Graph Coloring

Alexander Golovnev

Outline

Map Coloring

Graph Coloring

Bounds on the Chromatic Number

Applications

South America

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Thus, there must be at least 4 colors

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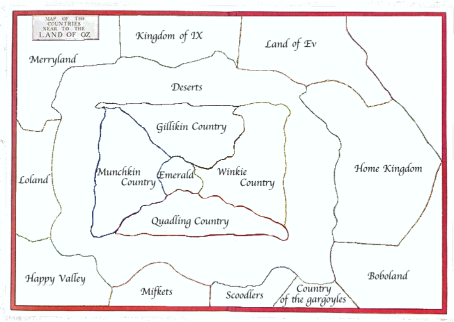
The Land of Oz



The Land of Oz

Emerald,
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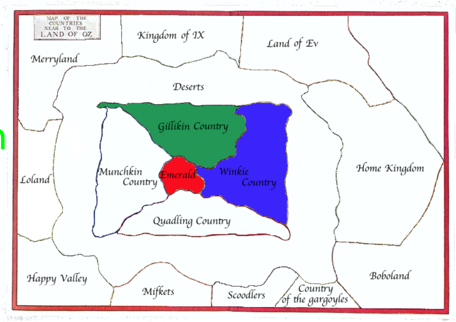

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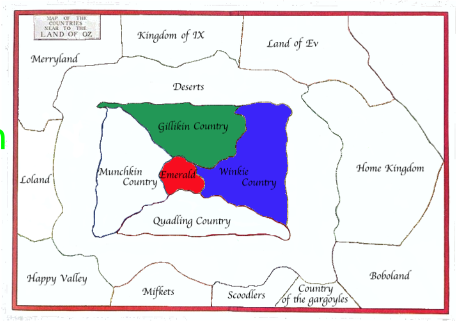

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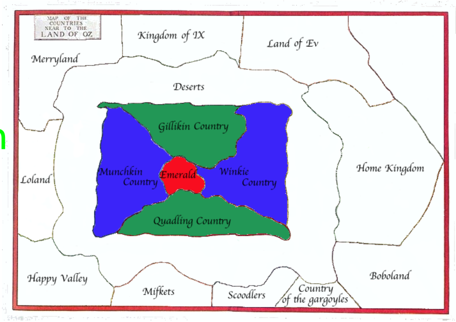

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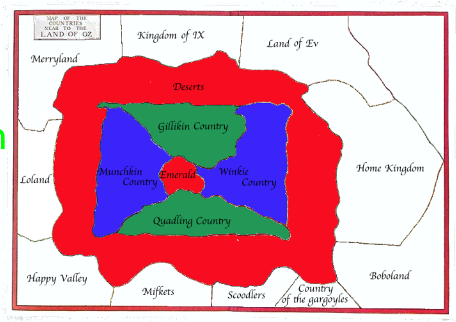

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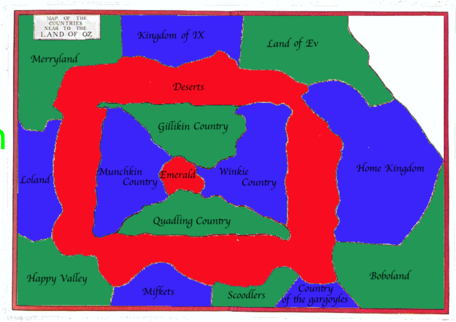

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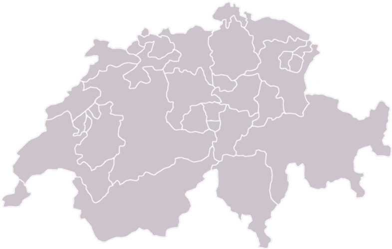
Swiss Cantons



Swiss Cantons

Requires
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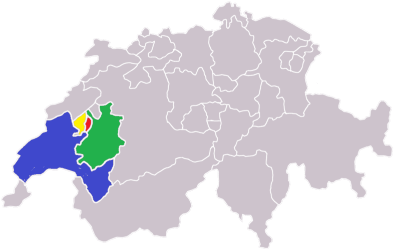

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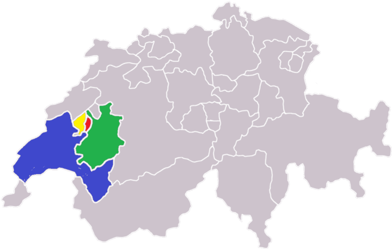

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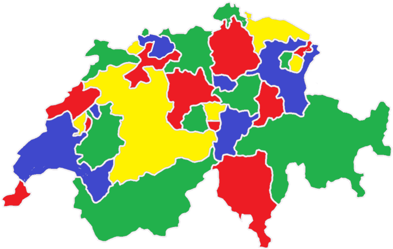

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Four Color Theorem

Theorem (Appel, Haken, 1976)

Every map can be colored with 4 colors.

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- Proved using a computer.
- Computer checked almost 2000 graphs.
- Robertson, Sanders, Seymour, and Thomas gave a much simpler proof in 1997 (still using a computer search).

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Graph Coloring

- A graph coloring is a coloring of the graph vertices s.t. no pair of adjacent vertices share the same color.

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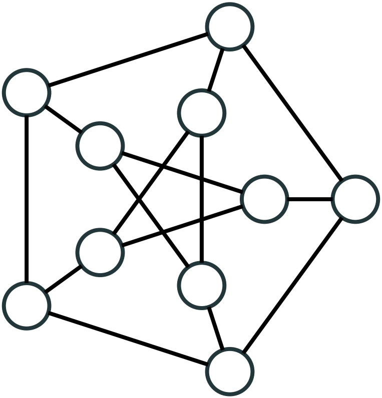
- The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the graph.

Chromatic Number



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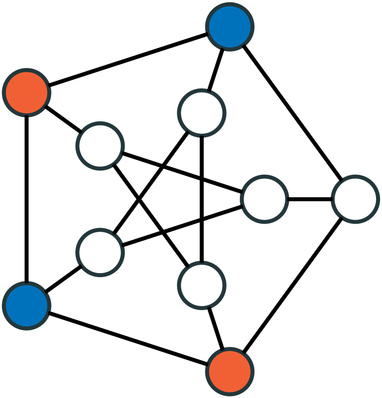

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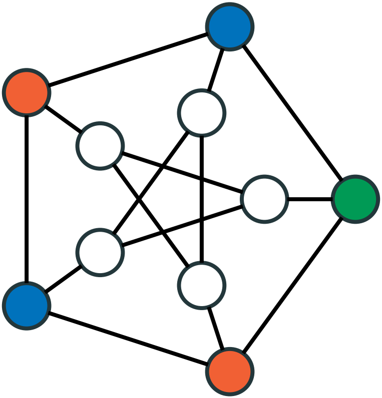

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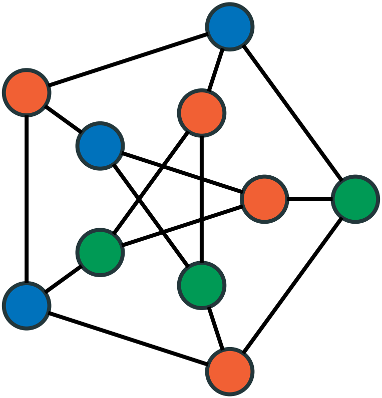
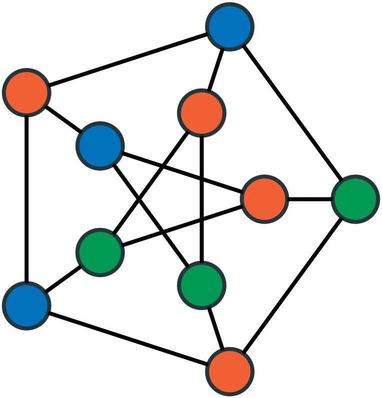


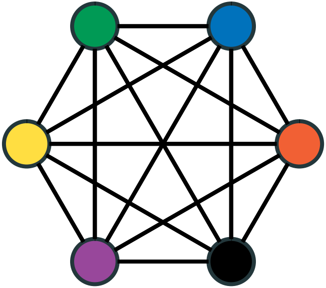
Chromatic Number

Chromatic number is 3



Full Graphs

The chromatic number of K_n is n.

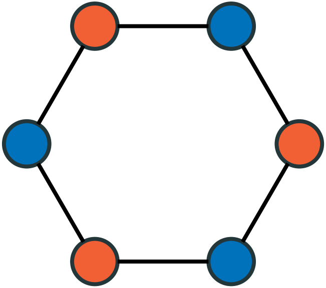


Path Graphs

For $n > 1$, the chromatic number of P_n is 2.

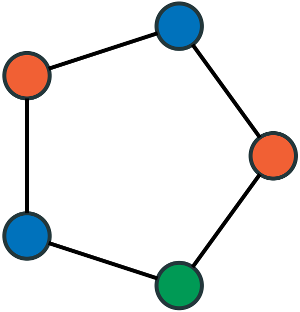
Cycle Graphs

For even n, the chromatic number of C_n is 2.



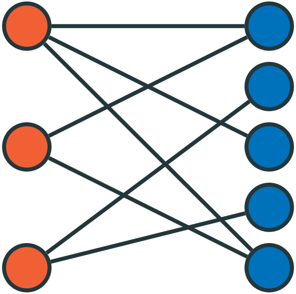
Cycle Graphs

For odd $n > 2$, the chromatic number of C_n is 3.



Bipartite Graphs

The chromatic number of a bipartite graph (with at least 1 edge) is 2.



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Every map corresponds to a planar graph, every planar graph can be formed from a map.

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Theorem (Weak Version)

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- Induction on the number of vertices n.

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- Induction on the number of vertices n.
- Base case. $n \leq 6$: can color with 6 colors.
- Induction assumption. All planar graphs on k vertices can be colored with 6 colors.
- Induction step. We'll show that any graph on $k + 1$ vertices can be colored with 6 colors.

Six Color Theorem. Proof

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Every planar graph contains a vertex v of degree at most 5.

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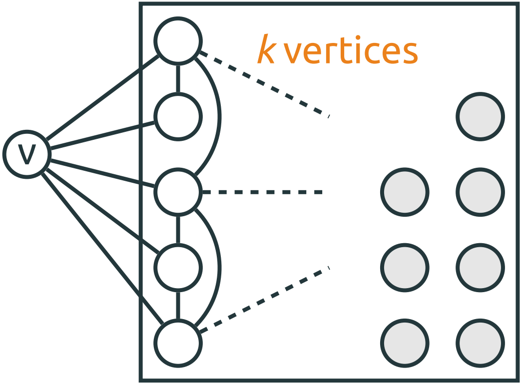

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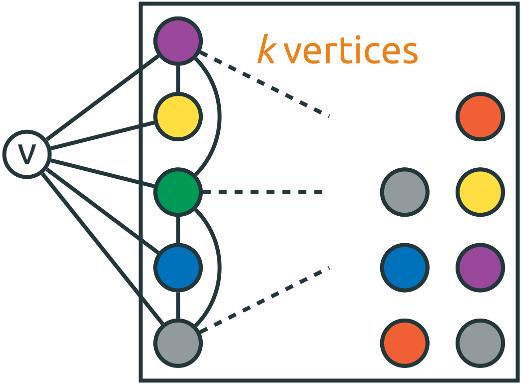

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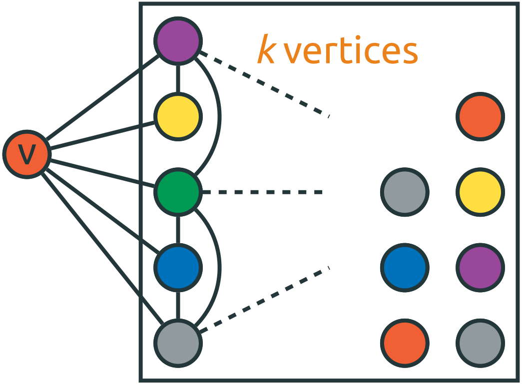

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Greedy Coloring

A graph G of maximum degree Δ can be colored with $\Delta + 1$ colors.

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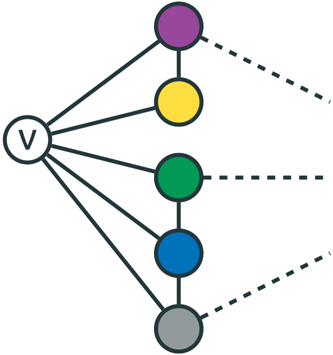

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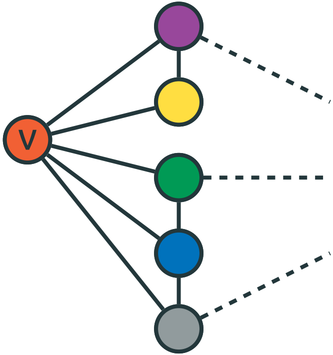

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Theorem (Brooks, 1941)

A graph G of maximum degree Δ can be colored with Δ colors, unless G is full (K_n) or a cycle of odd length (C_{2k+1}).

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Exam schedule

- Each student takes an exam in each of her courses
- All students in one course take the exam together
- One student cannot take two exams per day
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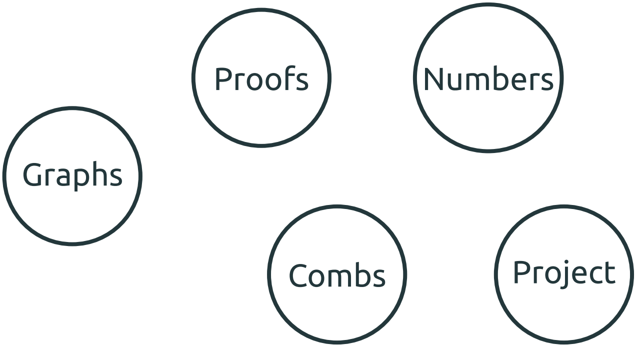

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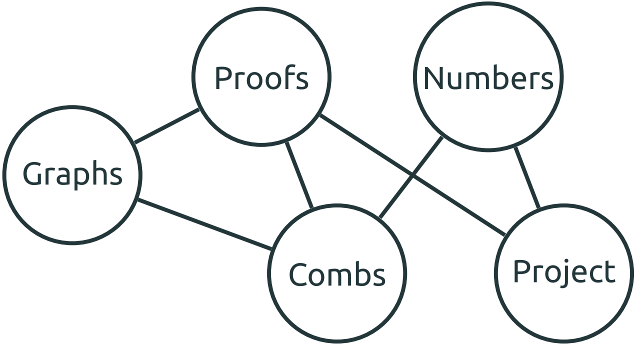

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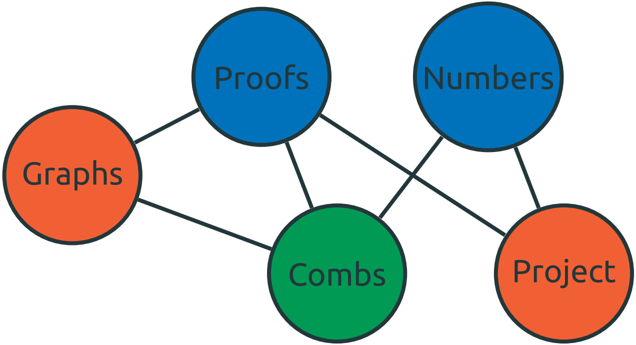
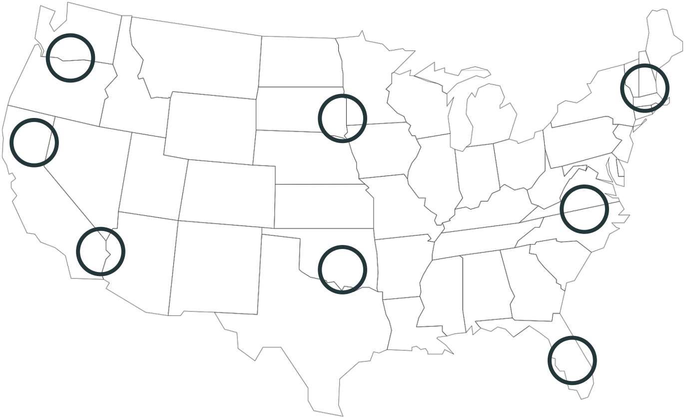


Bandwidth allocation

Different stations are allowed to use the same frequency if they are far apart. What is an optimal assignment of frequencies to stations?



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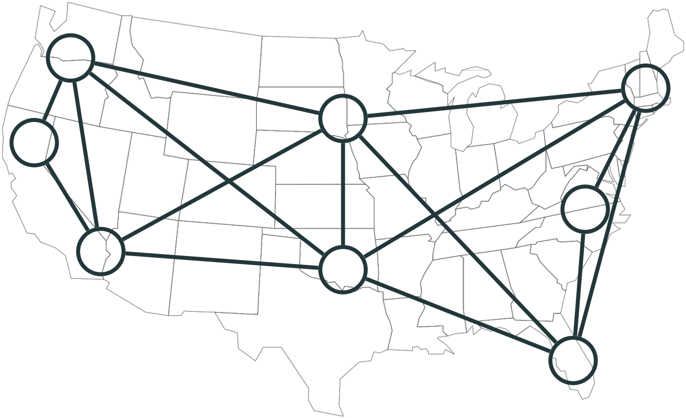

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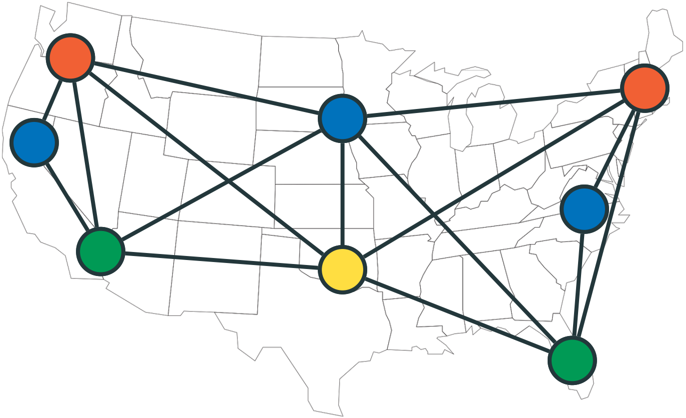

Other Applications

- Scheduling Problems
- Register Allocation
- Sudoku puzzles
- Taxis scheduling
- …