

Even and Odd Numbers

Vladimir Podolskii

Computer Science Department, Higher School of Economics

Outline

Even and Odd Numbers

Piece on a Chessboard

Summing up Digits

Switching Signs

Advanced Signs Switching

Even and Odd Numbers

- Even numbers are integer numbers divisible by 2

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Even and Odd Numbers

- The properties of a number to be even and odd are important **invariants**



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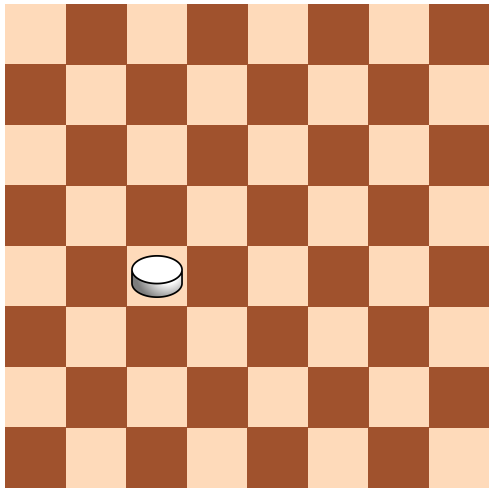
Piece on a Chessboard

Puzzle

A piece on a chessboard can move to any cell adjacent by edge to the current one. Can it return to the original position after 17 moves? What about 18 moves?

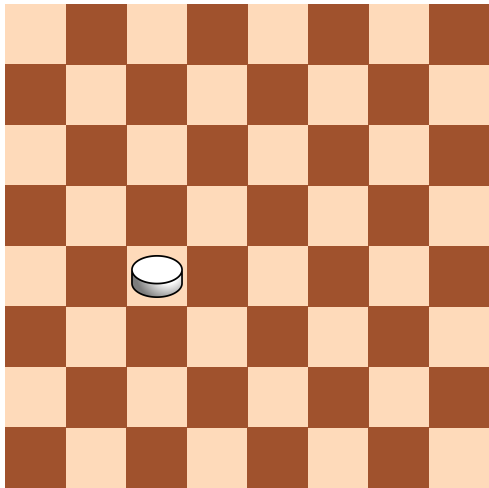
Piece on a Chessboard

Let's start with a more simple case of 18 moves



Piece on a Chessboard

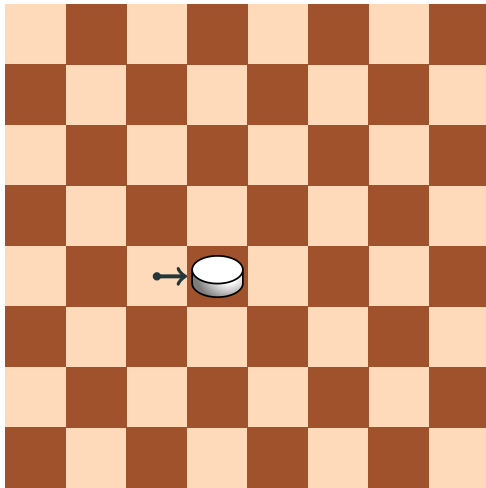
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We can return after 2 steps

Piece on a Chessboard

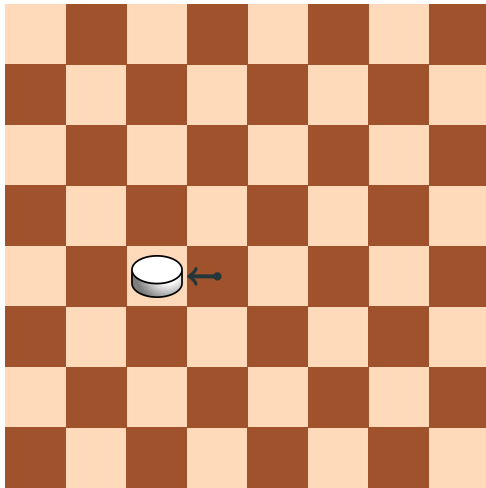
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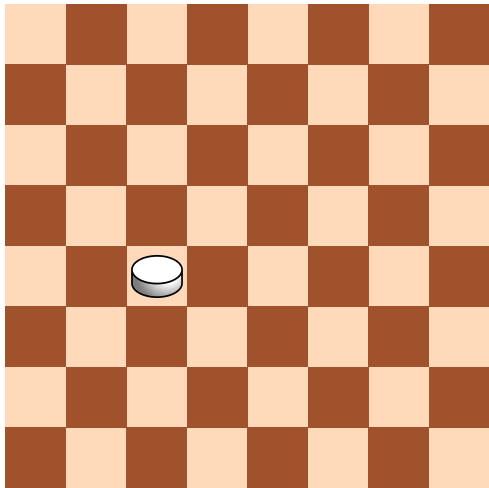
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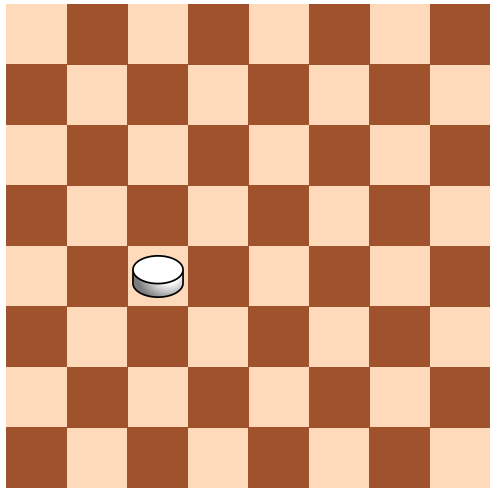


We can return after 2
steps

Repeat 9 times

Piece on a Chessboard

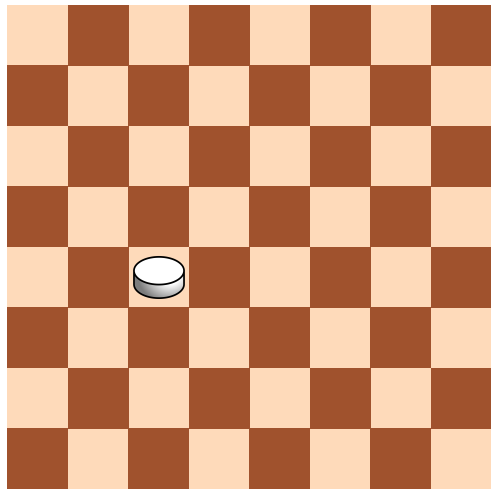
The same argument does not work for 17 steps



Indeed, 17 is odd

Piece on a Chessboard

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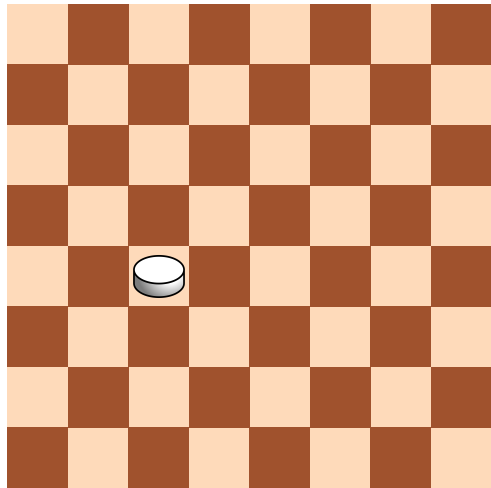
Indeed, 17 is odd

Observation: after even number of steps a piece is on the light field

After odd number of steps it is on the dark field

Piece on a Chessboard

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Indeed, 17 is odd

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After odd number of steps it is on the dark field

No way to get back after odd number of steps

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Is it possible to place signs in the expression $\pm 1 \pm 2 \pm \dots \pm 9$ to get as a result the sum 100? Can we get 2?

$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9$$

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- Let's start with 100

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$$\pm 1 \pm 2 \pm 3 \pm 4 \pm 5 \pm 6 \pm 7 \pm 8 \pm 9$$

- Let's start with 100
- We get the largest sum if place '+' everywhere

Summing up Digits

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$$+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9$$

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$$+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

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Summing up Digits

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$$+ 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

- Let's start with 100
- We get the largest sum if place '+' everywhere
- So we cannot get anything greater than 45

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- Note that there are 5 even numbers and 5 odd numbers in the sequence

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- What about the sum 2?
- Note that there are 5 even numbers and 5 odd numbers in the sequence
- No matter how we put the signs, the sum will be odd

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- What about the sum 2?
- Note that there are 5 even numbers and 5 odd numbers in the sequence
- No matter how we put the signs, the sum will be odd
- So the sum 2 is impossible

Summing up Digits

Puzzle

Which sums is it possible to get by place signs in the expression $\pm 1 \pm 2 \pm \dots \pm 9$?

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- We have seen two obstacles: the sum is **odd** and the sum is **at most 45** (and symmetrically, **at least -45**)

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- But what about other sums?

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Which sums is it possible to get by place signs in the expression $\pm 1 \pm 2 \pm \dots \pm 9$?

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- But what about other sums?
- It turns out, **all sums satisfying restrictions are possible!**
We will not discuss it, but you can try to show it yourself

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Which sums is it possible to get by place signs in the expression $\pm 1 \pm 2 \pm \dots \pm 9$?

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- But what about other sums?
- It turns out, **all sums satisfying restrictions are possible!**
We will not discuss it, but you can try to show it yourself
- For this consider the following **"greedy" algorithm**. Start from right to left and place the signs greedily:
if the current sum is less than the goal, increase the sum, otherwise decrease it

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Puzzle

We are given a 4×4 table. In left top corner there is '—' sign. In all other cells there are '+' signs. On one step we are allowed to switch all signs in some column or some row. Can we switch all signs to '+'?

—	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

Switching Signs

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

Switching Signs

-	+	+	+
+	+	+	+
+	+	+	+
+	+	+	+

- If we try to do it, we can notice that the number of minuses is always odd

Switching Signs

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-	-	-	-

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+	+	+	+
+	+	+	+
+	+	+	+

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- This is an **invariant**!

Switching Signs

-	+	+	+
+	+	+	+
+	+	+	+
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- If we try to do it, we can notice that the number of minuses is always odd
- This is an **invariant**!
- Cannot switch to all '+' since then the number of '-' would be even (zero)

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We are given a 4×4 table. In all corners there are '—' signs. In all other cells there are '+' signs. On one step we are allowed to switch all signs in some column or some row. Can we switch all signs to '+'?

—	+	+	—
+	+	+	+
+	+	+	+
—	+	+	—

Advanced Signs Switching

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

Advanced Signs Switching

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- Now, this looks way more tricky.

Advanced Signs Switching

-	+	+	-
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Advanced Signs Switching

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- If you know about residues modulo 4 (we will learn later), you can try to use them.

Advanced Signs Switching

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- Now, this looks way more tricky. The old solution does not work
- If you know about residues modulo 4 (we will learn later), you can try to use them. But it also does not help!
- Yet, there is a very simple solution
- If you do not know it, think a bit more before watching further for a better experience

Advanced Signs Switching

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-	+	+	-

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- The idea is simple. Let's look at the small part of the problem

Advanced Signs Switching

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- The idea is simple. Let's look at the small part of the problem
- If we switch a column or a row in the large table, we switch a row or a column in the small too (or do nothing)

Advanced Signs Switching

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- To solve a big problem we have to solve this 2×2 problem

Advanced Signs Switching

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- To solve a big problem we have to solve this 2×2 problem
- And 2×2 problem is unsolvable by the old argument!

Advanced Signs Switching

-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

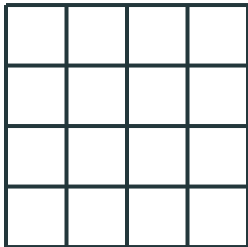
- We have found powerful obstacles: each 2×2 square should have even number of '-' for the puzzle to be solvable

Advanced Signs Switching

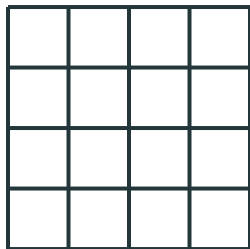
-	+	+	-
+	+	+	+
+	+	+	+
-	+	+	-

- We have found powerful obstacles: each 2×2 square should have even number of '-' for the puzzle to be solvable
- In fact these are **the only obstacles**! That is, if all 2×2 squares have even number of '-', it is possible to switch all signs to '+'.

Hint

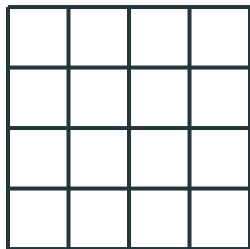


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- Switch the first row and the first column to all '+'

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+			
+			
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+			
+			
+			

- Assume that **our invariant** holds: each 2×2 square has even number of '—'
- Switch the first row and the first column to all '+'
- Use invariant to show that the rest of the table is filled with '+'

Conclusion

- **Invariants** are important

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- Can take many forms: numbers, “being even or odd”, equations, inequalities, etc.

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- **Double counting**: a special case

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- **Invariants** are important
- Help to prove **impossibility**
- And also to establish **termination** and **running time** bounds (important)
- Can take many forms: numbers, “being even or odd”, equations, inequalities, etc.
- **Double counting**: a special case
- This is not all: there are other forms of invariants (e.g. residues); we will see some more later on