Connected Components

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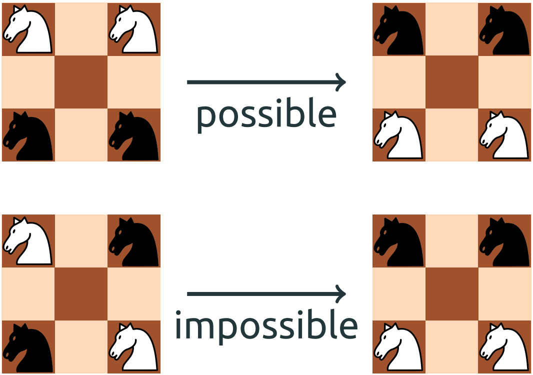
The Heaviest Stone

There are n stones of different weights. An expert knows the weights and wants to convince the court that a particular stone is the heaviest one. For this, he repeatedly uses a pan balance to compare the weights of some two stones. What is the minimum number of comparisons required?

Guarini Puzzle, Revisited



can we check this automatically
instead of manually?

Hm...

- What do these two unrelated puzzles have in common?

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- They both can be solved by analyzing connected components of an underlying graph!

Outline

Connected Components

Guarini Puzzle: Program

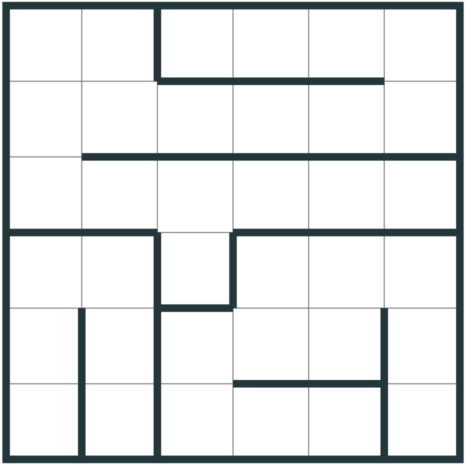
Lower Bound

The Heaviest Stone

Directed Acyclic Graphs

Strongly Connected Components

Connected Components in a Maze



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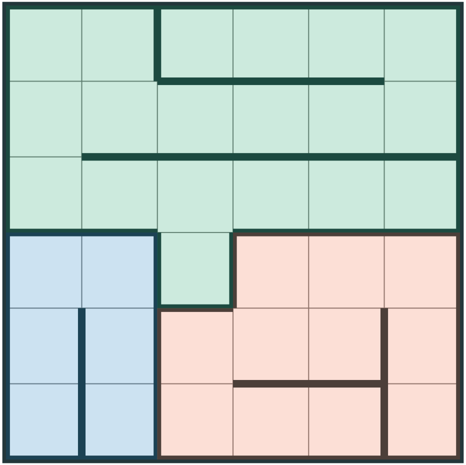

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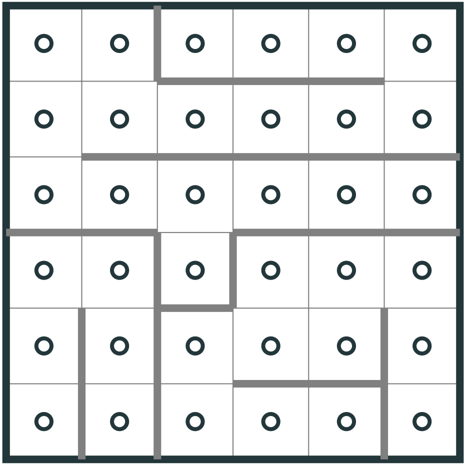

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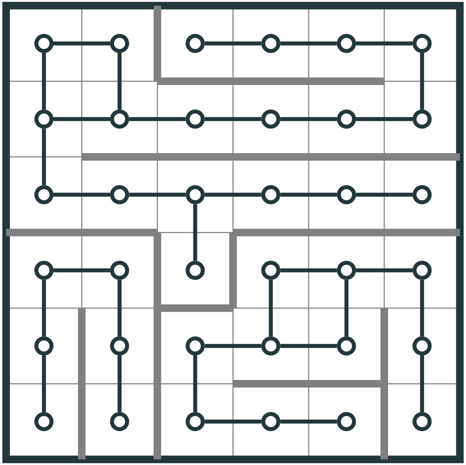

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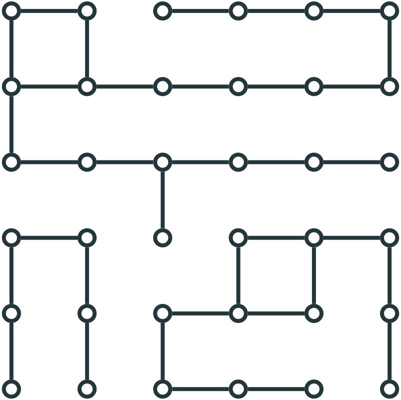

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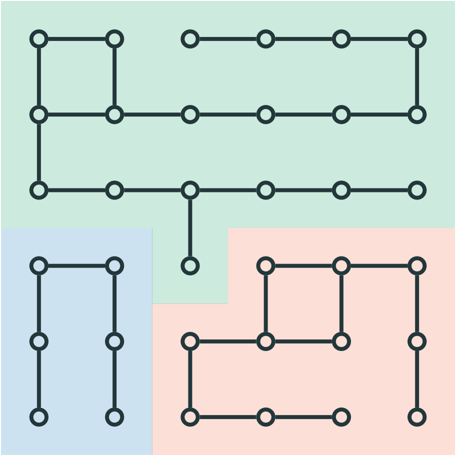

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- Consider an *undirected* graph
- Two nodes are connected, if there is a path between them
- It is transitive: if u and v are connected and v and w are connected, then u and w are connected, too
- A graph is connected, if any two of its nodes are connected. In other words, there is a path between any two of its nodes

Connected Components

The nodes of any undirected graph can be partitioned into subsets called connected components:

- Any node belongs to exactly one connected component
- Any two nodes from the same connected component are connected
- Any two nodes from different connected components are not connected

Examples

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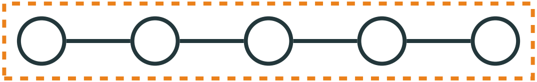
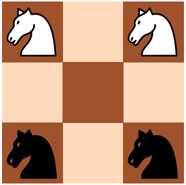
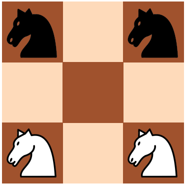


Outline

Revisiting the Guarini Puzzle





Given two configurations, check whether one is reachable from the other one

Graph of Configurations

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- Join two nodes by an edge if their configurations are within a single move from each other

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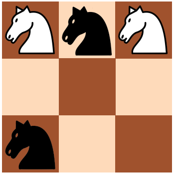

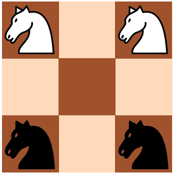

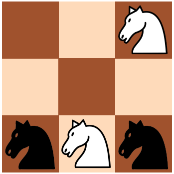

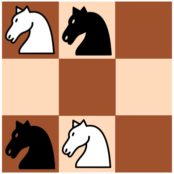

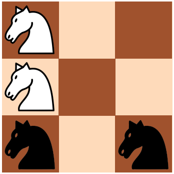

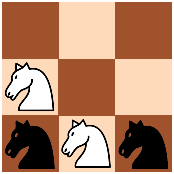

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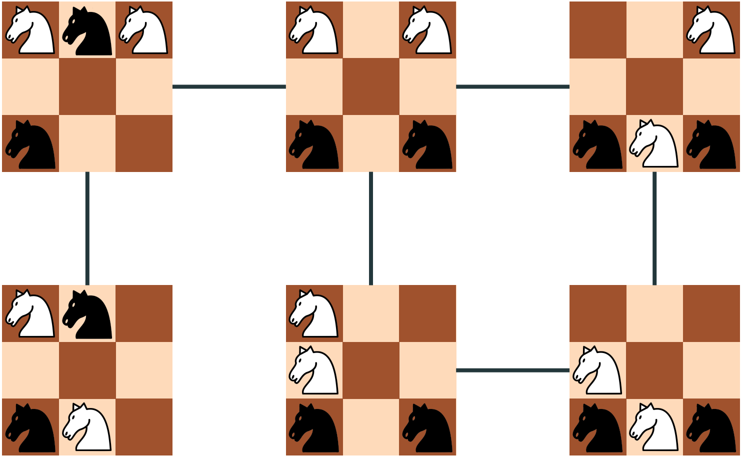

Solution

Then, one configuration is reachable from the other one, if and only if they belong to the same connected component!

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Lower Bound

Theorem

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- The theorem is useless for graphs with $|E| \geq |V|$

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- Each time when we add a new edge, $|V| - |E|$ decreases by 1
- At the same time, the number of connected components either decreases by 1 or stays the same

Illustration

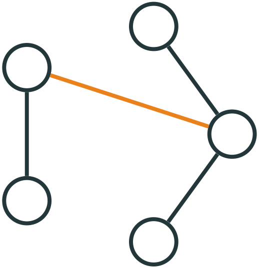

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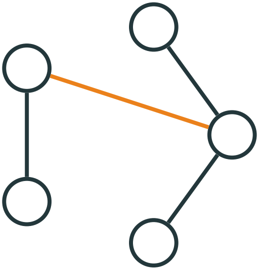
Illustration



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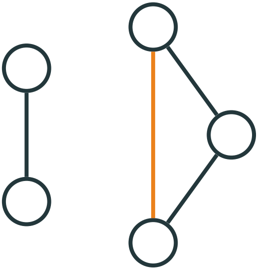

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- but is it optimal?
- yes!

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- But this means that the court is still not sure about the heaviest stone!

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DAGs

Definition

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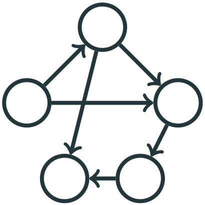

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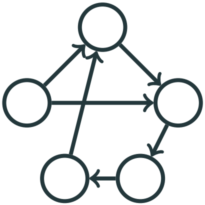

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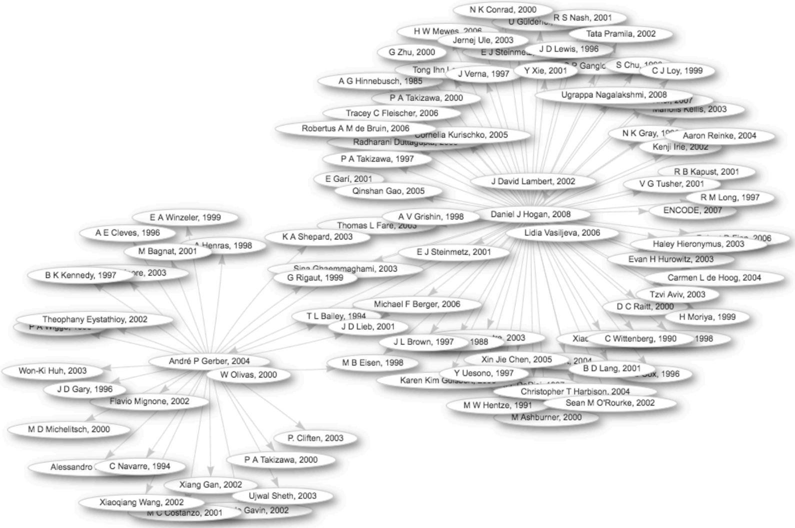
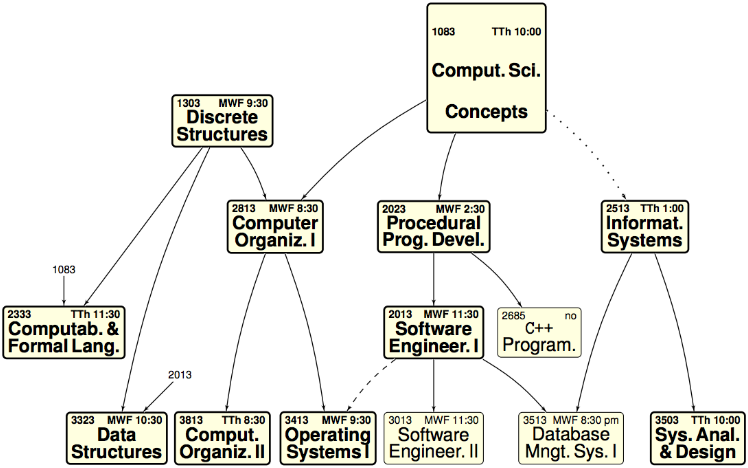
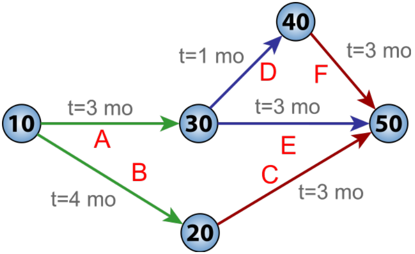
Citation Graph



Prerequisite Graph



Dependency Graph



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- We want to process jobs one by one
- How to find an order of jobs satisfying all constraints?
- If there is a cycle in the graph, then there is no such order
- It turns out that this is the only obstacle: if the graph is acyclic, then there is an ordering of its vertices satisfying all the constraints!

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A topological ordering of a directed graph is an ordering of its vertices such that, for each edge (u, v), u comes before v.

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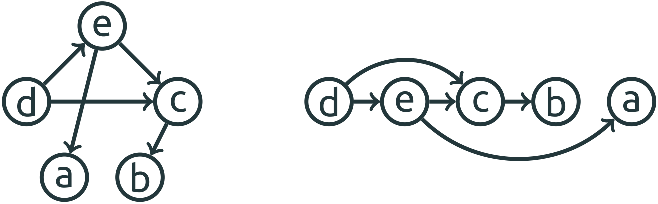

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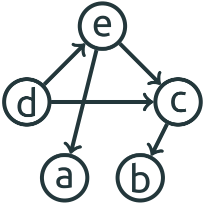
- We'll show that every DAG has a sink — a node with no outgoing edges
- Take a sink, put it to the end of the ordering, remove it from the graph (this keeps the graph acyclic), and repeat

Example



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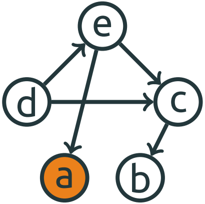

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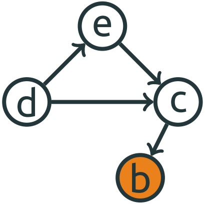

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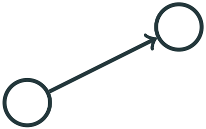

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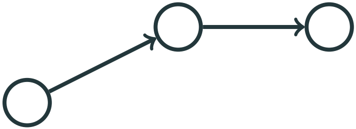

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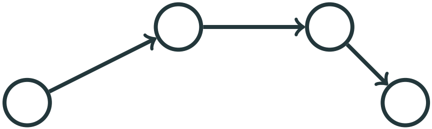

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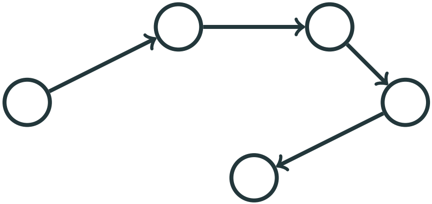

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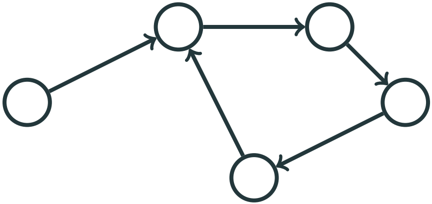

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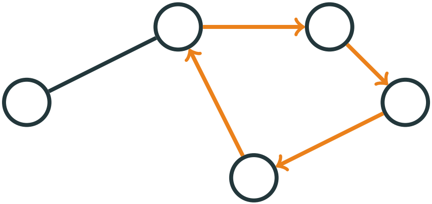

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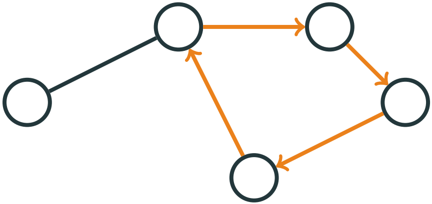

- A contradiction!

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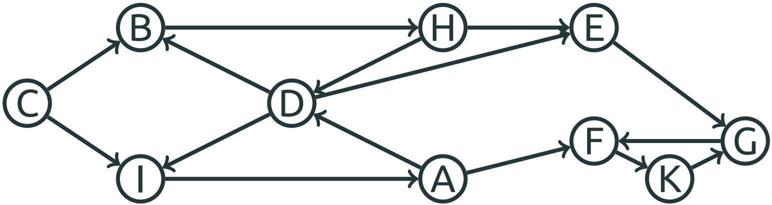
Is This Graph Connected?



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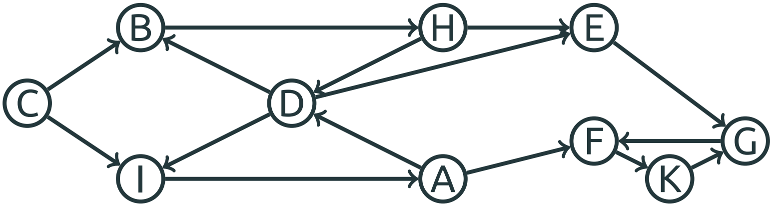

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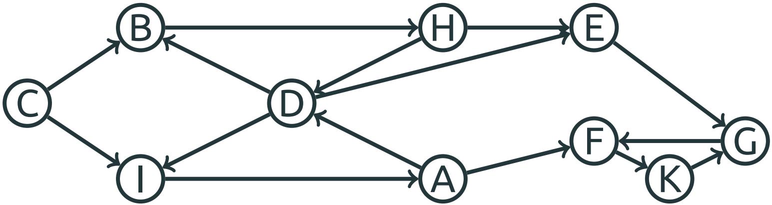

- On one hand, this graph is connected: it cannot be “pulled apart”
- On the other hand, it is not connected: e.g., there is no path from A to C

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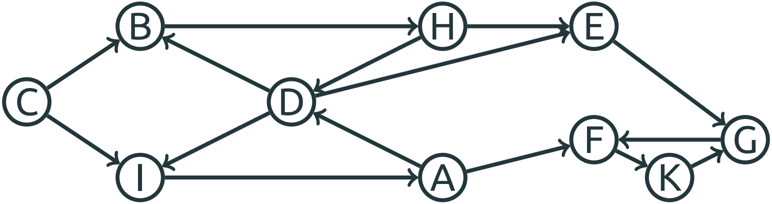

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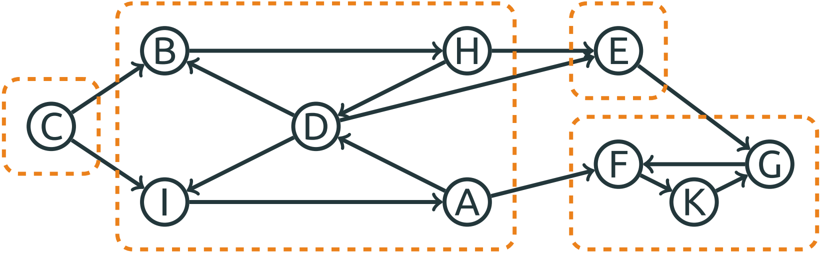

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