Mathematical Induction

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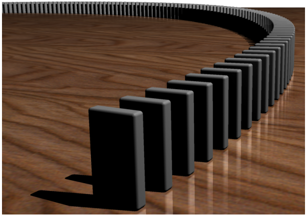
Mathematical Induction

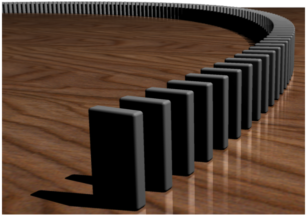
- A very powerful proof method



wikipedia.org

Mathematical Induction

- A very powerful proof method
- Falling dominos



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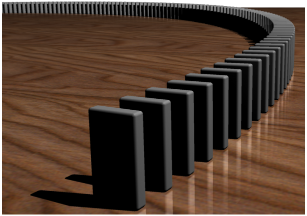
Mathematical Induction

- A very powerful proof method
- Falling dominos
- Check for $1, 2, 5, 100, 1000$ dominos



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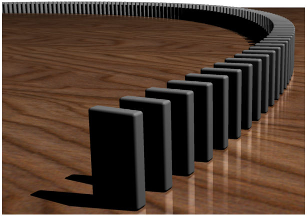
Mathematical Induction

- A very powerful proof method
- Falling dominos
- Check for $1, 2, 5, 100, 1000$ dominos
- How to prove for any number of dominos?



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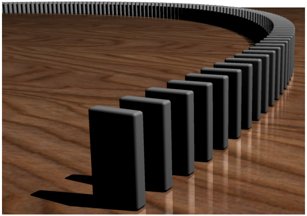
Mathematical Induction

- A very powerful proof method
- Falling dominos
- Check for $1, 2, 5, 100, 1000$ dominos
- How to prove for any number of dominos?
- Many computer science algorithms are proven using induction



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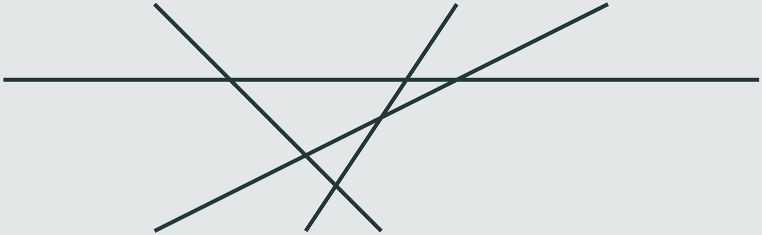
Outline

Problem

Several straight lines (at least three) cut a plane into pieces. Each line intersects with every other line, and all the intersection points are different. Prove that there is at least one triangular piece.



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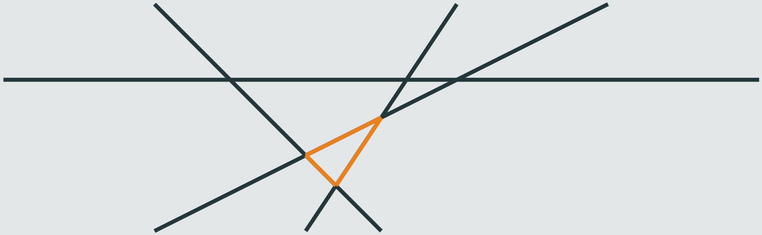

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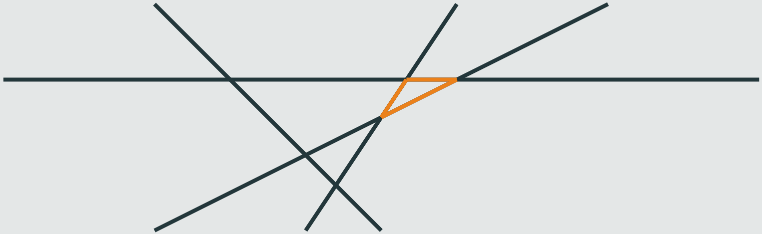

Proof Idea

- A triangle appears as soon as there are 3 lines

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- When we add more lines one by one, each time

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    - Either the same triangle remains...

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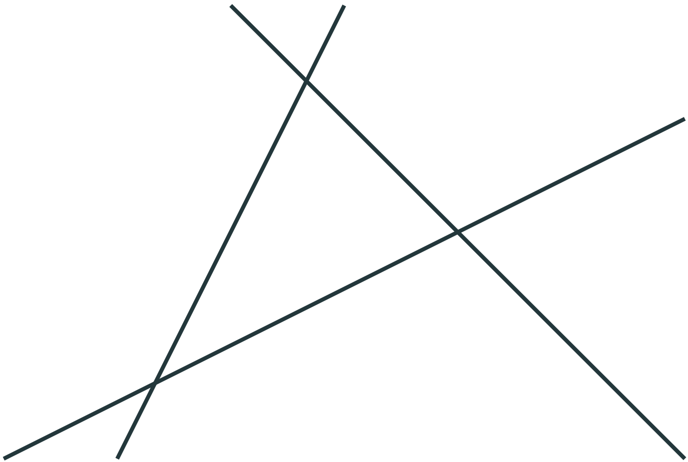
    - Either the same triangle remains...
    - Or a new one appears

Three Lines



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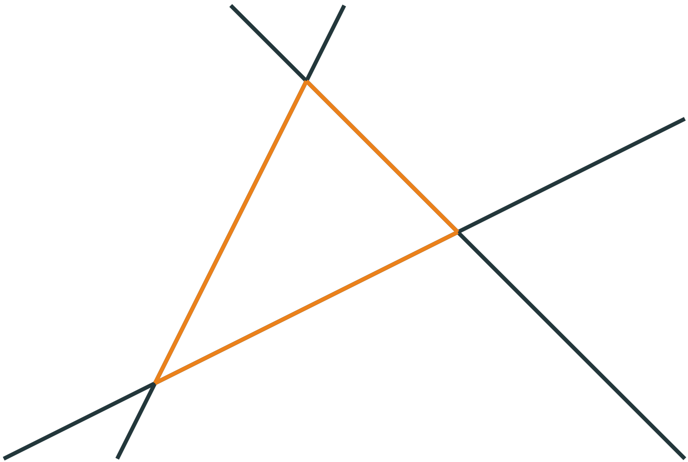
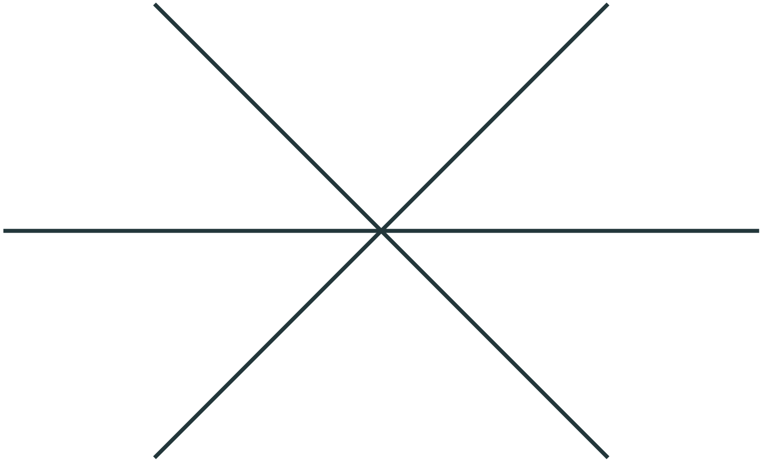


Three Lines - Bad Case



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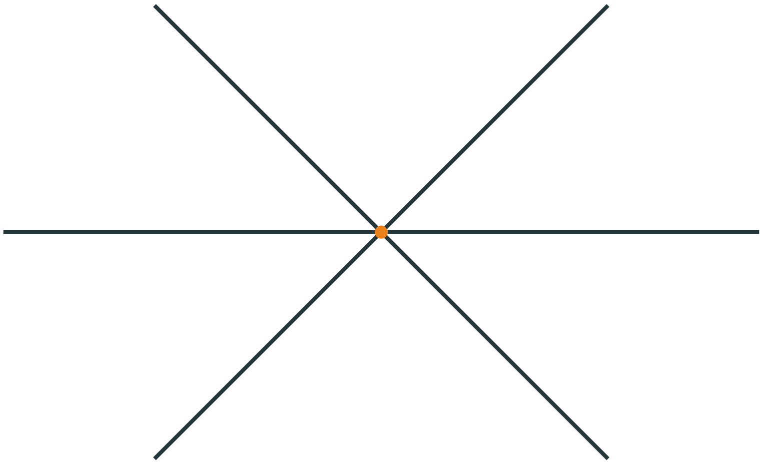
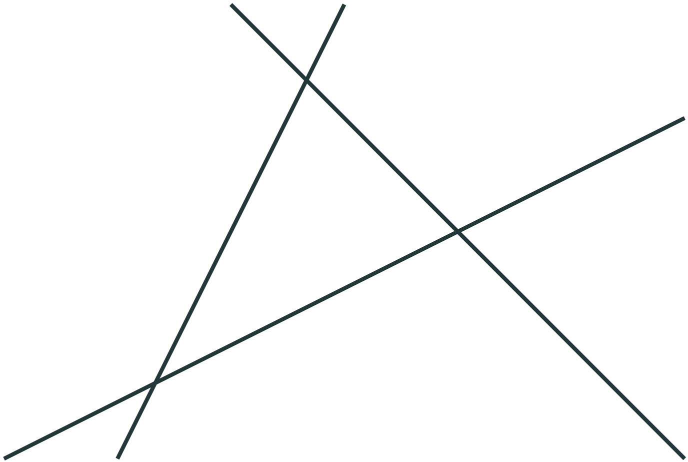


Adding One More Line



There is a triangle cut by three lines

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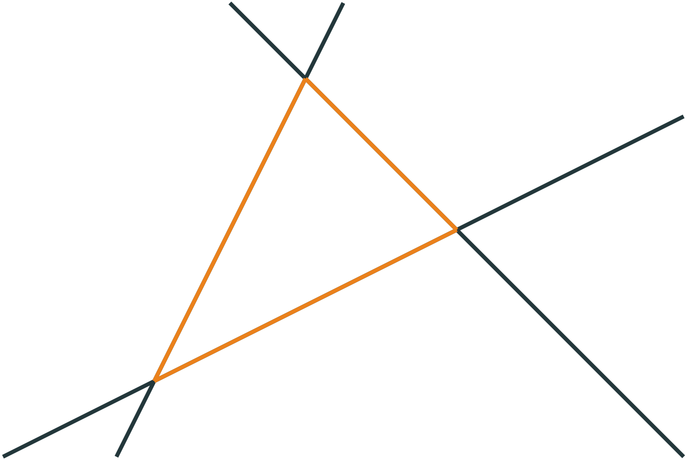

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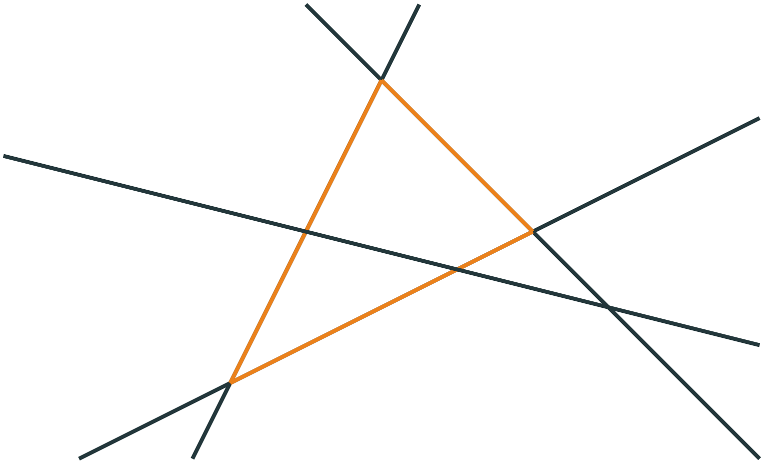
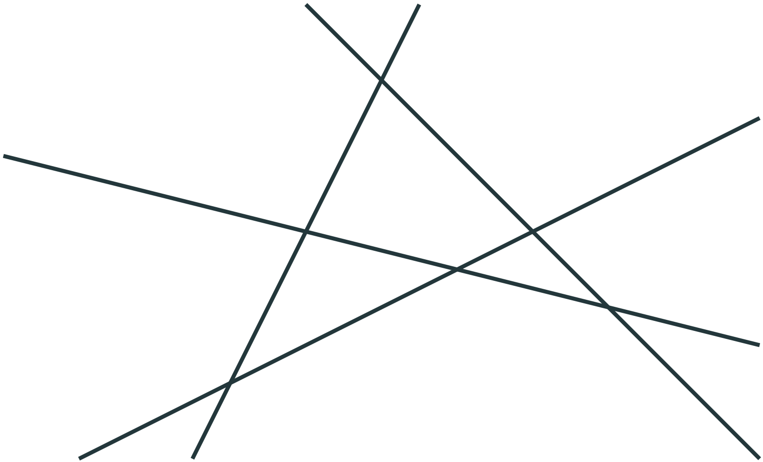


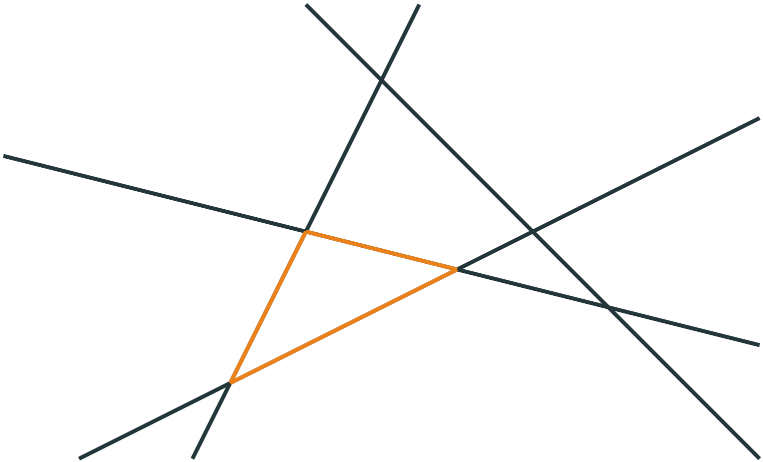
When the new line intersects the triangle,…

Adding One More Line



When the new line intersects the triangle,… a new triangle appears

Adding One More Line



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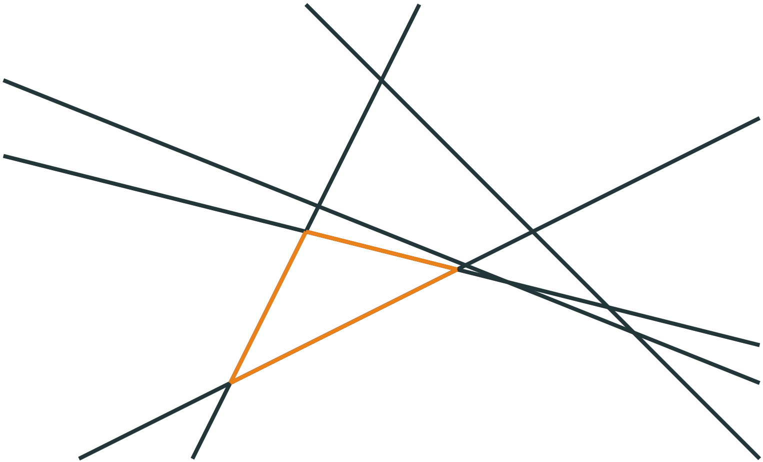
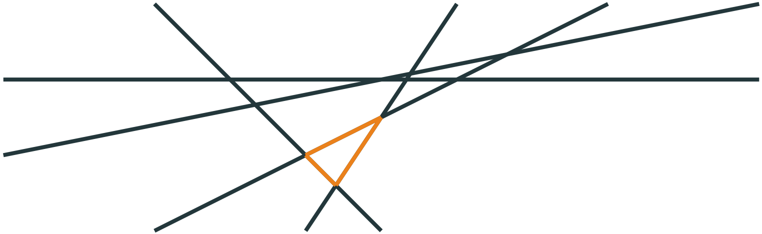
And One More Line

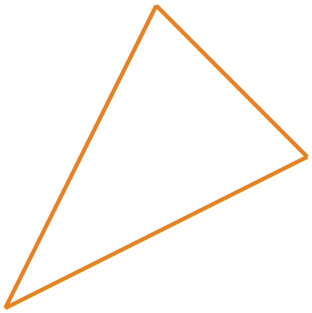


When the new line doesn't touch the triangle, the triangle remains intact

General Case

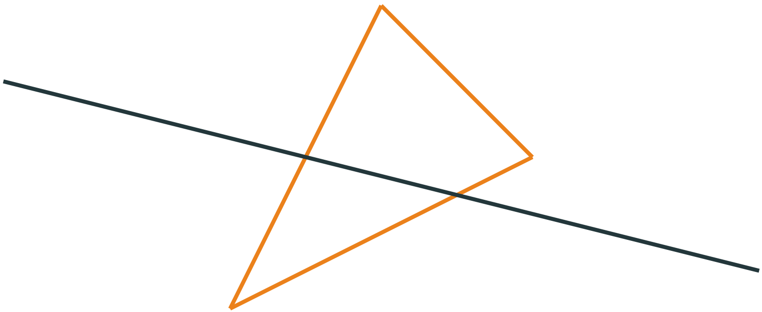


Adding a Line in General



There is a triangle

Adding a Line in General

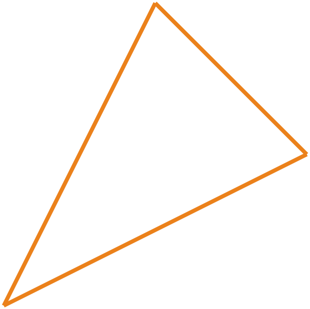


There is a triangle

Adding a Line in General

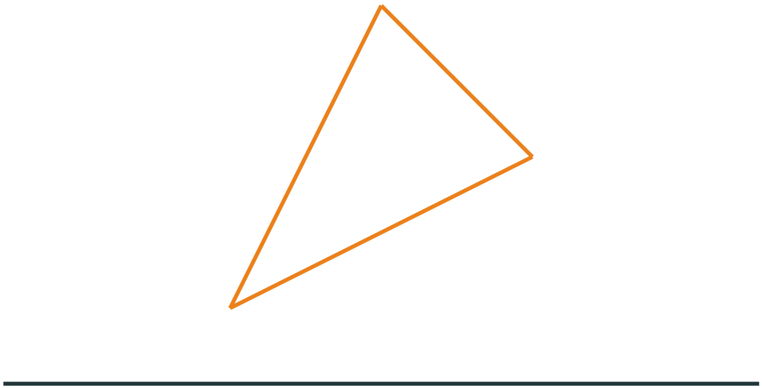
New triangle appears

Adding a Line in General



There is a triangle

Adding a Line in General



The triangle remains

Mathematical Induction

Theorem

For any $n \geq 3$ and any n straight lines on a plane, if every two lines intersect, and all the intersection points are different, there is a triangular piece among the pieces into which these lines cut the plane.

Proof Structure

Number of lines

3 4 5 6 n

Proof Structure

Number of lines

3 4 5 6 n

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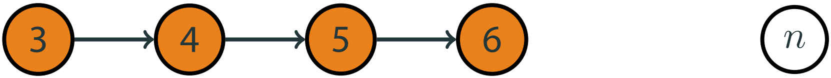

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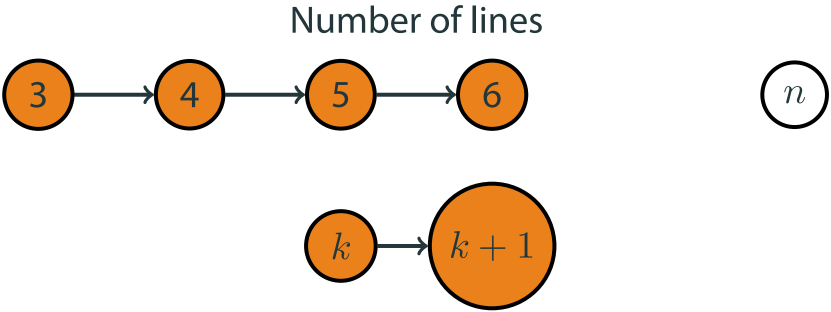

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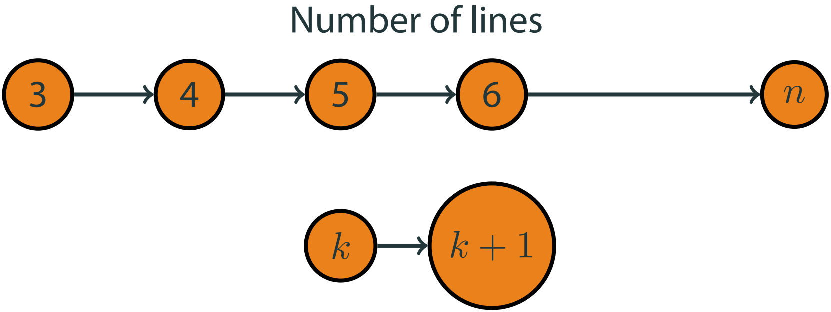

Mathematical Induction

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- Prove induction step from n to $n + 1$ — adding one more line in the general case

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- ...
- Profit!

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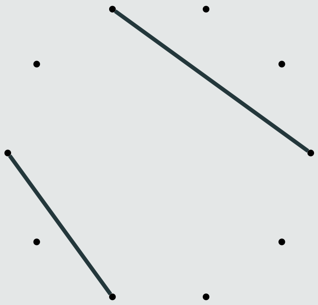
Outline

- Lines and Triangles
- Connecting Points
- Sums of Numbers
- Bernoulli's Inequality
- Coins
- Cutting a Triangle
- Flawed Induction Proofs
- Alternating Sum

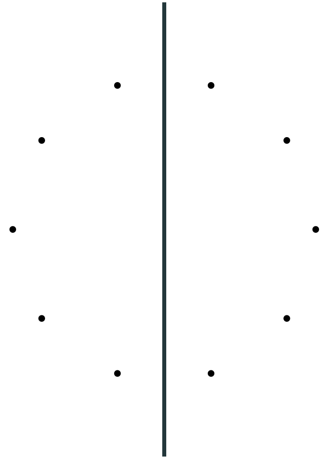
Problem

Connect some of these 10 points with segments, such that every point is connected with 5 other points.



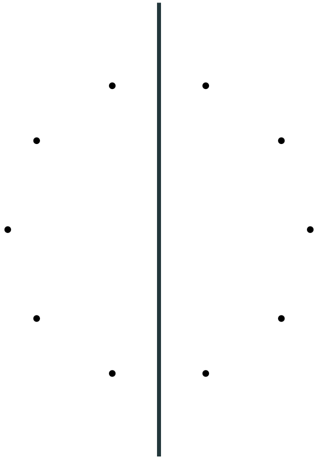
Solution

Separate the points into the left half and the right half. Each half has 5 points.



Solution

Connect each point from the left half to each point of the right half.



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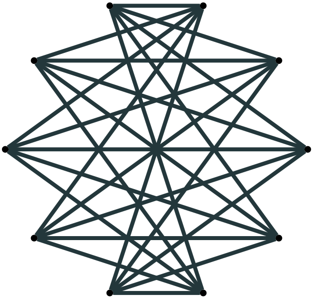
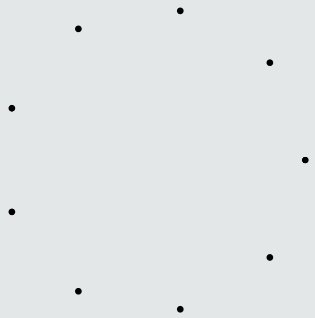


Problem

Now you are given 9 points. Can you connect some of them with segments so that each point is connected with 5 other points?



Even and Odd Numbers

Numbers $0, 2, 4, 6, 8, \ldots$ are called even, and numbers $1, 3, 5, 7, 9, \ldots$ are called odd.

Even numbers are divisible by 2, and odd numbers are not divisible by 2.

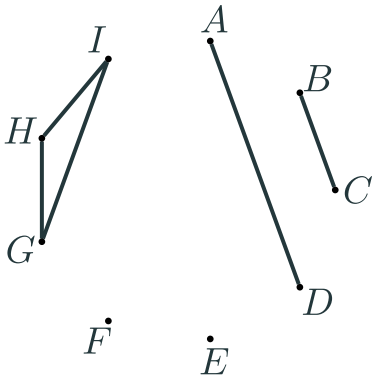
Neighbors

Let us call point B neighbor of point A if A and B are connected with a segment.

If B is a neighbor of A, then A is also a neighbor of B.

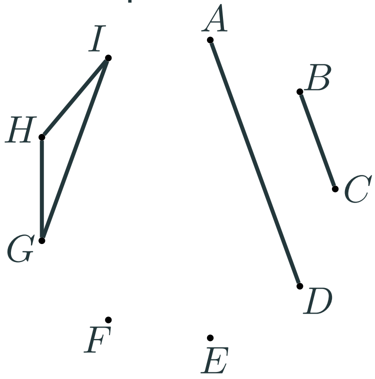
Even and Odd Points

Let us call a point even if it has even number of neighbors, otherwise we call this point odd.



Even and Odd Points

Let us call a point even if it has even number of neighbors, otherwise we call this point odd.
In the example below, points A, B, C and D are odd, and all the other points are even.



Even Number of Odd Points

Theorem

The number of odd points is always even, regardless of how many points and segments are there and which pairs of points are connected by segments.

Proof Idea

When there are no segments, there are no odd points, so the number of odd points is indeed even.

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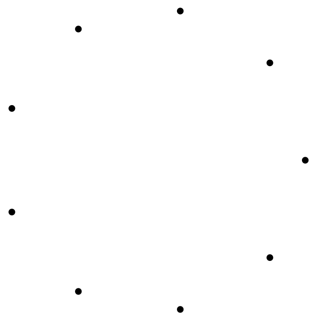
When there are no segments, there are no odd points, so the number of odd points is indeed even.

When we add segments one by one, the number of odd points either doesn't change, increases by 2 or decreases by 2. Thus the number of odd points stays even.

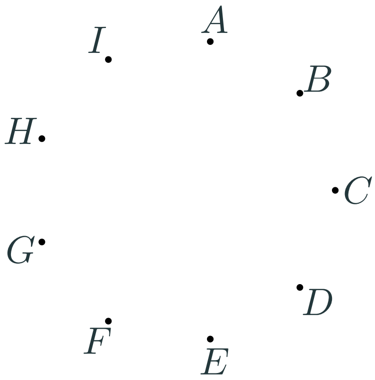
Easy Case: No Segments

When there are no segments, each point has 0 neighbors, so there are no odd points. The number of odd points is 0, which is even, so there is indeed even number of odd points.



Adding a Segment

Segment AB adds two odd points A and B.



Number of odd points: 0

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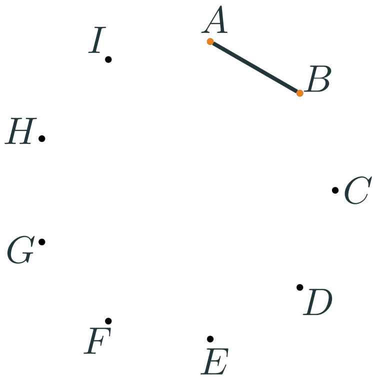
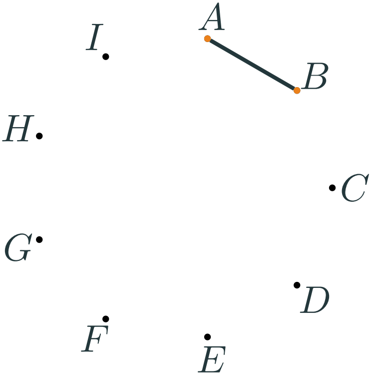


Number of odd points: 2

Adding a Segment

Segment HI adds two odd points H and I.



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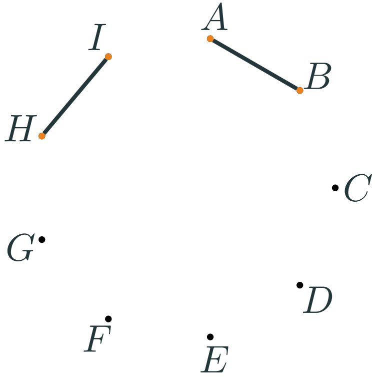
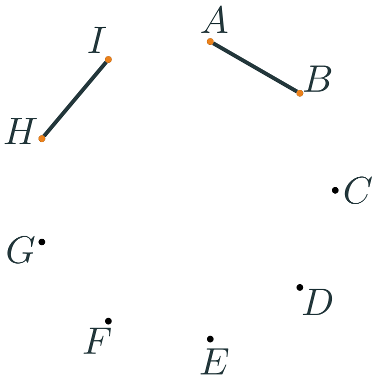


Number of odd points: 4

Adding a Segment

Segment BC makes B even and C odd.



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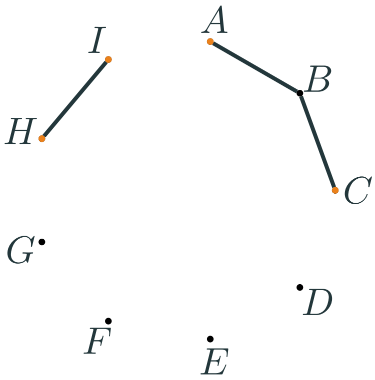

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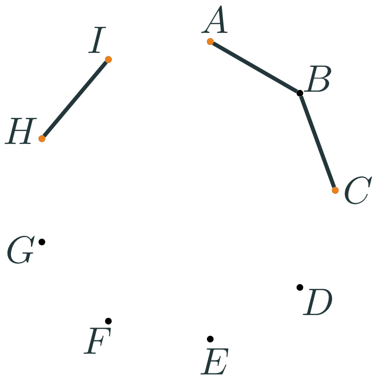
Adding a Segment

Segment AC makes A and C even.



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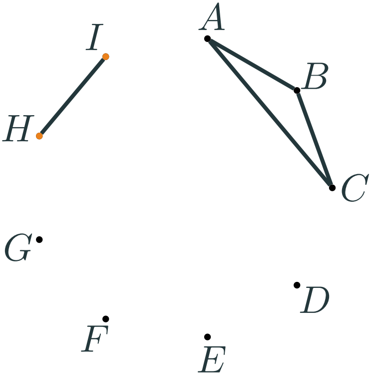

Number of odd points: 2

Adding a Segment in General

If A and B are even, segment AB makes them both odd and adds 2 odd points.

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If A and B are odd, segment AB makes them both even and removes 2 odd points.

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Adding a Segment in General

If A is even and B is odd, segment AB swaps them, keeping number of odd points the same.

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If A is even and B is odd, segment AB swaps them, keeping number of odd points the same.

Proof Structure

Number of segments

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Number of segments

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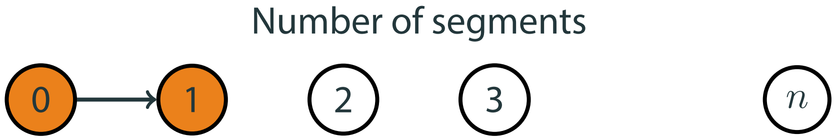

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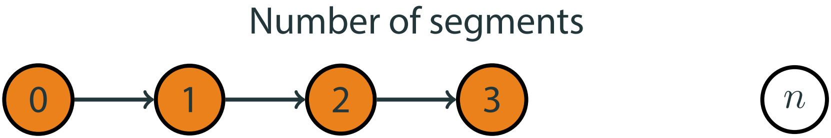

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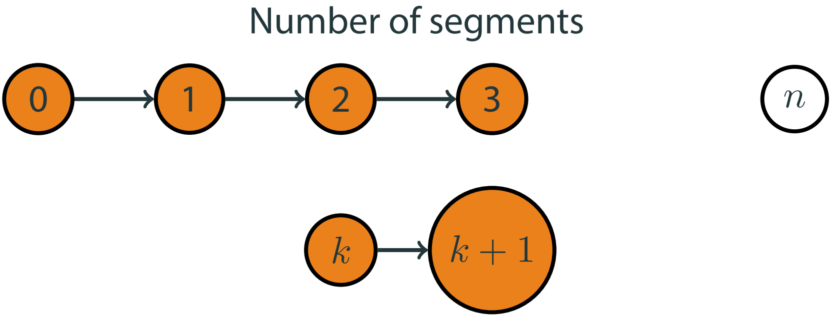

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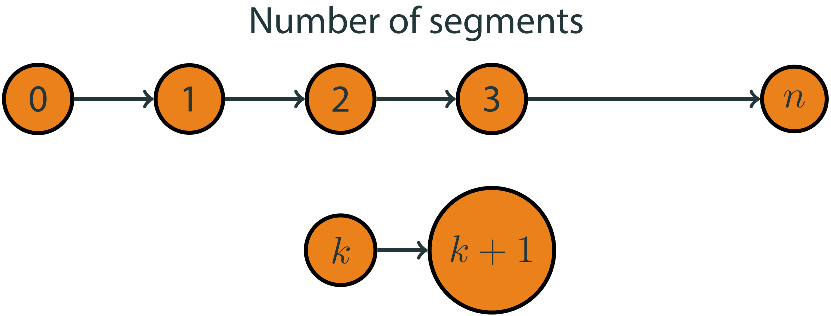

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9 Points — Solution

There are 9 points, and we want to draw some segments, so that every point has 5 neighbors. If we succeeded, all 9 points would be odd. But the number of odd points must be even, and 9 is not even. So, it is impossible!

Outline

- Lines and Triangles
- Connecting Points
- Sums of Numbers
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- Flawed Induction Proofs
- Alternating Sum

The Prince of Mathematicians

Carl Friedrich Gauss (1777–1855)

wikipedia.org

Gauss Teacher's Problem

Problem

What is the sum of numbers from 1 to 100?

General Case

Problem

What is the sum of integer numbers from 1 to n?

Theorem

The sum of integers from 1 to n is $\frac{n(n+1)}{2}$.

Proof by Induction

Induction base: $n = 1$

$$1 + 2 + \cdots + n = 1 = \frac{1 \cdot 2}{2}$$

Induction step: $n \to n + 1$

$$1 + 2 + \cdots + n + (n+1) \overset{!}{=} \frac{n(n+1)}{2} + (n+1) =$$

$$= \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

How to come up with this formula in the first place?

Gauss's Idea

$$\begin{aligned} S &= 1 &&+2+\ldots &&+99+100 \\ S &= 100 &&+99+\ldots &&+2+1 \\ 2S &= 101 &&+101+\ldots &&+101+101 \\ S &= \frac{100\cdot 101}{2} = 5050 \end{aligned}$$

If we changed 100 to n here, it would be another way to prove the same formula, without induction.

Outline

- Lines and Triangles
- Connecting Points
- Sums of Numbers
- Bernoulli's Inequality
- Coins
- Cutting a Triangle
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Problem

You start with $\$1\ 000$ and earn 2% of what you have every day. Will you ever get more than $\$1\ 000\ 000$?

On day 1, you have $\$1\ 000$

On day 2, you have $\$1\ 000 \cdot 1.02$

On day 3, you have

$\$1\ 000 \cdot 1.02 \cdot 1.02 = \$1\ 000 \cdot 1.02^2$

$\dots$

On day n, you have $\$1\ 000 \cdot 1.02^{n-1}$

Mathematical Statement

Problem

Is there such n that $1000 \cdot 1.02^n > 1000000$? Or, is there such n that $1.02^n > 1000$?

Bernoulli's Inequality

Theorem

For any $n \geq 0$ *and* $x > 0$, $(1+x)^n \geq 1 + nx$.

Proof by Induction

Induction base: $n = 0$

$$(1+x)^n = (1+x)^0 = 1 = 1 + 0x = 1 + nx$$

Proof by Induction

Induction base: $n = 0$

$$(1+x)^n = (1+x)^0 = 1 = 1 + 0x = 1 + nx$$

Induction step: $n \to n+1$

$$\begin{aligned}(1+x)^{n+1} = (1+x)^n(1+x) \overset{!}{\geq} (1+nx)(1+x) = \\ = 1 + nx + x + nx^2 > 1 + (n+1)x\end{aligned}$$

$\square$

Solution

$$n = 50000$$

$$\begin{aligned} 1.02^{50000} = (1+0.02)^{50000} \geq 1 + 50000 \cdot 0.02 = \\ = 1 + 1000 > 1000 \end{aligned}$$

Complex Percentage

In fact, $1.02^{349} = 1003.36730\ldots > 1000$

Complex Percentage

In fact, $1.02^{349} = 1003.36730\ldots > 1000$

If you have $\$1\ 000$ on January 1st and increase it by 2% every day, you will get more than $\$1\ 000\ 000$ by the end of the year!

Outline

Problem

You have an unlimited supply of 4 cents and 5 cents coins. Prove that for any $n \geq 12$, you can give change of n cents using these coins.

Proof by Induction

Induction base: $n = 12$

Indeed, $12 = 3 \cdot 4$, so using just 4 cents coins is enough.

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Induction step: $n \to n + 1$

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Indeed, $12 = 3 \cdot 4$, so using just 4 cents coins is enough.

Induction step: $n \to n + 1$

???

It is unclear how to prove that we can give change of $n + 1$ cents assuming that we can give change of n and we have more 4 cents and 5 cents coins.

Complete Induction

Induction base: $n = 12, n = 13, n = 14$ and $n = 15$

$$12 = 3 \cdot 4$$

$$13 = 2 \cdot 4 + 1 \cdot 5$$

$$14 = 2 \cdot 5 + 1 \cdot 4$$

$$15 = 3 \cdot 5$$

Complete Induction

Induction step: $n, n-1, n-2, n-3 \to n+1$

If we know that $n-3$ can be given with 4 cents and 5 cents coins $n-3 = a \cdot 4 + b \cdot 5$, then $n+1$ also can be given with these coins:

$$
\begin{aligned}
n+1 = (n-3) + 4 = a \cdot 4 + b \cdot 5 + 4 = \\
= (a+1) \cdot 4 + b \cdot 5
\end{aligned}
$$

$\square$

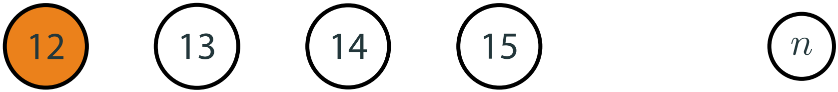
Proof Structure

Change amount

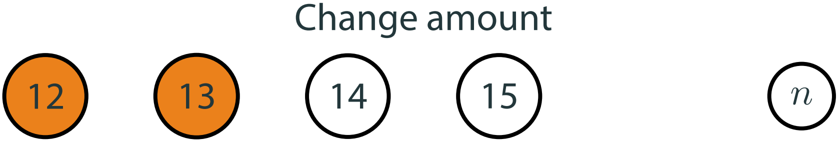
12 13 14 15 n

Proof Structure

Change amount

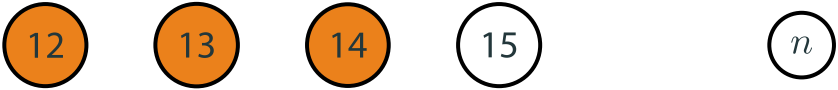


Proof Structure



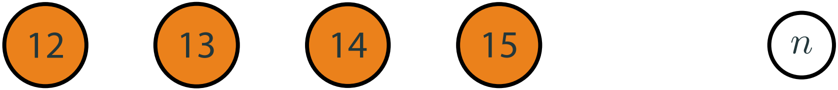
Proof Structure

Change amount

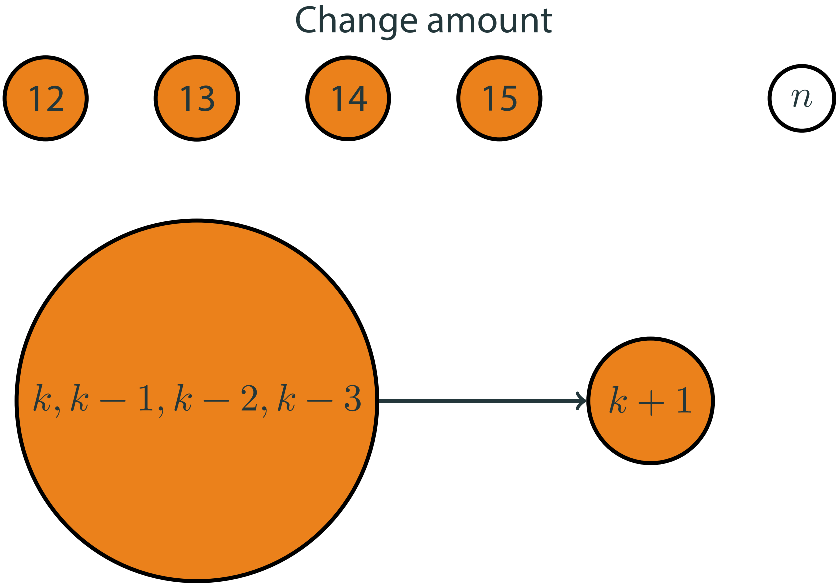


Proof Structure

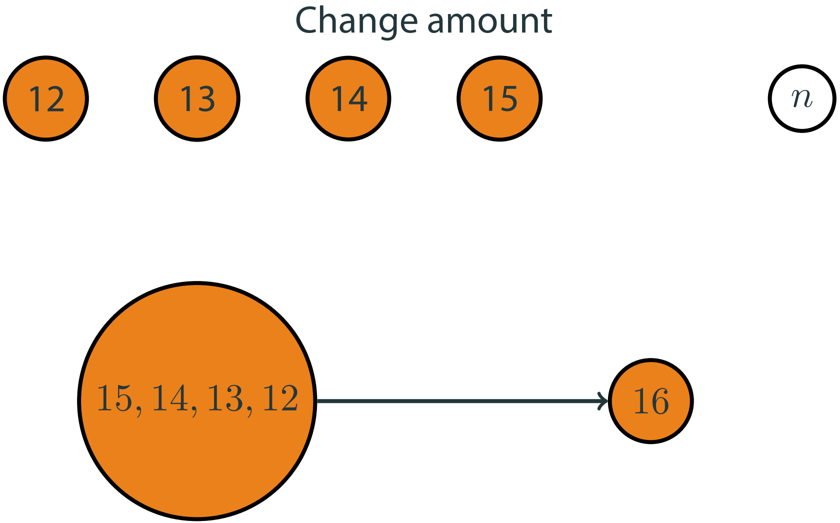
Change amount



Proof Structure

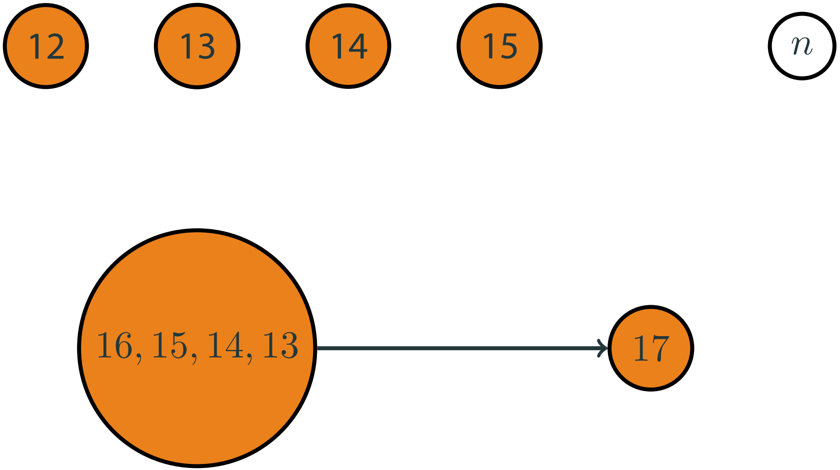


Proof Structure

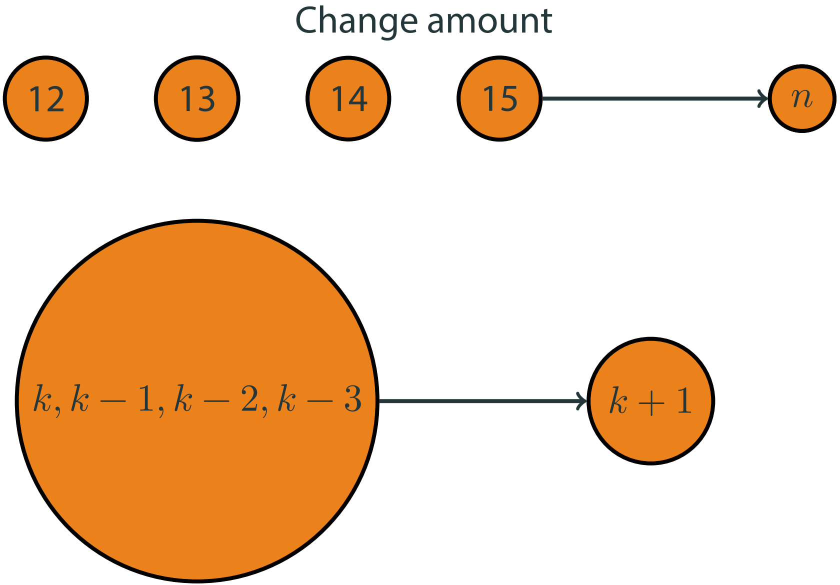


Proof Structure

Change amount



Proof Structure



Complete Induction

Induction base: prove for the first k values of n

Induction step: if the statement is true for $n, n-1, n-2, \dots, n-k+1$, prove it for $n+1$.

Complete Induction 2

Induction base: prove for the first k values of n

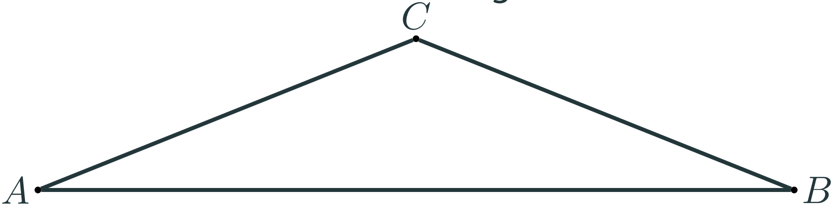
Induction step: if the statement is true for all previous n, prove it for $n + 1$.

If we explicitly assume in the induction step, for example, that the statement is true for $n - 10$, then we need $k \geq 10$.

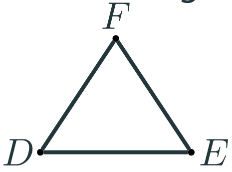
Outline

Acute and Obtuse Triangles

Obtuse triangle



Acute triangle



Problem

Is it possible to cut an obtuse triangle into several acute triangles?

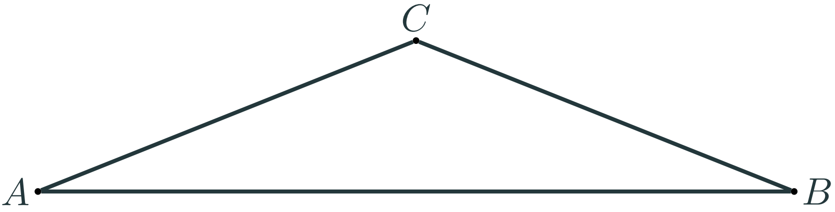
Theorem

If an obtuse triangle is cut into $n \geq 1$ triangular pieces, at least one of the pieces is obtuse.

This sounds very similar to the problem about cutting a plane by lines. Let us try to prove it in a similar way using Mathematical Induction.

Proof by Induction

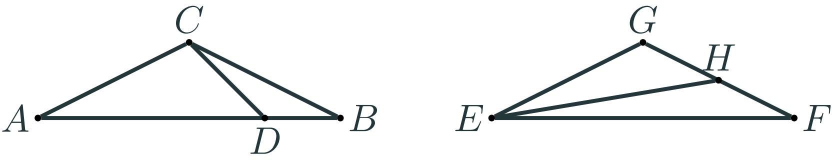
Induction base: $n = 1$. The initial triangle is obtuse, so if there is only one piece, it is also obtuse.



Proof by Induction

Induction step: $n \to n+1$.

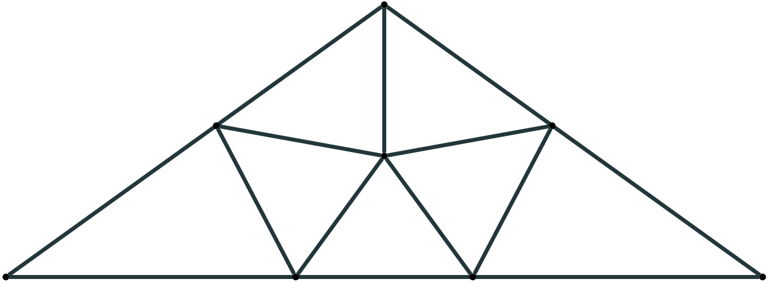
By assumption of induction, there is an obtuse piece. If we cut it into two triangles, at least one of them is obtuse, so an obtuse piece remains.



This Proof is Wrong

Can you spot what was wrong in this proof?

Example



Wrong Induction Step

The induction step assumed that if we cut a triangle into several triangular pieces, we can do it by several steps of cutting a triangular piece into two triangles. However, this is not always possible.

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Flawed Proofs by Induction

- Induction is a very powerful method
- But as any power, you should apply it with care
- We will see that often proofs by induction only seem to be correct

Theorem

For any $n \geq 1$ people, they are all of the same age.

Proof by Induction

Induction base: $n = 1$

Obviously, the statement is true for just one person.

Induction step: $n \to n + 1$

By the assumption of induction, the first n people are of the same age. Also, by the same assumption, the last n people are of the same age. Then all $n + 1$ people are of the same age as the middle $n - 1$ people. $\square$

Can you spot what was wrong in this proof?

The induction step breaks for $n = 1 \to n + 1 = 2$: indeed, among $n + 1 = 2$ people, the first $n = 1$ is of the same age, and the last $n = 1$ is of the same age, but these two people can be of different ages, because the middle $n - 1$ people are actually 0 people.

Theorem

For any integer $n \geq 0$, $5n = 0$.

Proof by Induction

Induction base: $n = 0$

Indeed, $5n = 5 \cdot 0 = 0$

Induction step: $n \to n + 1$

Write $n + 1 = i + j$ where i and j are non-negative integers up to n. Then
$5(n + 1) = 5(i + j) = 5i + 5j = 0 + 0 = 0$. $\square$

Can you spot what was wrong in this proof?

The induction step is wrong for $n = 0 \to n + 1 = 1$. Indeed, it is impossible to write $n + 1 = 1$ as a sum $i + j$ of two non-negative integers up to $n = 0$, because then both i and j would have to be 0, and $0 + 0 = 0 < 1$.

Outline

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Problem

Prove that

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{99}-\frac{1}{100}=$$
$$=\frac{1}{51}+\frac{1}{52}+\cdots+\frac{1}{100}$$

Generalization

Let's solve a more general problem:

Problem

Prove that

$$1 - \frac{1}{2} + \cdots + \frac{1}{2k-1} - \frac{1}{2k} =$$
$$= \frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k}$$

For $k = 50$, it is the same as the initial problem.

Proof by Induction

Induction base: $k = 1$

$$1 - \frac{1}{2} = \frac{1}{2}$$

Proof by Induction

Induction step: $k \to k+1$

Let's see what changes in the left and the right part when k increases by one.

Two new summands are added in the left part when k increases by one: $\frac{1}{2k+1} - \frac{1}{2(k+1)}$

Proof by Induction

Right part:

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}=$$
$$=\left(-\frac{1}{k+1}+\frac{1}{k+1}\right)+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}=$$
$$\left(\frac{1}{k+1}+\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k}\right)+\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}$$

So, right part changes by $\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}=\frac{1}{2k+1}-\frac{1}{2(k+1)}$, and the left part changes by the same amount. Thus, left part and right part are the same initially, and they change by the same value, so they stay the same.

Conclusion

- Mathematical Induction is a powerful proof method
- Reformulate problem in mathematical terms
- Prove induction base
- Prove induction step
- Before induction, we often need to come up with a formula somehow
- Sometimes, generalization is needed before induction