Conditional Probability

Alexander Shen

LIRMM / CNRS & University of Montpellier

Outline

What is Conditional Probability?

How Reliable Is the Test?

Bayes' Theorem

Conditional Probability: A Paradox

Past and Future

Independence

Demographic Example

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A, B, C form a queue in the random order (all orderings are equiprobable). What is the probability of the events X = "A is the second", Y = "A is before B in the queue"? What are the probabilities $\Pr[X \mid Y]$ and $\Pr[Y \mid X]$?

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- test is quite reliable, isn't it?

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- only $1/12$ of test-positives are ill

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- $\Pr[T] = \Pr[T \text{and } D] + \Pr[T \text{and not } D] =$ $0.009 + 0.099 = 0.108$

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- $\Pr[T] = \Pr[T \text{and } D] + \Pr[T \text{and not } D] =$ $0.009 + 0.099 = 0.108$
- $\Pr[D \mid T] = \Pr[D \text{and } T] / \Pr[T] =$ $0.009/0.108 = 9/108 = 1/12 =$ $0.0833\ldots \approx 8.3\%$

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$$= \Pr[B] \cdot \Pr[A \mid B] + (1 - \Pr[B]) \cdot \Pr[A \mid \text{not } B]$$

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- three components of Bayes' reasoning

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- "foreign address make the scam hypothesis much more probable, because it appears in scam messages much more often than in general"

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- take-home message from Bayes: *if condition B increases the probability of A by factor k, then condition A increases the probability of B by the same factor k*

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- event C: *at least one tail*
- contains HT, TH, TT: probability $3/4$
- event E: *two tails*
- contains TT: probability $1/4$
- conditional probability:

$$\Pr[E \mid C] = \frac{\Pr[E \text{ and } C]}{\Pr[C]} = \frac{\Pr[E]}{\Pr[C]} = \frac{1}{3}$$

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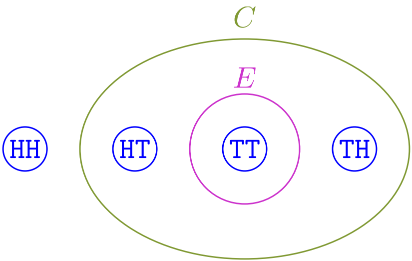

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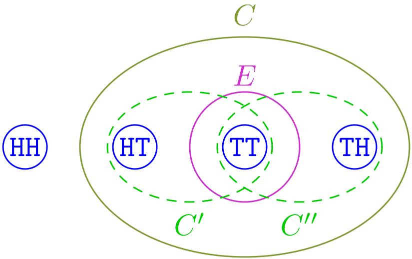

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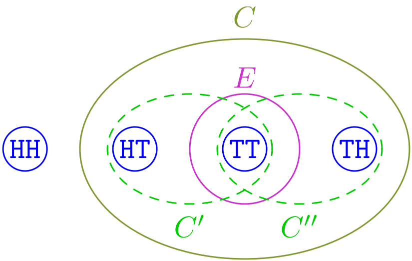

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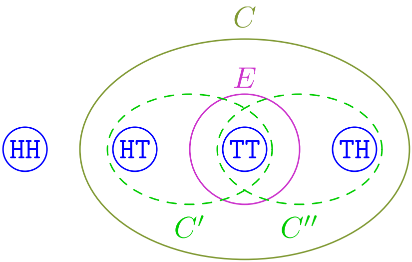

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increase / decrease / remain unchanged

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- crazy statistician brings the bomb to a plane because he thinks two bombs on the same plane are quite improbable

Edgar Poe Says

Nothing… is more difficult than to convince the merely general reader that the fact of sixes having been thrown twice in succession…is sufficient cause for betting the largest odds that sixes will not be thrown in the third attempt. A suggestion to this effect is usually rejected by the intellect at once. It does not appear that the two throws which have been completed, and which lie now absolutely in the Past, can have influence upon the throw which exists only in the Future. The chance for throwing sixes seems to be precisely as it was at any ordinary time… And this is a reflection which appears so exceedingly obvious that attempts to controvert it are received more frequently with a derisive smile than with any thing like respectful attention. The error here involved – a gross error redolent of mischief – I cannot pretend to expose within the limits assigned me at present… (Edgar Poe, 'The Mystery of Marie Roget', 1850)

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- "condition B makes A more probable"
- then condition A makes B more probable (symmetry)
- Bayes' formula: the factor is the same:

$$\Pr[B \mid A] = \frac{\Pr[A \mid B]}{\Pr[A]} \Pr[B]$$

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- $\Pr[\text{ ill } \mid \text{visiting a doctor}] > \Pr[\text{ ill }]$