

L_p spaces

• $\|f\|_2 := \sqrt{\int_0^\infty |f(x)|^2 dx}$ L_2 norm

$$\|p_1(x) - p_2(x)\|_2 = \sqrt{\int_0^\infty |p_1(x) - p_2(x)|^2 dx}$$

$$\|p_1(x) - p_2(x)\|_1 = \int_0^\infty |p_1(x) - p_2(x)| dx$$

$$\|p_1(x)\|_1 = \int_0^\infty |p_1(x)| dx = \int_0^\infty p_1(x) dx = 1$$

$$\|p_1(x) - p_2(x)\| \leq \|p_1(x)\| + \|p_2(x)\|$$
$$= 1 + 1 = 2$$

• Kullback-Leibler divergence

$$D_{p||q} := \sum_{n=0}^{\infty} p(n) \log \left[\frac{p(n)}{q(n)} \right]$$

$p(n)$: true probability dist.

$q(n)$: approx prob. dist.

Interpretation:

'amount of information lost' by using approx. prob. dist.

Ex: Birth-death process

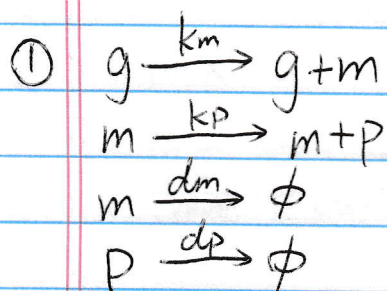
$$g \xrightarrow{km} m+g$$
$$m \xrightarrow{dm} \phi$$

$$p(m) = \frac{\mu^m e^{-\mu}}{m!}$$

$$q(m) \cong \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(m-\mu)^2}{2\mu}}$$

$$D_{p||q} = \sum_{m=0}^{\infty} \frac{\mu^m e^{-\mu}}{m!} \log \left[\frac{\mu^m e^{-\mu}}{m!} \sqrt{2\pi\mu} e^{\frac{(m-\mu)^2}{2\mu}} \right]$$

As μ increases, expect $D_{p||q}$ to decrease



② SDEs:

$$\begin{aligned}
 \dot{m} &= [k_m g - d_m m] + \sqrt{k_m g + d_m m} \Gamma_m(t) \\
 \dot{p} &= [k_p m - d_p p] + \sqrt{k_p m + d_p p} \Gamma_p(t)
 \end{aligned}$$

③ Assume mRNA is at QSS:

$$0 = \dot{m} = k_m g - d_m m = 0$$

$$\Rightarrow m = \frac{k_m g}{d_m} = \text{constant}$$

④ New SDE:

$$\dot{p} = \left[k_p \frac{k_m g}{d_m} - d_p p \right] + \sqrt{k_p \frac{k_m g}{d_m} + d_p p} \Gamma_p(t)$$

⑤ Approximate p as being not too far from equilibrium:

$$p = \underbrace{\bar{p}}_{\text{mean}} + \underbrace{\Delta p}_{\text{deviation}} \quad \text{with } \frac{\Delta p}{\bar{p}} \ll 1$$

The mean/ss is where $\dot{p} = 0$,

$$\begin{aligned}
 \text{i.e.} \quad 0 &= \dot{p} = k_p \frac{k_m g}{d_m} - d_p \bar{p} \\
 \Rightarrow p_{ss} = \bar{p} &= \frac{k_p k_m g}{d_p d_m}
 \end{aligned}$$

⑥ Use Taylor series:

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$$

$$\begin{aligned} \sqrt{\frac{k_p k_{mg}}{d_m} + d_p P} &= \sqrt{d_p} \sqrt{\frac{k_p k_{mg}}{d_p d_m} + P} \\ &= \sqrt{d_p} \sqrt{P_{ss} + P} \\ &= \sqrt{d_p} \sqrt{P_{ss} + P_{ss} + \Delta P} \\ &= \sqrt{d_p} \sqrt{2P_{ss} + \Delta P} \\ &= \sqrt{2d_p P_{ss}} \sqrt{1 + \frac{\Delta P}{2P_{ss}}} \\ &\approx \sqrt{2d_p P_{ss}} \left[1 + \frac{1}{2} \left(\frac{\Delta P}{2P_{ss}} \right) - \frac{1}{8} \left(\frac{\Delta P}{2P_{ss}} \right)^2 \right] \\ &= \sqrt{2d_p P_{ss}} \left[1 + \frac{1}{4P_{ss}} \Delta P - \frac{1}{32P_{ss}^2} \Delta P^2 \right] \\ &= \sqrt{2d_p P_{ss}} \left[1 + \frac{1}{4P_{ss}} (P - P_{ss}) - \frac{1}{32P_{ss}^2} (P - P_{ss})^2 \right] \\ &\approx \sqrt{2d_p P_{ss}} \left[1 + \frac{P}{4P_{ss}} - \frac{1}{4} \right] \\ &= \frac{3}{4} \sqrt{2d_p P_{ss}} + \frac{\sqrt{2d_p P_{ss}}}{4P_{ss}} P \end{aligned}$$

If HAD to write:

$$\text{noise} \approx \sigma_a$$

$$\sigma_a \approx \frac{3}{4} \sqrt{2d_p P_{ss}} + \frac{\sqrt{2d_p P_{ss}}}{4} = \sqrt{2d_p P_{ss}}$$

$$\text{noise} \approx \sigma_m P$$

$$\text{noise} \approx \frac{3}{4} \frac{\sqrt{2d_p P_{ss}}}{P_{ss}} P + \frac{\sqrt{2d_p P_{ss}}}{4P_{ss}} P$$

$$\sigma_m \approx \frac{\sqrt{2d_p P_{ss}}}{P_{ss}}$$