Cliques and Independent Sets

Alexander Golovnev

Outline

Graph Cliques

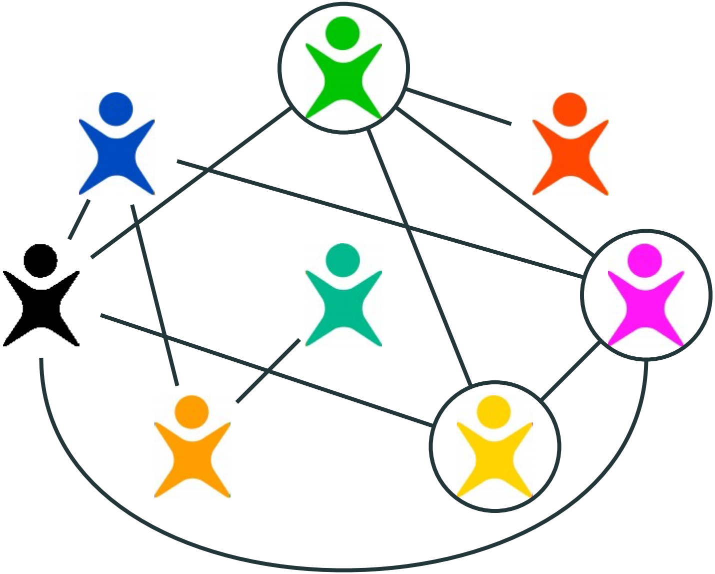
Cliques and Independent Sets

Connections to Coloring

Mantel's Theorem

Friendship Graph

Friendship Graph



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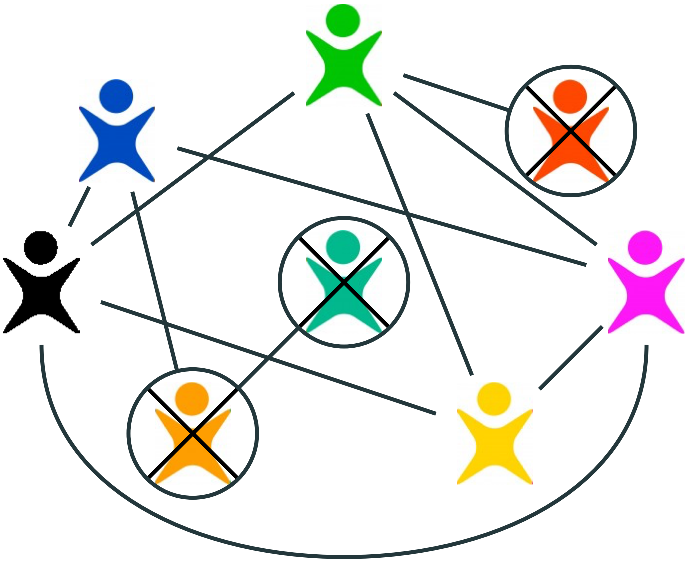

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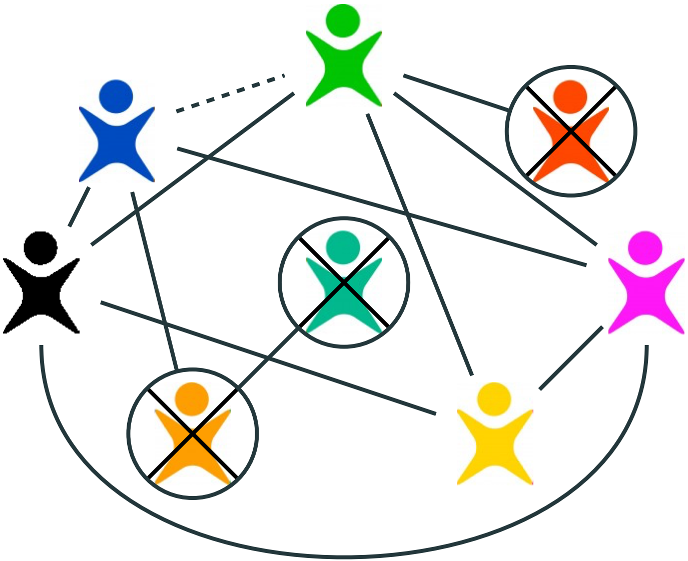

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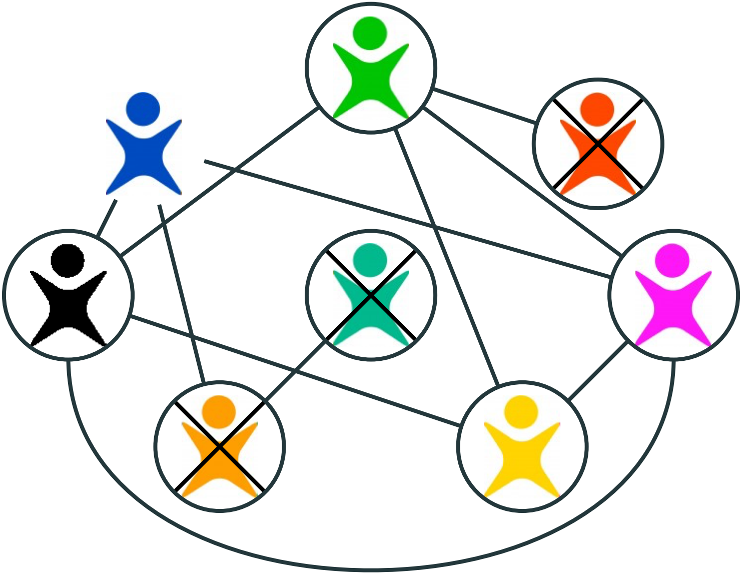

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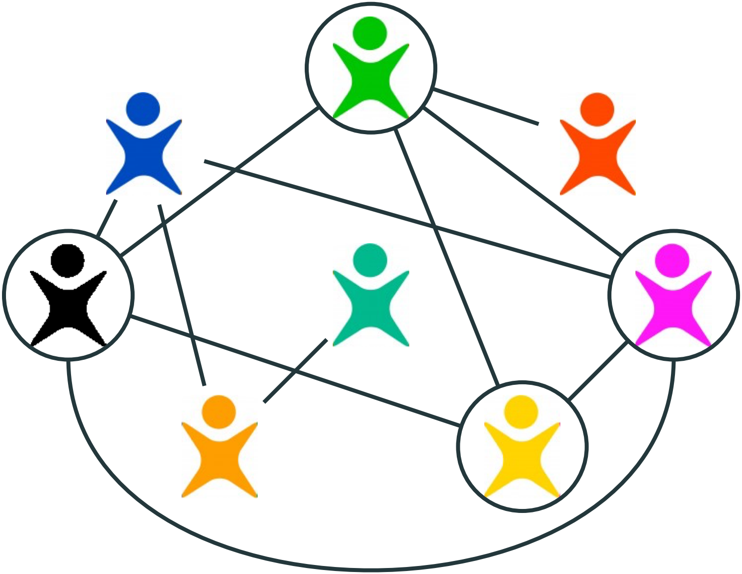

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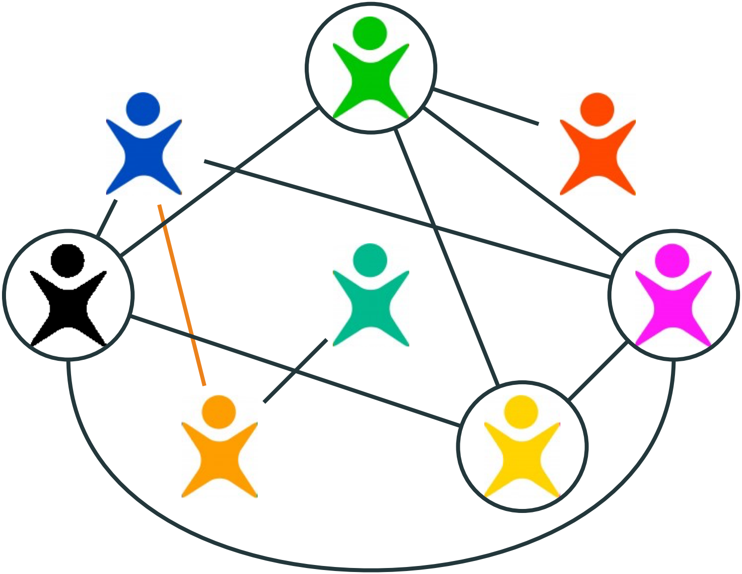

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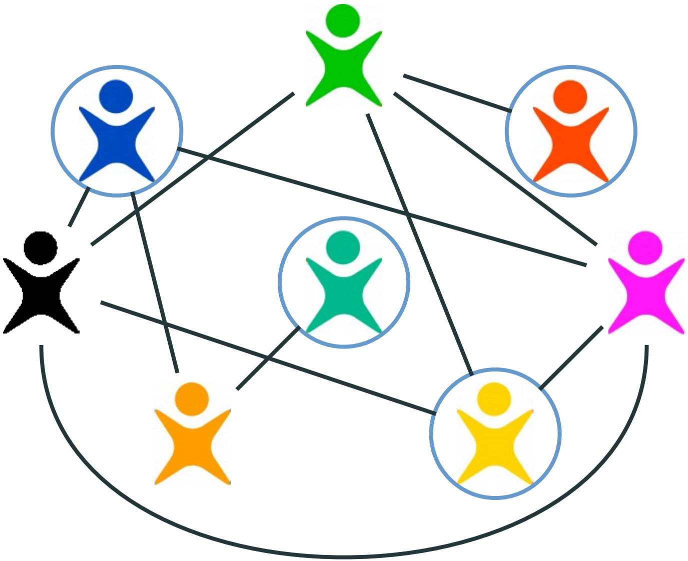
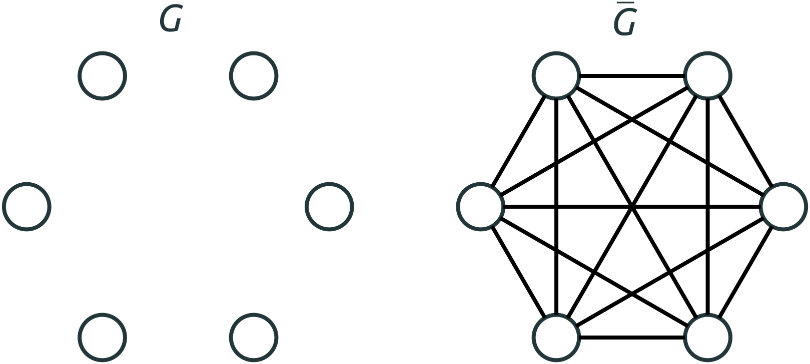


Complement Graph



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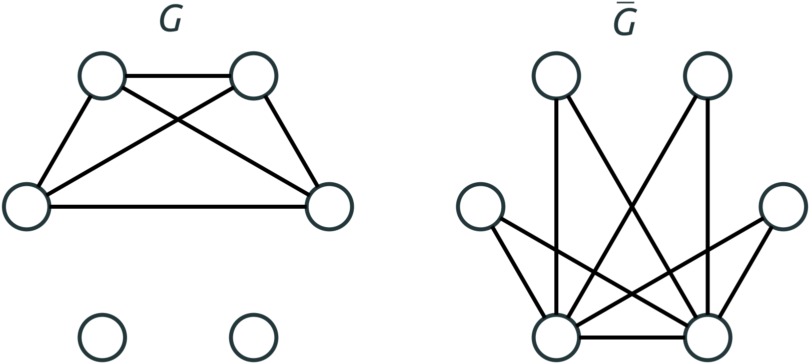

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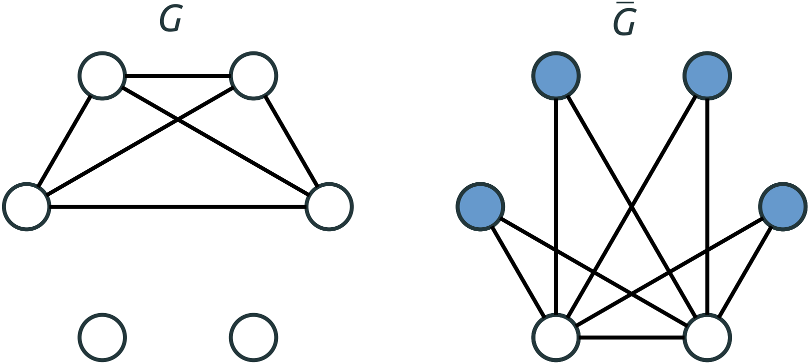

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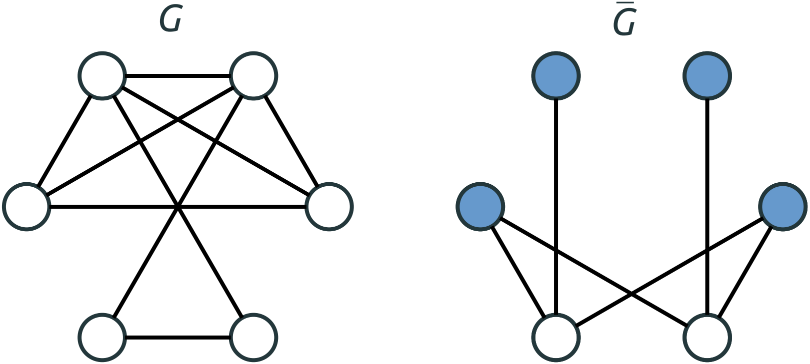

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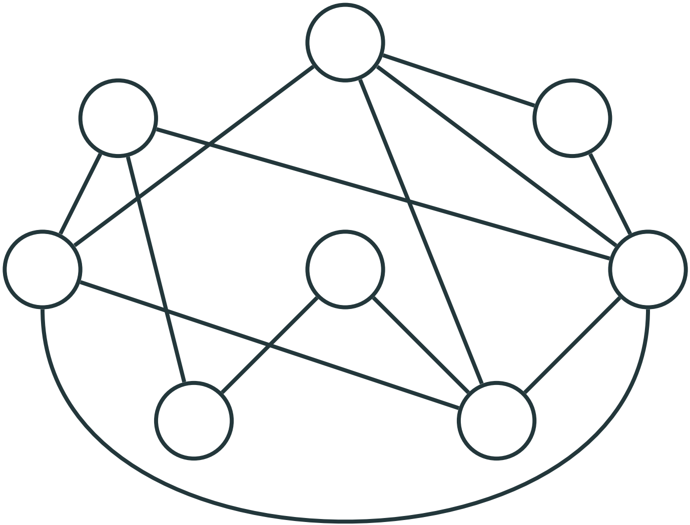
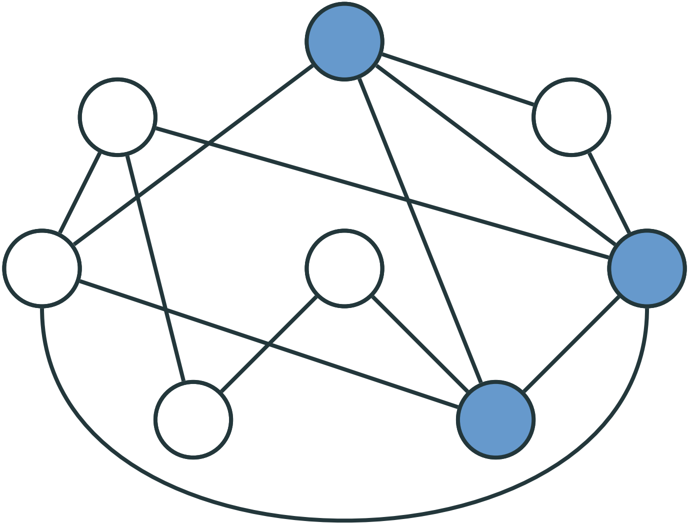
- The Clique Number $\omega(G)$ of a graph G is the number of vertices in its maximum clique.

Cliques: Examples



Cliques: Examples

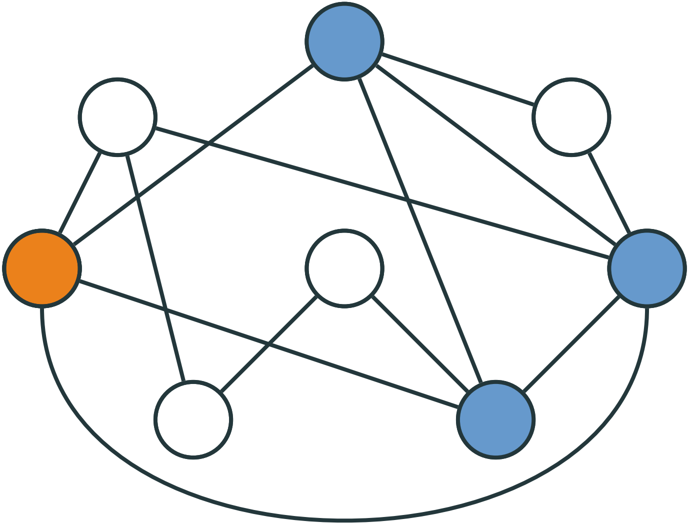
A Clique



Cliques: Examples

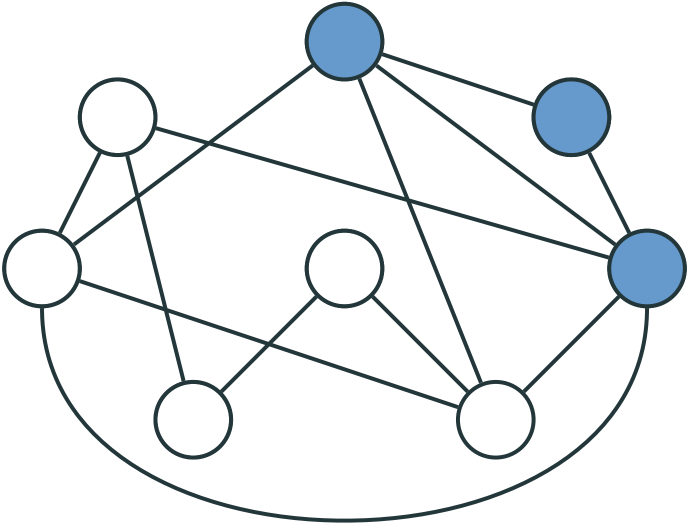
A Clique

Not a Maximal Clique



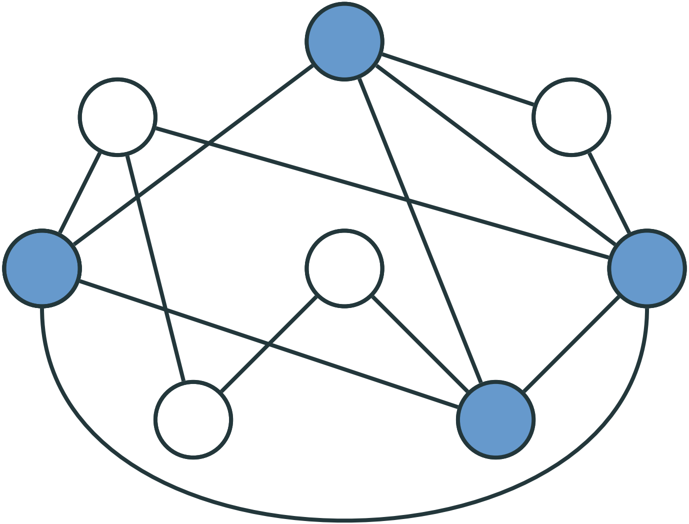
Cliques: Examples

A Maximal Clique: cannot be extended to a larger clique



Cliques: Examples

A Maximum Clique: there are no cliques with more than 4 vertices



Independent Sets

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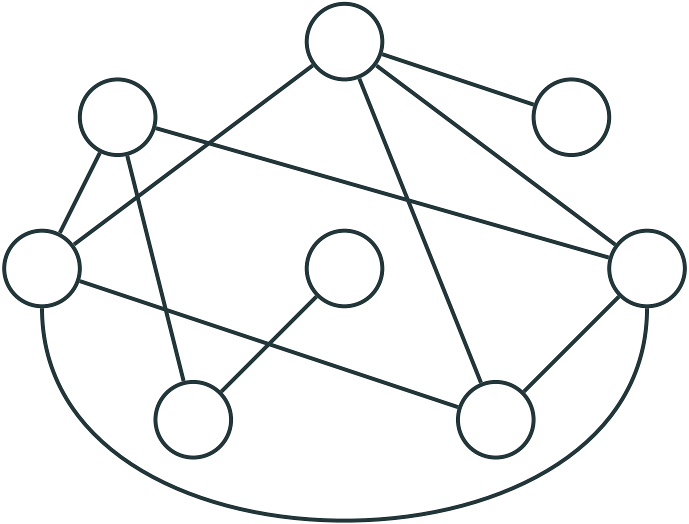
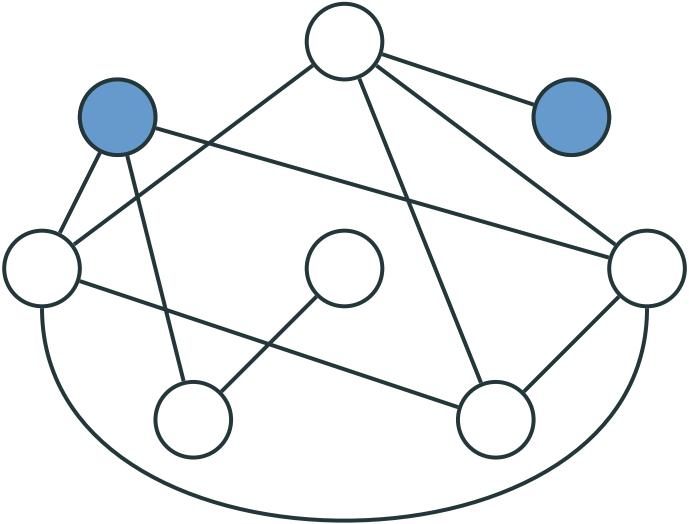
- The Independence Number $\alpha(G)$ of a graph G is the number of vertices in its maximum IS.

Independent Sets: Examples



Independent Sets: Examples

An IS



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An IS

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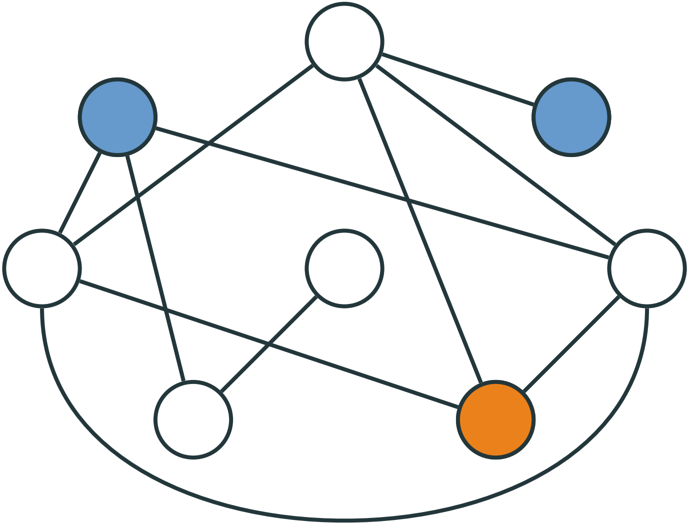
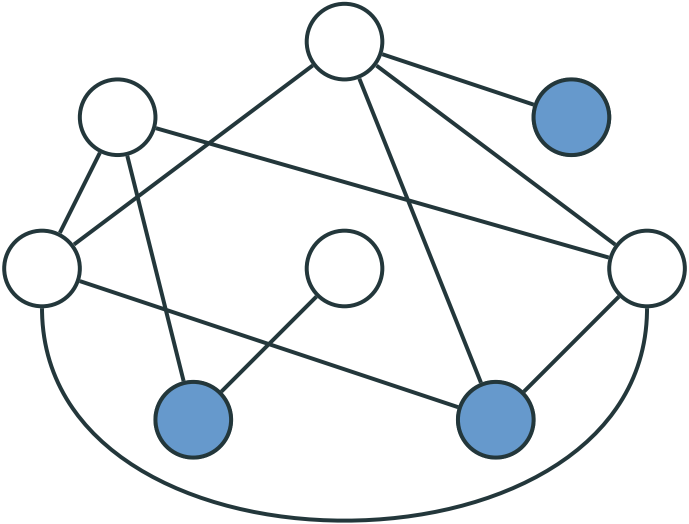


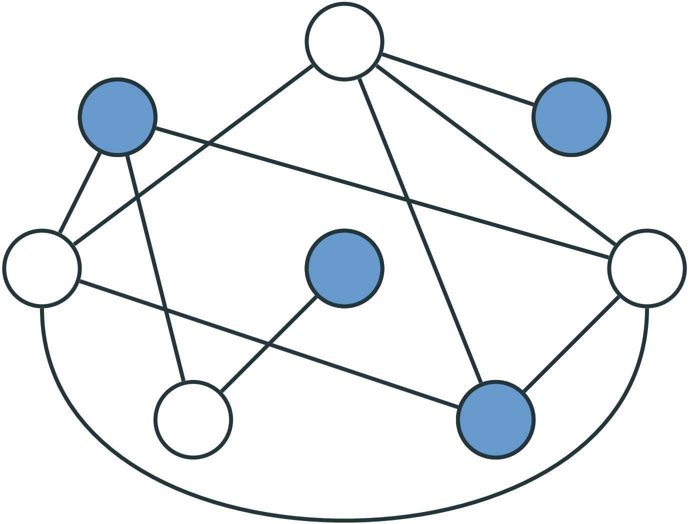
Independent Sets: Examples

A Maximal IS: cannot be extended to a larger IS

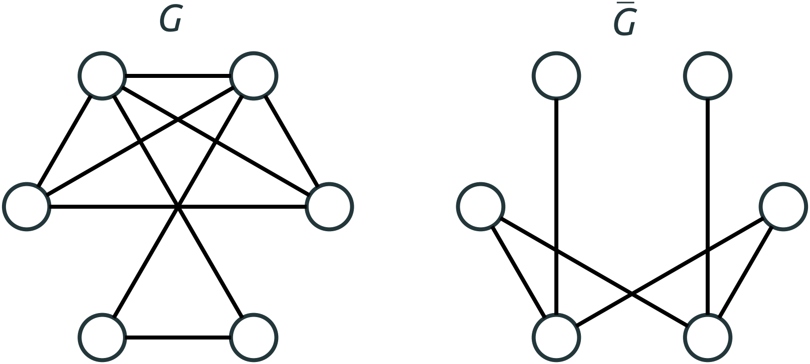


Independent Sets: Examples

A Maximum IS: there are no IS's with more than 4 vertices



Cliques and IS's



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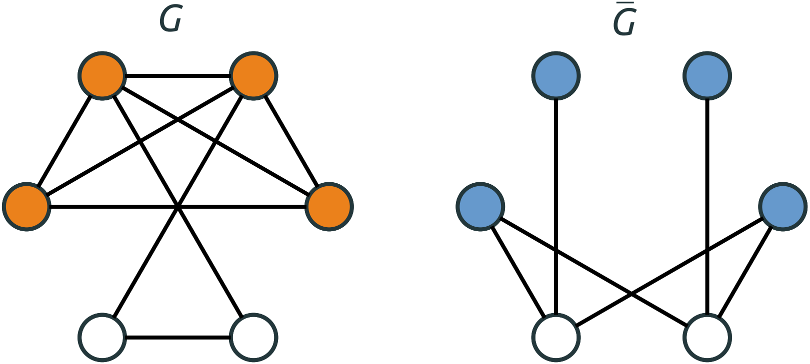

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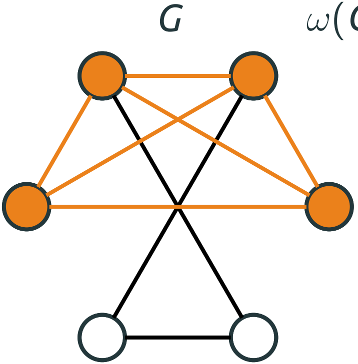
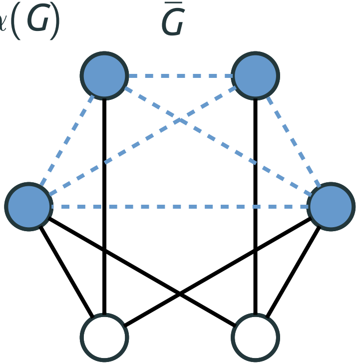


$$\omega(G) = \alpha(\bar{G})$$



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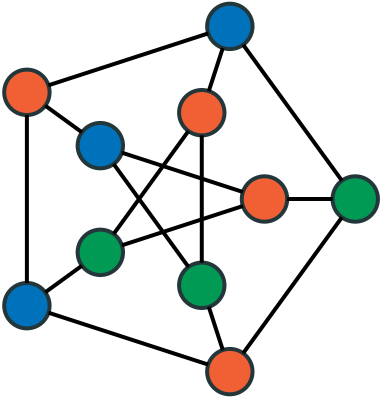
Connections to Coloring

Mantel's Theorem

Coloring and IS's

Each color forms an independent set



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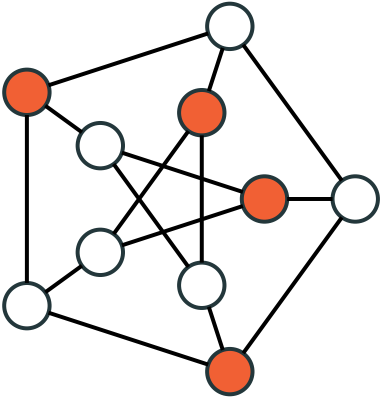

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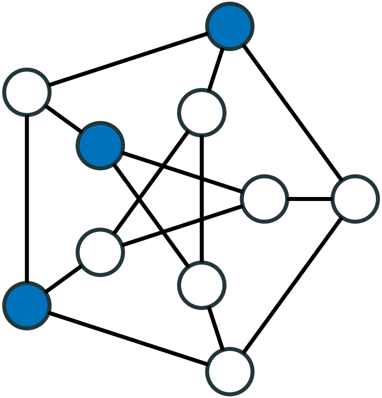

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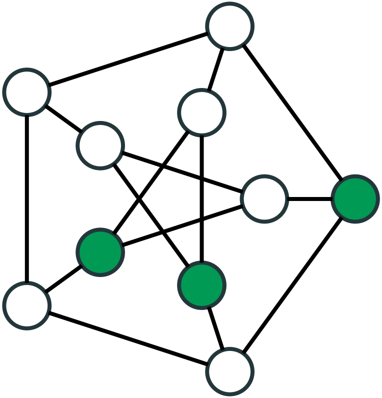

Chromatic and Independence Numbers

Theorem

For every graph G on n vertices it holds that:

$$\chi(G) \cdot \alpha(G) \geq n \,.$$

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Proof:

- n vertices can be partitioned into $\chi(G)$ IS's
- Each IS is of size $\leq \alpha(G)$
- Therefore, $n \leq \chi(G) \cdot \alpha(G)$

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- Induction step will prove the statement for all graphs of size $\leq n$. Step of size 2, this is why we did base cases for $n = 1, 2$

Mantel's Theorem. Proof

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Tight Example: $K_{n/r,\ldots,n/r}$