Euclid’s Algorithm

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Outline

Greatest Common Divisor

Euclid’s Algorithm

Extended Euclid’s Algorithm

Greatest Common Divisor

Definition

The greatest common divisor, $\gcd(a, b)$, of integers a and b (not both equal to zero) is the largest integer that divides both a and b

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Convention

We assume that a and b are non-negative

First Application: Computing Inverses

Given integers a and n, how to find an integer k such that $ak \equiv 1 \mod n$? Basic primitive in modern crypto protocols, used billions times per day

pexels.com

Second Application: Computing Fractions

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print(Fraction(31, 177) + Fraction(29, 59))
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```

```
2/3
```

Naive Algorithm

To find the greatest common divisor, simply try all numbers and select the largest one

Naive Algorithm: Code

```
def gcd(a, b):
  assert a >= 0 and b >= 0 and a + b > 0

  if a == 0 or b == 0:
    return max(a, b)

  for d in range(min(a, b), 0, -1):
    if a % d == 0 and b % d == 0:
      return d

  return 1
```

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  return 1
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print(gcd(24, 16))
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```
8
```

Naive Algorithm: Analysis

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- If $\gcd(a, b) = 1$, the algorithm will perform $\min\{a, b\}$ divisions
- Modern laptops perform roughly one billion (10^9) operations per second
- On your laptop, the call

  ```
  print(gcd(790933790547, 1849639579327))
  ```

  will take more than one minute

Supercomputer

If *a* and *b* consist of hundreds of digits (typical case for crypto protocols), even on a supercomputer with quadrillion operations per second this algorithm will run for more than thousand years

Next Parts

Euclid's algorithm:
efficient algorithm for computing the greatest common divisor of two integers

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Proof

$$\Rightarrow \text{ if } a = dp \text{ and } b = dq\text{, then}$$
$$a - b = d(p - q)$$

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Proof

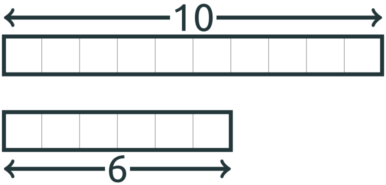
$\Rightarrow$ if $a = dp$ and $b = dq$, then
$a - b = d(p - q)$

$\Leftarrow$ if $a - b = dp$ and $b = dq$, then
$a = d(p + q)$

$\square$

Euclid's Lemma: Pictorially

2 divides 10 and 6, hence 2 divides $4 = 10 - 6$



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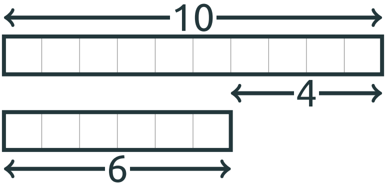

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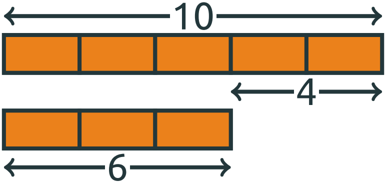

Algorithm

```
def gcd(a, b):
  assert a >= 0 and b >= 0 and a + b > 0

  while a > 0 and b > 0:
    if a >= b:
      a = a - b
    else:
      b = b - a

  return max(a, b)
```

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- Still, there are inputs where this code is too slow:

```
gcd(790933790548, 7)
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- Reason: the code will subtract 7 billions of times!
- Idea: what is left is the reminder modulo 7

Euclid’s Algorithm

```
def gcd(a, b):
  assert a >= 0 and b >= 0 and a + b > 0

  while a > 0 and b > 0:
    if a >= b:
      a = a % b
    else:
      b = b % a

  return max(a, b)
```

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- Already quite fast: if *a* and *b* are 100 digits long, the number of iterations of the `while` loop is at most 660

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- Each iteration is a division

Code with Logging

```
def gcd(a, b):
  assert a >= 0 and b >= 0 and a + b > 0

  while a > 0 and b > 0:
    print("gcd({},␣{})=".format(a, b))
    if a >= b:
      a = a % b
    else:
      b = b % a

  print("gcd({},␣{})=".format(a, b))
  return max(a, b)

print(gcd(790933790547, 1849639579327))
```

Code with Logging: Output

```
gcd(790933790547, 1849639579327)=
gcd(790933790547, 267771998233)=
gcd(255389794081, 267771998233)=
gcd(255389794081, 12382204152)=
gcd(7745711041, 12382204152)=
gcd(7745711041, 4636493111)=
gcd(3109217930, 4636493111)=
gcd(3109217930, 1527275181)=
gcd(54667568, 1527275181)=
gcd(54667568, 51250845)=
gcd(3416723, 51250845)=
gcd(3416723, 0)=
3416723
```

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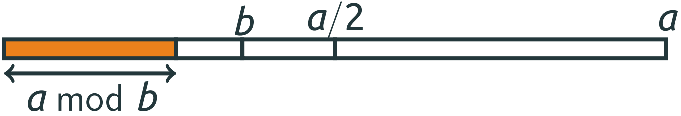
Lemma

Let $a \geq b > 0$. Then $(a \bmod b) < a/2$.

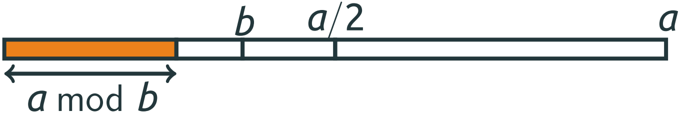
Proof

- If $b \leq a/2$, then $(a \bmod b) < b \leq a/2$.

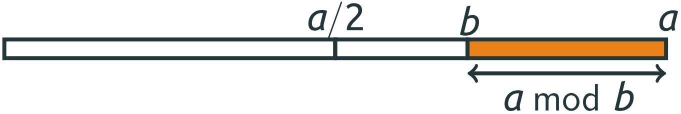


Proof

- If $b \leq a/2$, then $(a \bmod b) < b \leq a/2$.



- If $b > a/2$, then $(a \bmod b) = a - b < a/2$.



$\square$

Final Analysis

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- Hence, at each iteration, either a or b is dropped by at least a factor of 2
- Thus, the total number of iterations is at most $\log_2 a + \log_2 b$
- If a consists of less than 5 000 decimal digits (i.e., $a < 10^{5\,000}$), then $\log_2 a < 16\,610$

Compact Code

```python
def gcd(a, b):
  assert a >= b and b >= 0 and a + b > 0
  return gcd(b, a % b) if b > 0 else a
```

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- Somebody computed the greatest common divisor of a and b and wants to convince you that it is equal to d

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Certificate

- Somebody computed the greatest common divisor of a and b and wants to convince you that it is equal to d
- You can check that d divides both a and b, but this only shows that d is a common divisor of a and b, but does not guarantees that is the greatest one
- It turns out that it is enough to represent d as $ax + by$ (for integers x and y)!

Test

Lemma

If d divides a and b and $d = ax + by$ for integers x and y, then $d = \gcd(a, b)$.

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Lemma

If d divides a and b and $d = ax + by$ for integers x and y, then $d = \gcd(a, b)$.

Proof

- d is a common divisor of a and b, hence $d \leq \gcd(a, b)$
- $\gcd(a, b)$ divides both a and b, hence it also divides $d = ax + by$, and hence $\gcd(a, b) \leq d$ $\square$

Examples

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- $\gcd(7, 5) = 1 = 7 \cdot (-2) + 5 \cdot 3$
- $\gcd(391, 299) = 23 = 391 \cdot (-3) + 299 \cdot 4$

Examples

- $\gcd(10, 6) = 2 = 10 \cdot (-1) + 6 \cdot 2$
- $\gcd(7, 5) = 1 = 7 \cdot (-2) + 5 \cdot 3$
- $\gcd(391, 299) = 23 = 391 \cdot (-3) + 299 \cdot 4$
- $\gcd(239, 201) = 1 = 239 \cdot (-37) + 201 \cdot 44$

Extending Euclid's Algorithm

- Recall that Euclid's algorithm uses the fact that, for $a \geq b$, $\gcd(a, b) = \gcd(b, a \bmod b)$

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- Assume that $d = \gcd(b, a \bmod b)$ and that $d = bp + (a \bmod b)q$
- Then

$$
\begin{aligned}
d &= bp + (a \bmod b)q \\
&= bp + \left(a - \left\lfloor \frac{a}{b} \right\rfloor b\right)q \\
&= aq + b\left(p - \left\lfloor \frac{a}{b} \right\rfloor q\right)
\end{aligned}
$$

Extended Euclid's Algorithm

```
# returns gcd(a,b), x, y: gcd(a,b)=ax+by
def extended_gcd(a, b):
  assert a >= b and b >= 0 and a + b > 0

  if b == 0:
    d, x, y = a, 1, 0
  else:
    (d, p, q) = extended_gcd(b, a % b)
    x = q
    y = p - q * (a // b)

  assert a % d == 0 and b % d == 0
  assert d == a * x + b * y
  return (d, x, y)
```