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For a single gene with Nth order self-regulation, the Gillespie noise prescription yields the SDE

$$\begin{aligned}\dot{p} =& \frac{k_p G}{d_m} \ \frac{k_0 + k_1 c_1 p + k_2 c_2 p^2 + \cdots + k_N c_N p^N}{1 + c_1 p + c_2 p^2 + \cdots + c_N p^N} - d_p p \\ & + \sqrt{\frac{k_p G}{d_m} \ \frac{k_0 + k_1 c_1 p + k_2 c_2 p^2 + \cdots + k_N c_N p^N}{1 + c_1 p + c_2 p^2 + \cdots + c_N p^N} + d_p p + 2G \ \frac{b_{10} c_1 p + b_{21} c_2 p^2 + \cdots + b_{N,N-1} c_N p^N}{1 + c_1 p + c_2 p^2 + \cdots + c_N p^N}} \ \eta(t) \ .\end{aligned}$$

1 Exact 1D steady state Fokker-Planck solution

For Langevin dynamics that proceed according to

$$\dot{x} = f(x) + g(x)\eta(t) \ ,$$

the 1D steady state Fokker-Planck equation for $P_{ss}(x)$ reads

$$0 = -\frac{d}{dx}\left[f(x) P_{ss}(x)\right] + \frac{1}{2}\frac{d^2}{dx^2}\left[g(x)^2 P_{ss}(x)\right] \ .$$

Integrating, we have

$$C = f(x) P_{ss}(x) - \frac{1}{2}\frac{d}{dx}\left[g(x)^2 P_{ss}(x)\right]$$

for some constant C. But we need $P_{ss}(x)$ and its derivative to go to zero as x goes to infinity; hence, we can see that the right hand side must approach zero for large x, which means that C must be zero.

Now we have

$$f(x) P_{ss}(x) - g(x) g'(x) P_{ss}(x) - \frac{1}{2} g(x)^2 P_{ss}'(x) = 0 \ ,$$

or equivalently,

$$P_{ss}'(x) + \left[\frac{g(x)g'(x) - f(x)}{g(x)^2/2}\right] P_{ss}(x) = 0 \ .$$

Define the function

$$\boxed{V(x) := \int \frac{g(x)g'(x) - f(x)}{g(x)^2/2} \ dx} \ .$$

Now our equation is

$$P'_{ss}(x) + V'(x) P_{ss}(x) = 0 \ ,$$

and its solution is clearly

$$\boxed{P_{ss}(x) = N e^{-V(x)} = \frac{e^{-V(x)}}{\int_0^\infty e^{-V(x)} \ dx}}$$

where we are using N as shorthand for the normalization constant, and we normalize over $[0, \infty)$ since protein concentrations can only be nonnegative.

2 Additive noise results

We will solve the specific cases $N = 0, 1, 2$. Writing a general result depends on being able to find a nice expression for the integral of $f(p)$, which I wasn't able to find in other cases.

First, note that if $g(x) = \sigma$ for some $\sigma > 0$, our definition of $V(x)$ reduces to

$$V(x) := -\frac{1}{\sigma^2/2} \int f(x) \ dx \ .$$

2.1 Single gene with no regulation and additive noise

Our Langevin equation reads

$$\dot{p} = \frac{k_p G k_0}{d_m} - d_p p + \sigma \ \eta(t) \ ,$$

so

$$f(p) = \frac{k_p G k_0}{d_m} - d_p p \ .$$

Then

$$V_0(x) = -\frac{1}{\sigma^2/2} \int \frac{k_p G k_0}{d_m} - d_p p \ dp = \frac{1}{\sigma^2/2} \left[-\frac{k_p G k_0}{d_m} p + \frac{d_p}{2} p^2 \right] \ ,$$

so

$$\boxed{P^0_{ss}(x) = N \exp \left[-\frac{\left(\frac{d_p}{2} p^2 - \frac{k_p G k_0}{d_m} p \right)}{\sigma^2/2} \right]} \ .$$

2.2 Single gene with first order feedback and additive noise

Our Langevin equation reads

$$\dot{p} = \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p}{1 + c_1 p} - d_p p + \sigma\ \eta(t)\ ,$$

so

$$f(p) = \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p}{1 + c_1 p} - d_p p\ .$$

Then

$$\begin{aligned} V_1(x) &= -\frac{1}{\sigma^2/2}\int \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p}{1 + c_1 p} - d_p p\ dp \\ &= \frac{1}{\sigma^2/2}\left[\frac{d_p}{2}p^2 - \frac{k_p G}{d_m}\frac{k_1(1+c_1 p)}{c_1} - \frac{k_p G}{d_m}\frac{(k_0-k_1)\log(1+c_1 p)}{c_1}\right]\ , \end{aligned}$$

so

$$\boxed{P_{ss}^1(x) = N\exp\left[-\frac{\left(\frac{d_p}{2}p^2 - \frac{k_p G}{d_m}\frac{k_1(1+c_1 p)}{c_1} - \frac{k_p G}{d_m}\frac{(k_0-k_1)\log(1+c_1 p)}{c_1}\right)}{\sigma^2/2}\right]}\ .$$

2.3 Single gene with second order feedback and additive noise

Our Langevin equation reads

$$\dot{p} = \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p + k_2 c_2 p^2}{1 + c_1 p + c_2 p^2} - d_p p + \sigma\ \eta(t)\ ,$$

so

$$f(p) = \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p + k_2 c_2 p^2}{1 + c_1 p + c_2 p^2} - d_p p\ .$$

Then

$$\begin{aligned} V_2(x) =& -\frac{1}{\sigma^2/2}\int \frac{k_p G}{d_m}\ \frac{k_0 + k_1 c_1 p + k_2 c_2 p^2}{1 + c_1 p + c_2 p^2} - d_p p\ dp \\ =& \frac{d_p}{2}p^2 - \frac{k_p G}{d_m}k_2 p - \frac{k_p G}{d_m}\frac{\left[c_1^2(k_2-k_1) + 2c_2(k_0-k_2)\right]\tan^{-1}\left[\frac{c_1+2c_2 p}{\sqrt{4c_2-c_1^2}}\right]}{c_2\sqrt{4c_2-c_1^2}} \\ &- \frac{k_p G}{d_m}\frac{c_1(k_1-k_2)\log(1+c_1 p+c_2 p^2)}{2c_2}\ , \end{aligned}$$

so

$$\boxed{\frac{P_{ss}^2(x)}{N} = e^{-\frac{\left(\frac{d_p}{2}p^2 - \frac{k_p G}{d_m}k_2 p - \frac{k_p G}{d_m}\frac{\left[c_1^2(k_2-k_1)+2c_2(k_0-k_2)\right]\tan^{-1}\left[\frac{c_1+2c_2 p}{\sqrt{4c_2-c_1^2}}\right]}{c_2\sqrt{4c_2-c_1^2}} - \frac{k_p G}{d_m}\frac{c_1(k_1-k_2)\log(1+c_1 p+c_2 p^2)}{2c_2}\right)}{\sigma^2/2}}}\ .$$