

Connected Components

Alexander S. Kulikov

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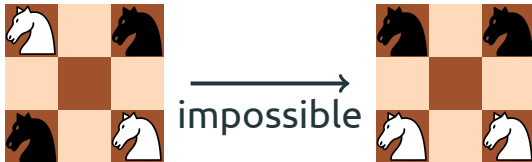
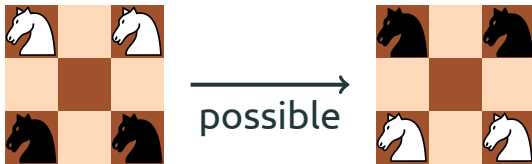
The Heaviest Stone



There are n stones of different weights. An expert knows the weights and wants to convince the court that a particular stone is the

heaviest one. For this, he repeatedly uses a pan balance to compare the weights of some two stones. **What is the minimum number of comparisons required?**

Guarini Puzzle, Revisited



can we check this automatically
instead of manually?

Hm...

- What do these two unrelated puzzles have in common?

Hm...

- What do these two unrelated puzzles have in common?
- They both can be solved by analyzing **connected components** of an underlying graph!

Outline

Connected Components

Guarini Puzzle: Program

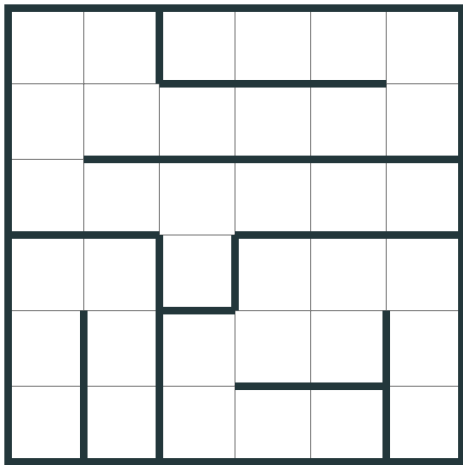
Lower Bound

The Heaviest Stone

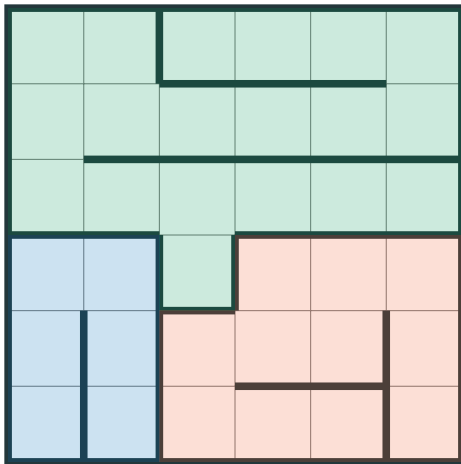
Directed Acyclic Graphs

Strongly Connected Components

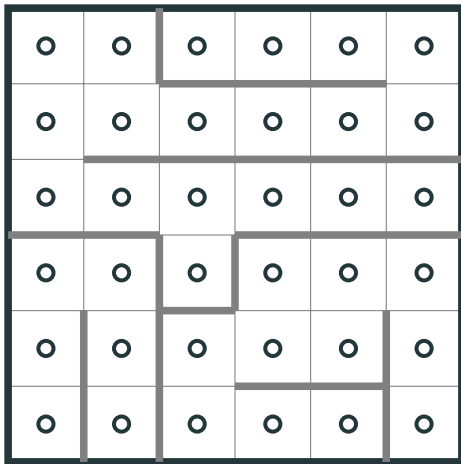
Connected Components in a Maze



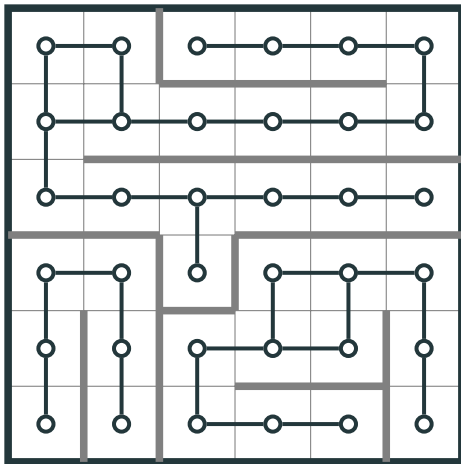
Connected Components in a Maze



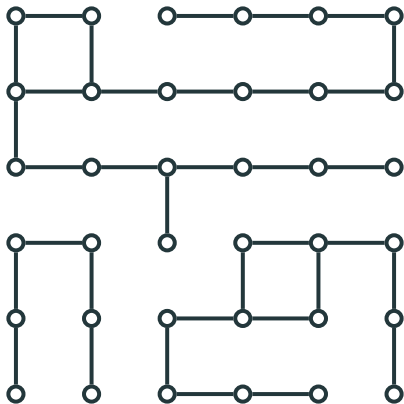
Connected Components in a Maze



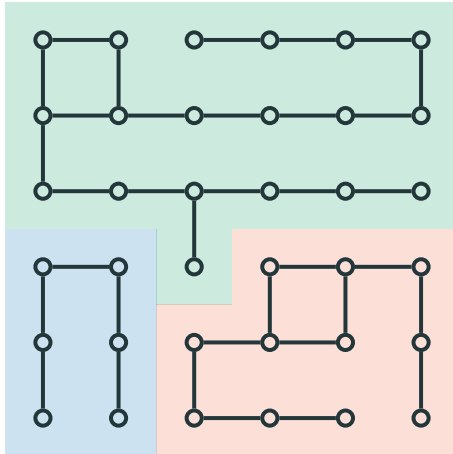
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Connected Components in a Maze



Connected Components in a Maze



Connected Graphs

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Connected Graphs

- Consider an *undirected* graph
- Two nodes are **connected**, if there is a path between them
- It is transitive: if u and v are connected and v and w are connected, then u and w are connected, too
- A graph is **connected**, if any two of its nodes are connected. In other words, there is a path between any two of its nodes

Connected Components

The nodes of any undirected graph can be partitioned into subsets called **connected components**:

- Any node belongs to exactly one connected component
- Any two nodes from the same connected component are connected
- Any two nodes from different connected components are not connected

Examples



Examples



Examples



Examples



Examples



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Connected Components

Guarini Puzzle: Program

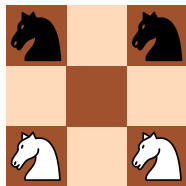
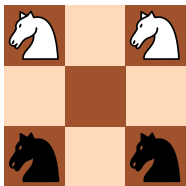
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Revisiting the Guarini Puzzle



Given two configurations, check whether one is reachable from the other one

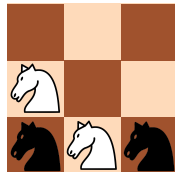
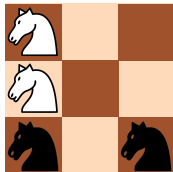
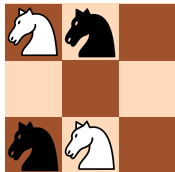
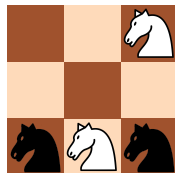
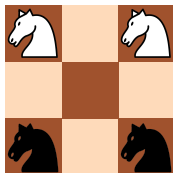
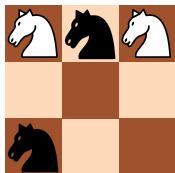
Graph of Configurations

- Consider a graph where the set of nodes is the set of all configurations, i.e., all possible 3×3 boards with two white knights and two black knights

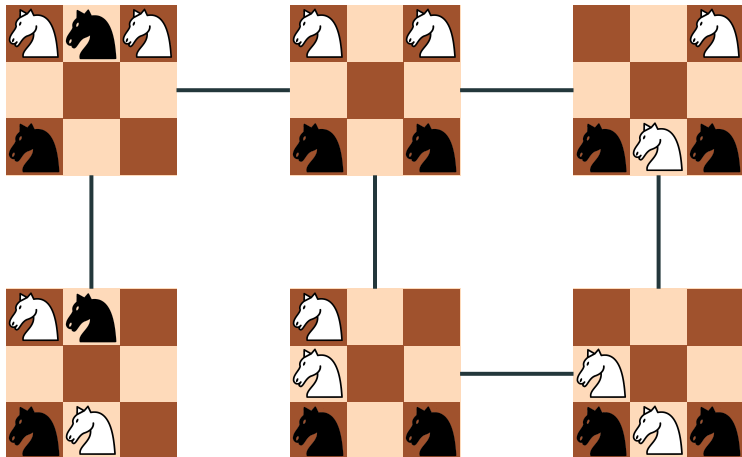
Graph of Configurations

- Consider a graph where the set of nodes is the set of all configurations, i.e., all possible 3×3 boards with two white knights and two black knights
- Join two nodes by an edge if their configurations are within a single move from each other

Graph of Configurations



Graph of Configurations



Solution

Then, one configuration is reachable from the other one, if and only if they belong to the same connected component!

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Theorem

An undirected graph $G(V, E)$ has at least $|V| - |E|$ connected components.

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- The theorem is useless for graphs with $|E| \geq |V|$

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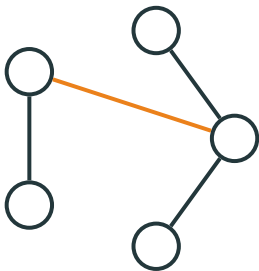
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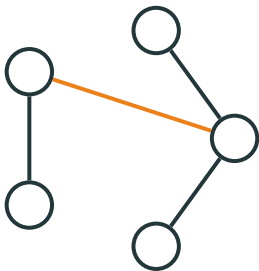
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- Each time when we add a new edge, $|V| - |E|$ decreases by 1
- At the same time, the number of connected components either decreases by 1 or stays the same

Illustration

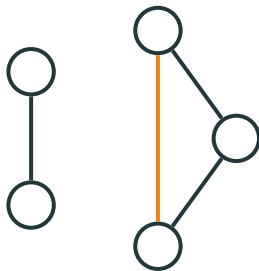


decreases

Illustration



decreases



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There are n stones of different weights. An expert knows the weights and wants to convince the court that a particular stone is the

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- but is it optimal?
- yes!

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- Note that we are not even interested in the results of comparisons performed by the expert
- If there were less than $n - 1$ comparisons, then the graph contains at least two connected components
- But this means that the court is still not sure about the heaviest stone!

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Strongly Connected Components

DAGs

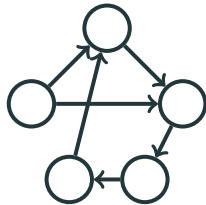
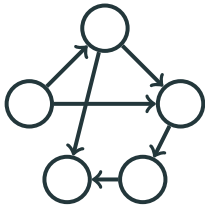
Definition

A **directed acyclic graph**, or simply a DAG, is a directed graph without cycles.

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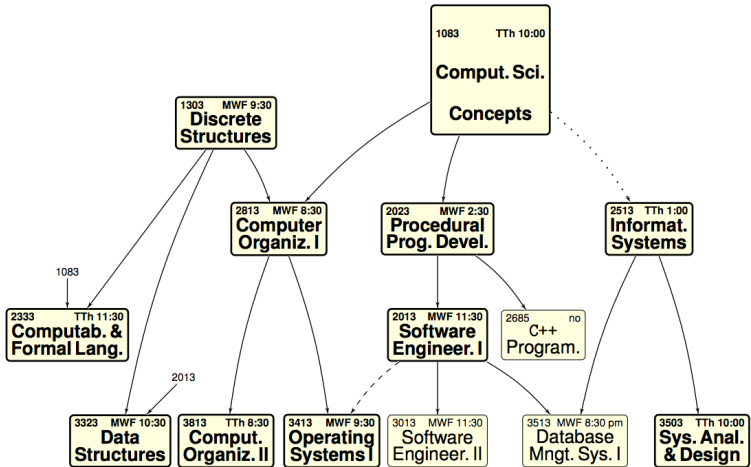


The diagram illustrates a network of scientists and their research groups. The nodes are represented by ovals containing names and years, and the edges represent collaborative or influential connections. The network is highly interconnected, with many nodes having multiple incoming and outgoing edges. The layout is somewhat circular, with nodes arranged in concentric layers around a central cluster.

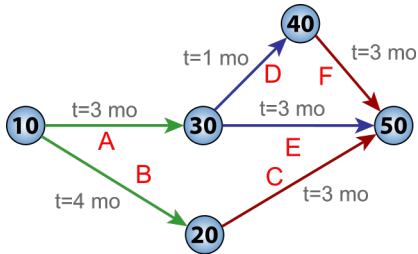
Key nodes and connections include:

- Central Nodes:** Daniel J Hogan, 2008; Lidia Vasiljeva, 2006; E J Steinmetz, 2001; A V Grishin, 1998; Thomas L Fare, 2003; K A Shepard, 2003; Michael F Berger, 2006; T L Bailey, 1994; J D Lieb, 2001; J L Brown, 1997; 1988; 2003; Xin Jie Chen, 2005; 2008; B D Lang, 2001; Y Uesono, 1997; Karen Kim, 2003; Christopher T Harbison, 2004; Sean M O'Rourke, 2002; M W Hentze, 1991; M Ashburner, 2000; M C Costanzo, 2001; Gavin, 2002; Ujwal Sheth, 2003; Xiang Gan, 2002; Xiaoqiang Wang, 2002; Alessandro C Navarre, 1994; P A Takizawa, 2000; P. Ciften, 2003; Flavio Mignone, 2002; J D Gary, 1996; M D Michelitsch, 2000; Won-Ki Huh, 2003; Theophany Eystathiou, 2002; B K Kennedy, 1997; M Bagnat, 2001; A Henras, 1998; A E Clives, 1996; E A Winzeler, 1999.
- Peripheral Nodes:** N K Conrad, 2000; R S Nash, 2001; H W Mewes, 2000; U Guldert, 2000; Tata Pramila, 2002; Jernej Ule, 2003; E J Steinmetz, 1997; J D Lewis, 1996; G Zhu, 2000; Tong Inn, 1997; J Verna, 1997; Y Xie, 2001; Gangli, 2000; S Chu, 1999; C J Loy, 1999; A G Hinnebusch, 1985; P A Takizawa, 2000; Tracey C Fleischer, 2006; Robertus A M de Bruin, 2006; Cornelia Kurischko, 2005; Radharani Duttagupta, 2005; P A Takizawa, 1997; E Garl, 2001; Qinsan Gao, 2005; J David Lambert, 2002; N K Gray, 1999; Aaron Reinke, 2004; Kenji Irie, 2002; R B Kapust, 2001; V G Tusher, 2001; R M Long, 1997; ENCODE, 2007; Hsueh-Feng Chen, 2006; Hsueh-Feng Chen, 2006; Evan H Hurowitz, 2003; Carmen L de Hoog, 2004; Tzvi Aviv, 2003; D C Raitt, 2000; H Moriya, 1999; Xiac, 1990; 1998; C Wittenberg, 1990; 1998; J L Brown, 1997; 1988; 2003; Xin Jie Chen, 2005; 2008; B D Lang, 2001; Y Uesono, 1997; Karen Kim, 2003; Christopher T Harbison, 2004; Sean M O'Rourke, 2002; M W Hentze, 1991; M Ashburner, 2000; M C Costanzo, 2001; Gavin, 2002; Ujwal Sheth, 2003; Xiang Gan, 2002; Xiaoqiang Wang, 2002; Alessandro C Navarre, 1994; P A Takizawa, 2000; P. Ciften, 2003; Flavio Mignone, 2002; J D Gary, 1996; M D Michelitsch, 2000; Won-Ki Huh, 2003; Theophany Eystathiou, 2002; B K Kennedy, 1997; M Bagnat, 2001; A Henras, 1998; A E Clives, 1996; E A Winzeler, 1999.

Prerequisite Graph



Dependency Graph



Dependency Graph

- Consider the following (directed) dependency graph: nodes are jobs, there is a directed edge from A to B if the job A must be processed before B

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- How to find an order of jobs satisfying all constraints?
- If there is a cycle in the graph, then there is no such order
- It turns out that this is the only obstacle: if the graph is acyclic, then there is an ordering of its vertices satisfying all the constraints!

Topological Ordering

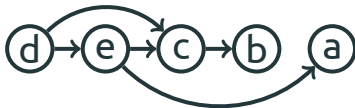
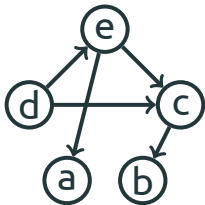
Definition

A **topological ordering** of a directed graph is an ordering of its vertices such that, for each edge (u, v) , u comes before v .

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Every DAG Can Be Ordered

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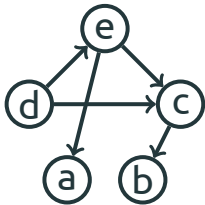
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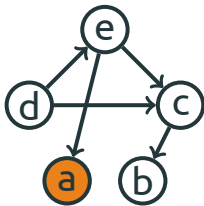
Proof

- We'll show that every DAG has a **sink** — a node with no outgoing edges
- Take a sink, put it to the end of the ordering, remove it from the graph (this keeps the graph acyclic), and repeat

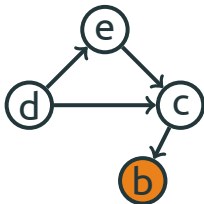
Example



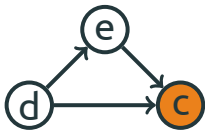
Example



Example



Example



Example



Example

d

d

e

c

b

a

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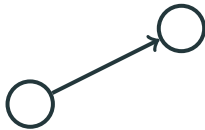
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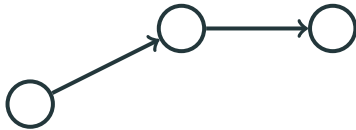
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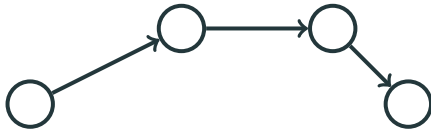
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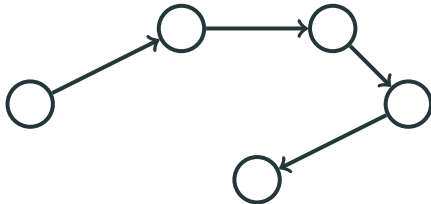
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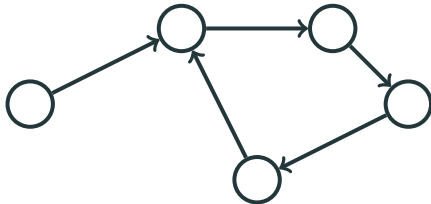
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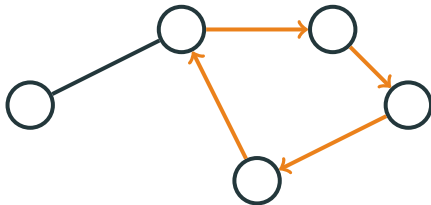
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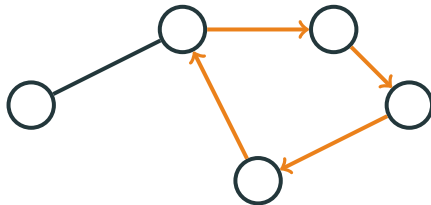
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- A contradiction!

Outline

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Guarini Puzzle: Program

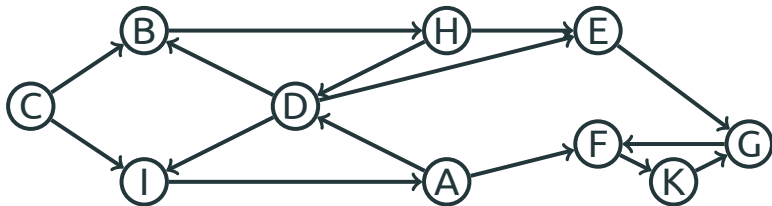
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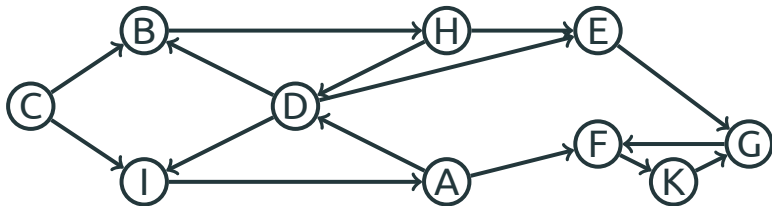
Directed Acyclic Graphs

Strongly Connected Components

Is This Graph Connected?

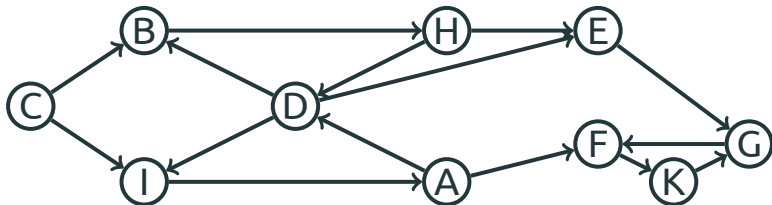


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- On one hand, this graph is connected: it cannot be “pulled apart”
- On the other hand, it is not connected: e.g., there is no path from A to C

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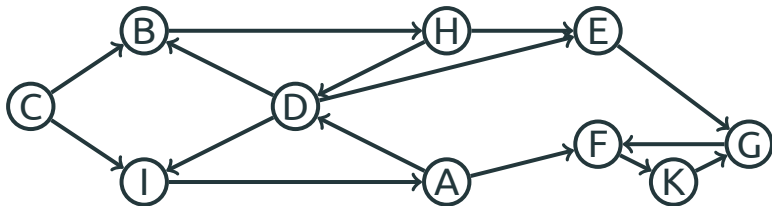
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 - every node belongs to exactly one SCC
 - nodes from the same SCC are connected

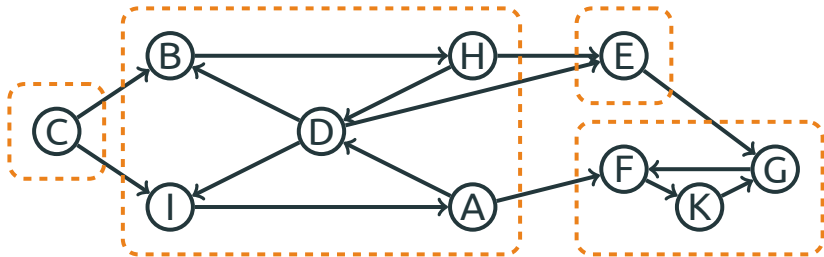
Strongly Connected Components

- In a directed graph, nodes u, v are **connected**, if there is a path from u to v *and* a path from v to u
- Nodes of any directed graph can be partitioned into subsets called **strongly connected components** (SCCs):
 - every node belongs to exactly one SCC
 - nodes from the same SCC are connected
 - nodes from different SCCs are not connected

Example



Example



Example

