Division by 2

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Outline

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Binary System

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- Consider division of integers by 2

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- If the remainder of a is 1, a is not divisible by 2
- These are odd numbers

Sums of Even and Odd Numbers

Splitting in pairs

Suppose there are two classes with a and b students respectively. We unite the classes and would like to split all students into pairs to work on a project. Is it possible if a is even and b is odd? What if both a and b are even? What if both a and b are odd?

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- If a is even and b is odd we can split in pairs all students in the first class and all students except one in the second class
- So there is one student left and the answer is no

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- Then we can pair two remaining students

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- So the sum is $2 \times (q_1 + q_2)$ and is even

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Suppose we know the remainders of a and b when divided by 2. Can we deduce the remainder of $a + b$?

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- So we have the following table for the remainders of $a + b$ when dividing by 2

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- So we have the following table for the remainders of $a + b$ when dividing by 2
- Rows are remainders of a and columns are remainders of b

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Suppose we know the remainder of a when divided by 2.
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- It is the same
- Indeed, a is divisible by 2 iff $-a$ is divisible by 2

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- So, for subtraction we have the same table

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- So the product $a \times b$ is divisible by 2 and so is even

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Suppose we know the remainders of a and b when divided by 2. Can we deduce the remainder of $a \times b$?

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- If both a and b are odd they have the form $2 \times q_1 + 1$ and $2 \times q_2 + 1$
- So the product is $(q_1 \times 2 + 1) \times (q_2 \times 2 + 1)$ $= (q_1 \times q_2 \times 2 + q_1 + q_2) \times 2 + 1$ and is odd

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Operations on Remainders

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What is the remainder of $374 \times (419 + 267 \times 38) - 625$ when divided by 2?

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  $0 \times (1 + 1 \times 0) - 1$

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- So the expression in the problem has remainder 1

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- In general, digits are multiplied by powers of 10
- Each non-negative integer number has a unique representation

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- Binary system is important since computers use it

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- This number is equal to $1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = (2^3 + 2^2 + 2^1) \times 2 + 1$

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- So the remainder is 1 and the quotient is 1110 in binary

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- However, the proof is not hard; one can just combine our previous Lemma and induction
- This is also a way to construct binary representation for a given number: apply previous lemma recursively

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- Observe that to represent 2^n we need $n+1$ bits: 2^4 is 10000
- And 2^n is the smallest number that requires $n+1$ bits: $2^4 - 1 = 2^3 + 2^2 + 2^1 + 1$ is 1111
- The number of bits that is needed to represent a is determined by the largest n such that $2^n \leq a$: it is equal to $n+1$

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- In particular, $\log_2 2^n = n$
- Binary logarithm has a clear meaning in terms of binary representation

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The number of bits needed to represent an integer a in binary representation is equal to $\lfloor \log_2 a \rfloor + 1$

- By $\lfloor x \rfloor$ we denote the largest integer that is less or equal to (possibly non-integer) x

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- Why the lemma is true?
- Recall that the number of bits in binary representation of a is $n + 1$ where n is the largest number such that $2^n \leq a$
- Applying logarithm to both sides of inequality we obtain that n is the largest number such that $n \leq \log_2 a$