Applications

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Outline

Least Common Multiple

Diophantine Equations: Examples

Diophantine Equations: Theorem

Modular Division

Least Common Multiple

Definition

The least common multiple, $\operatorname{lcm}(a, b)$, of integers a and b (both different from zero) is the smallest positive integer that is divisible by both a and b

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Convention

We assume that a and b are positive

Working with Fractions

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$$\frac{31}{177} + \frac{29}{118} = \frac{31 \cdot 2}{177 \cdot 2} + \frac{29 \cdot 3}{118 \cdot 3}$$

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$$\frac{31}{177} + \frac{29}{118} = \frac{31 \cdot 2}{177 \cdot 2} + \frac{29 \cdot 3}{118 \cdot 3} = \frac{149}{354}$$

Naive Algorithm

Clearly, $a \cdot b$ is divisible by a and b. To find the *least* common multiple, simply try all numbers up to $a \cdot b$ and select the smallest one

Naive Algorithm: Code

```
def lcm(a, b):
  assert a > 0 and b > 0

  for d in range(1, a * b + 1):
    if d % a == 0 and d % b == 0:
      return d
```

Naive Algorithm: Code

```
def lcm(a, b):
  assert a > 0 and b > 0

  for d in range(1, a * b + 1):
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```

```
print(lcm(24, 16))
```

```
48
```

Naive Algorithm: Analysis

- If $\text{lcm}(a, b) = a \cdot b$, the algorithm will perform $a \cdot b$ divisions

Naive Algorithm: Analysis

- If $\text{lcm}(a, b) = a \cdot b$, the algorithm will perform $a \cdot b$ divisions
- Again, very slow: the call

```
print(lcm(531441, 262144))
```

will take more than one minute

Euclid's algorithm

Can we use efficient Euclid's algorithm to compute the least common multiple, too?

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Can we use efficient Euclid's algorithm to compute the least common multiple, too?

Yes!

$$\mathrm{lcm}(a, b) \cdot \gcd(a, b) = a \cdot b$$

lcm and gcd

Lemma

If $a, b > 0$, then $\mathrm{lcm}(a, b) = ab/\gcd(a, b)$

lcm and gcd

Lemma

If $a, b > 0$, then $\operatorname{lcm}(a, b) = ab/\gcd(a, b)$

Proof

- Let $d = \gcd(a, b)$, $a = dp$, $b = dq$

$\square$

lcm and gcd

Lemma

If $a, b > 0$, then $\mathrm{lcm}(a, b) = ab/\mathrm{gcd}(a, b)$

Proof

- Let $d = \mathrm{gcd}(a, b)$, $a = dp$, $b = dq$
- Then $m = ab/d = pb = qa$ is a multiple of a and b

$\square$

lcm and gcd

Lemma

If $a, b > 0$, then $\mathrm{lcm}(a, b) = ab/\gcd(a, b)$

Proof

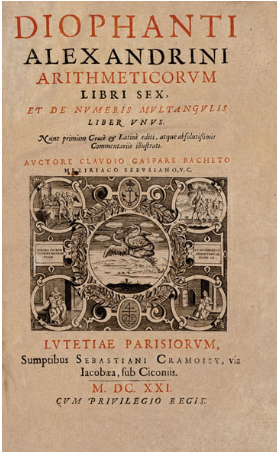
- Let $d = \gcd(a, b)$, $a = dp$, $b = dq$
- Then $m = ab/d = pb = qa$ is a multiple of a and b
- If there was a smaller multiple $\overline{m} < m$, then $\bar{d} = ab/\overline{m} > d$ would be a common divisor: $a/\bar{d} = \overline{m}/b$, $b/\bar{d} = \overline{m}/a$

$\square$

Outline

Diophantine Equations

A Diophantine equation is an equation where only integer solutions are allowed


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX,
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
Nunc primùm Græcè & Latinè editi, atque abſolutiſſimis
Commentariis illuſtrati.
AVCTORE CLAVDIO GASPARE BACHETO
MEZIRIACO SEBVSIANO, V.C.
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via
Iacobæa, ſub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS.

Diophantine Equations: Examples

- 1 apple costs 22 pesos

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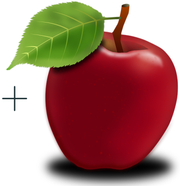

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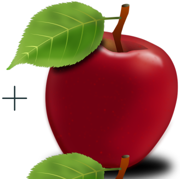
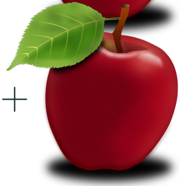


$\times 14 =$


$\times 4 +$


Diophantine Equations: Examples

RECEIPT

[redacted] Lamps

Each	$49.36
Total	$[redacted]7.28

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- x lamps, each 4936 ¢

Diophantine Equations: Examples

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- $100 \le y < 1000$

Diophantine Equations: Examples

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- $4936x = 1000y + 728$, x, y are non-negative integers, $100 \leq y < 1000$

Diophantine Equations: Examples

RECEIPT	
98 Lamps	
Each	\$49.36
Total	\$4837.28

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- $100 \leq y < 1000$
- Total: $1000y + 728$ ¢
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    - $187 \cdot 3 + 55 \cdot (-8) = 121$
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    - No solutions!

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  - $187 \cdot (-2) + 55 \cdot 9 = 121$
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- $187x + 55y = 45$, x and y are integers
  - No solutions!

When does a Diophantine equation have solutions?

Outline

Solutions of Diophantine Equations

Theorem

Given integers a, b, c (at least one of a and $b \neq 0$), the Diophantine equation

$$ax + by = c$$

has a solution (where x and y are integers) if and only if

$$\gcd(a, b) \mid c\,.$$

Proof of Theorem

Proof

Let $d = \gcd(a, b)$

Proof of Theorem

Proof

$$
\begin{aligned}
& \text{Let } d = \gcd(a, b) \\
\Rightarrow\ & a = dp \text{ and } b = dq \text{, thus,} \\
& c = ax + by = d(px + qy)
\end{aligned}
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& c = ax + by = d(px + qy) \\
\Leftarrow\ & \text{Extended Euclid's algorithm:} \\
& a\bar{x} + b\bar{y} = d
\end{aligned}
$$

Proof of Theorem

Proof

Let $d = \gcd(a, b)$

$\Rightarrow$ $a = dp$ and $b = dq$, thus,

$c = ax + by = d(px + qy)$

$\Leftarrow$ Extended Euclid's algorithm:

$a\bar{x} + b\bar{y} = d$

If $c = td$, then $x = t \cdot \bar{x}, y = t \cdot \bar{y}$:

$ax + by = t(a\bar{x} + b\bar{y}) = td = c$ $\square$

Finding a Solution

- $10x + 6y = 14$

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  - $x = 9, y = -12$

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  - $-69 = 391 \cdot (-3) \cdot (-3) + 299 \cdot 4 \cdot (-3)$
  - $x = 9, y = -12$
  - But $x = -4, y = 5$ is also a solution. How do we find all solutions?

Euclid’s Lemma

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If $n \mid ab$ and $\gcd(a, n) = 1$, then $n \mid b$.

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- $axb + nyb = b$

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Proof

- From Extended Euclid's algorithm (a, n):
- $ax + ny = 1$
- $axb + nyb = b$
- From $ab = kn$,
  $b = axb + nyb = n(xk + yb)$ $\square$

Finding All Solution

Theorem

Let $\gcd(a, b) = d$, $a = dp$, $b = dq$. If (x_0, y_0) is a solution of the Diophantine equation $ax + by = c$:

$$ax_0 + by_0 = c\,,$$

then all the solutions have the form

$$a(x_0 + tq) + b(y_0 - tp) = c\,,$$

where t is an arbitrary integer.

Proof of Theorem

Proof

- $a = dp, b = dq, ax_0 + by_0 = c$

Proof of Theorem

Proof

- $a = dp, b = dq, ax_0 + by_0 = c$
- For any integer t,

$$\begin{aligned} &a(x_0 + tq) + b(y_0 - tp) \\ &= ax_0 + by_0 + t(aq - bp) \\ &= c + t(dpq - dpq) = c \end{aligned}$$

is a solution

Proof of Theorem

Proof (continued)

- Consider 2 solutions: (x_1, y_1) and (x_2, y_2)

Proof of Theorem

Proof (continued)

- Consider 2 solutions: (x_1, y_1) and (x_2, y_2)
- $a(x_1 - x_2) + b(y_1 - y_2) = c - c = 0$

Proof of Theorem

Proof (continued)

- Consider 2 solutions: (x_1, y_1) and (x_2, y_2)
- $a(x_1 - x_2) + b(y_1 - y_2) = c - c = 0$
- $p(x_1 - x_2) + q(y_1 - y_2) = 0$

Proof of Theorem

Proof (continued)

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- $\gcd(p, q) = 1$

Proof of Theorem

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- $\gcd(p, q) = 1$
- By Euclid's lemma: $x_1 - x_2 = tq$

Proof of Theorem

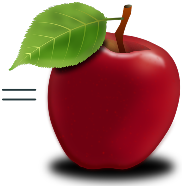
Proof (continued)

- Consider 2 solutions: (x_1, y_1) and (x_2, y_2)
- $a(x_1 - x_2) + b(y_1 - y_2) = c - c = 0$
- $p(x_1 - x_2) + q(y_1 - y_2) = 0$
- $\gcd(p, q) = 1$
- By Euclid's lemma: $x_1 - x_2 = tq$
- Then $y_1 - y_2 = -tp$ $\square$

Example

$\times 9 +$

$\times 1 =$



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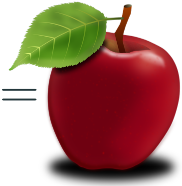

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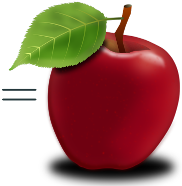

- $3x + 5y = 22$
- $x_0 = 9, y_0 = -1$

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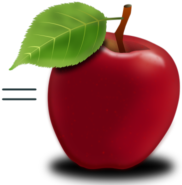

- $3x + 5y = 22$
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- $a = 3, b = 5, d = 1$

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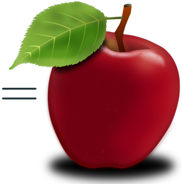

- $3x + 5y = 22$
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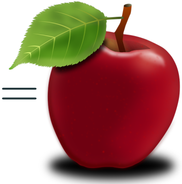

- $3x + 5y = 22$
- $x_0 = 9, y_0 = -1$
- $a = 3, b = 5, d = 1$
- $a = dp, b = dq, p = 3, q = 5$
- All solutions:

$$x = x_0 + tq = 9 + 5t$$

$$y = y_0 - tp = -1 - 3t$$

Example

- All solutions:

$$\begin{aligned} x &= x_0 + tq = 9 + 5t \\ y &= y_0 - tp = -1 - 3t \end{aligned}$$

Example

- All solutions:

$$x = x_0 + tq = 9 + 5t$$
$$y = y_0 - tp = -1 - 3t$$

- If we want $x \geq 0$ and $y \leq 0$, then take

$$9 + 5t \geq 0$$
$$-1 - 3t \leq 0$$

Example

- All solutions:

$$x = x_0 + tq = 9 + 5t$$
$$y = y_0 - tp = -1 - 3t$$

- If we want $x \geq 0$ and $y \leq 0$, then take

$$9 + 5t \geq 0$$
$$-1 - 3t \leq 0$$

- That is, $t \geq -1/3$, or $t \geq 0$

Outline

Least Common Multiple

Diophantine Equations: Examples

Diophantine Equations: Theorem

Modular Division

Division mod 7

$\times$	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Division mod 7

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0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

- Given $a \neq 0$ and b, there exists x such that $a \times x \equiv b \pmod 7$

Division mod 7

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0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

- Given $a \neq 0$ and b, there exists x such that $a \times x \equiv b \pmod 7$
- x plays the role of modular division $x = b/a \pmod 7$

Division mod 6

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Division mod 6

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
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3	0	3	0	3	0	3
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- But there is no x s.t. $3 \times x \equiv 1 \pmod{6}$
- We can't divide 1 by 3 modulo 6!

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- Indeed, for every b,

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Uniqueness of Inverses

Lemma

If a has a multiplicative inverse, then it is unique

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Proof

If x and y are multiplicative inverses of a, then

$$x = x \times (a \times y) = (x \times a) \times y = y$$

$\square$

Existence of Inverses

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a has a multiplicative inverse modulo n if and only if $\gcd(a, n) = 1$

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- For fixed a and n, this Diophantine equation has a solution (x) iff $\gcd(a, n) \mid 1$ $\square$

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  - Now $x \equiv b/a \equiv b \times s \pmod{n}$

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