed with the remaining 18 subgroups.



(a) Hotelling's T^2 After Removal



(b) Confidence Ellipse (After Cleaning)

Figure 4: Post-cleaning monitoring: All points are in control, except subgroup 6 (from x_2).

After removal, the multivariate chart showed stable behavior except for subgroup 6, which had already been identified in the univariate chart for x_2. While this point could also be removed, we retained it here to emphasize the importance of understanding both univariate and multivariate sources of variation.

Implications for Phase I Studies

This analysis demonstrates that multivariate methods like Hotelling's T^2 and confidence ellipses can detect subtle joint shifts missed by univariate charts. In Phase I studies, it is recommended to use at least 25 rational subgroups for reliable estimation. Given that $p = 2$ variables are monitored here with only 20 subgroups, the stability assessment should be interpreted with caution. Nonetheless, the clean post-removal chart suggests that — excluding known anomalies — the process is approximately in control with a stable mean vector and covariance structure.

4.2 Phase II Monitoring with Hotelling's T^2 Control Chart

Hotelling's T^2 control chart can be used in both **Phase I** and **Phase II** monitoring. In Phase I, it helps identify outliers and estimate the in-control process parameters (mean vector and covariance matrix). In Phase II, it evaluates whether the production process remains stable over time using those previously estimated parameters.

To illustrate the use of a Phase II multivariate control chart, we build upon the results from Phase I where subgroups 10 and 20 were removed after being flagged as out-of-control in the T^2 chart. The remaining subgroups were used to estimate the in-control mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$.

We now simulate 20 new subgroups (each of size $n = 4$) and use them as Phase II observations. The goal is to determine whether these new data points remain in control. The chart is constructed using the `mqcc()` function from the `qcc` package in R, which computes and visualizes the Hotelling T^2 statistics for each subgroup.

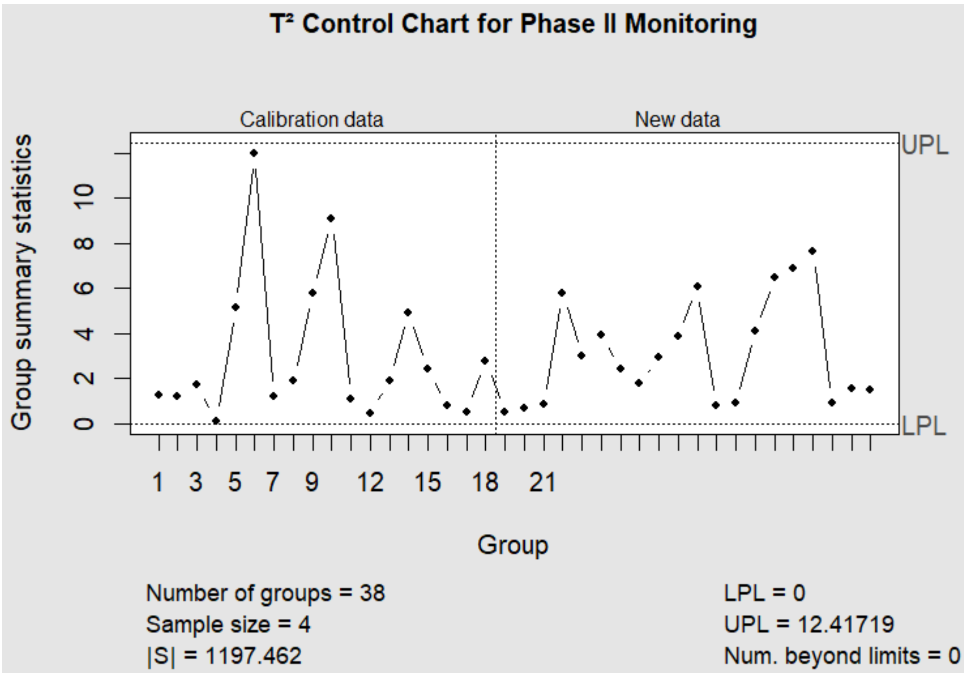


Figure 5: Phase II Hotelling T^2 Control Chart for Monitoring New Subgroups

For Phase II, the upper control limit (UCL) is adjusted using the F-distribution, as defined in Montgomery (Equation 11):

$$UCL = \frac{p(m+1)(n-1)}{m(n-p)} F_{\alpha,p,m(n-1)-p+1}$$

where p is the number of variables, m is the number of Phase I subgroups used to estimate the parameters, and α is the significance level.

Interpretation

As shown in Figure 5, all Phase II observations fall within the control limits, indicating no significant shift in the mean vector of the process. This confirms that the process remains in statistical control with respect to the multivariate mean.

While univariate control charts may miss subtle interactions, the T^2 chart incorporates the correlation structure between variables—offering a more holistic view. Nevertheless, for detecting smaller or gradual shifts, MEWMA charts (discussed next) may offer greater sensitivity due to their cumulative monitoring properties.

4.3 Phase II Monitoring with the MEWMA Control Chart

While the Hotelling's T^2 chart is effective at detecting large shifts in the multivariate mean, it is less sensitive to small or moderate changes. In contrast, the Multivariate Exponentially Weighted Moving Average (MEWMA) chart incorporates past observations through exponential weighting, making it more responsive to subtle process drifts.

To illustrate this, we simulate 30 observations from a bivariate normal distribution. The first 15 belong to the in-control Phase I process, and the remaining 15 exhibit a small shift in the mean vector. The covariance matrix remains constant throughout:

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 1 \end{bmatrix}, \quad \boldsymbol{\mu}_{\text{Phase I}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{\mu}_{\text{Phase II}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The MEWMA chart is constructed using the following statistic:

$$\mathbf{Z}_i = \lambda \mathbf{X}_i + (1-\lambda)\mathbf{Z}_{i-1}, \quad \mathbf{Z}_1 = \lambda \mathbf{X}_1$$

$$T_i = \mathbf{Z}_i' \boldsymbol{\Sigma}_{\mathbf{Z}_i}^{-1} \mathbf{Z}_i$$

The control limit for Phase II is based on the chi-squared distribution with $p = 2$ variables and significance level $\alpha = 0.01$:

$$\text{UCL} = \chi^2_{1-\alpha,p} = \chi^2_{0.99,2} \approx 9.21$$

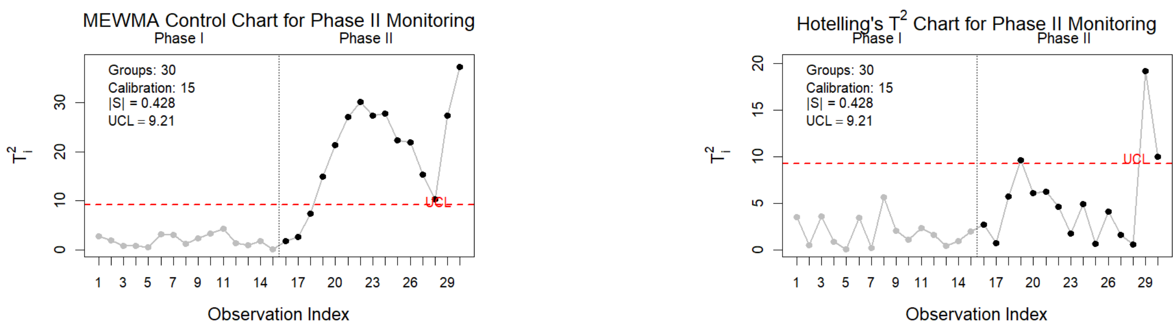


Figure 6: Comparison of MEWMA chart (left) and Hotelling's T^2 chart (right) for Phase II Monitoring. MEWMA detects the subtle shift earlier and more consistently.

This comparison emphasizes that while Hotelling's T^2 is useful for detecting significant shifts in mean vectors, MEWMA charts provide a more sensitive tool for identifying gradual drifts. This is particularly useful in continuous Phase II process monitoring when early intervention is critical.

5 Conclusion

In this paper, I explored how multivariate control charts like Hotelling's T^2 and MEWMA work, both in theory and in practice. These charts are useful when we need to track several variables that are connected. I also showed examples using simulated and real data to explain how these charts can help find unusual behavior in processes.

By applying these methods to both simulated and real datasets, we saw how multivariate charts offer a more complete picture compared to univariate charts. Each method has its own strengths, and knowing when to use them is important for better process control. With the help of R packages like `qcc` and `MSQC`, it is also quite practical to implement these tools in real-world situations.

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