Starting to Count

Vladimir Podolskii

Computer Science Department, Higher School of Economics

Outline

Why counting

Rule of Sum

How Not to Use Rule of Sum

Convenient Language: Sets

Generalizing Rule of Sum

Why Counting?

- Counting is one of the basic tasks in mathematics

Why Counting?

- Counting is one of the basic tasks in mathematics
- Goal: to tell how many objects are there without actually counting them one by one

Why Counting?

- Counting is one of the basic tasks in mathematics
- Goal: to tell how many objects are there without actually counting them one by one
- Used a lot in other parts of mathematics and applications

Why Counting?

- Counting is one of the basic tasks in mathematics
- Goal: to tell how many objects are there without actually counting them one by one
- Used a lot in other parts of mathematics and applications
- Important application: counting number of steps of algorithms

Why Counting?

- Counting is one of the basic tasks in mathematics
- Goal: to tell how many objects are there without actually counting them one by one
- Used a lot in other parts of mathematics and applications
- Important application: counting number of steps of algorithms
- Important application: computing probabilities

Why Counting?

We have already encountered counting in the course “What is a Proof?”:

- Estimating the running time of algorithms

Why Counting?

We have already encountered counting in the course "What is a Proof?":

- Estimating the running time of algorithms
- Applying pigeonhole principle

Real Life Example, Preview

wikimedia.org

- Suppose a country, a state or a region introduces a new format of a license plate

Real Life Example, Preview

wikimedia.org

- Suppose a country, a state or a region introduces a new format of a license plate
- Russian example: 3 digits, 3 letters; 78 is a regional code

Real Life Example, Preview

wikimedia.org

- Suppose a country, a state or a region introduces a new format of a license plate
- Russian example: 3 digits, 3 letters; 78 is a regional code
- We have: 10 options for digits, 12 options for letters (they must look similar to Latin letters; Cyrillic letters like Щ, Ю, Й are not allowed)

Real Life Example, Preview

wikimedia.org

- Suppose a country, a state or a region introduces a new format of a license plate
- Russian example: 3 digits, 3 letters; 78 is a regional code
- We have: 10 options for digits, 12 options for letters (they must look similar to Latin letters; Cyrillic letters like Щ, Ю, Й are not allowed)
- Will we have enough plates for everyone?

Outline

Why counting

Rule of Sum

How Not to Use Rule of Sum

Convenient Language: Sets

Generalizing Rule of Sum

Rule of Sum

Rule of Sum

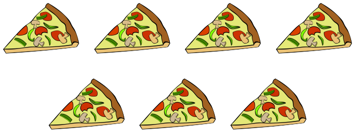
If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

Rule of Sum

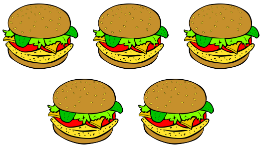
Rule of Sum

If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

Pizza places



Burger places

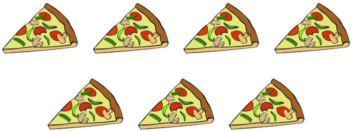


Rule of Sum

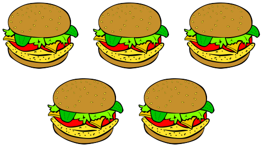
Rule of Sum

If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

Pizza places



Burger places

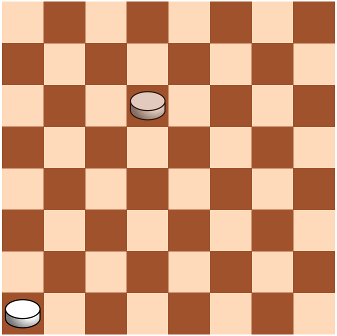


7+5=12 places to eat in total

Piece on a Chessboard

Piece on a chessboard

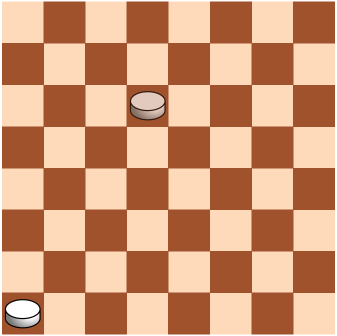
A piece stays in the bottom left corner of a chessboard. In one move it can move one step to the right or one step up. How many moves are needed to get to the position on the picture?



Piece on a Chessboard

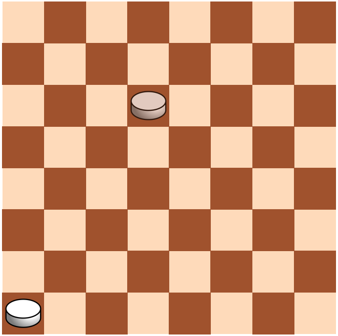
Piece on a chessboard

A piece stays in the bottom left corner of a chessboard. In one move it can move one step to the right or one step up. How many moves are needed to get to the position on the picture?

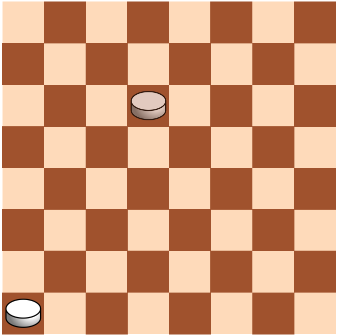


We have seen this problem in the first course

Piece on a Chessboard

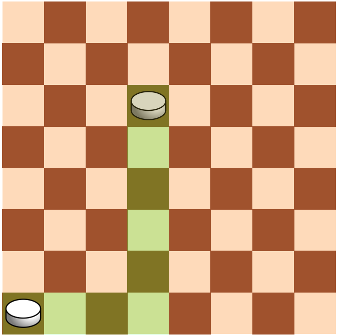


Piece on a Chessboard



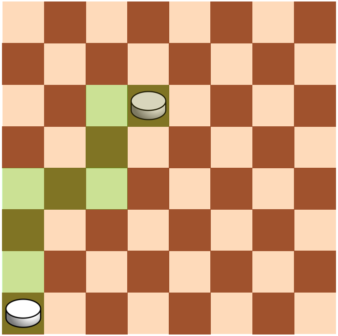
- We can do it in several ways

Piece on a Chessboard



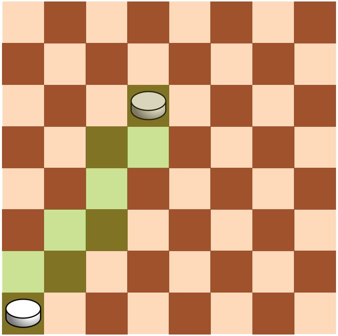
- We can do it in several ways

Piece on a Chessboard



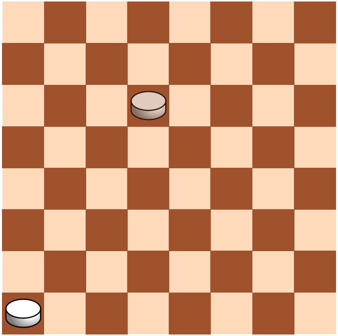
- We can do it in several ways

Piece on a Chessboard



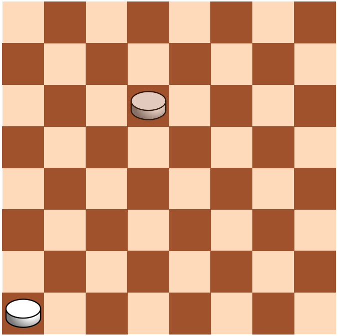
- We can do it in several ways

Piece on a Chessboard



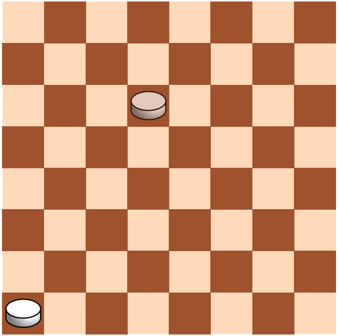
- We can do it in several ways
- In all cases we need 8 moves

Piece on a Chessboard



- We can do it in several ways
- In all cases we need 8 moves
- This is not a coincidence

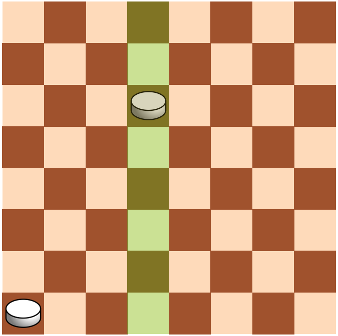
Piece on a Chessboard



- We can do it in several ways
- In all cases we need 8 moves
- This is not a coincidence

1. There are two types of moves: move right and move up

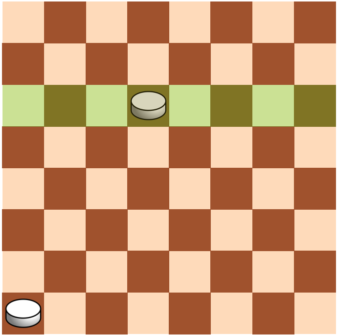
Piece on a Chessboard



- We can do it in several ways
- In all cases we need 8 moves
- This is not a coincidence

1. There are two types of moves: move right and move up
2. To get to the column 4 we need 3 moves to the right

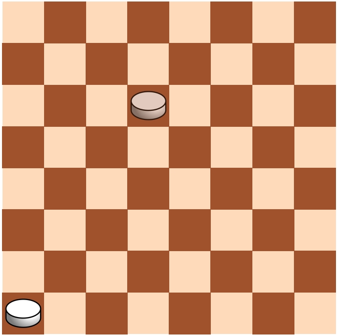
Piece on a Chessboard



- We can do it in several ways
- In all cases we need 8 moves
- This is not a coincidence

1. There are two types of moves: move right and move up
2. To get to the column 4 we need 3 moves to the right
3. To get to the row 6 we need 5 moves up

Piece on a Chessboard



- We can do it in several ways
- In all cases we need 8 moves
- This is not a coincidence

1. There are two types of moves: move right and move up
2. To get to the column 4 we need 3 moves to the right
3. To get to the row 6 we need 5 moves up
4. In total we need 3+5=8 moves

Outline

Why counting

Rule of Sum

How Not to Use Rule of Sum

Convenient Language: Sets

Generalizing Rule of Sum

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?
- The answer is 7!

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let’s attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?
- The answer is 7!
- What happened?

1 2 3 4 5 6 7 8 9 10

Divisibility by 2 or 3

Problem

Count all integers from 1 to 10 that are divisible by 2 or by 3

- Let's attempt to apply the rule of sum
- There are 5 numbers divisible by 2: 2, 4, 6, 8, 10
- There are 3 numbers divisible by 3: 3, 6, 9
- The answer should be 5+3=8
- But what if we count directly?
- The answer is 7!
- What happened? Number 6 causes problems!

1	2	3	4	5	6	7	8	9	10

Rule of Sum Revisited

Rule of Sum

If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

- Important lesson: in the rule of sum no object should belong to both classes!

Outline

Why counting

Rule of Sum

How Not to Use Rule of Sum

Convenient Language: Sets

Generalizing Rule of Sum

Convenient Language: Sets

- Set is an arbitrary group of arbitrary objects

Convenient Language: Sets

- Set is an arbitrary group of arbitrary objects
- We will denote sets by capital letters: A, B, C, S, etc.

Convenient Language: Sets

- Set is an arbitrary group of arbitrary objects
- We will denote sets by capital letters: A, B, C, S, etc.
- Sets can be given by the list of their elements: $S = \{0, 1, 2, 3\}$; this set consists of four elements, 0, 1, 2, 3

Convenient Language: Sets

- Set is an arbitrary group of arbitrary objects
- We will denote sets by capital letters: A, B, C, S, etc.
- Sets can be given by the list of their elements: $S = \{0, 1, 2, 3\}$; this set consists of four elements, 0, 1, 2, 3
- The order of elements is not important: $\{0, 1, 2, 3\} = \{2, 0, 3, 1\}$

Convenient Language: Sets

- Set is an arbitrary group of arbitrary objects
- We will denote sets by capital letters: A, B, C, S, etc.
- Sets can be given by the list of their elements: $S = \{0, 1, 2, 3\}$; this set consists of four elements, 0, 1, 2, 3
- The order of elements is not important: $\{0, 1, 2, 3\} = \{2, 0, 3, 1\}$
- Repetitions in the list of elements are not important: $\{0, 1, 2, 3\} = \{1, 0, 1, 3, 2, 3\}$

Convenient Language: Sets

- For us sets provide a convenient language

Convenient Language: Sets

- For us sets provide a convenient language
- In Mathematics sets play an important and fundamental role

Convenient Language: Sets

- For us sets provide a convenient language
- In Mathematics sets play an important and fundamental role
- For us sets can consist of anything:
  $S = \{0, \sqrt{2}, \text{ Isaac Newton}, \text{ a leprechaun}\}$

Convenient Language: Sets

- For us sets provide a convenient language
- In Mathematics sets play an important and fundamental role
- For us sets can consist of anything:
  $S = \{0, \sqrt{2}, \text{ Isaac Newton}, \text{ a leprechaun}\}$
- However, there are pitfalls

Convenient Language: Sets

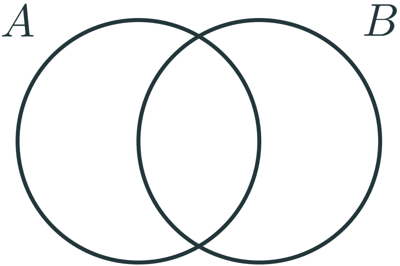
- For us sets provide a convenient language
- In Mathematics sets play an important and fundamental role
- For us sets can consist of anything: $S = \{0, \sqrt{2}, \text{ Isaac Newton}, \text{ a leprechaun}\}$
- However, there are pitfalls
- “Set consisting of all sets” is a dangerous construction

Convenient Language: Sets

- For us sets provide a convenient language
- In Mathematics sets play an important and fundamental role
- For us sets can consist of anything: $S = \{0, \sqrt{2}, \text{ Isaac Newton}, \text{ a leprechaun}\}$
- However, there are pitfalls
- "Set consisting of all sets" is a dangerous construction
- We will not encounter these difficulties in the course and will not discuss them

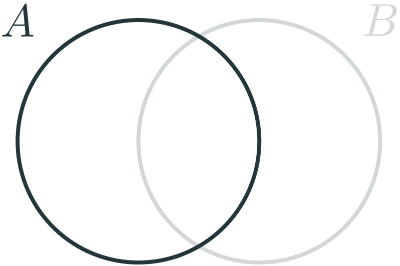
Venn Diagrams

- A convenient way to view sets is by the means of Venn Diagrams



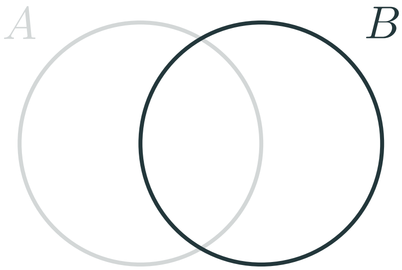
Venn Diagrams

- A convenient way to view sets is by the means of Venn Diagrams
- Elements of A are within the left circle



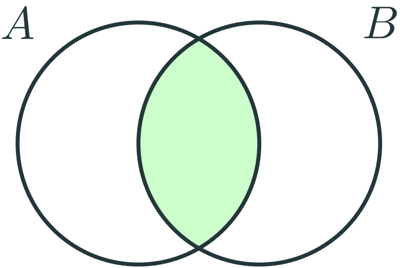
Venn Diagrams

- A convenient way to view sets is by the means of Venn Diagrams
- Elements of A are within the left circle
- Elements of B are within the right circle

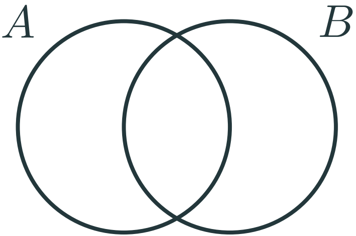


Venn Diagrams

- A convenient way to view sets is by the means of Venn Diagrams
- Elements of A are within the left circle
- Elements of B are within the right circle
- Intersection corresponds to elements belonging to both sets

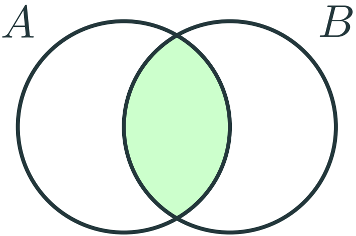


Useful Notations



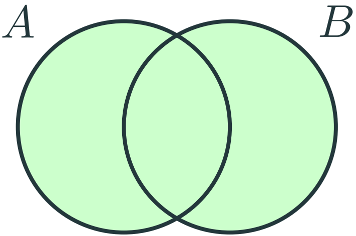
- Suppose we have two sets A and B

Useful Notations



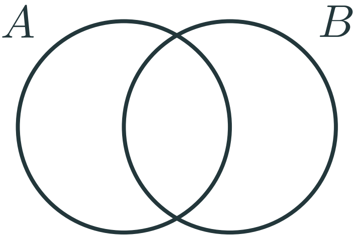
- Suppose we have two sets A and B
- The set $A \cap B$ is an intersection of these sets: it consists of elements belonging to both sets

Useful Notations



- Suppose we have two sets A and B
- The set $A \cap B$ is an intersection of these sets: it consists of elements belonging to both sets
- The set $A \cup B$ is a union of these sets: it consists of elements belonging to at least one of the sets

Useful Notations



- Suppose we have two sets A and B
- The set $A \cap B$ is an intersection of these sets: it consists of elements belonging to both sets
- The set $A \cup B$ is a union of these sets: it consists of elements belonging to at least one of the sets
- The number of elements in the set A is $|A|$ (can be infinite)

Outline

Why counting

Rule of Sum

How Not to Use Rule of Sum

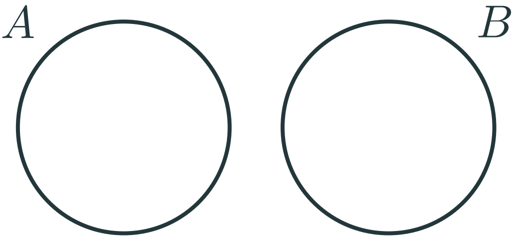
Convenient Language: Sets

Generalizing Rule of Sum

Rule of Sum in the Set Language

Rule of Sum

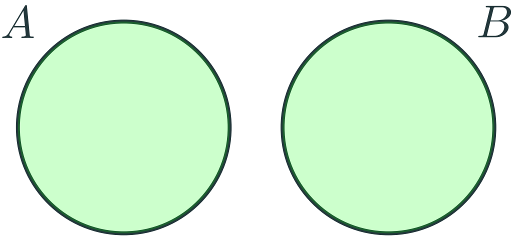
If there is a set A with k elements, a set B with n elements and these sets do not have common elements, then the set $A \cup B$ has $n + k$ elements



Rule of Sum in the Set Language

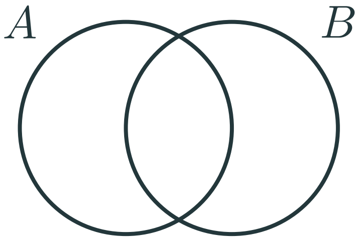
Rule of Sum

If there is a set A with k elements, a set B with n elements and these sets do not have common elements, then the set $A \cup B$ has $n + k$ elements



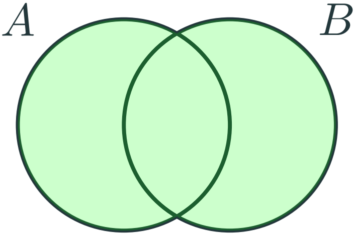
Generalized Rule of Sum

But what if we want to count $|A \cup B|$ in the setting below?

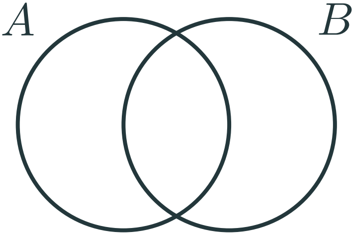


Generalized Rule of Sum

But what if we want to count $|A \cup B|$ in the setting below?

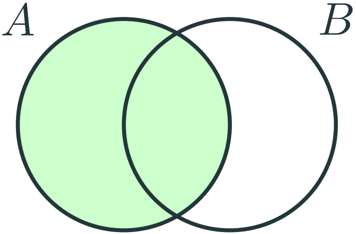


Generalized Rule of Sum



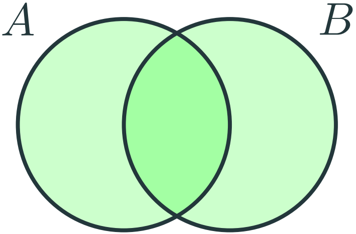
- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong

Generalized Rule of Sum



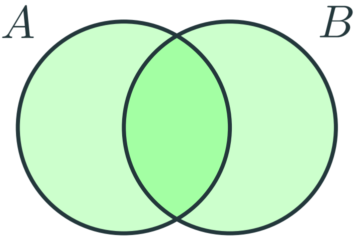
- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong

Generalized Rule of Sum



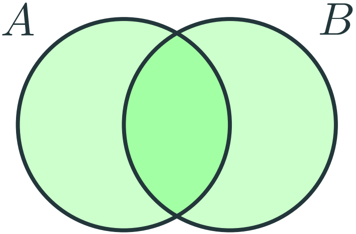
- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong

Generalized Rule of Sum



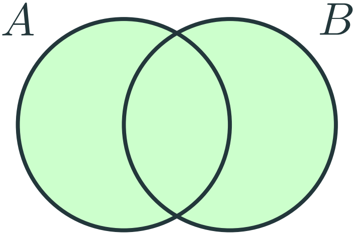
- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong
- We count elements that belong to both A and B twice

Generalized Rule of Sum



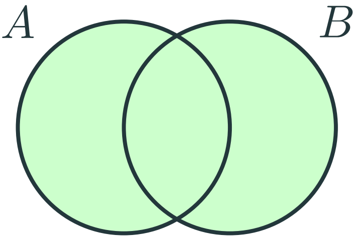
- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong
- We count elements that belong to both A and B twice
- So let's subtract them now!

Generalized Rule of Sum



- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong
- We count elements that belong to both A and B twice
- So let's subtract them now!

Generalized Rule of Sum



- If we just consider $|A| + |B|$ as in the rule of sum, then we will be wrong
- We count elements that belong to both A and B twice
- So let's subtract them now!
- This gives the right result: $|A \cup B| = |A| + |B| - |A \cap B|$

Summary

- Counting starts with simple things

Summary

- Counting starts with simple things
- But even rule of sum can be tricky

Summary

- Counting starts with simple things
- But even rule of sum can be tricky
- Next we will see how to build something more involved from the basic building blocks