15-puzzle

Alexander Shen

LIRMM / CNRS, University of Montpellier

Outline

The Game

Permutations

Proof: The Difficult Part

Mission Impossible

Classify a Permutation

15-Puzzle

By Micha L. Rieser, `https://commons.wikimedia.org/w/index.php?curid=3104433`

Game Rules

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- reverse: return to the standard position from a given one
- try yourself: if you succeed, we pay you $1000

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- a *proof* of impossibility existed
- challenge: reinvent this proof

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- cat and permutations: `https://www.youtube.com/watch?v=MDhsTOnd3J8`

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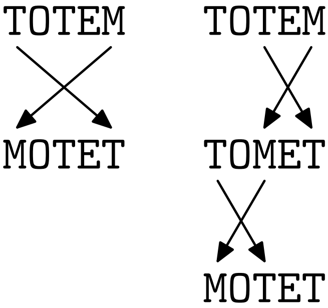
- $n \to n + 2$: transposition: twice = nothing
- conjecture: permutations are of two types: even or odd

A Counterexample?



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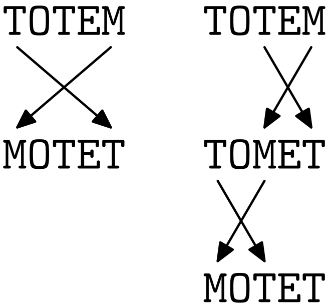

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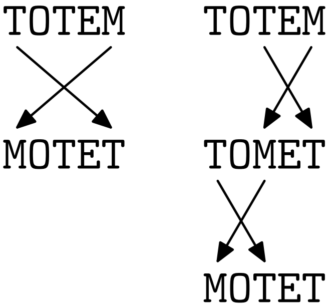

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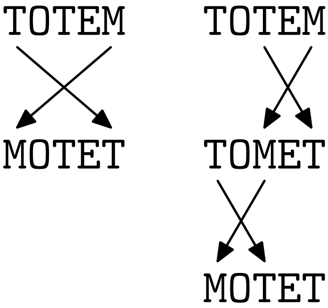

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spoiler: two `T`'s are mixed (we assumed that all letters are different)

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- some permutations can be obtained only by even number of transpositions, while others can be obtained only by odd number of transpositions

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- note that *neighbor* transposition does not change order in other pairs

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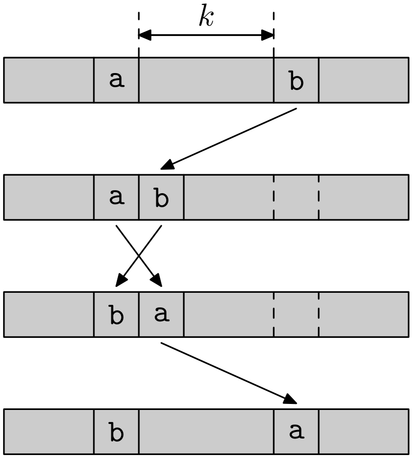

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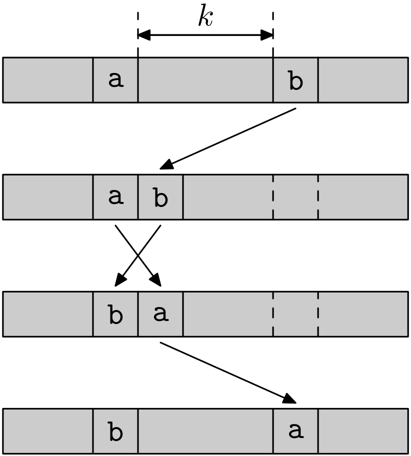

$k + 1 + k = 2k + 1$ neighbor transpositions

$= 1 \pmod 2$

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- generalization: permutations, transpositions
- classification theorem: some permutations need even number of transpositions, other require an odd number
- OK — so what?

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- each move is a transposition of an empty cell and some its neighbor
- the required exchange requires an *odd* number of moves
- but for other reasons it requires an *even* number of moves (why?)
- so the required exchange is impossible

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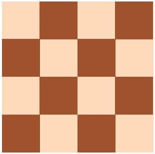

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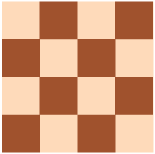

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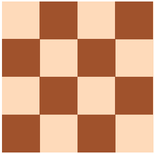

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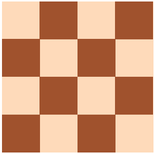

each move changes the color of the empty cell. bringing it back requires an even number of moves. No risk to lose money!

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Programming Assignment

An array $a[1], \ldots, a[n]$ is given, containing a permutation of numbers $1, \ldots, n$.

Output 0 if the permutation is even (requires an even number of transpositions) and 1 otherwise.

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- …to see if this number is even or odd

Implementation?

```
// sorting a[1]..a[n]
sign=0 // sign = the number of transpositions mod 2
s=0 // first s elements are at the right places
while (s<n){
  u=s+1; t=u; //a[t] is minimal among a[s+1]..a[u];
  while (u<n){
    u=u+1;
    if a[u]<a[t] {t=u;}
  }
  // a[t] is minimal among a[s+1]..a[n]
  tmp=a[s+1]; a[s+1]=a[t]; a[t]=tmp; sign=1-sign;
}
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What is wrong with this code?

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- $O(n^2)$ (two nested loops)
- More efficient algorithm?
- Hint: use $O(n \log n)$ sorting to get $O(n \log n)$ algorithm
- Challenge: can you think of $O(n)$ algorithm?

Project

A position in 15-puzzle is given (the empty space is in the right corner, the other numbers are listed in the book order, so the standard position is represented as $1, 2, 3, 4, \ldots, 14, 15$). Find out whether the puzzle is solvable; if yes, give the sequence of moves (for each move, just write the number of a piece that is moving at that step).