Stochastic Gene Expression Project: Homework 1
Due January 18th at 2:15 PM

Problem 1: Solving the 1D steady state Fokker-Planck equation.

In the continuous limit, the probability distribution $P(x,t)$ of our stochastic system satisfies the Fokker-Planck equation. The Fokker-Planck equation corresponding to the one-dimensional SDE

$$\dot{x} = f(x) + g(x)\eta(t) \tag{1}$$

is

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[f(x)P(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[g(x)^2 P(x,t)\right] \ . \tag{2}$$

In this problem, you will work out how to solve for the *steady state* probability distribution $P_{ss}(x)$ in one dimension.

(a) Note that, if we wanted to solve for $P(x,t)$, then we have to solve a two variable PDE. If we just want to know $P_{ss}(x) := \lim_{t\to\infty} P(x,t)$, then we can remove the time variable, making the equation significantly easier to solve.

At steady state, the probability distribution $P(x,t)$ shouldn't change, because we have waited long enough for the system to 'settle down'. Given this, what can we say about $\frac{\partial P(x,t)}{\partial t}$ in the limit as $t \to \infty$?

(b) Use the condition from part (a) to write an equation for $P_{ss}(x)$ with only one variable.

(c) The equation you wrote for part (b) should be a second-order (in $P_{ss}(x)$) ODE of the form

$$0 = \frac{d}{dx}\left[J(x)\right]$$

for some function $J(x)$. Integrate both sides to get a first-order ODE.

(d) To get biologically or physically realistic solutions, we want $\lim_{x\to\infty} P_{ss}(x) = 0$ and $\lim_{x\to\infty} P'_{ss}(x) = 0$; in other words, P_{ss} and its derivative must vanish at infinity, or else there is some finite probability that our system has infinitely many particles in it. Given this fact, what is the value of $J(\infty)$? Use this to simplify your equation from part (c).

(e) Now write your first-order ODE in the form

$$P'_{ss}(x) + F(x)P_{ss}(x) = 0 \ .$$

What is the function $F(x)$ in terms of $f(x)$, $g(x)$, and $g'(x)$?

(f) Show that the previous ODE is solved by

$$P_{ss}(x) = N \exp(-V(x)) \ ,$$

where

$$V(x) := \int_0^x F(x) dx$$

and N is an arbitrary constant that we can treat as a normalization constant. In other words, since N is arbitrary, we can choose the value of N so that

$$\int_0^\infty P_{ss}(x) \ dx = 1 \ ,$$

as must be true for a probability density function.

All of this means that we can numerically find $P_{ss}(x)$ for a 1D system by doing two things: (i) doing a numerical integral to calculate $V(x)$, and then (ii) choosing a constant N to normalize our solution. This is much faster than the Monte Carlo simulation method, but our ability to construct a solution in this way seems to be a special consequence of our system being one-dimensional.

Problem 2: The Euler-Maruyama time step in two dimensions.

Look at the file `euler_maruyama.py`, which is in the Github folder labeled `week_jan_14`. Given the 1D Euler-Maruyama time step, finish the code for the 2D Euler-Maruyama time step in that file.