Handshaking Lemma

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Outline

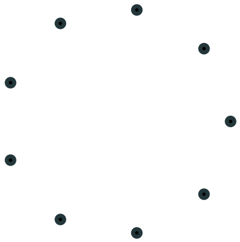
Handshaking Lemma

Total Degree

Revisiting a Puzzle

Puzzle

Is it possible to connect some pairs of nine points by segments so that each point is connected to five other points?



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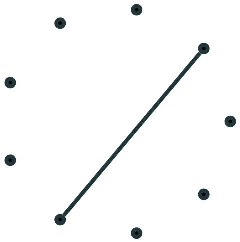

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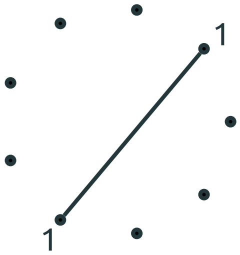

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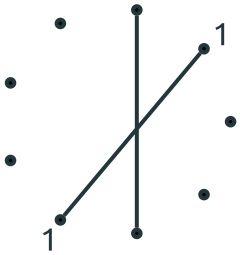

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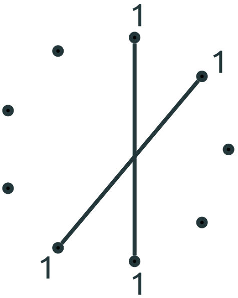

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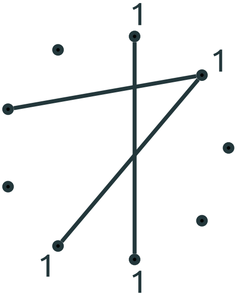

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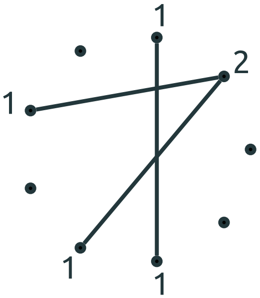

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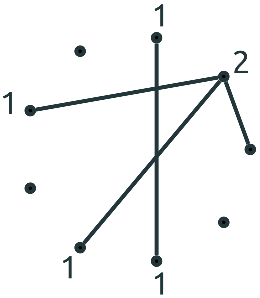

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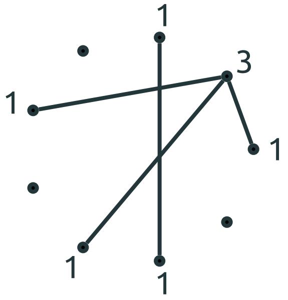

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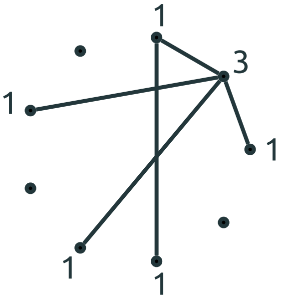

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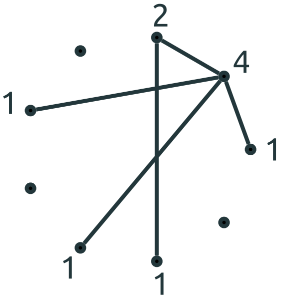

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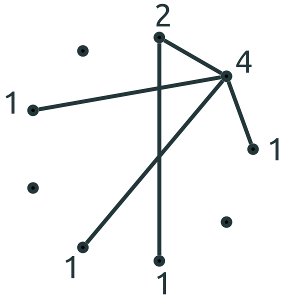

the number of odd points is always even, hence we will never reach a situation when there are 9 points of degree 5

Handshaking

- Before a business meeting, some people shook hands. Then the number of people who made an odd number of handshakes is even

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- Before a business meeting, some people shook hands. Then the number of people who made an odd number of handshakes is even
- In graph terms: A graph has an even number of odd nodes

Degree Sum Formula

Lemma

For any graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges:

$$\sum_{v \in V} \mathrm{degree}(v) = 2 \cdot |E| \,.$$

Degree Sum Formula

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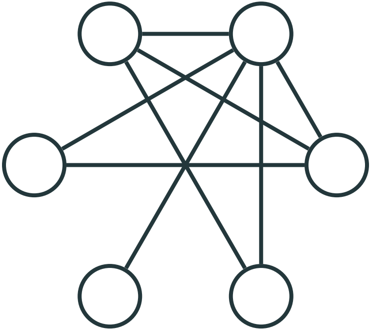
For any graph $G(V, E)$, the sum of degrees of all its nodes is twice the number of edges:

$$\sum_{v \in V} \mathrm{degree}(v) = 2 \cdot |E| \, .$$

Implies the handshaking lemma: if a graph had an odd number of odd nodes, then the sum of degrees would be also odd.

Example



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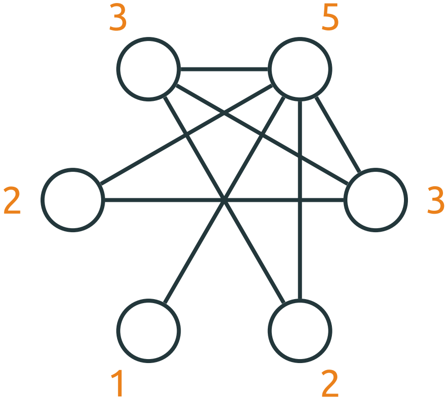

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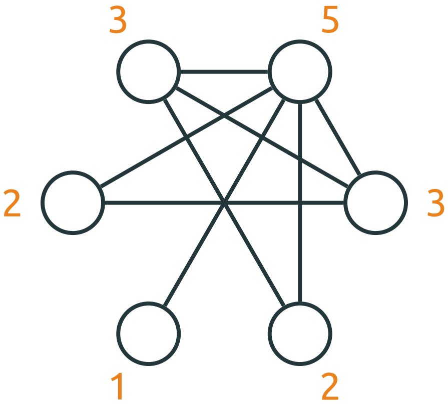
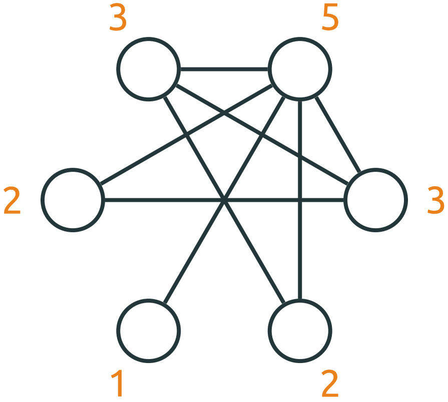


$$3 + 5 + 3 + 2 + 1 + 2 = 2 \cdot 8$$

Proof



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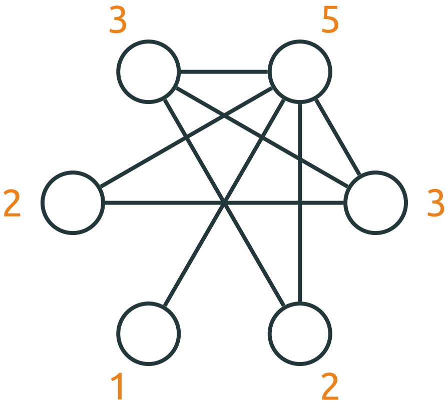

remove all edges

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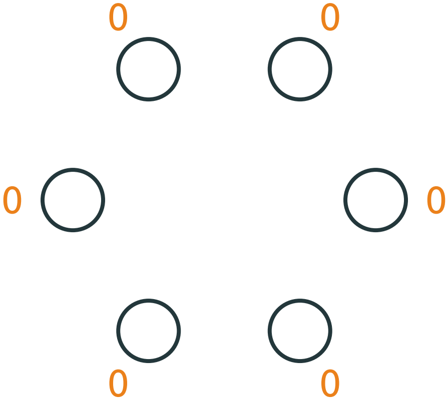

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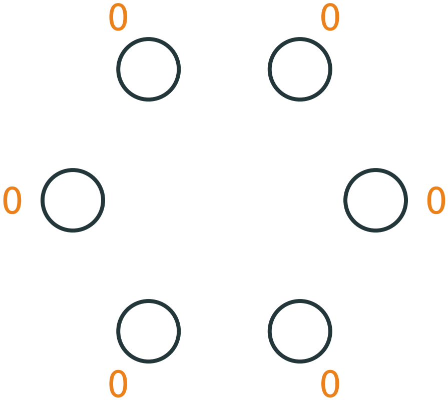

now, start adding them back, one by one

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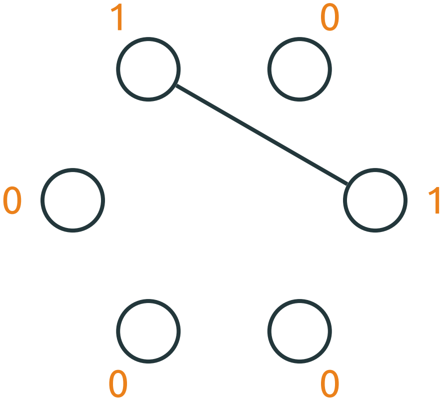

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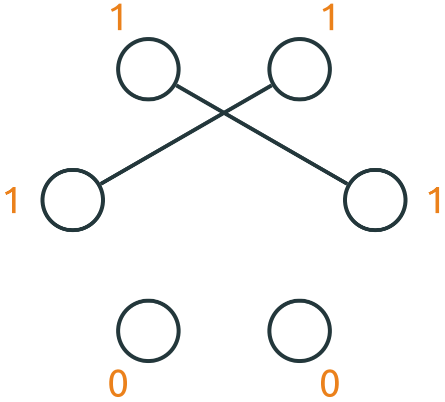

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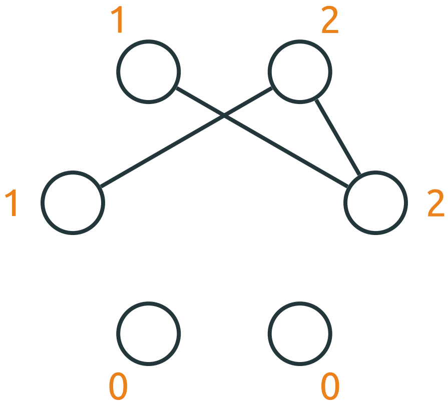

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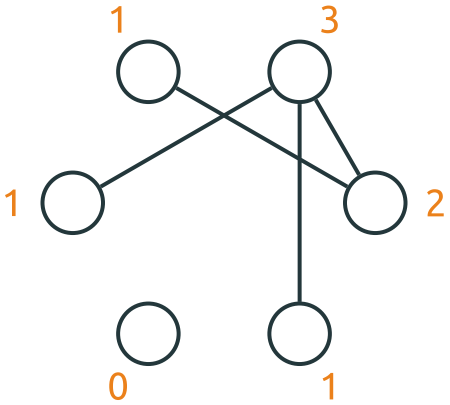

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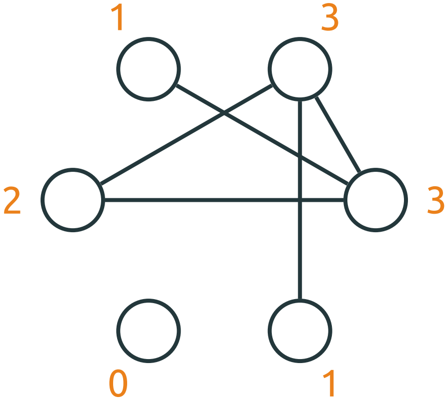

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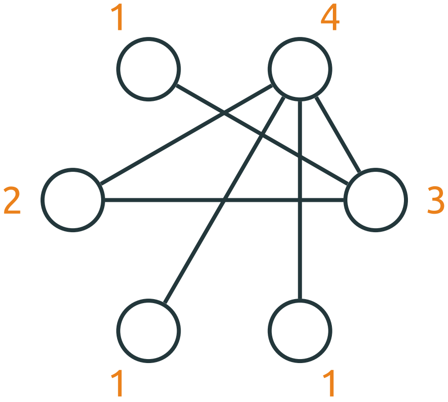

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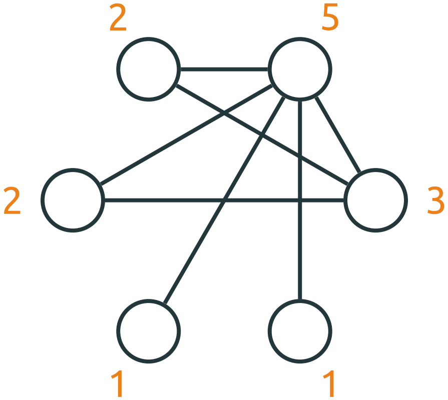

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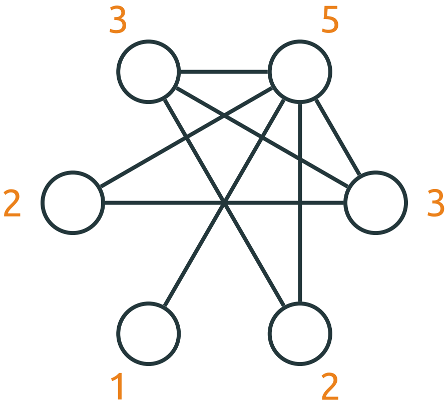

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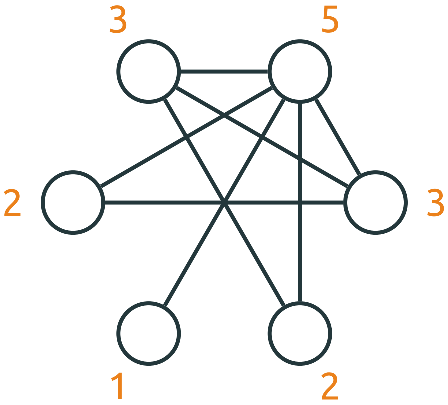
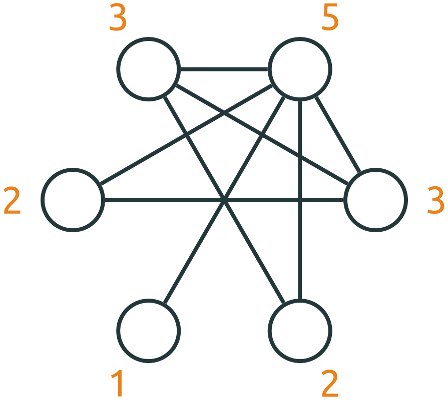


each edge contributes 2 to the sum of degrees

Proof



as well as to twice the number of edges!

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- Induction step: when we add an edge, the sum of degrees increases by 2

Outline

Handshaking Lemma

Total Degree

Exam

Problem

At an exam, each of 20 students solved 3 problems. Each problem was solved by 5 students. What was the number of problems?

https://commons.wikimedia.org/wiki/File%3AATC_Admission_Exam.JPG

Solution

- Assume that each student wrote
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- Then, there are $20 \times 3 = 60$ pieces of paper
- For each problem, let's stack together all its five solutions
- Thus, the number of problems is $60/5 = 12$

In Graph Terms

20 students

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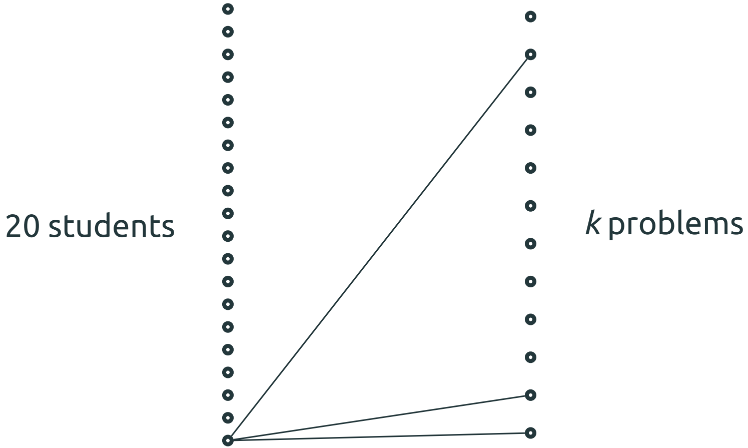

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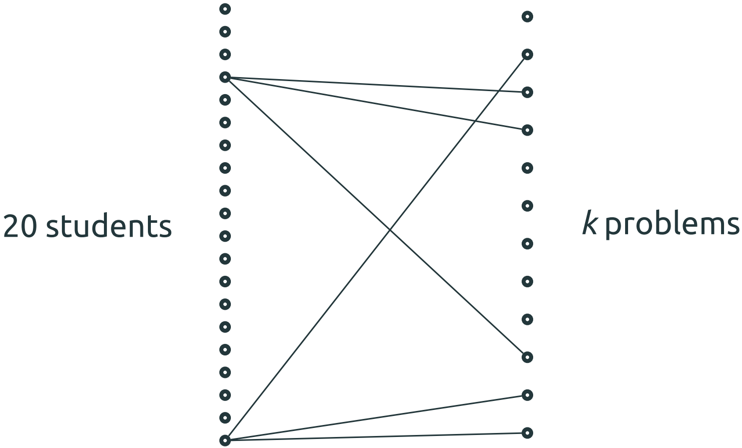

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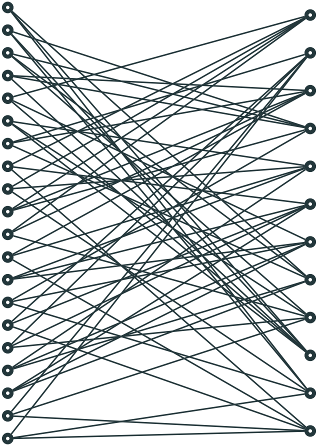
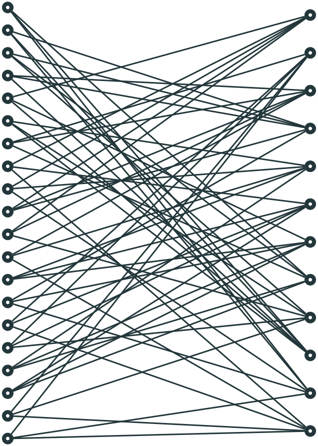


k problems

In Graph Terms

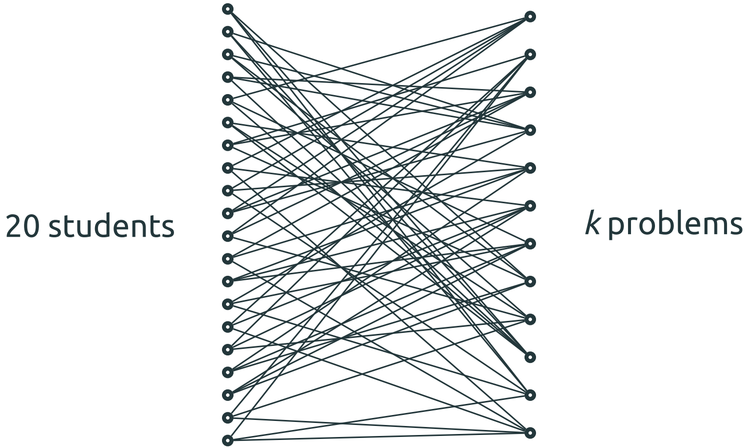
20 students



k problems

$$20 \times 3 = |E| = 5 \times k$$

In Graph Terms



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- We used the double counting technique to solve the problem:
    - on one hand, the number of edges is equal to the total degree of the left part (i.e., the sum of all degree of the nodes from the left
    - on the other hand, it is equal to the total degree of the right part