Stable matchings

Alexander Shen

LIRMM / CNRS, University of Montpellier. France

Outline

University Entrance in Russia

Old system:

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- many applications
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- graph theory: “perfect matching”
- minimal requirement: stability

Unstable Matching

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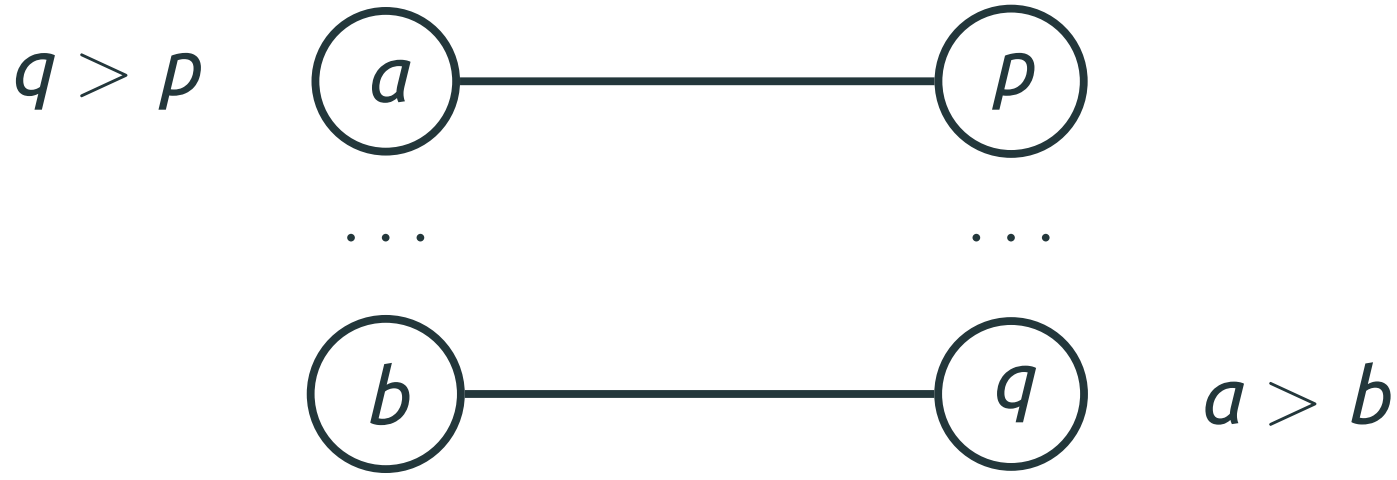

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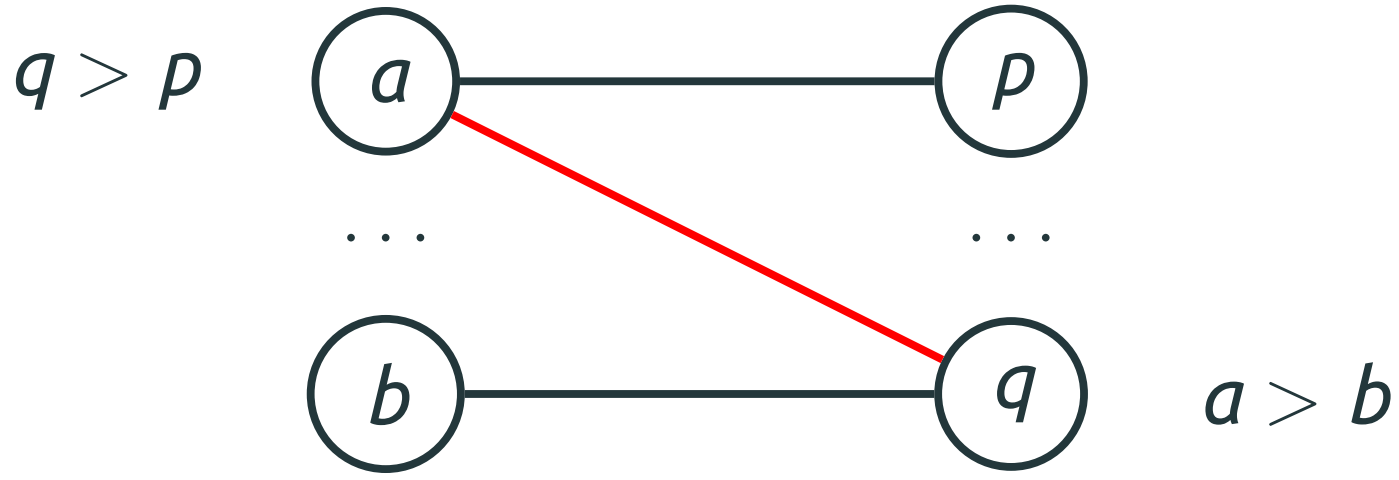

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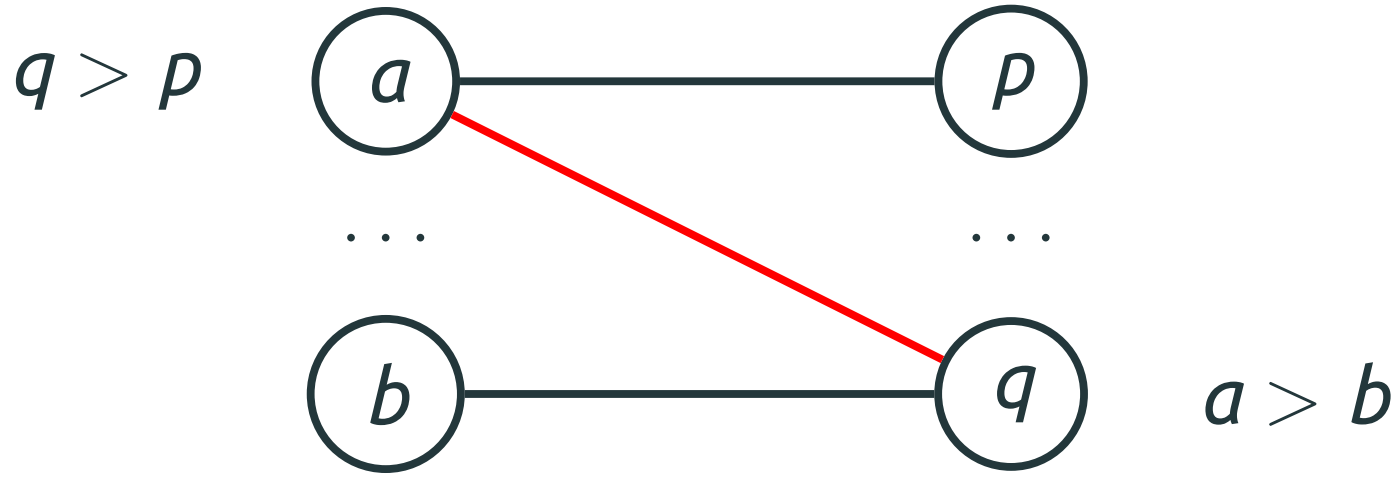

Is there a stable matching?

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- “Roth recognized that Shapley’s theoretical results could clarify the functioning of important markets in practice… helped redesign existing institutions for matching new doctors with hospitals, students with schools, and organ donors with patients.”

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- American Mathematical Monthly
  `https://www.jstor.org/stable/2312726`

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- unique stable marriage: why?

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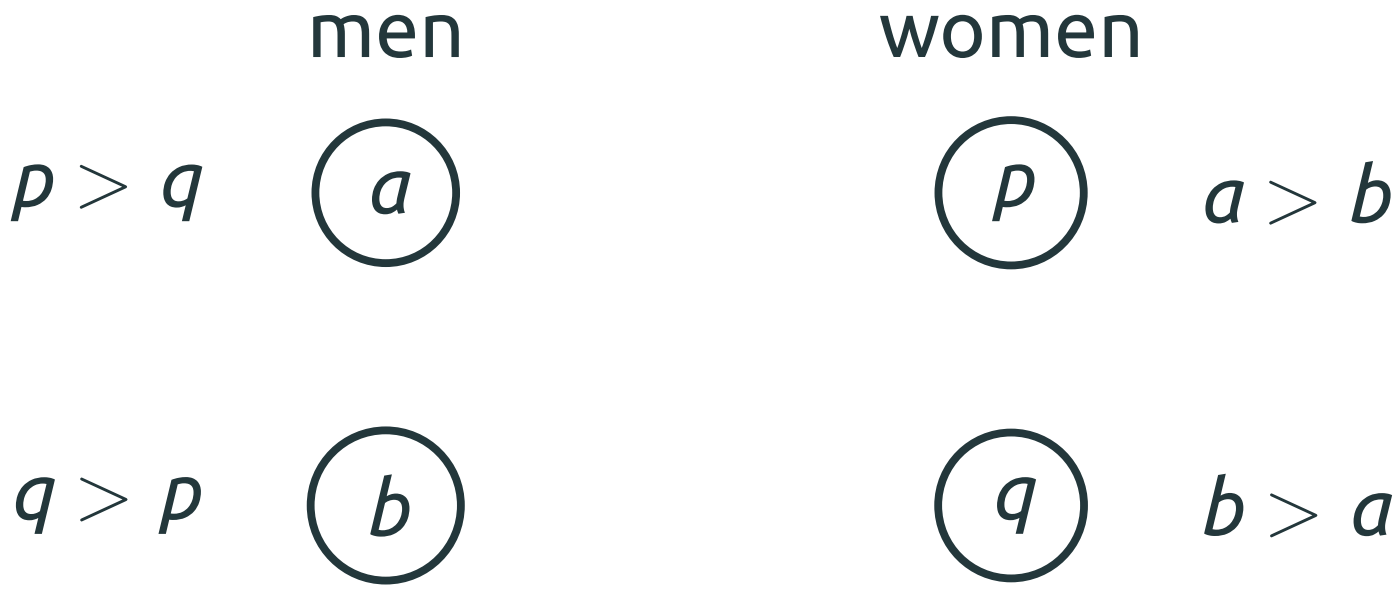
- induction
- recursive algorithm
- symmetry

$n = 2$: **Everyone Is Happy**



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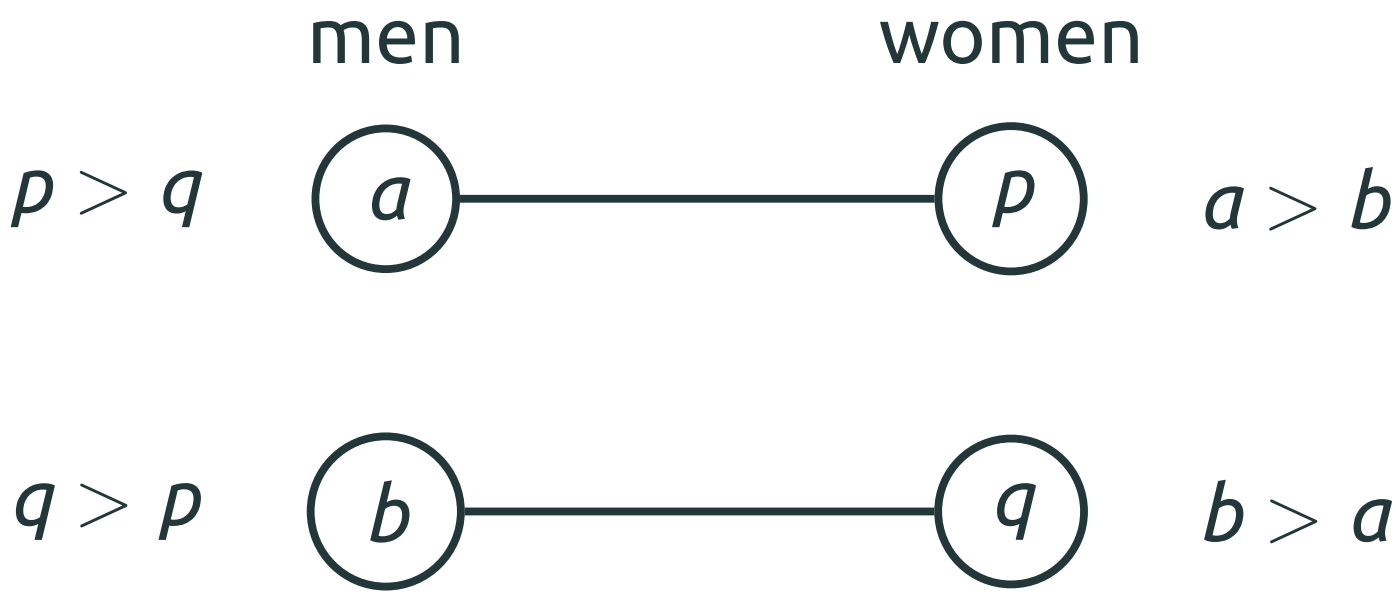
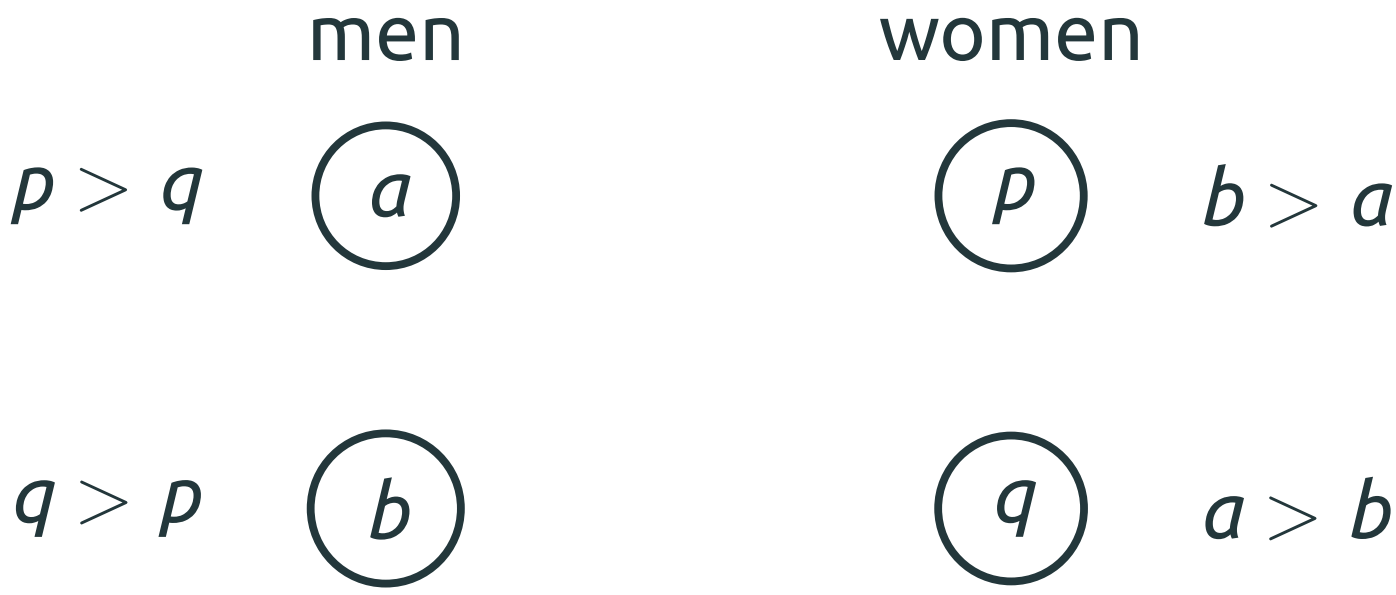


$n = 2$: Two Stable Matchings



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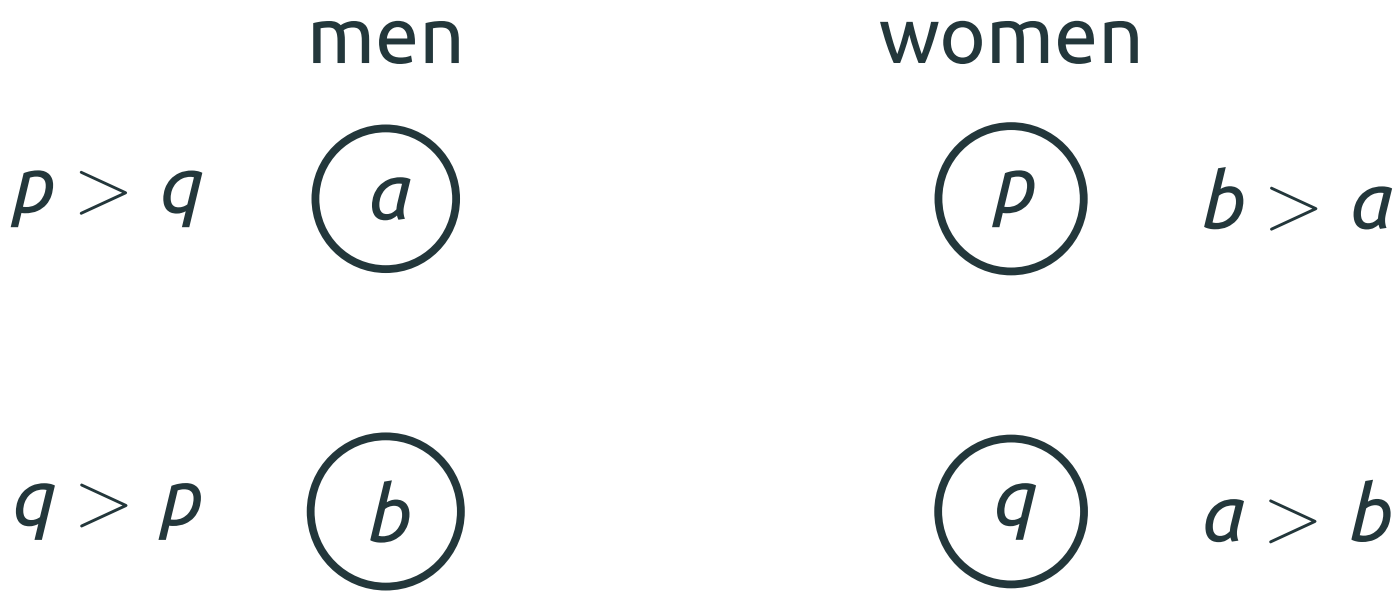

two bad but stable matchings

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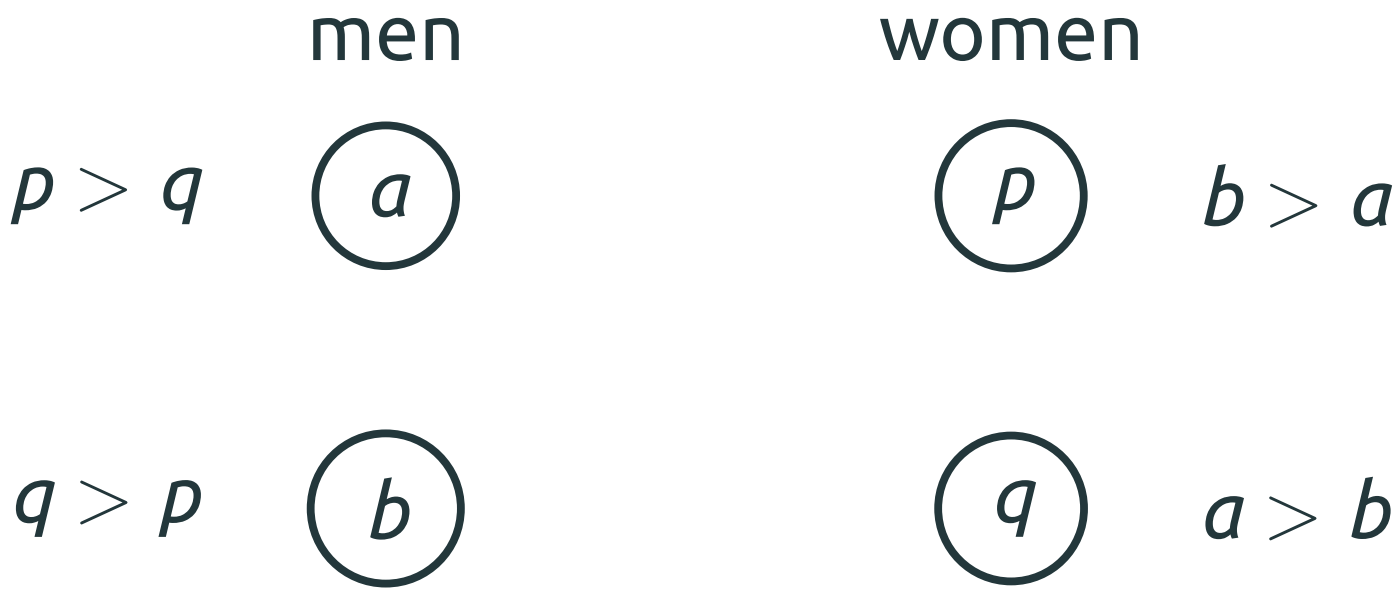

two bad but stable matchings

H. Heine / R. Schumann (“Dichterliebe”)

`https://www.youtube.com/watch?v=UzfWyCLmHLc`

Outline

Laissez-Faire…

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- “laissez-faire” approach:

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- repeat until a stable matching is obtained

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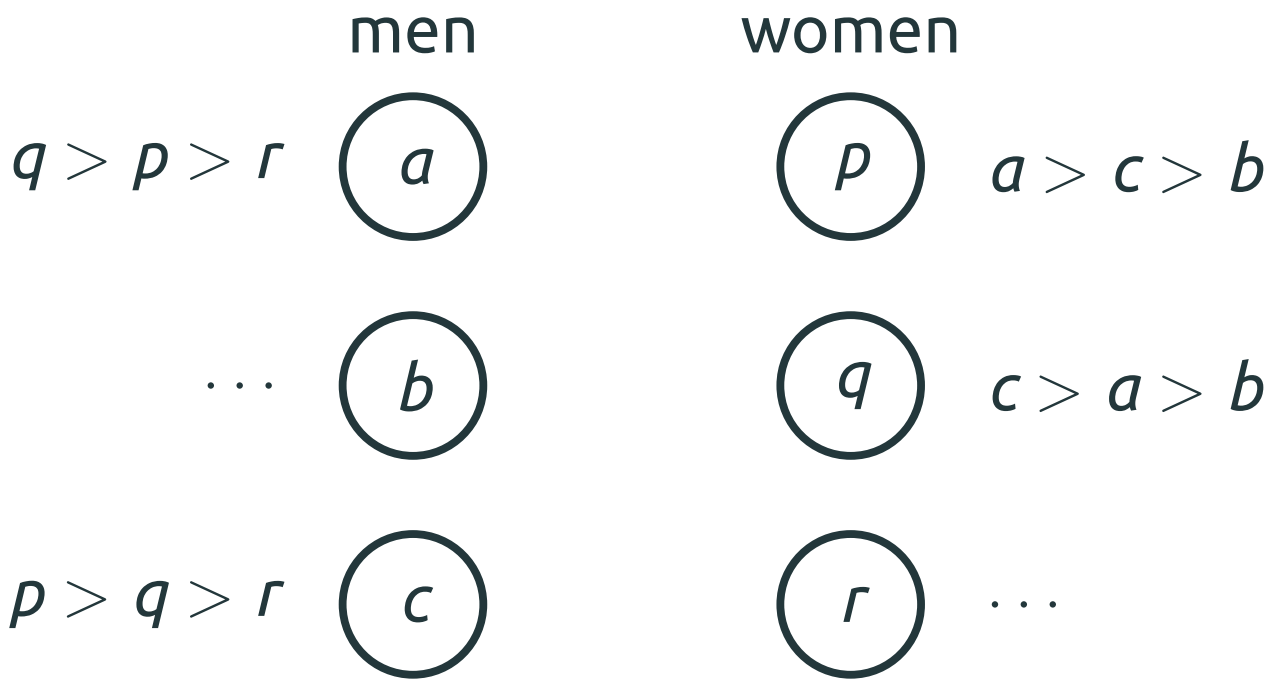

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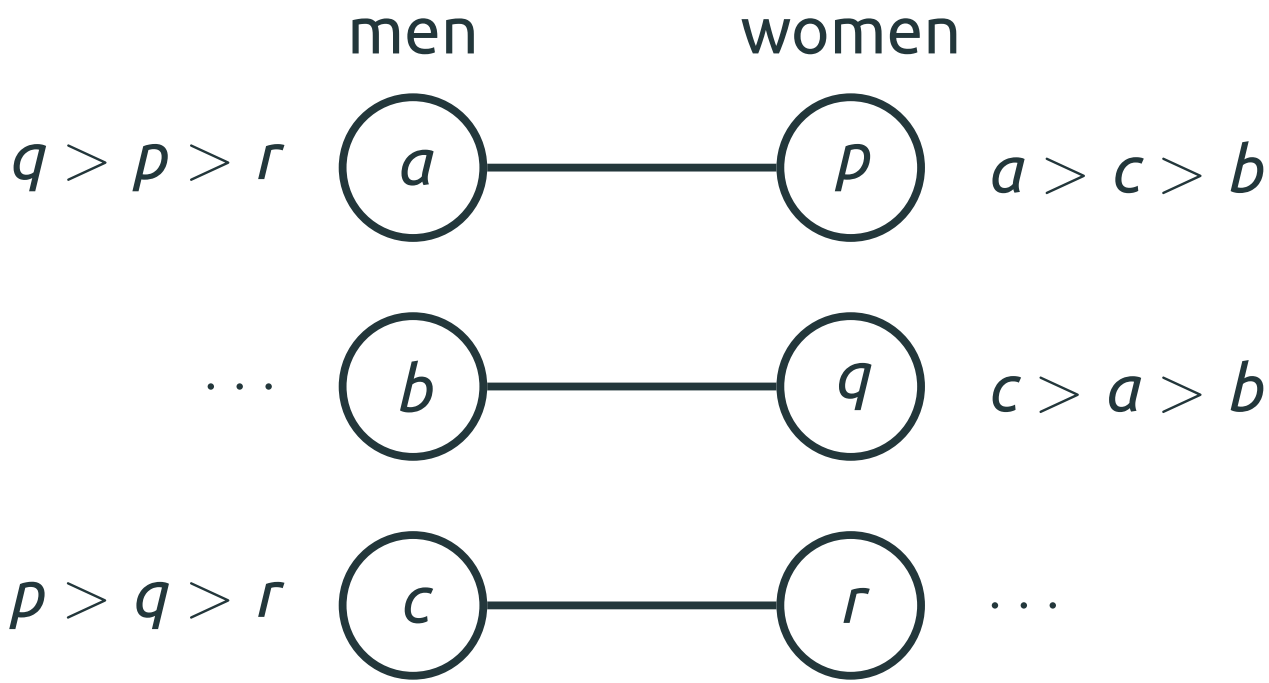

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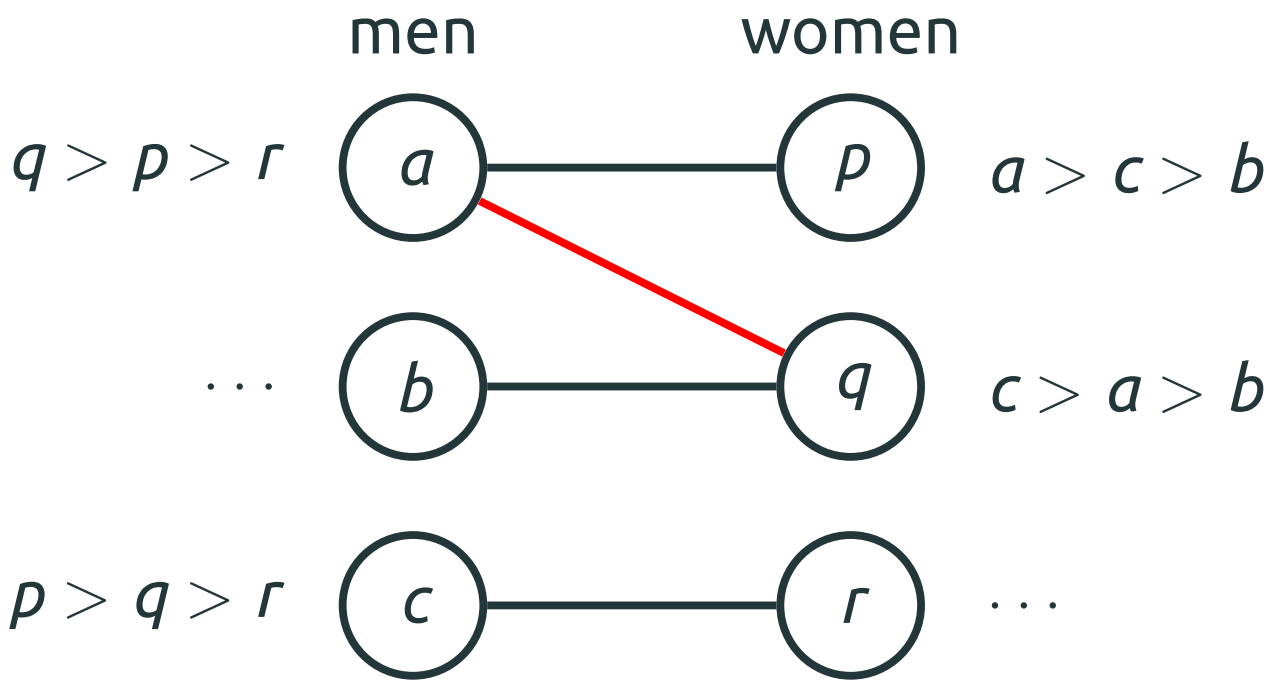

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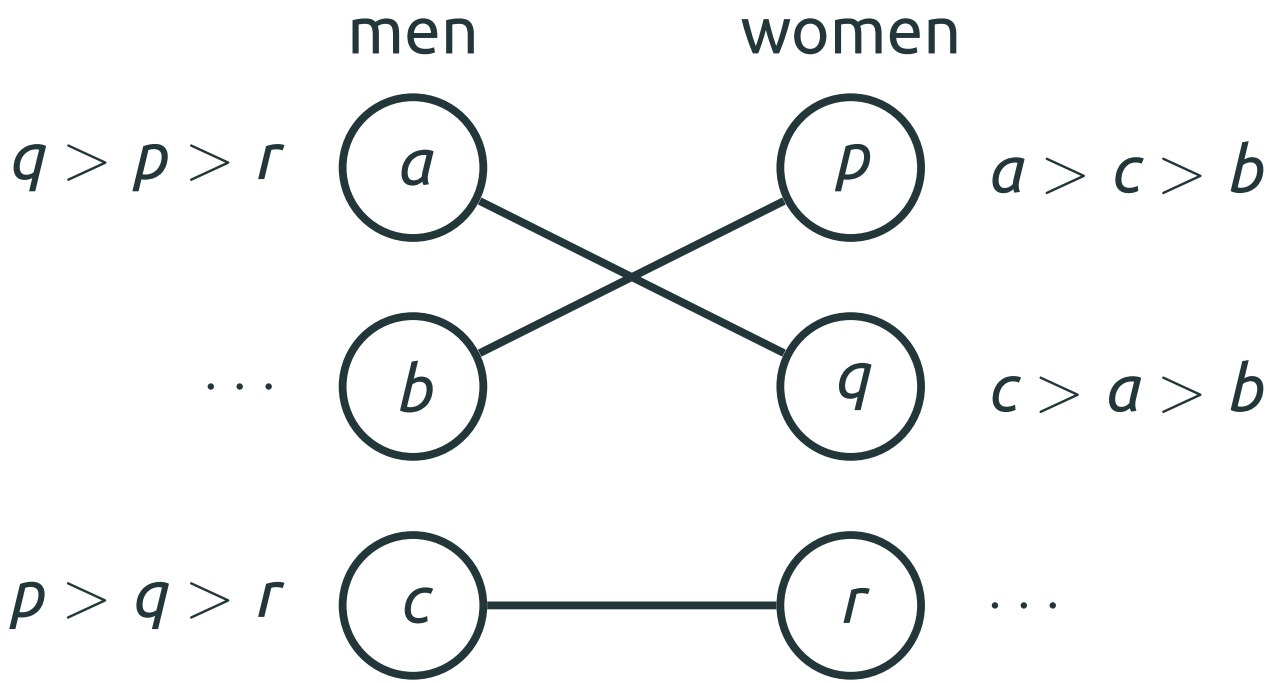

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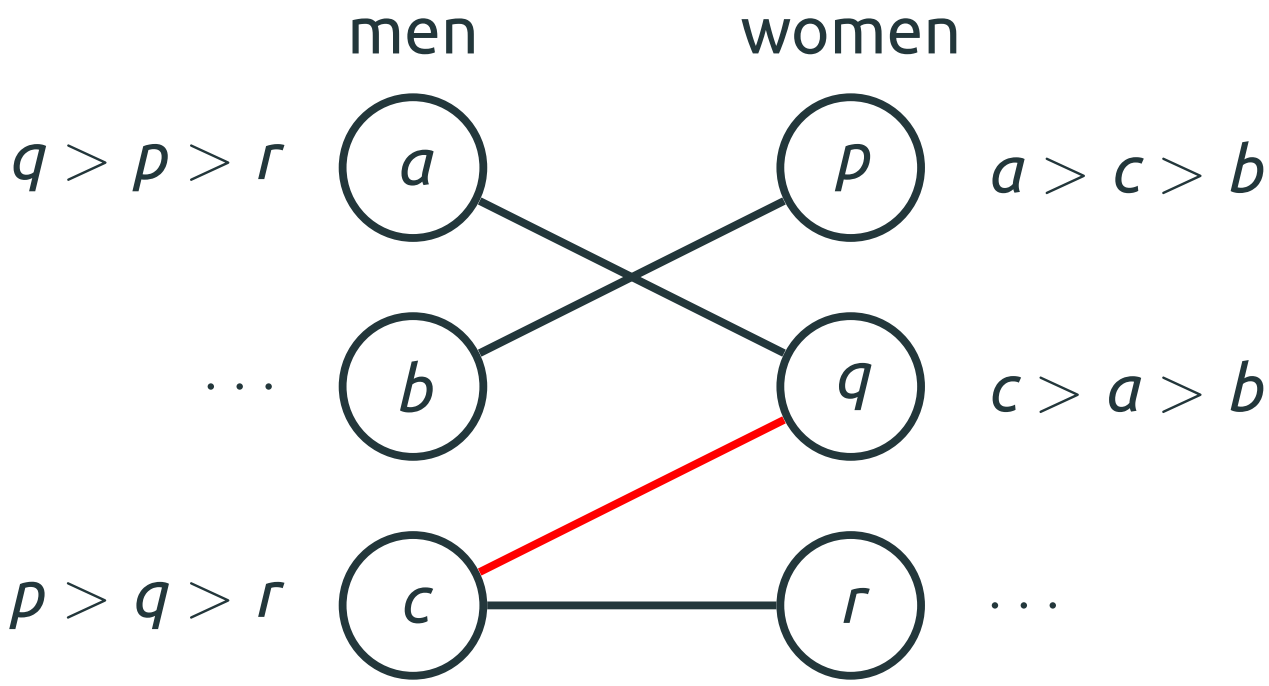

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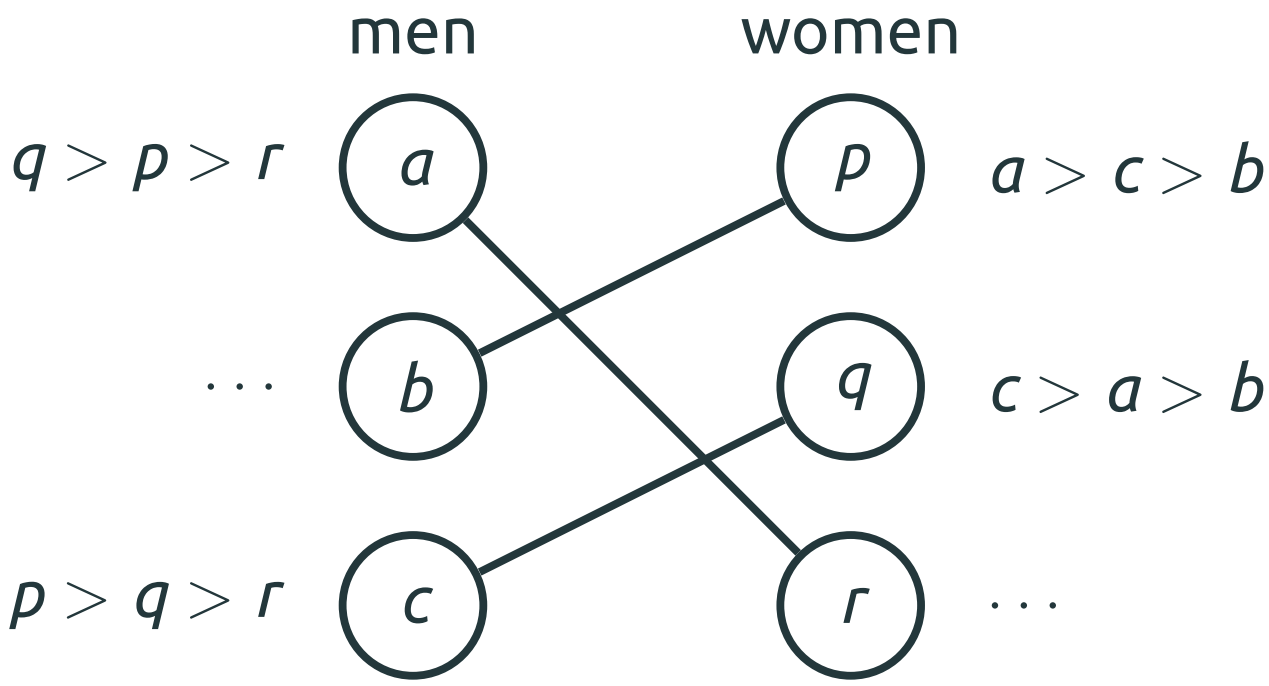

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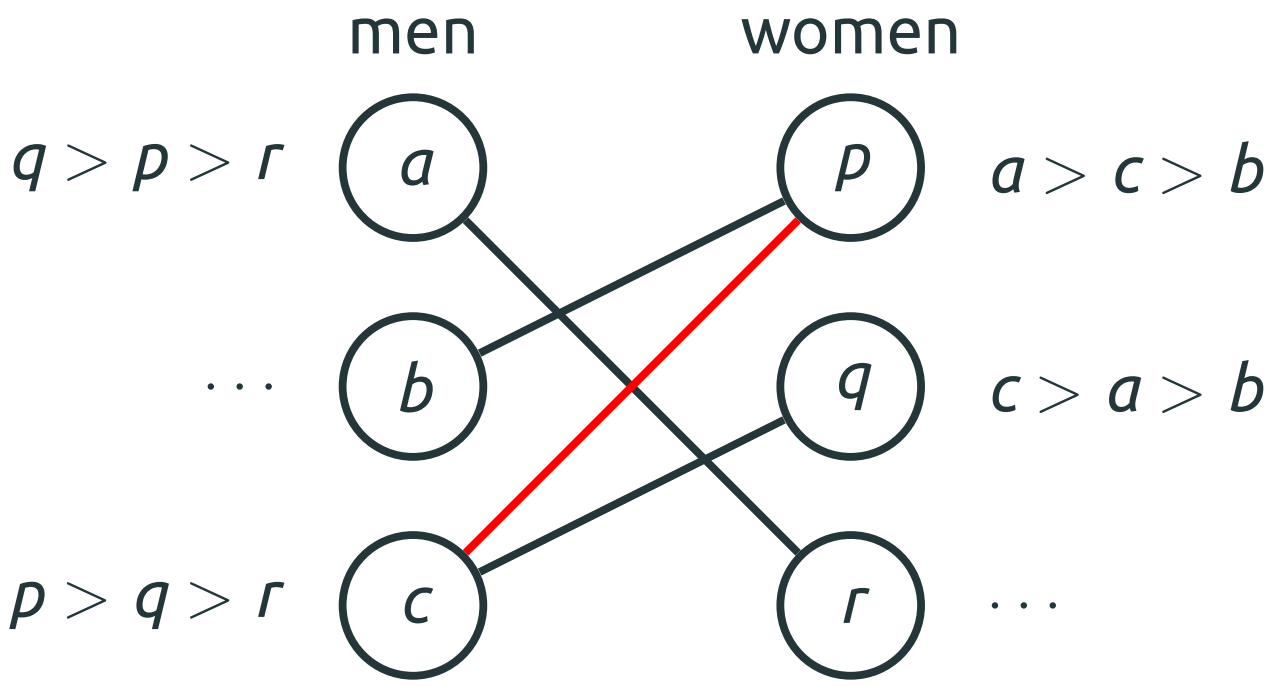

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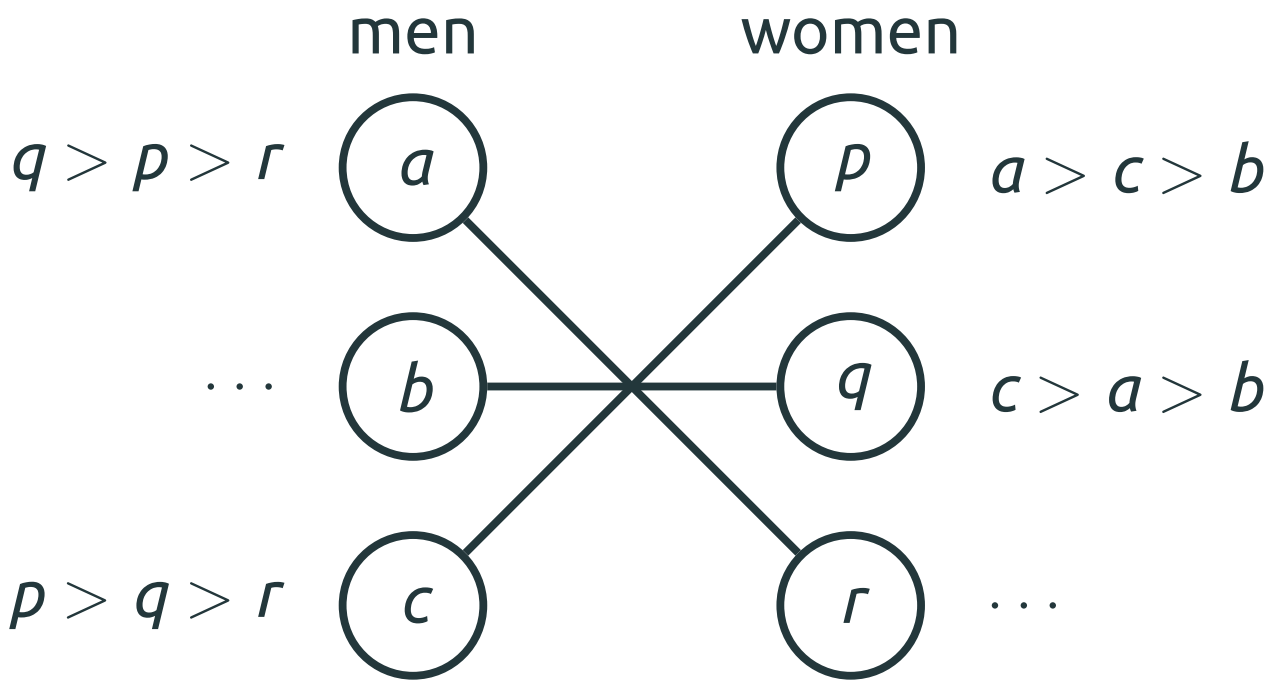

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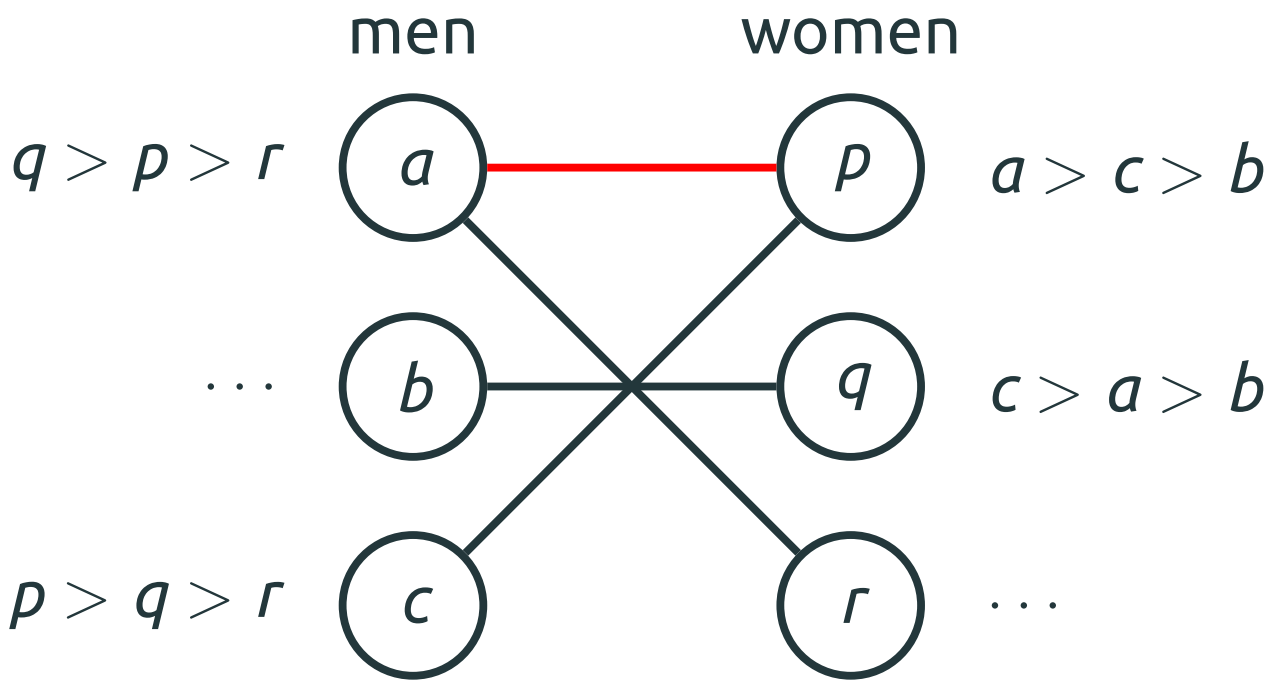

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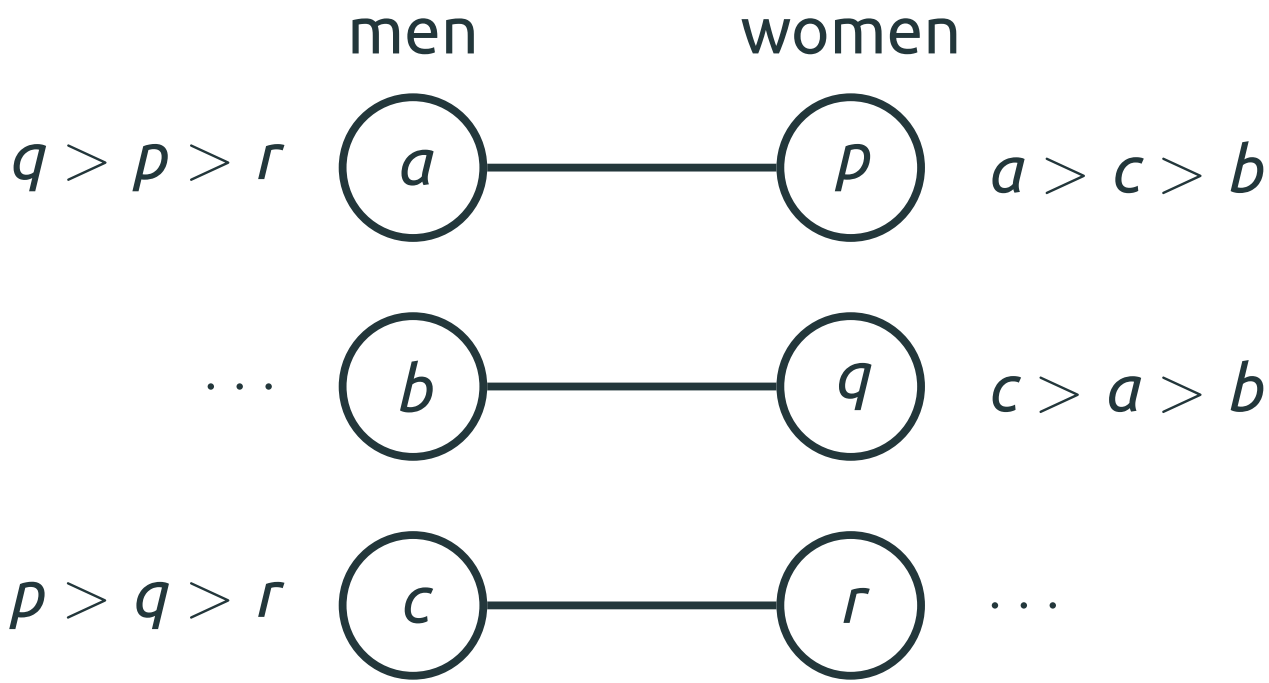

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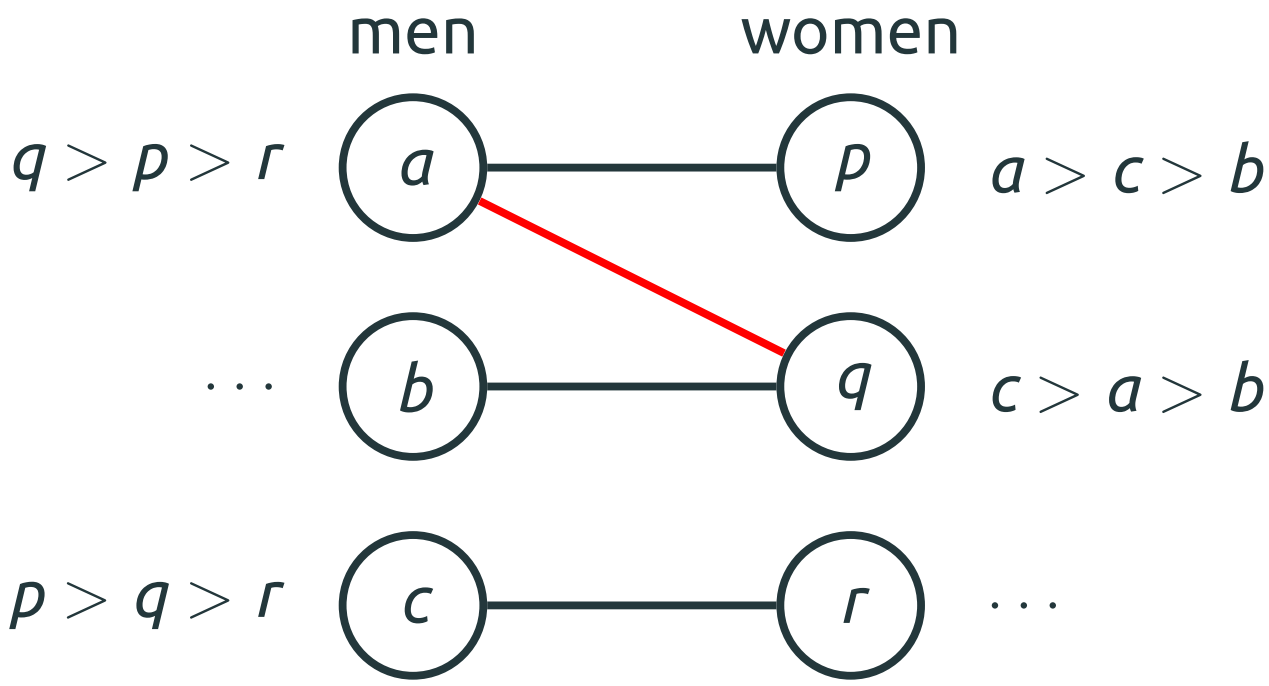

Outline

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- contradiction (#women = #men)

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- corollary: non-determinism inessential
- corollary: no collisions if for every man the best possible partner (in stable matchings) is chosen
- reformulation: if m was rejected (in any way) by w during the algorithm, (m, w) cannot appear in a stable matching

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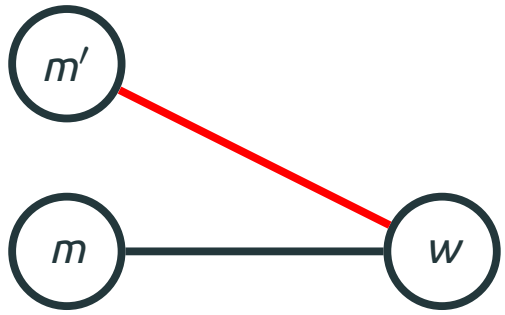
- but w rejected/left m because $w : m' > m$



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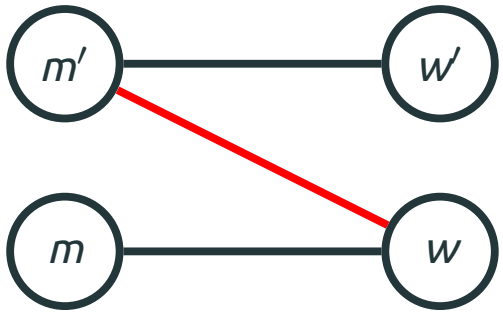

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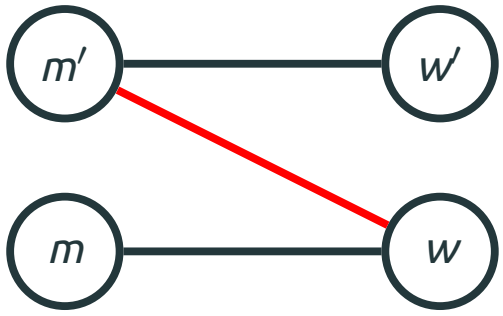

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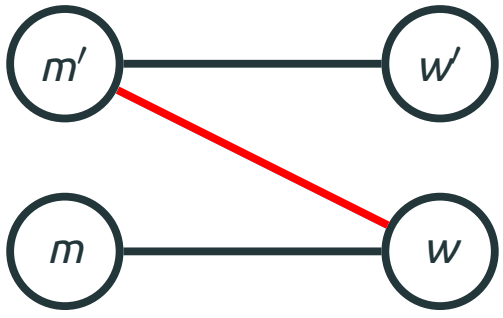
- m' was paired with w during the algorithm



Proof

- **claim**: if m was rejected by w during the algorithm, (m, w) cannot appear in a stable matching
- induction by the rejection moment
- (m, w) in a stable matching
- but w rejected/left m because $w : m' > m$
- (m', w') in (the same) stable matching
- $m' : w' > w$ (otherwise not stable)
- m' was paired with w during the algorithm
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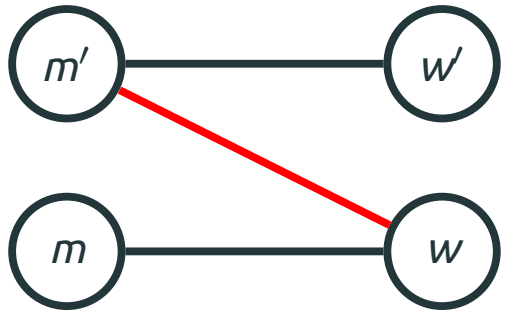

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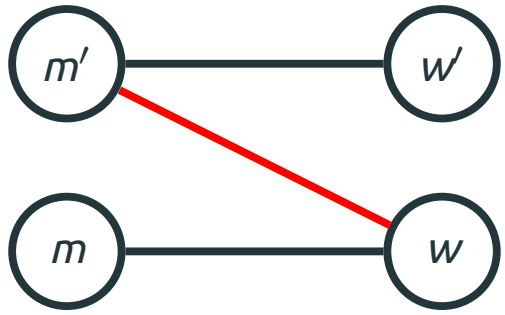

Outline

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- reformulation: if (m, w) is in some stable matching, the algorithm cannot pair w with some m' that is better (for w)

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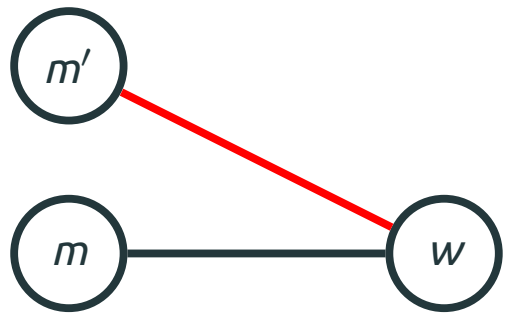

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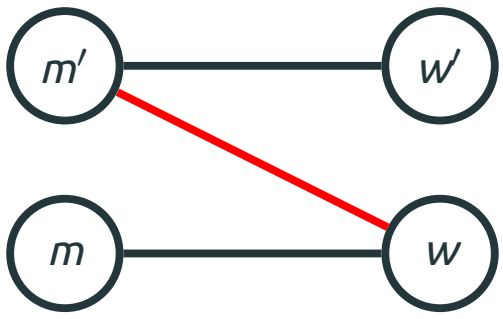

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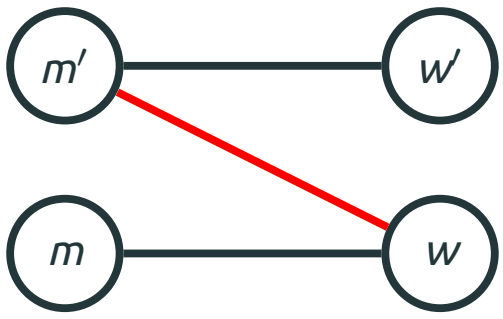

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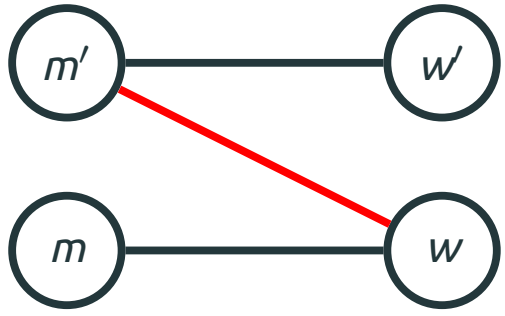

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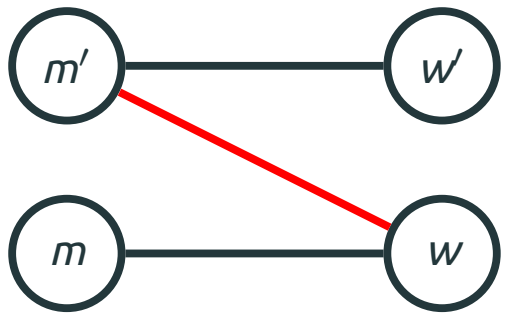

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- economics!