Homework 1

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Exercise 1

1) A rectangle is an intersection of halfspaces (polyhedra) defined from:

1. $b^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i \qquad b^{(i)T} x \leq \beta_i$

$\forall i = 1 \dots n$

2. $a^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \leftarrow i \qquad a^{(i)T} x \leq -\alpha_i$

Hence it is convex.

2) $H = \{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$. Let $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in H$ and $0 \leq \theta \leq 1$ then:

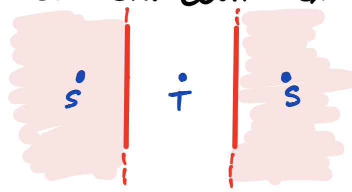
$$\theta \binom{x_1}{y_1} + (1-\theta)\binom{x_2}{y_2} \in H \iff (\theta x_1 + (1-\theta)x_2)(\theta y_1 + (1-\theta)y_2) \geq 1$$
$$\iff \theta^2 \underbrace{x_1 y_1}_{\geq 1} + (1-\theta)^2 \underbrace{x_2 y_2}_{\geq 1} + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \geq 1$$
$$\iff \theta^2 + (1-\theta)^2 + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \geq 1$$
$$\iff \cancel{1} - 2\theta(1-\theta) + \theta(1-\theta)(x_1 y_2 + x_2 y_1) \geq \cancel{1}$$
$$\iff \underbrace{\theta(1-\theta)}_{\geq 0}[-2 + x_1 y_2 + x_2 y_1] \geq 0$$
$$\iff x_1 y_2 + x_2 y_1 \geq 2$$
$$\iff x_1 y_1 y_2 + x_2 y_1^2 \geq 2 y_1$$
$$\iff x_2 y_2 + x_2^2 y_1^2 - 2 y_1 x_2 \geq 0$$
$$\iff x_2^2 y_1^2 - 2 y_1 x_2 + 1 \geq 0$$
$$\iff (x_2 y_1 - 1)^2 \geq 0 \quad \text{QED. } H \text{ is convex.}$$

3) Fix $y \in S$ and consider $C_y = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$ then we want to show C_y is a half space. If so then the intersection of C_y on all $y \in S$ is the set $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S\}$ and hence it is convex.

- Show C_y is a halfspace

$$\|x - x_0\|_2 \leq \|x - y\|_2 \iff \|x - x_0\|_2^2 \leq \|x - y\|_2^2$$
$$\iff (x - x_0)^T (x - x_0) \leq (x - y)^T (x - y)$$
$$\iff \cancel{x^T x} - 2 x_0^T x + x_0^T x_0 \leq \cancel{x^T x} - 2 y^T x + y^T y$$
$$\iff 2\underbrace{(y^T - x_0^T)}_{a^T} x \leq \underbrace{y^T y - x_0^T x_0}_{b}$$

4) The set is not-convex. Counterexample in $\mathbb{R}^2$:



Disjoint sets can not be convex.

5) Let $f_y : \mathbb{R}^n \to \mathbb{R}^n$ st $f_y(x) = x + y$ where $y \in S_2 \subseteq \mathbb{R}^n$. We know that since S_1 is convex and f_y is an affine function then $f^{-1}(S_1)$ is convex. $f_y^{-1}(S_1) = \{x \mid x+y \in S_1\}$.

The set $\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} f^{-1}(S_1)$. Intersection of convex sets is convex.

Exercise 2

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$. $\nabla^2 f(x_1,x_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For $\forall x, y \in \mathbb{R}^2$ then $(x \ y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2xy$, which is neither always positive nor always negative. Hence f is neither convex nor concave.

But the sublevel set of $-f$ is convex and dom f is convex then f is quasiconcave.

- $S_\alpha = \{x \in \text{dom} f \mid f(x_1,x_2) = x_1 x_2 \geq \alpha\}$ is convex since by Ex.1.2 H is convex and S_α is an affine transformation of H.

2) $f(x_1, x_2) = \dfrac{1}{x_1 x_2} = e^{-\ln x_1 - \ln x_2}$ $\quad \forall x_1, x_2 \in \mathbb{R}^2_{++}$

$-\ln x_1 - \ln x_2$ is convex ($\ln(x)$ is concave) and exp of convex is convex.

3) $f(x_1, x_2) = \dfrac{x_1}{x_2}$ on $\mathbb{R}^2_{++}$. $\nabla^2 f(x_1,x_2) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$ whose eigenvalues are $\dfrac{x_1 \pm \sqrt{x_1^2 + x_2^2}}{x_2^3}$ which are neither always positive nor always negative so f is neither convex nor concave. H is quasilinear since sublevel set of f and sublevel set of $-f$ are halfspaces.

- $S_\alpha = \{x \in \text{dom} f \mid f(x_1,x_2) = \frac{x_1}{x_2} \geq \alpha\} = \{x \in \text{dom} f \mid \overbrace{(1 \ \ -\alpha)}^{a^T}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq \overset{b}{0}\}$

  $S_\alpha = \{x \in \text{dom} f \mid f(x_1,x_2) = \frac{x_1}{x_2} \leq -\alpha\} = \{x \in \text{dom} f \mid (-1 \ \ \alpha)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \geq 0\}$

4) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ on $\mathbb{R}^2_{++}$ and $0 \leq \alpha \leq 1$.
Let $g : \mathbb{R} \to \mathbb{R}$ st $g(x) = x^{\alpha}$ (i.e g is concave) and since
$f(x_1, x_2) = x_2 \cdot g\left(\frac{x_1}{x_2}\right)$ by perspective we can conclude that f is concave.

Exercise 3

1) $f(X) = \mathrm{Tr}(X^{-1})$ on $\mathrm{dom} f = S^n_{++}$

$$g(t) = \mathrm{Tr}\left((X + tV)^{-1}\right) = \mathrm{Tr}\left(\left(X\left(Id + tX^{-1/2} V X^{-1/2}\right)\right)^{-1}\right)$$

$$= \mathrm{Tr}\left(X^{-1} \cdot \underbrace{(Id + tD)^{-1}}_{\text{diagonalizable}}\right)$$ where D is the diagonal matrix with containing the eigenvalues of $X^{-1/2} V X^{-1/2}$

$$= \sum_{i=1}^{n} X_{ii}^{-1} \cdot \frac{1}{\underbrace{1+\lambda_i t}_{\text{convex on } t}}$$

Since $g(t)$ is a linear combination of convex functions then $f(X)$ is convex.

2) $f(X, y) = y^T X^{-1} y$ on $\mathrm{dom} f = S^n_{++} \times \mathbb{R}^n$.

$$= 2 \sup_x \left(y^T x - \frac{1}{2} x^T X x \right)$$

Since suprimum and linear combinations preserve convexity then f is convex.

3) $f(X) = \sum_{i=1}^{n} \sigma_i(X)$ on $\mathrm{dom} f = S^n$. Since $X \in S^n$ then the singular values are equal to the eigenvalues.

$f(X) = \sum_{i=1}^{n} \sigma_i(X) = \sum_{i=1}^{n} \lambda_i = \mathrm{Tr}(X)$. $\mathrm{Tr}(X)$ is convex hence $f(X)$ is convex.