Linearity of Expectation

Vladimir Podolskii

Computer Science Department, Higher School of Economics

Outline

Linearity of Expectation

Birthday Problem

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \ldots, a_k$, the outcomes for g are $b_1, \ldots, b_k$, the probabilities are $p_1, \ldots, p_k$

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \dots, a_k$, the outcomes for g are $b_1, \dots, b_k$, the probabilities are $p_1, \dots, p_k$
- Consider $f + g$

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \dots, a_k$, the outcomes for g are $b_1, \dots, b_k$, the probabilities are $p_1, \dots, p_k$
- Consider $f + g$
- It is also a random variable over the same probability distribution

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \dots, a_k$, the outcomes for g are $b_1, \dots, b_k$, the probabilities are $p_1, \dots, p_k$
- Consider $f + g$
- It is also a random variable over the same probability distribution
- Values of $f + g$ are $a_1 + b_1, \dots, a_k + b_k$

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \ldots, a_k$, the outcomes for g are $b_1, \ldots, b_k$, the probabilities are $p_1, \ldots, p_k$
- Consider $f + g$
- It is also a random variable over the same probability distribution
- Values of $f + g$ are $a_1 + b_1, \ldots, a_k + b_k$
- Can we say anything about the expectation of $f + g$?

Linearity of Expectation

- Suppose there are two random variables f and g over the same probability space
- The outcomes for f are $a_1, \ldots, a_k$, the outcomes for g are $b_1, \ldots, b_k$, the probabilities are $p_1, \ldots, p_k$
- Consider $f + g$
- It is also a random variable over the same probability distribution
- Values of $f + g$ are $a_1 + b_1, \ldots, a_k + b_k$
- Can we say anything about the expectation of $f + g$? Yes!

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space. Then

$$\mathsf{E}(f+g) = \mathsf{E}f + \mathsf{E}g$$

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space. Then

$$\mathsf{E}(f+g) = \mathsf{E}f + \mathsf{E}g$$

- Indeed, we have

$$\mathsf{E}(f+g) = (f_1+g_1)p_1 + ... + (f_k+g_k)p_k$$

$$= (f_1p_1 + ... + f_kp_k) + (g_1p_1 + ... + g_kp_k) = \mathsf{E}f + \mathsf{E}g$$

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space. Then

$$\mathsf{E}(f+g) = \mathsf{E}f + \mathsf{E}g$$

- Linearity is a very useful property

Linearity of Expectation

Linearity of Expectation

Suppose there are random variables f and g on the same probability space. Then

$$\mathsf{E}(f+g) = \mathsf{E}f + \mathsf{E}g$$

- Linearity is a very useful property
- Greatly simplifies computation of expectations

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- If we compute it directly, we need to calculate probabilities of all values of the sum

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- If we compute it directly, we need to calculate probabilities of all values of the sum
- Not hard, but requires time

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the first dice

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the first dice
- f_2 is an outcome of the second dice

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the first dice
- f_2 is an outcome of the second dice
- We are interested in $\mathsf{E}(f_1 + f_2)$

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the first dice
- f_2 is an outcome of the second dice
- We are interested in $\mathsf{E}(f_1 + f_2)$
- We already computed the expected value of one dice throw: $\mathsf{E}f_1 = \mathsf{E}f_2 = 3.5$

Linearity of Expectation, Examples

Problem

We throw two dices. What is the expected value of the sum of two numbers on them?

- Instead we can consider two random variables on our probability distribution
- f_1 is an outcome of the first dice
- f_2 is an outcome of the second dice
- We are interested in $\mathsf{E}(f_1 + f_2)$
- We already computed the expected value of one dice throw: $\mathsf{E}f_1 = \mathsf{E}f_2 = 3.5$
- Thus, $\mathsf{E}(f_1 + f_2) = \mathsf{E}f_1 + \mathsf{E}f_2 = 7$

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Again, can compute directly

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Again, can compute directly
- But requires computation of probabilities of all possible numbers of tails

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Again, can compute directly
- But requires computation of probabilities of all possible numbers of tails
- Need to recall Combinatorics, and so on...

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Again, can compute directly
- But requires computation of probabilities of all possible numbers of tails
- Need to recall Combinatorics, and so on...
- Linearity, on the other hand, can give the answer almost immediately

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Let f_i be an outcome of the i-th coin: it is 1 if the outcome is “tails” and it is 0 if it is “heads”

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Let f_i be an outcome of the i-th coin: it is 1 if the outcome is “tails” and it is 0 if it is “heads”
- We are interested in E$(f_1 + f_2 + f_3 + f_4 + f_5)$!

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Let f_i be an outcome of the i-th coin: it is 1 if the outcome is "tails" and it is 0 if it is "heads"
- We are interested in E$(f_1 + f_2 + f_3 + f_4 + f_5)$!
- It is easy to compute the expectation for a single coin: E$f_i = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

Linearity of Expectation, Examples

Problem

We toss a coin 5 times in a row. What is the expected number of tails?

- Let f_i be an outcome of the i-th coin: it is 1 if the outcome is “tails” and it is 0 if it is “heads”
- We are interested in $\mathsf{E}(f_1 + f_2 + f_3 + f_4 + f_5)$!
- It is easy to compute the expectation for a single coin: $\mathsf{E}f_i = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$
- Thus, $\mathsf{E}(f_1 + f_2 + f_3 + f_4 + f_5) = \mathsf{E}f_1 + \mathsf{E}f_2 + \mathsf{E}f_3 + \mathsf{E}f_4 + \mathsf{E}f_5 = 2.5$

Outline

Linearity of Expectation

Birthday Problem

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- If there are two people with the same birthday, they will contribute 1 to the number of pairs in the problem

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- If there are two people with the same birthday, they will contribute 1 to the number of pairs in the problem
- If there are three people with the same birthday, they form 3 pairs

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- If there are two people with the same birthday, they will contribute 1 to the number of pairs in the problem
- If there are three people with the same birthday, they form 3 pairs
- So they will contribute 3 to the number of pairs in the problem

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- Looks surprising: not many people
- But we will prove it!

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- Formalization is needed

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- Formalization is needed
- We assume that birthdays are distributed uniformly among 365 days of the year

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- Formalization is needed
- We assume that birthdays are distributed uniformly among 365 days of the year
- We will not discuss it, but a nonuniform distribution on days of the year only increases the expectation!

Birthday Problem

Birthday Problem

Consider 28 randomly chosen people. Consider the number of pairs (i,j) such that the i-th person has a birthday on the same day as the j-th person. Show that the expectation of this number is greater than 1

- Formalization is needed
- We assume that birthdays are distributed uniformly among 365 days of the year
- We will not discuss it, but a nonuniform distribution on days of the year only increases the expectation!
- People are chosen independently

Birthday Problem

- We will use the linearity of expectation; denote the number of pairs of people with the same birthday by f

Birthday Problem

- We will use the linearity of expectation; denote the number of pairs of people with the same birthday by f
- Enumerate people from 1 to 28; consider a random variable g_{ij} that is equal to 1 if persons i and j have birthday on the same day, and is equal to 0 otherwise

Birthday Problem

- We will use the linearity of expectation; denote the number of pairs of people with the same birthday by f
- Enumerate people from 1 to 28; consider a random variable g_{ij} that is equal to 1 if persons i and j have birthday on the same day, and is equal to 0 otherwise
- Observation: f is equal to the sum of g_{ij} over all (unordered) pairs of i and j!

Birthday Problem

- We will use the linearity of expectation; denote the number of pairs of people with the same birthday by f
- Enumerate people from 1 to 28; consider a random variable g_{ij} that is equal to 1 if persons i and j have birthday on the same day, and is equal to 0 otherwise
- Observation: f is equal to the sum of g_{ij} over all (unordered) pairs of i and j!
- Why?

Birthday Problem

Consider an example of 5 people

Five people: 1, 2, 3, 4, 5

Birthday Problem

Consider an example of 5 people

Five people: 1, 2, 3, 4, 5

List of all pairs:

{1,2}	{2,4}
{1,3}	{2,5}
{1,4}	{3,4}
{1,5}	{3,5}
{2,3}	{4,5}

Birthday Problem

Consider an example of 5 people

Five people: 1, 2, 3, 4, 5

List of all pairs:

{1,2}	$g_{1,2} = 0$	{2,4}	$g_{2,4} = 0$
{1,3}	$g_{1,3} = 1$	{2,5}	$g_{2,5} = 0$
{1,4}	$g_{1,4} = 0$	{3,4}	$g_{3,4} = 0$
{1,5}	$g_{1,5} = 0$	{3,5}	$g_{3,5} = 0$
{2,3}	$g_{2,3} = 0$	{4,5}	$g_{4,5} = 1$

Birthday Problem

Consider an example of 5 people

Five people: 1, 2, 3, 4, 5

List of all pairs:

{1,2}	$g_{1,2} = 0$	{2,4}	$g_{2,4} = 0$
{1,3}	$g_{1,3} = 1$	{2,5}	$g_{2,5} = 0$
{1,4}	$g_{1,4} = 0$	{3,4}	$g_{3,4} = 0$
{1,5}	$g_{1,5} = 0$	{3,5}	$g_{3,5} = 0$
{2,3}	$g_{2,3} = 0$	{4,5}	$g_{4,5} = 1$

Note that f is the number of pairs $\{i, j\}$ with $g_{ij} = 1$.

Birthday Problem

Consider an example of 5 people

Five people: 1, 2, 3, 4, 5

List of all pairs:

{1,2}	$g_{1,2} = 0$	{2,4}	$g_{2,4} = 0$
{1,3}	$g_{1,3} = 1$	{2,5}	$g_{2,5} = 0$
{1,4}	$g_{1,4} = 0$	{3,4}	$g_{3,4} = 0$
{1,5}	$g_{1,5} = 0$	{3,5}	$g_{3,5} = 0$
{2,3}	$g_{2,3} = 0$	{4,5}	$g_{4,5} = 1$

Note that f is the number of pairs $\{i, j\}$ with $g_{ij} = 1$.
The sum of g_{ij} is the same number!

Birthday Problem

Birthday Problem

Consider 28 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

Birthday Problem

Birthday Problem

Consider 28 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Let's get back to the proof

Birthday Problem

Birthday Problem

Consider 28 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Let's get back to the proof
- We know that Ef is equal to the sum of Eg_{ij} over all pairs $\{i, j\}$

Birthday Problem

Birthday Problem

Consider 28 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Let's get back to the proof
- We know that Ef is equal to the sum of Eg_{ij} over all pairs $\{i, j\}$
- We need to compute Eg_{ij}

Birthday Problem

Birthday Problem

Consider 28 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Let's get back to the proof
- We know that Ef is equal to the sum of Eg_{ij} over all pairs $\{i, j\}$
- We need to compute Eg_{ij}
- We also need to count how many pairs of i and j do we have

Birthday Problem

- Expectation of individual g_{ij} is easy to compute:

$$\mathsf{E}g_{ij} = 1 \times \frac{1}{365} + 0 \times \frac{364}{365} = \frac{1}{365}$$

Birthday Problem

- Expectation of individual g_{ij} is easy to compute:

$$\mathsf{E} g_{ij} = 1 \times \frac{1}{365} + 0 \times \frac{364}{365} = \frac{1}{365}$$

- Why $\frac{1}{365}$?

Birthday Problem

- Expectation of individual g_{ij} is easy to compute:

$$\mathsf{E} g_{ij} = 1 \times \frac{1}{365} + 0 \times \frac{364}{365} = \frac{1}{365}$$

- Why $\frac{1}{365}$?
- There are 365×365 outcomes for birthdays of two people

Birthday Problem

- Expectation of individual g_{ij} is easy to compute:

$$\mathsf{E}g_{ij} = 1 \times \frac{1}{365} + 0 \times \frac{364}{365} = \frac{1}{365}$$

- Why $\frac{1}{365}$?
- There are 365×365 outcomes for birthdays of two people
- And only 365 outcomes with birthdays on the same day

Birthday Problem

- How many pairs of i and j do we have?

Birthday Problem

- How many pairs of i and j do we have?
- There are 28 people in total

Birthday Problem

- How many pairs of i and j do we have?
- There are 28 people in total
- There are $\binom{28}{2} = \frac{28 \times 27}{2} = 378$ ways to choose an unordered pair among them

Birthday Problem

- How many pairs of i and j do we have?
- There are 28 people in total
- There are $\binom{28}{2} = \frac{28 \times 27}{2} = 378$ ways to choose an unordered pair among them
- Short reminder: we have 28 options for the first one in the pair, we have 27 options for the second one, and we counted each pair twice

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Finally, we have the following

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Finally, we have the following
- Ef is the sum of Eg_{ij} over all pairs $\{i, j\}$

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Finally, we have the following
- Ef is the sum of Eg_{ij} over all pairs $\{i, j\}$
- E$g_{ij} = \frac{1}{365}$

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Finally, we have the following
- Ef is the sum of Eg_{ij} over all pairs $\{i, j\}$
- E$g_{ij} = \frac{1}{365}$
- There are 378 pairs of people

Birthday Problem

Birthday Problem

Consider 27 people randomly chosen people. Consider the number of pairs of people among them having birthday on the same day. Show that the expectation of this number is greater than 1

- Finally, we have the following
- $\mathsf{E}f$ is the sum of $\mathsf{E}g_{ij}$ over all pairs $\{i, j\}$
- $\mathsf{E}g_{ij} = \frac{1}{365}$
- There are 378 pairs of people
- Overall, we have

$$\mathsf{E}f = 378 \times \frac{1}{365} > 1$$