Question 1

First of all we assume $n \geq 3$ otherwise it would be impossible. The first edge removed transforms a cycle graph into a linear connected path graph. The second edge removed then separated the graph into 2 connected components.

Question 2

Not it is not true. As we can see in Figure 1 the graphs have the same degree sequence $\{1, 2, 2, 2, 3\}$ however they are not isomorphic as graph A has a 3-loop and graph B has a 4-loop. Hence there exists no bijective mapping that could transform graph A into graph B (or vice-versa).

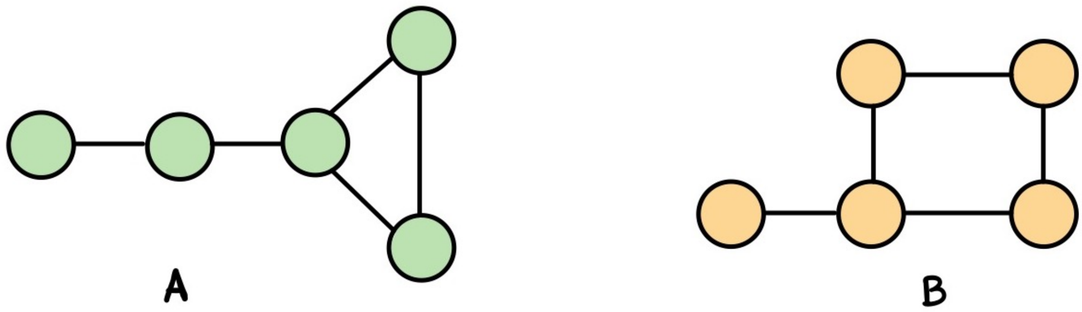

Figure 1: Two non-isomorphic graphs with the same degree sequence

Question 3

One notices that an undirected graph (without self-loops) of n nodes and $\frac{n(n-1)}{2}$ edges is a complete graph. Hence a graph with $\frac{n(n-1)}{2} - 1$ edges is a complete graph with an edge removed. For a complete graph since everything is connected there are no open triplets (i.e. 0), and the number of closed triplets would be $\binom{n}{3}$. Hence the global clustering coefficient is $\binom{n}{3}$. Now if we remove an edge we have transformed open triplets into closed ones. So we just need to calculate the number of open triples that got destroyed, which is $\binom{n-2}{1} = n - 2$ (since the destroyed edge has 2 nodes then from $n-2$ nodes we need to choose another node). Therefore the number of open triplets is $\binom{n}{3} - (n-2)$ and the number of closed triplets is $n-2$. The global clustering coefficient is hence:

$$\frac{\binom{n}{3} - (n-2)}{\binom{n}{3} - (n-2) + (n-2)} = \frac{\frac{n!}{(n-3)!3!} - \frac{(n-2)(n-3)!3!}{(n-3)!3!}}{\frac{n!}{(n-3)!3!}} = \frac{n! - (n-2)!3!}{n!} = 1 - 3 \cdot \binom{n}{2}^{-1} \tag{1}$$

Question 4

The multiplicity k of the eigenvalue 0 of L_{rw} (and 0 is the smallest eigenvalue by Proposition 3.5 in [1]) corresponds to the number of connected components in the graph represented by L_{rw} by Proposition 4 in [1]. Let these connected components be denoted by $A_1, \cdots, A_k$ then for any eigenvector v with eigenvalue 0 we have that

$$v = \sum_{i=1}^{k} \lambda_i 1_{A_i} \tag{2}$$

where 1_{A_i} is the indicator vector of the component A_i, i.e. $1_{A_i} = (f_1, \cdots, f_n)$ where $f_j = 1$ if $j \in A_i$ and 0 otherwise.

Question 5

The spectral clustering algorithm is a stochastic algorithm as it depends on the k-means algorithm which is a stochastic one as it is initializes the clusters randomly.

Question 6

First we have that $m_A = m_B = 8$ and $n_{c_A} = 2$ whereas $n_{c_B} = 3$. Table 1 then shows the different values for l_c and d_c for each graph G.

G	c	l	d
A	blue	4	9
	green	3	7
B	blue	1	4
	orange	2	8
	green	1	4

Table 1: G (graph), c (community), l number of edges and d the sum of degrees of nodes within community c

The modularity is then:

$$Q_A = \frac{4}{8} - (\frac{9}{2 \cdot 8})^2 + \frac{3}{8} - (\frac{7}{2 \cdot 8})^2 = \frac{47}{128} \simeq 0.367 \tag{3}$$

$$Q_B = \frac{1}{8} - (\frac{4}{2 \cdot 8})^2 + \frac{2}{8} - (\frac{8}{2 \cdot 8})^2 + \frac{1}{8} - (\frac{4}{2 \cdot 8})^2 = \frac{1}{8} = 0.125 \tag{4}$$

Graph A has a better community structure than graph B as the modularity is higher.

Question 7

$$\begin{aligned}
\phi(P_4) &= (3, 2, 1, 0, \cdots) \\
\phi(K_4) &= (6, 0, 0, 0, \cdots) \\
k(P_4, P_4) &= \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 14 \\
k(P_4, K_4) &= \begin{pmatrix} 3 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = 18 \\
k(K_4, K_4) &= \begin{pmatrix} 6 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} = 36
\end{aligned}$$

References

[1] Ulrike Von Luxburg. A tutorial on spectral clustering. *Statistics and computing*, 2007.