Combinations with Repetitions

Vladimir Podolskii

Computer Science Department, Higher School of Economics

Outline

- Review
- Salad
- Combinations with Repetitions

Review

We considered selections of k items out of n possible options

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	With repetitions	Without repetitions
Ordered		
Unordered		

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Unordered		{a, b}, {a, c}, {b, c}

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Current state of things:

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Example: Voting

There are k voters that vote for one of n candidates

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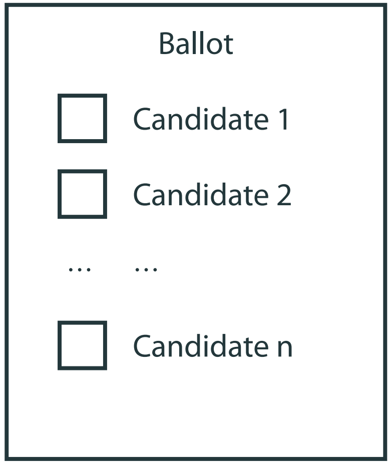

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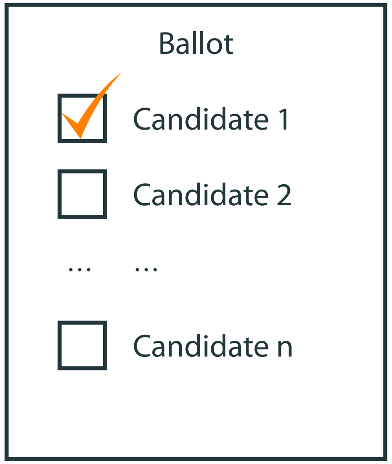

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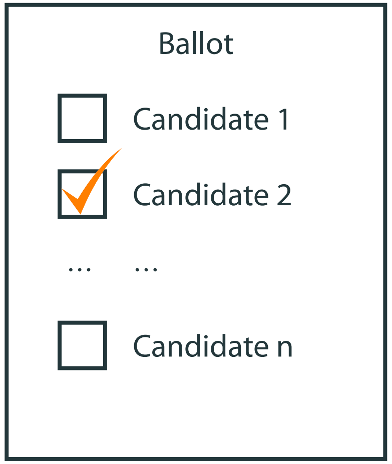

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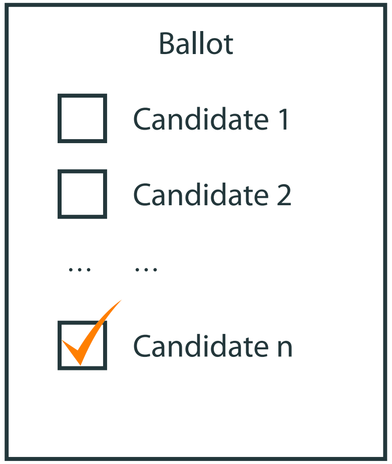

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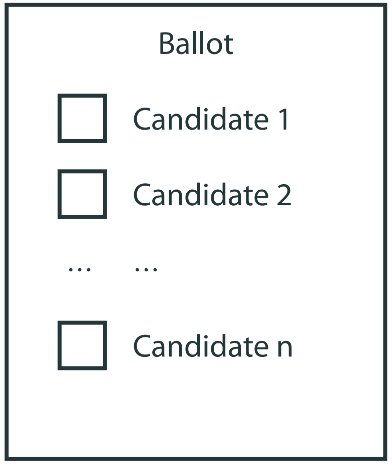
There are k voters that vote for one of n candidates

- All votes equally matter



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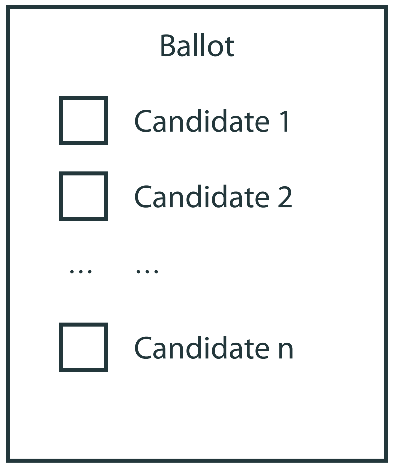

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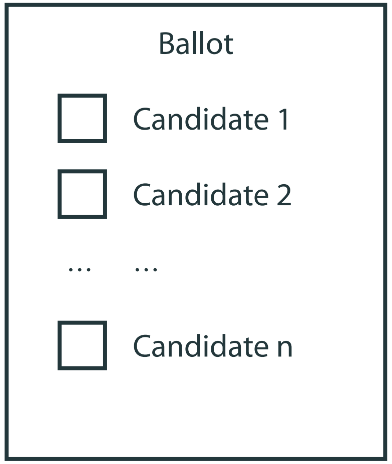

Example: Voting

There are k voters that vote for one of n candidates

- All votes equally matter
- So votes are unordered
- Candidates can be voted for several times
- So voters as a group pick k people out of n with repetitions

Ballot

☐ Candidate 1

☐ Candidate 2

… …

☐ Candidate n

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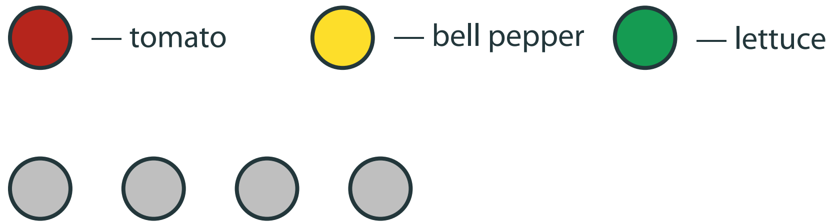
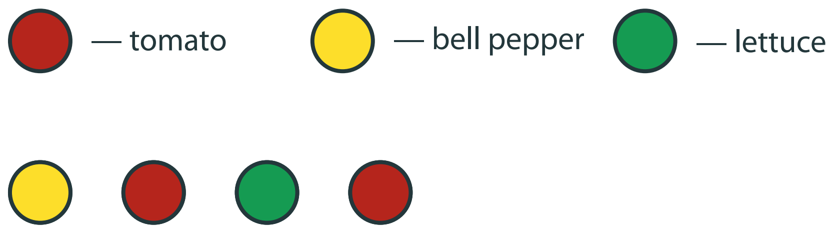
- We will list all possible salads, then count them
- But we want to do it wisely

Salad



Salad



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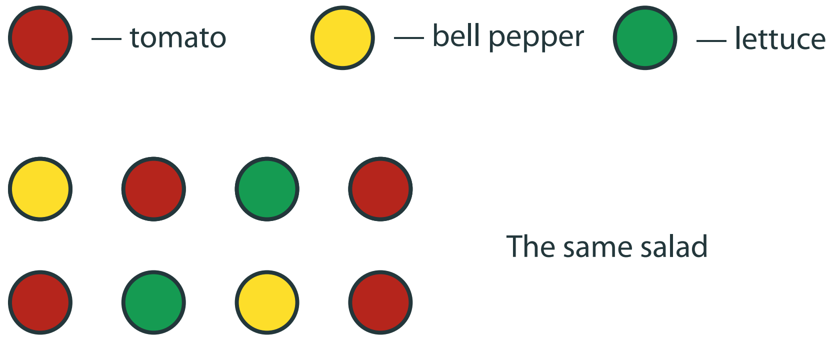

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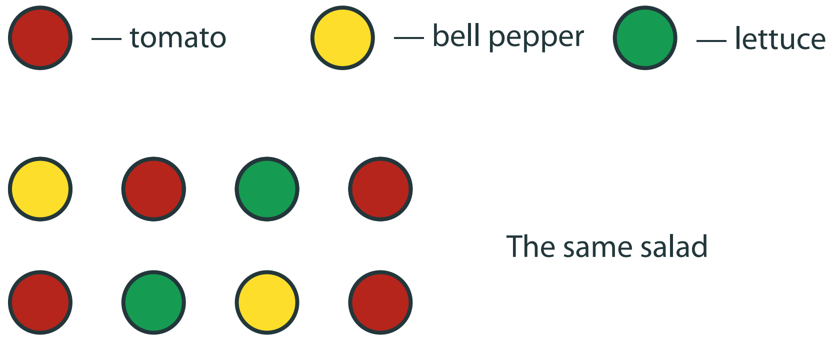

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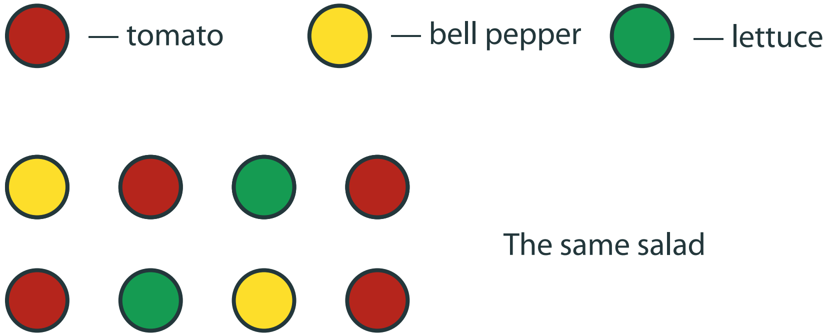
Salad



- The order does not matter
- So let’s draw tomatoes first, then bell peppers, then lettuce

Salad

— tomato — bell pepper — lettuce

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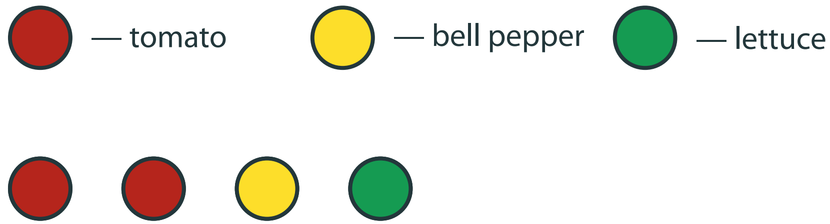

- The order does not matter
- So let's draw tomatoes first, then bell peppers, then lettuce
- Let's consider all possible numbers of tomatoes in the salad and count in each case separately

Salad

Salad

Case 1: 4 tomatoes

Salad

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Salad

Case 1: 4 tomatoes

- 4 tomatoes: 1 salad

Salad

Case 2: 3 tomatoes

- 4 tomatoes: 1 salad

Salad

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Salad

Case 2: 3 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads

Salad

Case 3: 2 tomatoes

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Salad

Case 4: 1 tomato

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Case 4: 1 tomato

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
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- 1 tomato: 4 salads

Salad

Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
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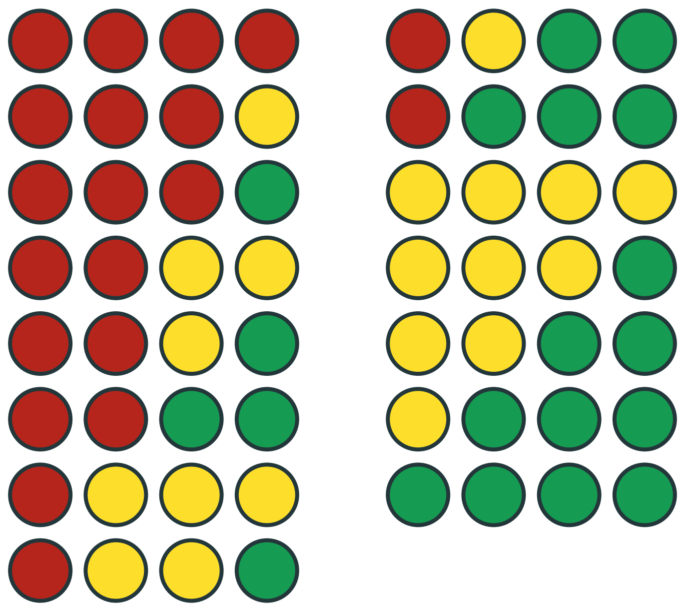
- 1 tomato: 4 salads
- 0 tomatoes: 5 salads

Salad

Case 5: 0 tomatoes

- 4 tomatoes: 1 salad
- 3 tomatoes: 2 salads
- 2 tomatoes: 3 salads
- 1 tomato: 4 salads
- 0 tomatoes: 5 salads
- In total: 15 salads

List of all Salads



Summary

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- Same structure for larger salads
- But more complicated for more ingredients
- Yet, the same strategy works for recursive counting for any salad size and any number of ingredients

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- But now we will obtain a formula

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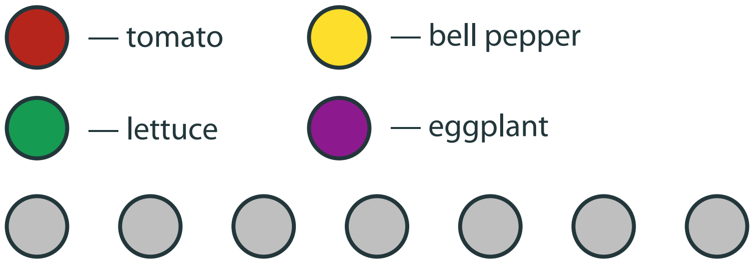
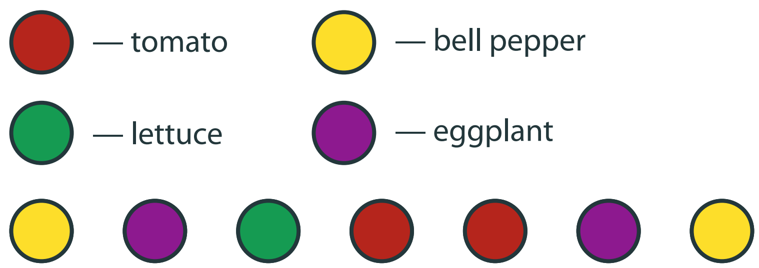
- We can use recursive counting here as well
- But now we will obtain a formula
- This will be a general solution

Large Salad



Large Salad



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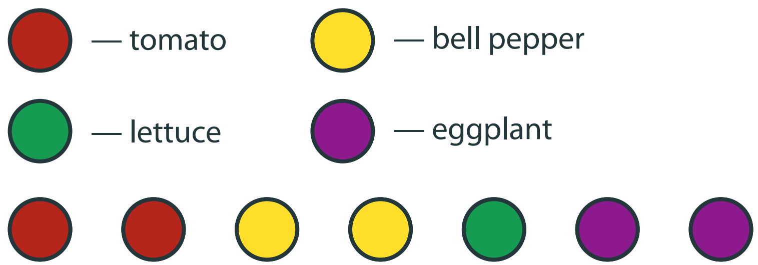
Large Salad



- The order does not matter
- Let’s list first tomatoes, then bell pepper, then lettuce, then eggplant

Large Salad

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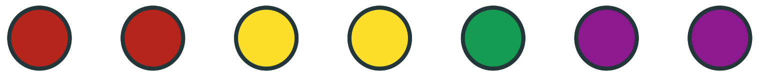
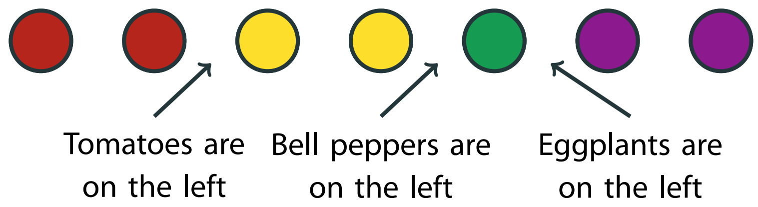


- Idea 1: to specify the list it is enough to indicate where the ingredients switch

Large Salad



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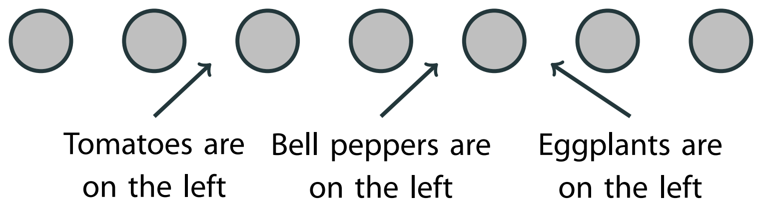

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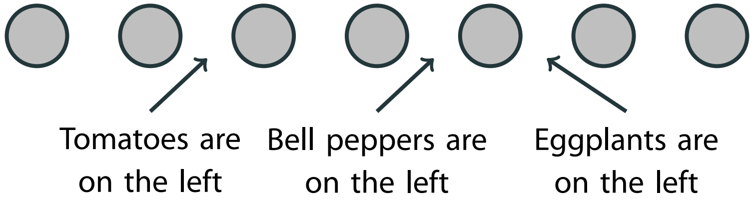
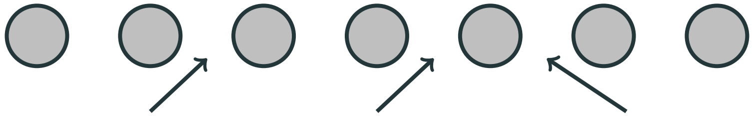


- Idea 1: to specify the list it is enough to indicate where the ingredients switch
- Idea 2: Do not even need the text descriptions

Large Salad



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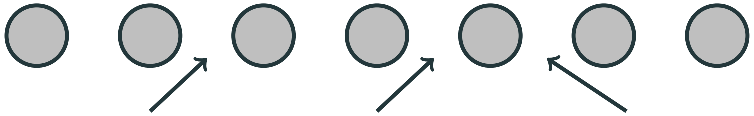

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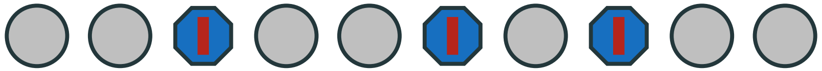

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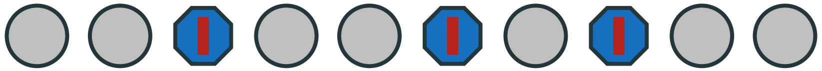
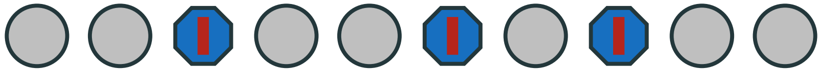


- Idea 1: to specify the list it is enough to indicate where the ingredients switch
- Idea 2: Do not even need the text descriptions
- Idea 3: Can represent places of switch as delimiter signs
- The salad can still be restored: tomatoes are on the left from the left delimiter, bell peppers are next, and so on

Large Salad



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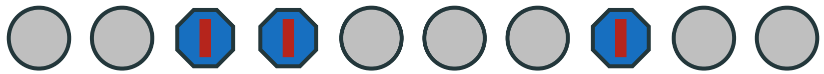

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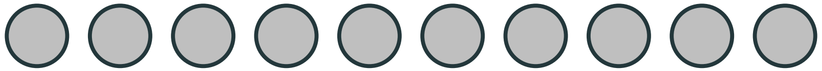

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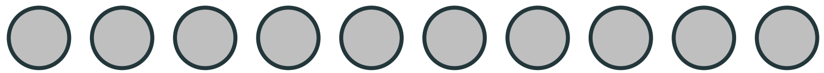

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- These are combinations! The answer to the problem is $\binom{10}{3} = 120$!

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Main ideas:

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- Order salad in a convenient way
- Salad is determined by delimiters between types of ingredients
- Place delimiters in the line with ingredients
- It is left to choose delimiters in the line — old problem

General case

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- Size of the combination = size of salad
- Number of objects = number of ingredients
- The same argument works
- Why $k+n-1$ and $n-1$?
- n ingredients mean $n-1$ delimiters; choosing $(n-1)$ element in the line of $k+(n-1)$ elements

Standard Settings

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Ordered	Tuples n^k	k-permutations $\frac{n!}{(n-k)!}$
Unordered	Combinations with repetitions $\binom{k+n-1}{n-1}$	Combinations $\binom{n}{k}$