Vertex Covers

Alexander Golovnev

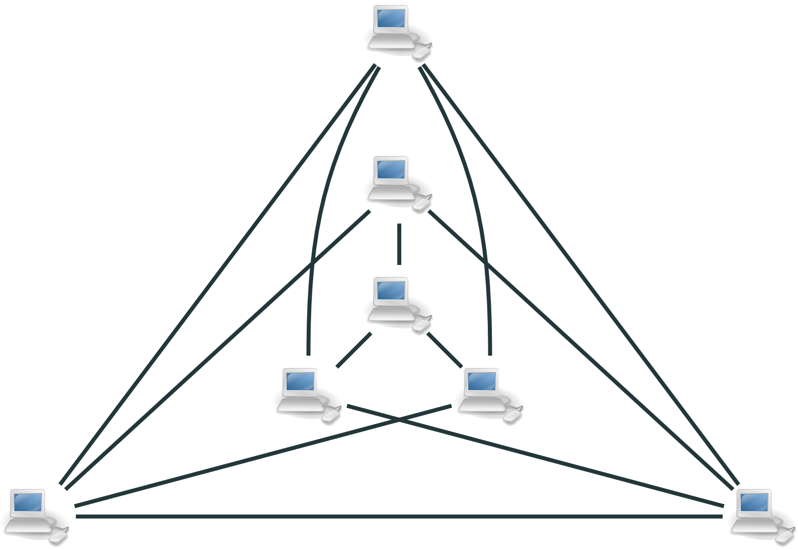
Outline

Antivirus System

Vertex Covers

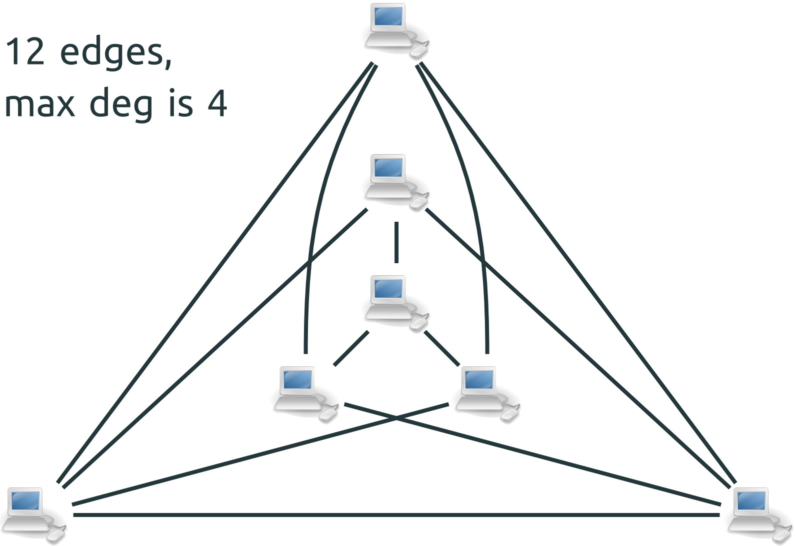
König's Theorem

Antivirus System



Antivirus System

12 edges,
max deg is 4



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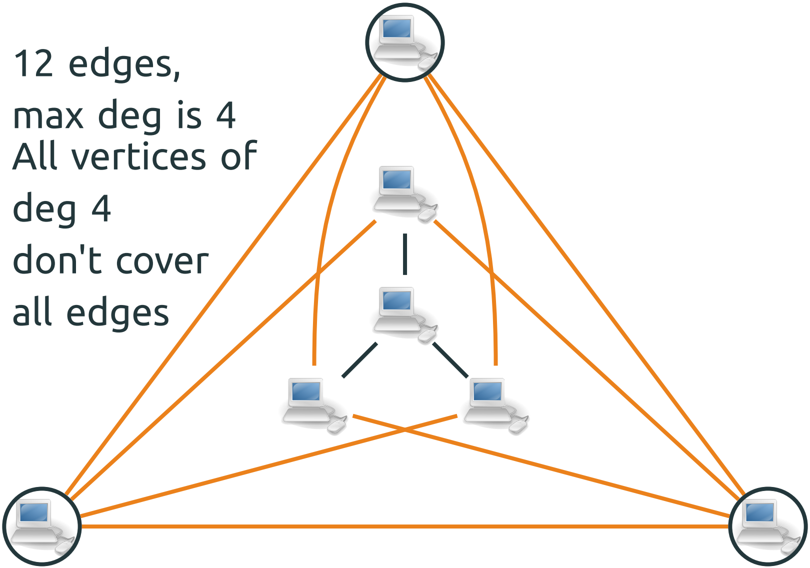

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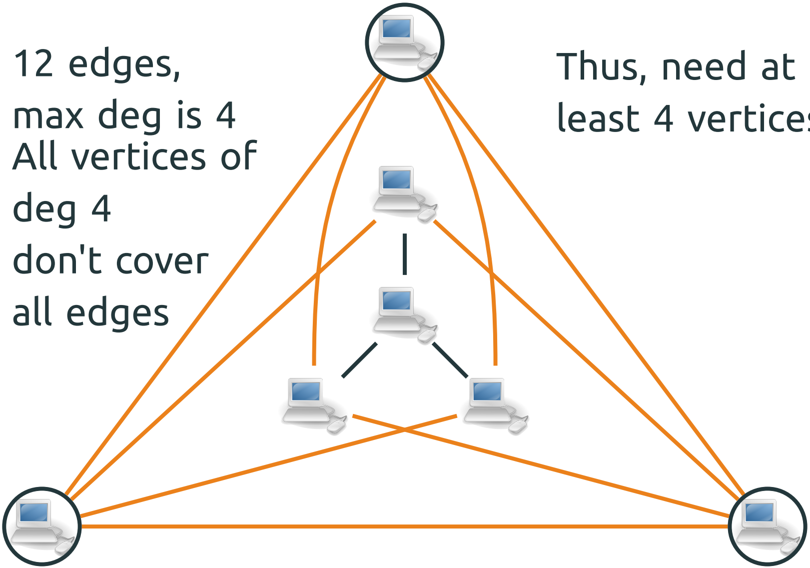

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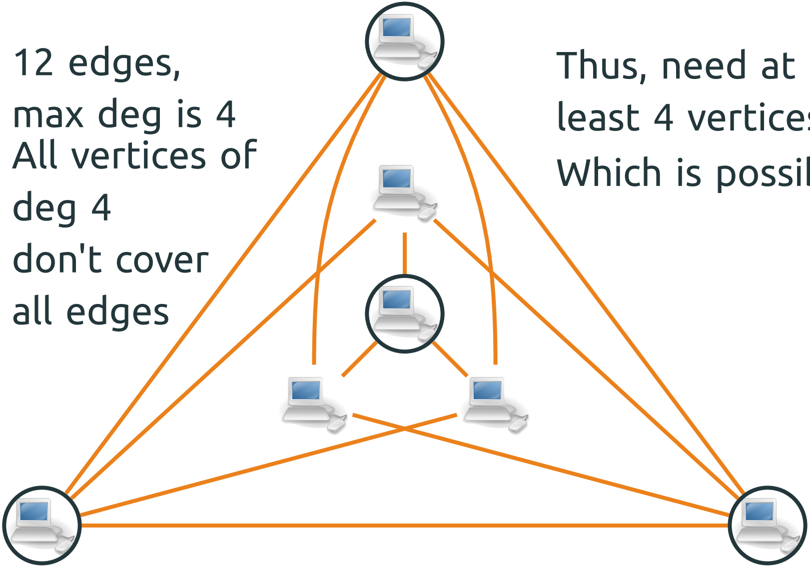

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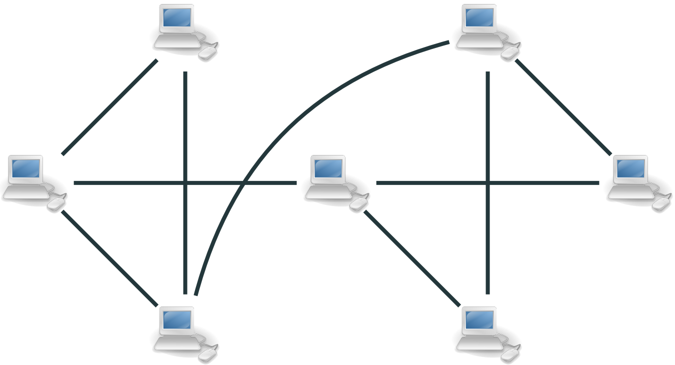

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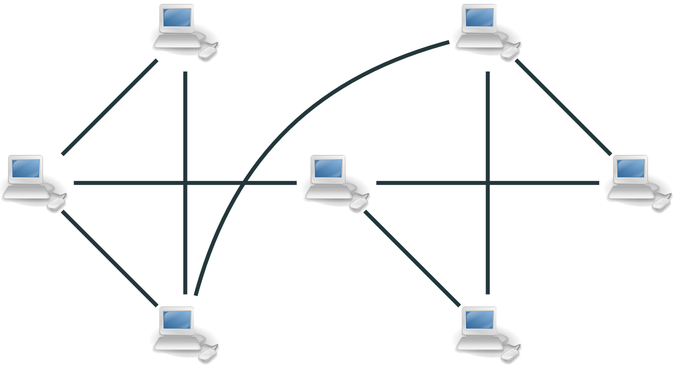

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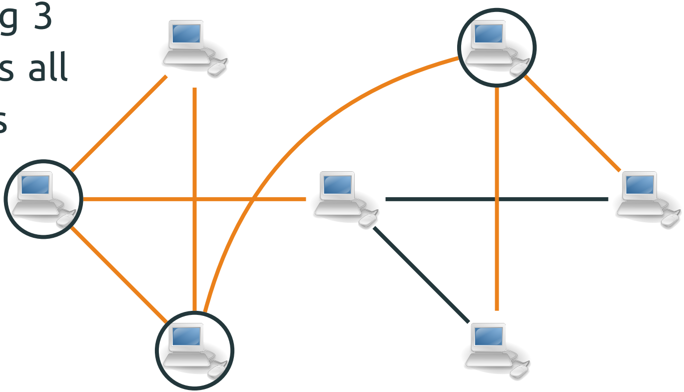

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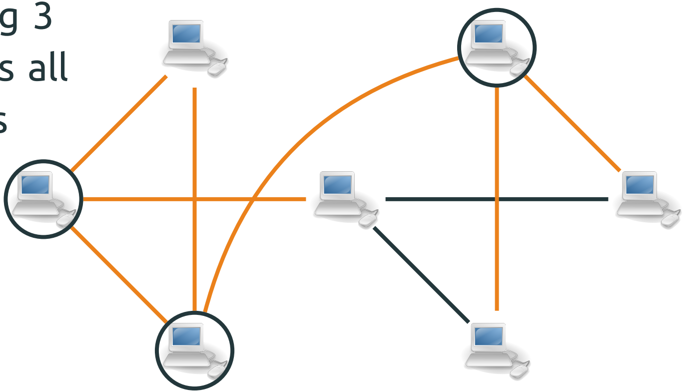

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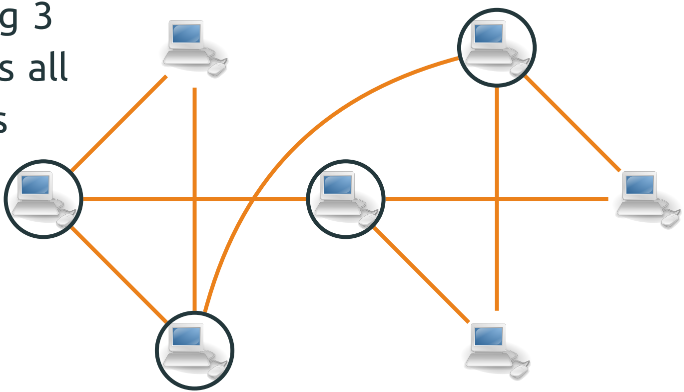

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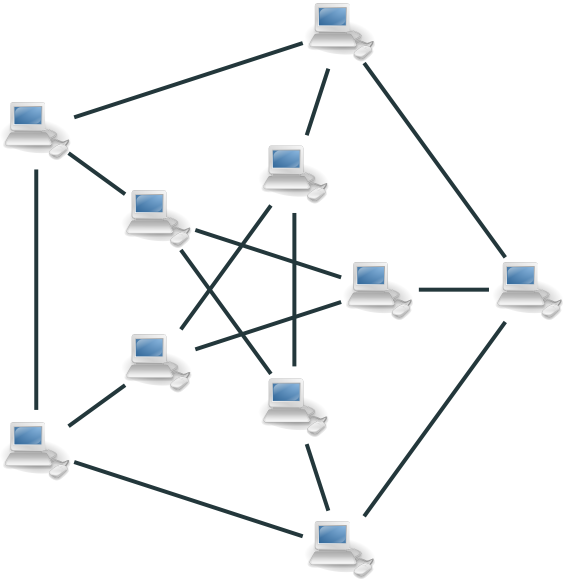

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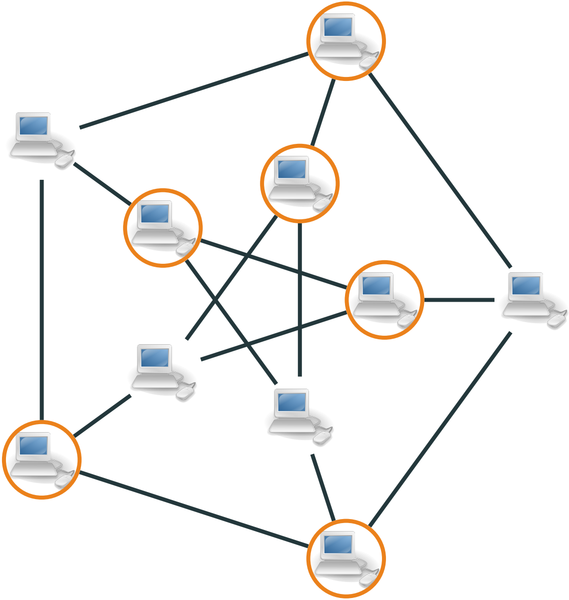

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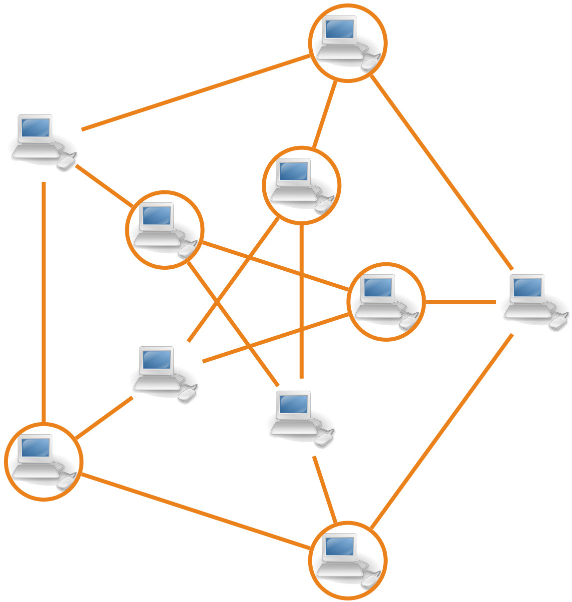

Outline

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Vertex Covers

König's Theorem

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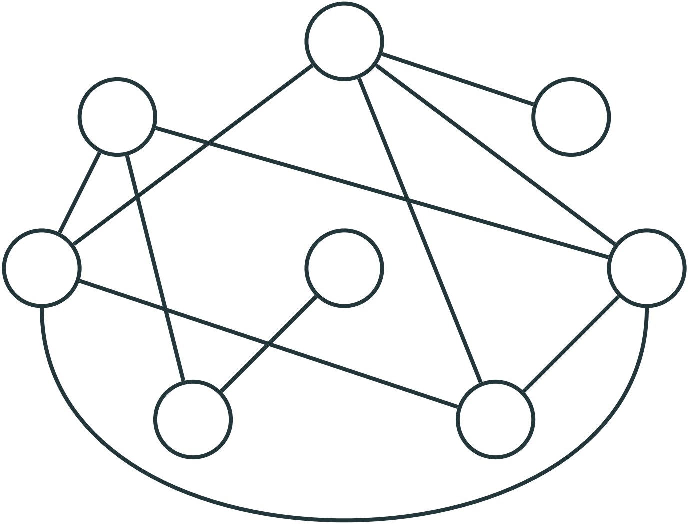
- The Size of a Minimum Vertex Cover is denoted by $\beta(G)$.

Vertex Covers: Examples



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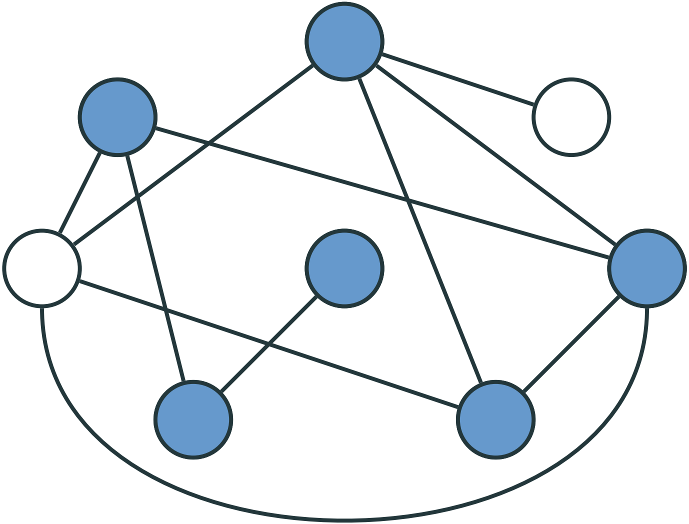
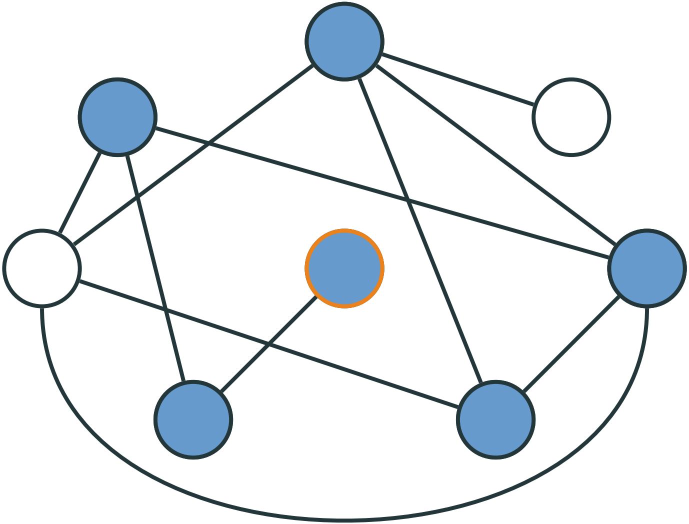
A Vertex Cover



Vertex Covers: Examples

A Vertex Cover

Not a Minimal VC



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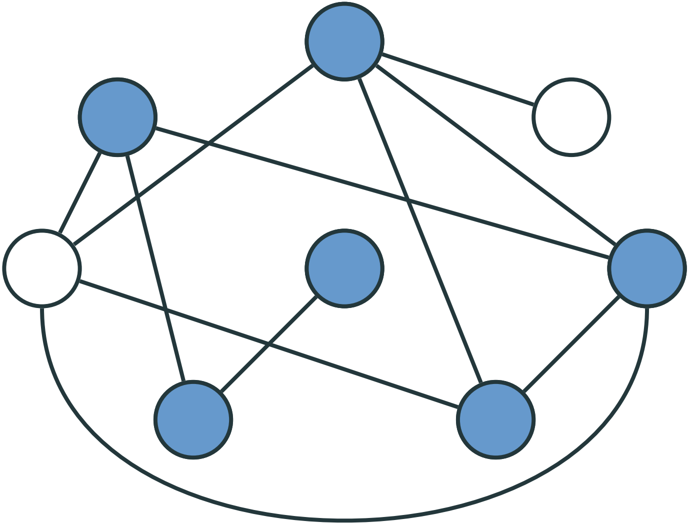

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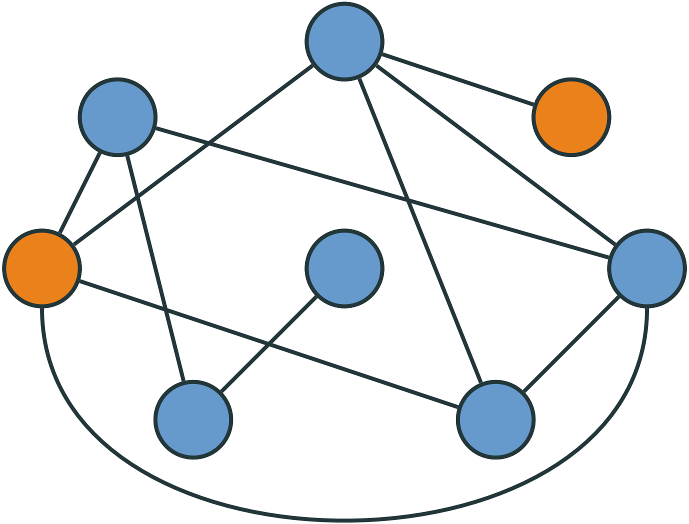

Vertex Covers: Examples

An Indepen-
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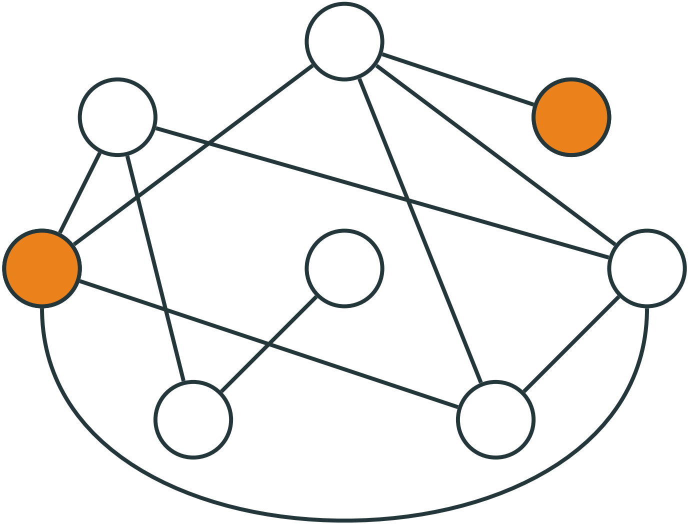
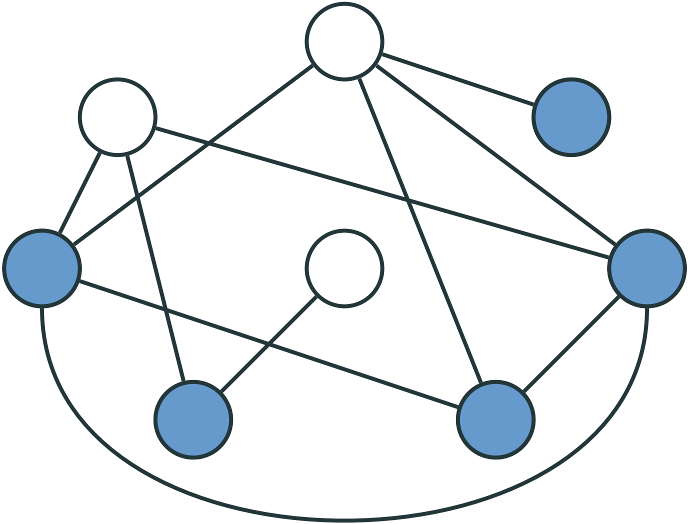


Vertex Covers: Examples

A Minimal VC



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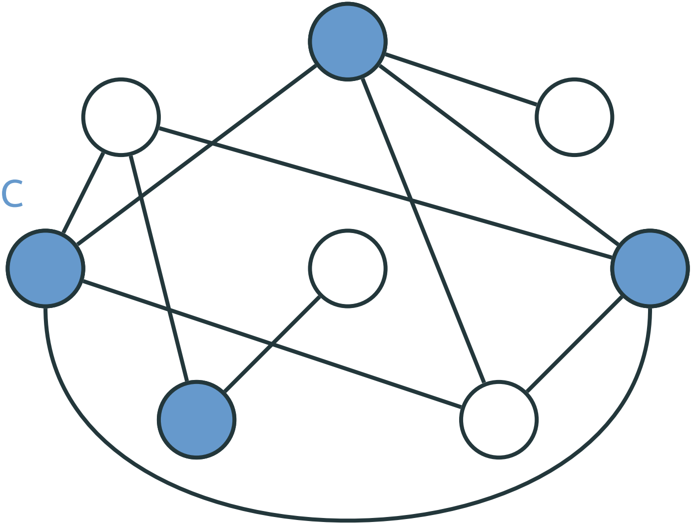

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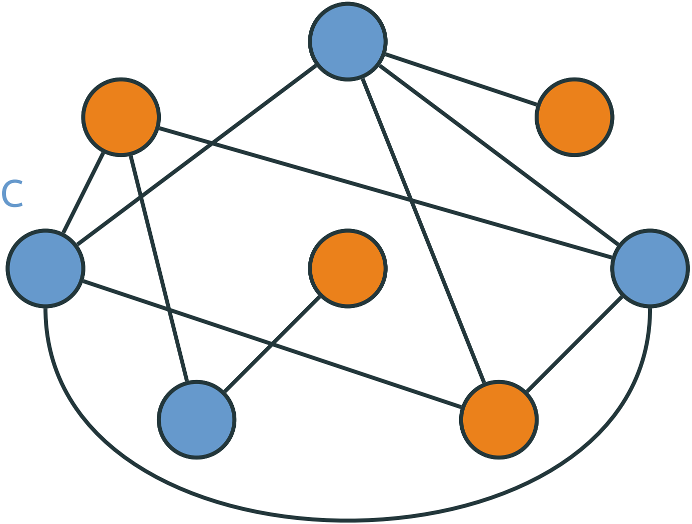
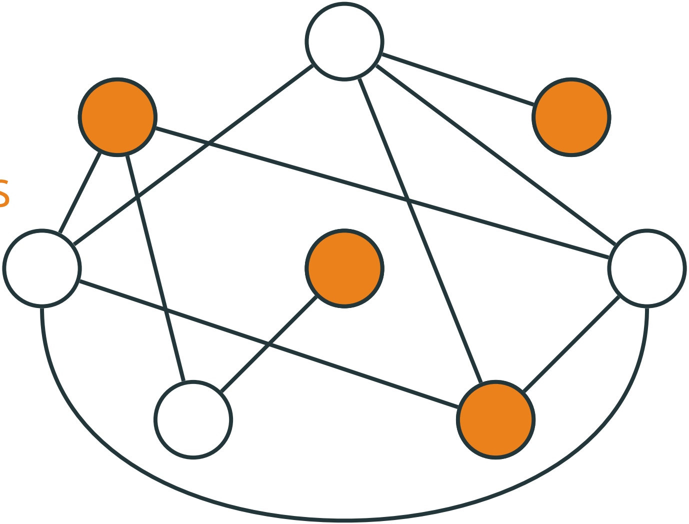


Vertex Covers: Examples

A Maximum IS



Vertex Covers and IS's

Fact

A set of vertices is a Vertex Cover if and only if its complement is an Independent Set

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Corollary

For every graph G on n vertices:

$$\beta(G) + \alpha(G) = n\ .$$

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Theorem (König, 1931)

In a bipartite graph, the number of edges in a Maximum Matching equals the number of vertices in a Minimum Vertex Cover.

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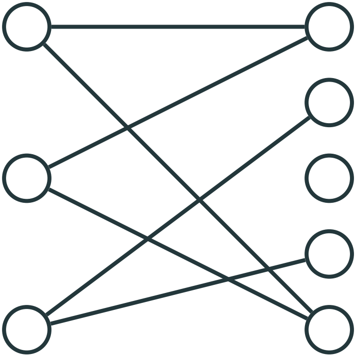

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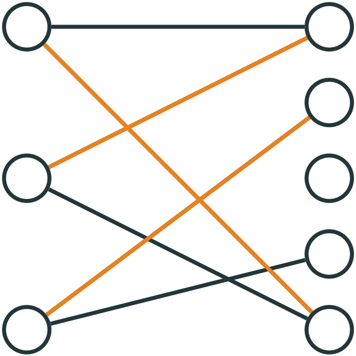

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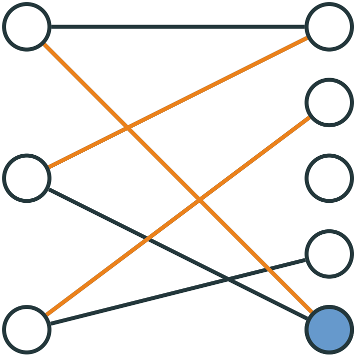

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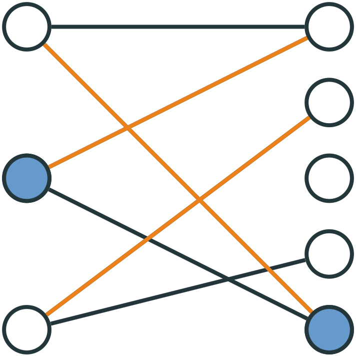

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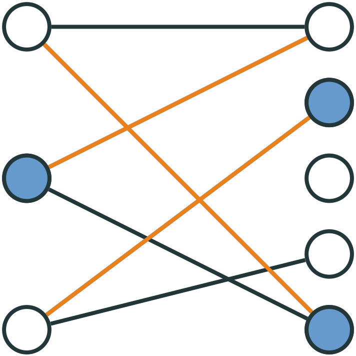

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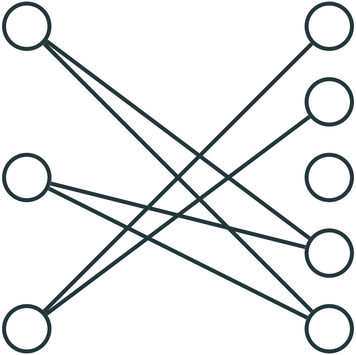

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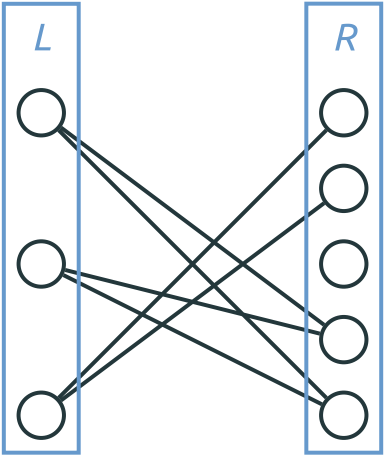

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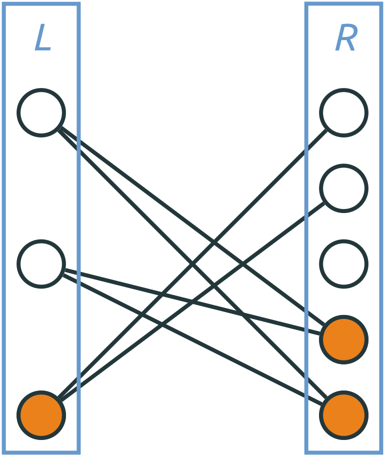

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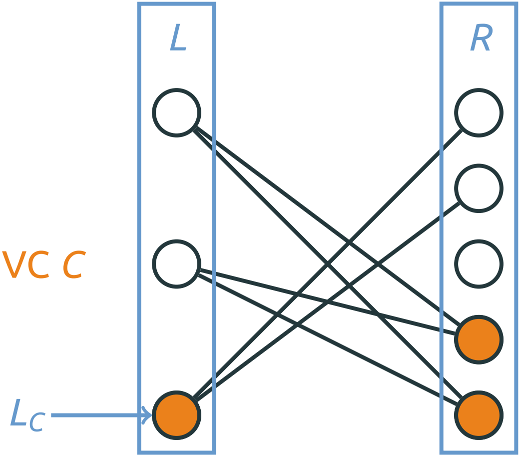

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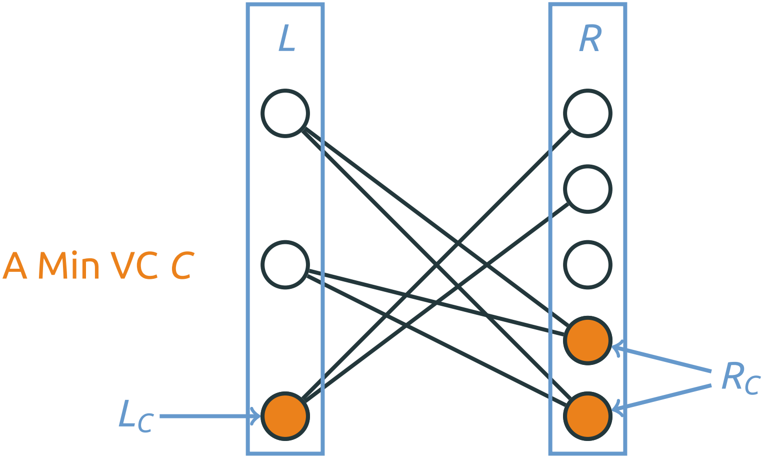

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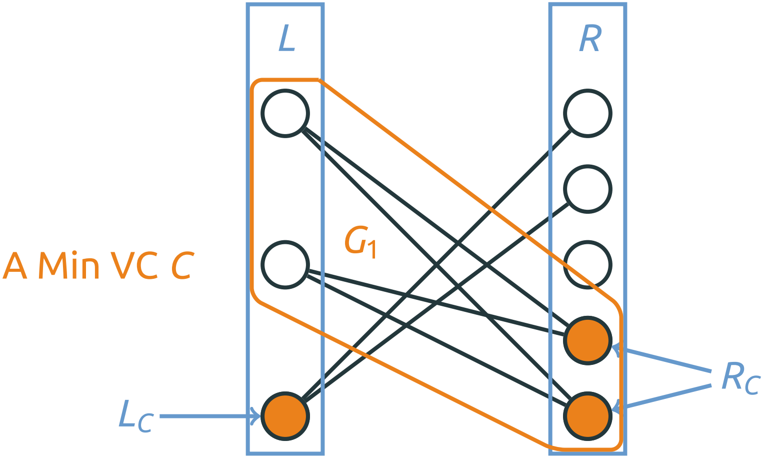

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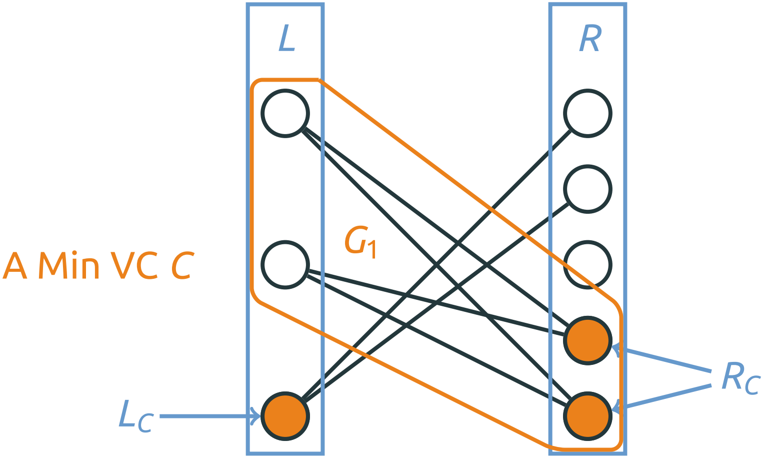

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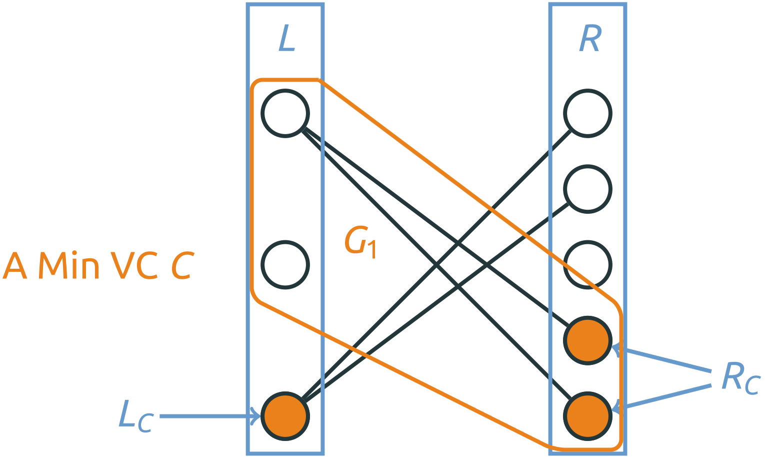

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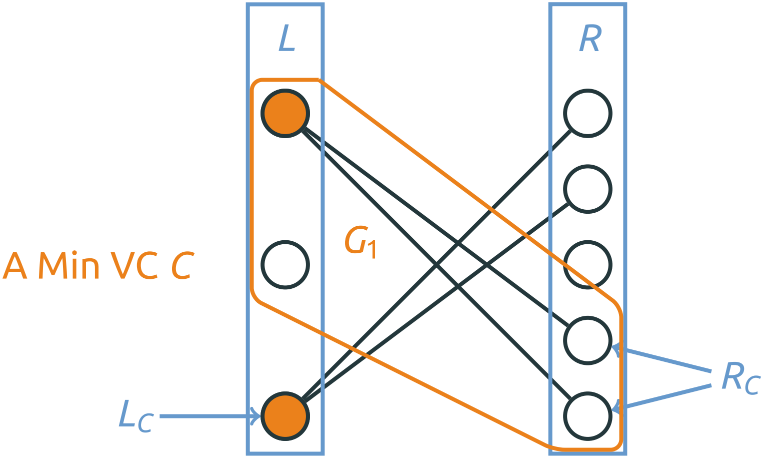

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- Similarly, $G_2 -$ the subgraph on $L_C \cup (R \setminus R_C)$ has a matching of size $|L_C|$
- Total matching size $|L_C| + |R_C| = |C|$