Algorithms and Programation

Megi Dervishi

January 4, 2020

Homework 4 (5/01/2020)

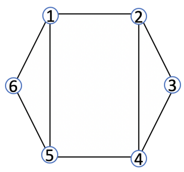
Exercise 1

Solution

(1)
We want to prove the following theorem:

$$G \text{ is } k\text{-edge-connected} \Leftrightarrow \forall (v_i, v_j),\ 1 \leq i, j \leq n,\ i \neq j \text{ are joined by } k\text{-pairwise edge disjoint paths}$$

Note: There is an ambiguity on the term "path" on the theorem which actually intends "at least k paths". For example the following graph is 2-edge-connected but there are 3 disjoint paths between vertex 1 and 5.



($\Leftarrow$) Pair of vertices are joined by k-pairwise edge disjoint paths means that there are k-paths none of which share an edge between all pairs of vertices. So it is trivial to see that if you delete an edge from each k-paths i.e. k edges, you have disconnected the graph. If you delete less than k-edge then since you have $k-$paths then there is at least 1 path that is not destroyed so the graph would still be connected. Hence if every pair of vertices are joined by k pairwise edge disjoint paths then you need to delete at least k edges to disconnect the graph i.e. G is k- edge connected.

($\Rightarrow$) For G to be k-edge connected at least k edges need to be cut in order to disconnect it. Let X be the set of minimum edges which if removed they disconnect the graph into two parts A and B. Consequently we have that $card(X) = k$. Let $\forall e \in G, c(e) = 1$. In other words "$G$ is k-edge connected" is equivalent to the minimum cut problem $\forall (s,t) \in A \times B$. So the value of the minimum cut is k since the minimum amount of edges you have to remove is k, $(card(X))$. By the max flow min cut theorem we have that the value of the minimum cut is equal to the value of the maximal flow i.e. k. Now we have two case:
case 1: Fix $(s,t) \in A \times B$. We know that the maximal flow from s to t is k. Suppose that there were less than k disjoint paths then the maximal flow would be strictly lower than k. So the value of maximal flow being k forces that there must be k disjoint paths.
case 2: Wlog, fix $(s,t) \in A$. Then either there exists another X' such that the removal of it would separate the graph into a new A', B' such that $s \in A'$ and $t \in B'$ or there exists no such set. If that's the case then by the contrapositive of the first implication proof we have that there are more than k paths. Hence at least k paths. In all cases there are at least k-pairwise edge disjoint paths $\forall s, t \in V$.

By double implication this concludes the proof.

(2) The algorithm takes as input an undirected graph $G = (V, E)$ and outputs the minimum edge cut X. We start by replacing each undirected edge in G with two oppositely directed edges both of capacity 1. Let G_f be the resulting graph. Then we choose an $s \in V$ at random and apply the Ford-Fulkerson algorithm $\forall t \in V \setminus \{s\}$ which will compute the maximum flow for all t. Then we take t_{min} which has the minimal maximal flow and its residual graph G_r. By the corollary in the lecture we are sure that we can compute the min-edge-cut from G_r (in linear time). The algorithm is correct and terminates.
Time complexity We are iterating over $n-1$ vertices the Ford-Fulkerson algorithm which has complexity

$\mathcal{O}(fm) = \mathcal{O}(mn)$ since $f \leq nC = n$ and $C = 1$. Finally we compute the min-edge-cut in $\mathcal{O}(n)$ which makes the time complexity of our algorithm be $\mathcal{O}(mn^2)$. So our algorithm runs in poly-time.

Exercise 2

Solution

First we show that the greedy algorithm terminates. Let S_E be the set of edges of the cut which initially is equal to $\emptyset$. Let $|e_v|$ be the cardinality of incident edges of a vertex v. We know that $|e_v| = |nc_v| + |c_v|$ where nc_v is the not-crossing edges and c_v is the crossing edges of v. Also by step b we have that $|c_v| < \frac{1}{2}|e_v|$. Assume we want to add in the S_E a new set of edges, call the new formed set S_E'. Hence we have that:

$$|S_E'| = |S_E| + |nc_v| - |c_v| = |S_E| + |e_v| - 2|c_v|$$

$$|S_E'| > |S_E| + |e_v| - 2\frac{1}{2}|e_v| = |S_E|$$

So the cardinality is increasing and since $|S_E'|$ is bounded above by $|E|$ then that means that the algorithm terminates.
Now we show that the algorithm is a 2-approximation of Max-Cut. Since this is a maximization problem then we need to prove that $ALG(P) \geq \rho(n)OPT(P)$ where $\rho(n) < 1$. Let MC be the max-cut i.e. the optimal solution and S_E the solution that the greedy algorithm outputs. We notice at step b that we add more than $\frac{1}{2}$ of the incident edges so in the end that would result in the following inequality $S_E \geq \frac{|E|}{2}$. Hence we have the following which proves that the greedy algorithm is 2-approximation of the Max-Cut:

$$|S_E| \geq \frac{1}{2}|E| \geq \frac{1}{2}|MC|$$

Finally we show that the randomized algorithm is also a 2-approximation of the Max-Cut. We keep the same notation. Then using the linearity of expectation we have that:

$$\begin{aligned}
\mathbb{E}[|S_E|] &= \sum_{(v_1,v_2)\in G.E} \mathbb{P}(v_1, v_2 \in A \times B \vee v_1, v_2 \in B \times A) = \sum_{(v_1,v_2)\in G.E} (\mathbb{P}(v_1, v_2 \in A \times B) + \mathbb{P}(v_1, v_2 \in B \times A)) \\
&= \sum_{(v_1,v_2)\in G.E} (\frac{1}{4} + \frac{1}{4}) = \frac{|E|}{2} \geq \frac{1}{2}|MC|
\end{aligned}$$

Exercise 3

Solution

"Become infinitely rich" means that there exists a sequence of currencies $c_1, \cdots c_k$ that when multiplying their exchange rates, the product will be bigger than 1 i.e.

$$R[i_1, i_2] \cdot R[i_2, i_3] \cdots R[i_{k-1}, i_k] \cdot R[i_k, i_1] > 1$$

We transform this sequence in the following way so we could apply the Bellman-Ford algorithm which returns False if there exists a negative cycle in the weighted graph of currencies:

$$\log(R[i_1, i_2] \cdot R[i_2, i_3] \cdots R[i_{k-1}, i_k] \cdot R[i_k, i_1]) > \log(1)$$

$$\log(R[i_1, i_2]) + \log(R[i_2, i_3]) + \cdots \log(+R[i_{k-1}, i_k]) + \log(R[i_k, i_1]) > 0$$

$$-\log(R[i_1, i_2]) - \log(R[i_2, i_3]) - \cdots \log(-R[i_{k-1}, i_k]) - \log(R[i_k, i_1]) < 0$$

Hence if the algorithm returns False then there exists such a sequence in the graph, i.e. it is time to become rich. The time complexity of the Bellman-Ford algorithm is $\mathcal{O}(VE)$. Assume we have n vertices then in a complete digraph we would have $\frac{n^2-1}{2}$ edges so the time complexity is $\mathcal{O}(n^3)$. The printing and transforming of such sequence back to its original form would take $\mathcal{O}(n)$. In conclusion this algorithm would be of time complexity $\mathcal{O}(n^3)$ and space complexity $\mathcal{O}(n^2)$ (for the storing of the table R).