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Measuring the Depth of the 4km Mine

I. Introduction

Before we attempt to conduct our measurement of the vertical depth of the mine using the proposed method of dropping a 1kg mass and timing its journey to the bottom, we must determine if this is a realistic strategy. We have completed an investigation into this method by modeling the theoretical behavior of the mass, taking into consideration factors such as drag, Coriolis force, and non-constant gravity. Additionally, using this initial investigation as a framework, we have conducted preliminary tests on the behavior of the mass in a mine of infinite depth, which we believe will offer crucial insights to any future work pursued by the company. This report summarizes our findings, and makes a recommendation as to our course of action based on the results.

II. Calculation of Fall Time

To perform our calculation of the expected fall time of the mass, we completed progressively more in depth calculations by considering more and more factors that would impact the result. We began by assuming a constant acceleration due to gravity (g_0) and no drag, with the following equations:

$$t = \sqrt{\frac{2}{ad}} \qquad\qquad g0 = \frac{GM_E}{R_E^2}$$

where a = g_0, d is the depth of the mine G is the gravitational constant, M_E is the mass of the Earth, and R_E is the radius of the Earth. From this calculation, we found a fall time of 28.6 seconds. Second, we modeled this problem using the following differential equation:

$$\frac{d^2y}{dt^2} = -g + \alpha|\frac{dy}{dt}|^\gamma$$

where y is the position of the mass, α and γ are constants, and g is the acceleration due to gravity. The second term on the right is the drag term. If we set the drag term equal to zero, a numerical solution to this equation yields a fall time of 28.6 seconds, the same as our first calculation.

However, using a constant value for acceleration due to gravity is incorrect. As the mass falls, there is less and less Earth matter pulling it down, and g is constantly decreasing. If we assume the Earth has a constant density, a more accurate approximation for the value of g is:

$$g(r) = g_0(\frac{r}{R_E})$$

where r is the mass's distance from the center of the Earth. If we assume a non constant g and re-solve the above differential equation, we get the same fall time of 28.6 seconds. We can therefore conclude that the effect of non-constant gravity is negligible in this scenario.

Finally, we must consider the effects of drag, or air resistance, on the fall time of the mass. We set γ = 2, and we calibrated the value of α to be about 0.004 by approximating the terminal velocity of the mass to be 50 m/s. Solving the differential equation again yields a fall time of 83.5 seconds.

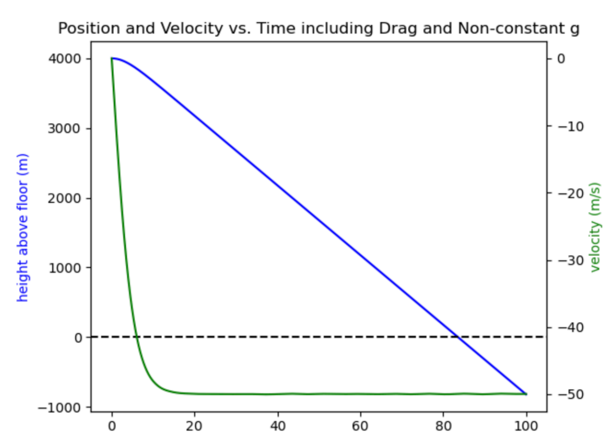

Fig. 1. In the plot, non-constant g is clear based on the non-linear velocity curve. The effect of drag is also evident, as the velocity curve levels off to 50 m/s, the terminal velocity. The position curve appears to turn linear around the same time, which we would expect when there is zero acceleration.

The effects of drag are therefore extremely significant, considering they increased the fall time nearly by a factor of three.

A plot of the position and velocity of the mass falling down the mine under these conditions is shown in Fig. 1. In the plot, non-constant g is clear based on the non-linear velocity curve. The effect of drag is also evident, as the velocity curve levels off to 50 m/s, the terminal velocity. The position curve appears to turn linear around the same time, which we would expect when there is zero acceleration.

III. Feasibility of Depth Measurement Approach

In order to evaluate the feasibility of this measurement approach, we must consider an additional effect on the mass, the Coriolis force. This "force" appears to cause a displacement of the mass in the direction of Earth's rotation, but in reality is due to the Earth spinning. If we add the Coriolis force into our differential equation, our result is the following:

$$\frac{d^2y}{dt} = -g_0\left(\frac{r}{R_E}\right) + \alpha\left|\frac{dy}{dt}\right|^{\gamma} - 2(\vec{\Omega} \times \vec{v})$$

where Ω is the angular velocity of the Earth at the equator, where our mine is located, and v is the velocity of the mass. The width of the mine is about five meters, so if we drop the mass in the center, it needs to travel 2.5 meters horizontally to hit the wall. If we neglect drag, we get our original fall time of 29.6 seconds, but we also see that the mass will hit the wall after 21.9 seconds, roughly eight seconds early. It will hit the wall at a depth of about 2352.9 meters, about halfway down the mine. We can therefore conclude that the mass will hit the wall before it reaches the bottom.

If we now consider drag, we will get the same fall time as before, 83.5 seconds, and we find that the mass will hit the wall at 29.6 seconds and at a depth of 1302.5 meters. Compared to no drag, the mass will fall for an extra eight seconds before it hits the wall, however it still hits the wall in almost a third of the time it would take to reach the bottom. The transverse position of the mass as a function of depth is shown in Figure 2. It is clear from the plot that the mass reaches a horizontal displacement of -2.5 meters nearly four times higher than the full depth we are attempting to measure.

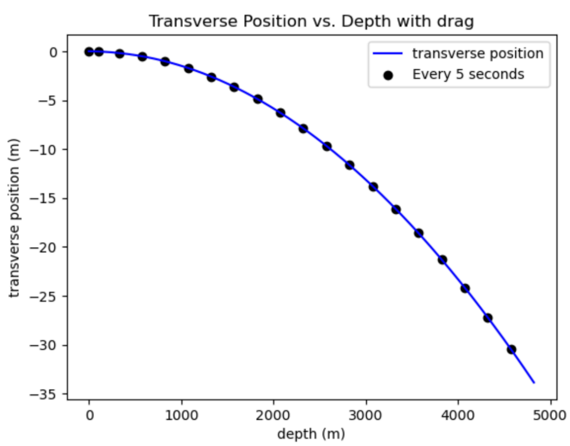

Fig. 2: The transverse position of the mass as a function of depth, assuming non-constant g and non-negligible drag. The black points represent the position of the mass every five seconds.

From these calculations, we can conclude that the mass will hit the wall before it hits the bottom, interfering with the fall time and our measurement of the depth of the mine. Therefore, we cannot recommend this method of measuring the depth of the mine. The result would be inaccurate, and a waste of company resources.

IV. Calculation of crossing times for homogeneous and non-homogeneous Earth

In the final part of our investigation, we expanded our original purpose to include a study of an infinitely deep mine, or a mine which goes all the way through the Earth, as well as an infinitely deep mine on the Moon. As the company digs deeper and deeper mines, this model will become more and more relevant to our efforts.

For this section, we considered a density distribution of the Earth and Moon using the model:

$$\rho(r) = \rho_n(1 - \frac{r^2}{R_E^2})^n$$

where ρ_n is a constant, r is the distance from the center, and n is a constant. Higher values of n correspond to a higher concentration of mass closer to the center of the Earth. We chose to ignore the effects of drag, and therefore we would expect the mass to oscillate continuously up and down the infinite mine. In particular, we considered n = 0, which corresponds to a uniform density, and n = 9, where mass is highly concentrated at the center. Figure 3 shows the position and velocity of the mass for several oscillations for both of these density distributions. It is clear from these graphs the effect of density on the path of the object. In particular, the slope of the position graph in between extrema is steeper, indicating a greater speed. Additionally, for n = 0, it would take the mass 2483.6 seconds to fall from one side of the Earth to the other. For n = 9, it would take 1887.7 seconds.

These numbers and the figure demonstrates that for larger values of n, the mass would have a shorter crossing time. This is what we expect, because for higher values of n the enclosed mass, and therefore the acceleration, will fall off more slowly, leading to a shorter crossing time.

For the Moon, we assumed a uniform density, and found a crossing time of 3249.9. Considering the Moon is 0.6 times the density of the Earth, we would expect it to have a longer crossing time, because there is less mass pulling the object down. A higher density therefore corresponds to a faster crossing time.

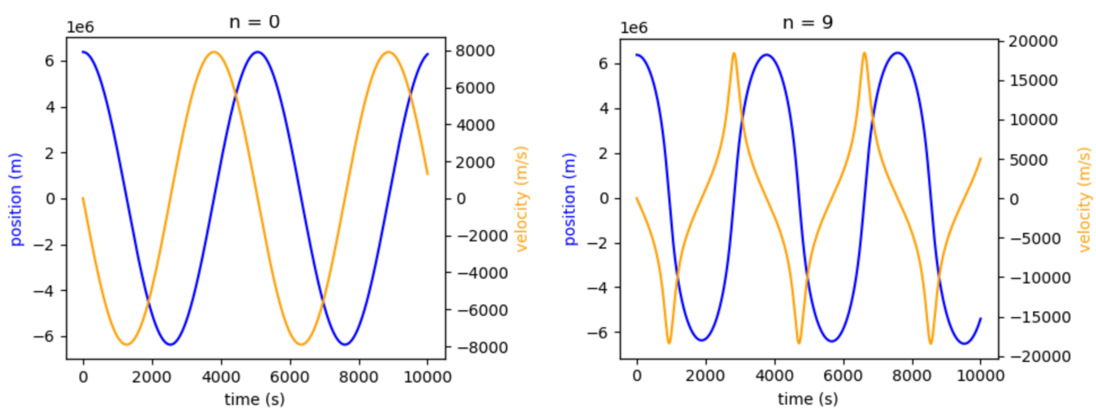

Figure 3: The position and velocity of the mass in an infinite mine for n = 0 and n = 9

V. Discussion and Future Work

Over the course of our investigation, we made several notable approximations in order to simplify our calculations. In our calculation of the fall time including drag and non-constant gravity, we approximated the values of γ and α. In future investigations, more accurate values for these constants should be measured and used, in particular we should use a more precise value for the terminal velocity of the mass. Another approximation we made was assuming a spherical Earth. In fact, the Earth is slightly oblate. This is significant for our mine, because it is located right on the equator. Additionally, our models for density were inaccurate. The Earth's density is not continuous as a function of distance from the center, rather the density is different in each distinct layer of the Earth. A more accurate model of Earth's density and shape should be used for future calculations, especially regarding further investigations into infinitely deep mines.

Considering the fact that our calculated fall time is nearly triple the time it would take to hit the wall, it is unlikely that any future, more precise calculation would yield results different enough from these to reverse our recommendation that this method of measuring the depth of the mine not be implemented. However, these differences are still notable, and future investigations should take them into account.