L_p spaces

- $\|f\|_2 := \sqrt{\int_0^\infty |f(x)|^2 dx}$ $\quad L_2$ norm

$$\|P_1(x) - P_2(x)\|_2 = \sqrt{\int_0^\infty |P_1(x) - P_2(x)|^2 dx}$$

$$\|P_1(x) - P_2(x)\|_1 = \int_0^\infty |P_1(x) - P_2(x)|\, dx$$

$$\|P_1(x)\|_1 = \int_0^\infty |P_1(x)|\, dx = \int_0^\infty P_1(x)\, dx = 1$$

$$\|P_1(x) - P_2(x)\| \leq \|P_1(x)\| + \|P_2(x)\|$$

$$= 1 + 1 = 2$$

- Kullback-Leibler divergence

$$D_{p\|q} := \sum_{n=0}^{\infty} p(n) \log\left[\frac{p(n)}{q(n)}\right]$$

$p(n)$: true probability dist.

$q(n)$: approx prob. dist.

Interpretation: 'amount of infomation lost' by using approx. prob. dist.

Ex: Birth-death process

$$g \xrightarrow{km} m + g$$

$$m \xrightarrow{dm} \phi$$

$$p(m) = \frac{\mu^m e^{-\mu}}{m!}$$

$$q(m) \cong \frac{1}{\sqrt{2\pi\mu}} e^{-\frac{(m-\mu)^2}{2\mu}}$$

$$D_{p\|q} = \sum_{m=0}^{\infty} \frac{\mu^m e^{-\mu}}{m!} \log\left[\frac{\mu^m e^{-\mu}}{m!} \sqrt{2\pi\mu}\, e^{\frac{(m-\mu)^2}{2\mu}}\right]$$

As μ increases, expect $D_{p\|q}$ to <u>decrease</u>

① $g \xrightarrow{k_m} g+m$

$m \xrightarrow{k_p} m+p$

$m \xrightarrow{d_m} \phi$

$p \xrightarrow{d_p} \phi$

② SDEs:

$$\dot{m} = [k_m g - d_m m] + \sqrt{k_m g + d_m m}\,\eta_m(t)$$

$$\dot{p} = [k_p m - d_p p] + \sqrt{k_p m + d_p p}\,\eta_p(t)$$

③ Assume mRNA is at QSS:

$$0 = \dot{m} = k_m g - d_m m = 0$$

$$\Rightarrow m = \frac{k_m g}{d_m} = \text{constant}$$

④ New SDE:

$$\dot{p} = [k_p \frac{k_m g}{d_m} - d_p p] + \sqrt{k_p \frac{k_m g}{d_m} + d_p p}\,\eta_p(t)$$

⑤ Approximate p as being not too far from equilibrium:

$$p = \underset{\text{mean}}{\bar{p}} + \underset{\text{deviation}}{\Delta p} \qquad \text{with } \frac{\Delta p}{\bar{p}} \ll 1$$

The mean/ss is where $\dot{p} = 0$,

i.e. $$0 = \dot{p} = k_p \frac{k_m g}{d_m} - d_p p$$

$$\Rightarrow p_{ss} = \bar{p} = \frac{k_p k_m g}{d_p d_m}$$

⑥ Use Taylor series:

$$\sqrt{1+x} \approx 1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

$$\sqrt{\frac{k_p k_m g}{d_m}+d_p P} = \sqrt{d_p}\sqrt{\frac{k_p k_m g}{d_p d_m}+P}$$

$$= \sqrt{d_p}\sqrt{P_{ss}+P}$$

$$= \sqrt{d_p}\sqrt{P_{ss}+P_{ss}+\Delta P}$$

$$= \sqrt{d_p}\sqrt{2P_{ss}+\Delta P}$$

$$= \sqrt{2d_p P_{ss}}\sqrt{1+\frac{\Delta P}{2P_{ss}}}$$

$$\approx \sqrt{2d_p P_{ss}}\left[1+\frac{1}{2}\left(\frac{\Delta P}{2P_{ss}}\right)-\frac{1}{8}\left(\frac{\Delta P}{2P_{ss}}\right)^2\right]$$

$$= \sqrt{2d_p P_{ss}}\left[1+\frac{1}{4P_{ss}}\cdot\Delta P-\frac{1}{32P_{ss}^2}\Delta P^2\right]$$

$$= \sqrt{2d_p P_{ss}}\left[1+\frac{1}{4P_{ss}}(P-P_{ss})-\frac{1}{32P_{ss}^2}(P-P_{ss})^2\right]$$

$$\approx \sqrt{2d_p P_{ss}}\left[1+\frac{P}{4P_{ss}}-\frac{1}{4}\right]$$

$$= \frac{3}{4}\sqrt{2d_p P_{ss}}+\frac{\sqrt{2d_p P_{ss}}}{4P_{ss}}P$$

If HAD to write:

noise $\approx \sigma_a$

$$\sigma_a \approx \frac{3}{4}\sqrt{2d_p P_{ss}}+\frac{\sqrt{2d_p P_{ss}}}{4} = \sqrt{2d_p P_{ss}}$$

noise $\approx \sigma_m P$

$$\text{noise} \approx \frac{3}{4}\frac{\sqrt{2d_p P_{ss}}}{P_{ss}}P+\frac{\sqrt{2d_p P_{ss}}}{4P_{ss}}P$$

$$\sigma_m \approx \frac{\sqrt{2d_p P_{ss}}}{P_{ss}}$$