Ramsey Numbers

Alexander Golovnev

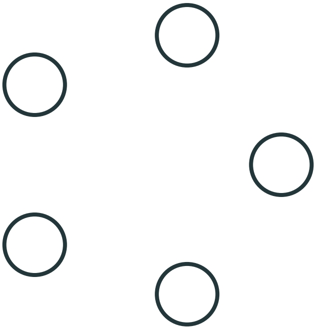
Outline

Balanced Graphs

Ramsey Numbers

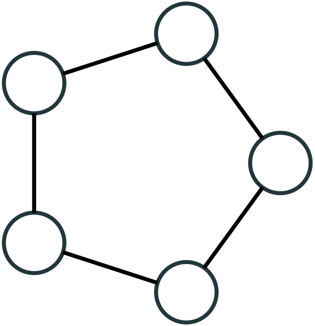
Existence of Ramsey Numbers

Balanced Graphs

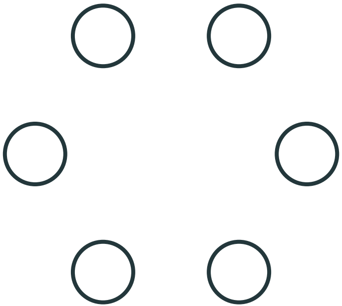


Balanced Graphs

The Clique Number and the Independence Number of C_5 are 2.

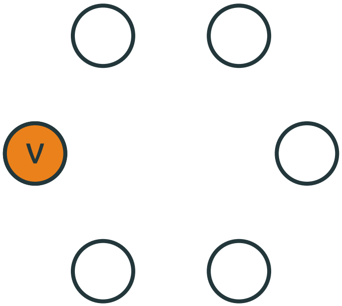


Balanced Graphs



Balanced Graphs

Consider an arbitrary vertex v of G

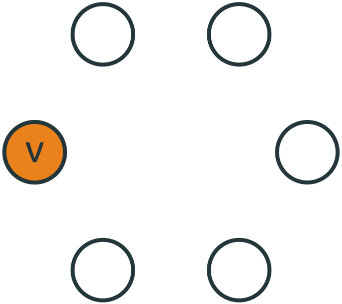


Balanced Graphs

Consider an arbitrary vertex v of G

Either v is connected to ≥ 3 vertices

Or v is not connected to ≥ 3 vertices



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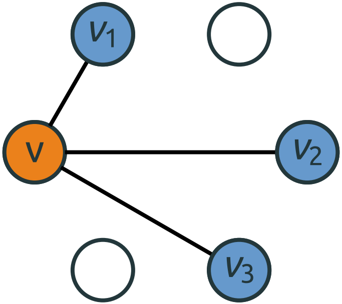

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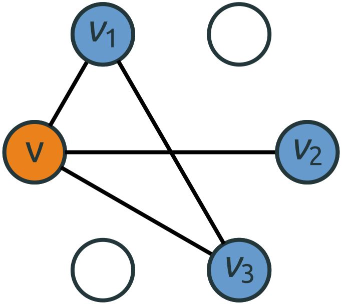

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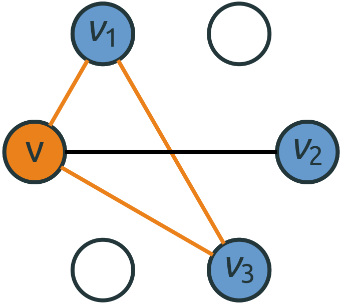

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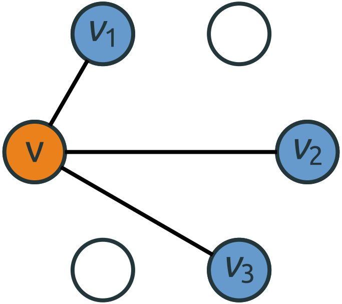

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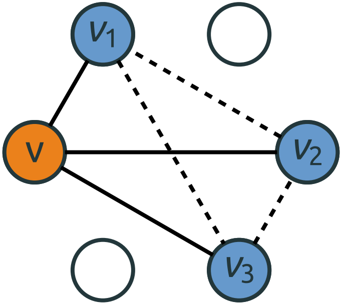

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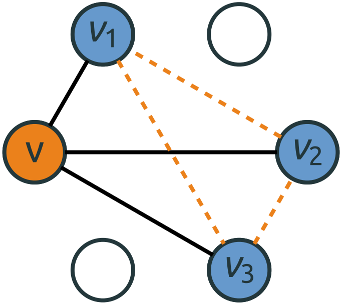

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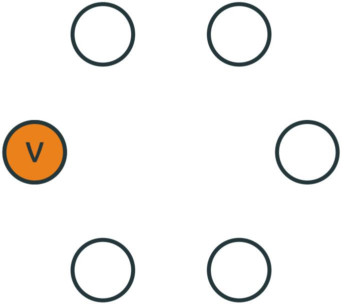

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Children Friendship

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- Consults Erdös, Turán and Sós
- This holds in any group of 18 and more people

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- $43 \leq R(5, 5) \leq 48$

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- Therefore, $R(k, \ell)$ always exists!

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- Pick an arbitrary vertex v. A – set of neighbors of v, B – the remaining vertices

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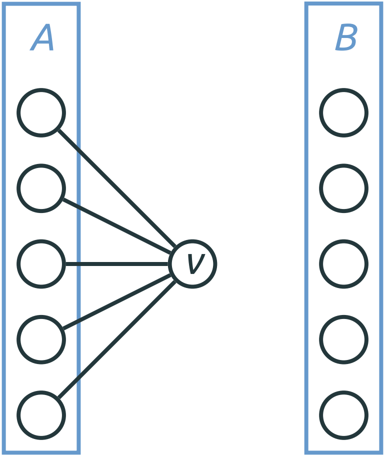

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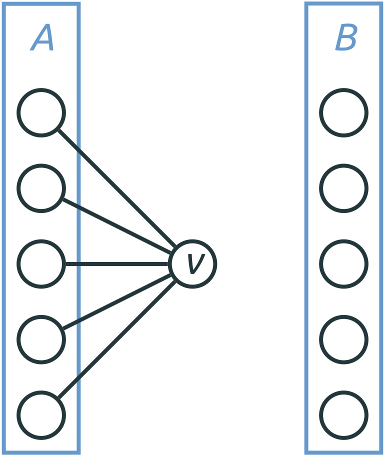

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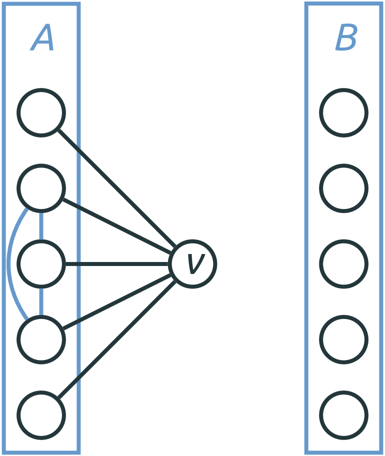

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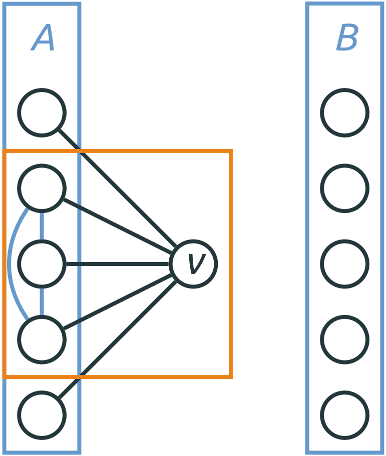
If $|A| \geq R(k-1, \ell)$, A contains a $k-1$-Clique

Then $|A| \cup v$ contains a k-Clique



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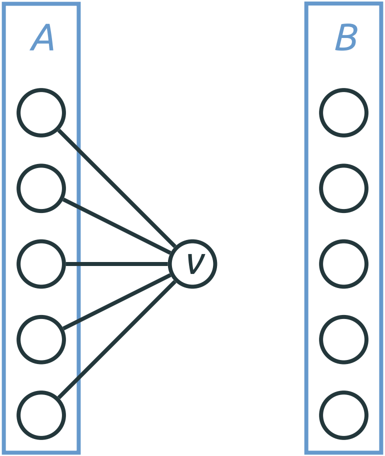

If $|B| \geq R(k, \ell - 1)$, B contains an $\ell - 1$-IS

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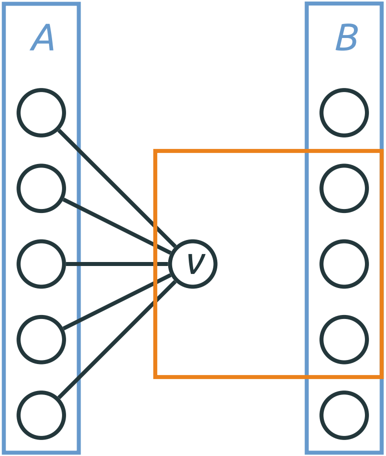

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Then $|B| \cup v$ contains an ℓ-IS