Reductio ad absurdum

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Outline

- Reductio ad Absurdum
- Balls in Boxes
- Numbers in Tables
- Pigeonhole Principle
- An (-1,0,1) Antimagic Square
- Handshakes

Reductio ad absurdum

- How to prove that something is true?

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- Show that the opposite is impossible!
- This method is called reductio ad absurdum or proof by contradiction
- One of the base methods of reasoning: is used everywhere
- Is often combined with other methods
- We will use constantly throughout our courses

Socratic Method

- Reductio ad absurdum is classic: used in Socratic method (Plato, ~400 BC)

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- Socrates revealed contradictions in his students believes by asking them questions step by step

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Example

Problem

There are boys and girls in the class. They are divided into two groups for the foreign language: there are students studying French, and there are students studying German. Each student picks one of the two languages. Show that there is a boy and a girl who study different languages.

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Seems impossible at first: we know basically nothing and we claim something nontrivial!

Solution

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Suppose the statement is wrong: there are no boy and girl studying different languages

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Everyone learns French!

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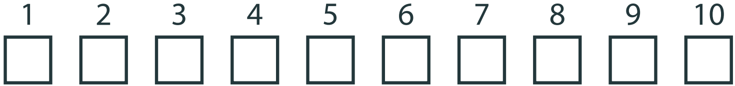
Balls in Boxes

Puzzle

We have 10 boxes filled with lots of white balls. We additionally have 30 black balls. We want to distinguish boxes by placing different number of black balls in all of them. Can we do that?

Suppose we can do it and let's see what happens
Let us enumerate all boxes in the increasing order of the number of black balls



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1	2	3	4	5	6	7	8	9	10
≥ 0									

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≥ 0	≥ 1	≥ 2							

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1	2	3	4	5	6	7	8	9	10
≥ 0	≥ 1	≥ 2	≥ 3	≥ 4					

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≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5				

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≥ 0	≥ 1	≥ 2	≥ 3	≥ 4	≥ 5	≥ 6			

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$$\begin{array}{cccccccccccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & 6 & & 7 & & 8 & & 9 & & 10 & \\ \boxed{\geq 0} & + & \boxed{\geq 1} & + & \boxed{\geq 2} & + & \boxed{\geq 3} & + & \boxed{\geq 4} & + & \boxed{\geq 5} & + & \boxed{\geq 6} & + & \boxed{\geq 7} & + & \boxed{\geq 8} & + & \boxed{\geq 9} & \geq 45 \end{array}$$

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There is a sequence of 10 cells, the leftmost contains number 1 and the rightmost cell contains 30. Is it possible to fill other cells with numbers in such a way that consecutive numbers differ by at most 3?

1									30

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1	4	7	10	13					30

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1	4	7	10	13	16				30

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1	4	7	10	13	16	19			30

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1	4	7	10	13	16	19	22		30

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1	4	7	10	13	16	19	22	25	30

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Numbers in the cells grow too slow

This is a very common trick to estimate the running time of some algorithm

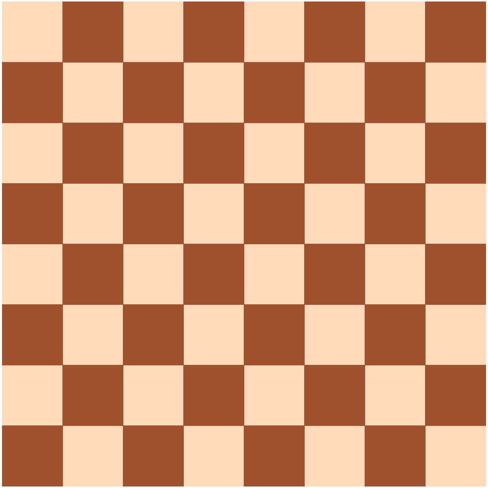
Numbers on the Chessboard

Puzzle

Is it possible to put numbers $1, 2, \ldots, 64$ on the chessboard in such a way that neighbors (sharing a side) differ by at most 4?

Again, suppose we can do it and let's see what happens

Numbers on the Chessboard



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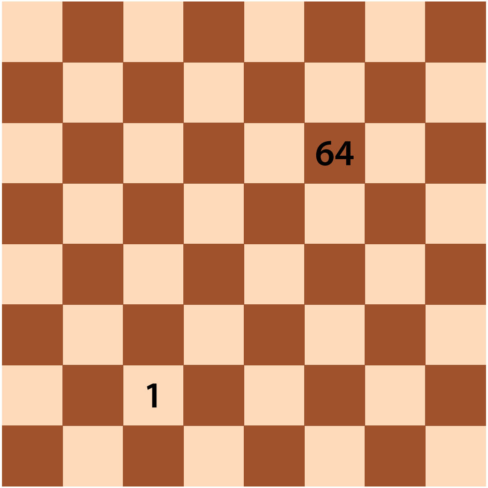

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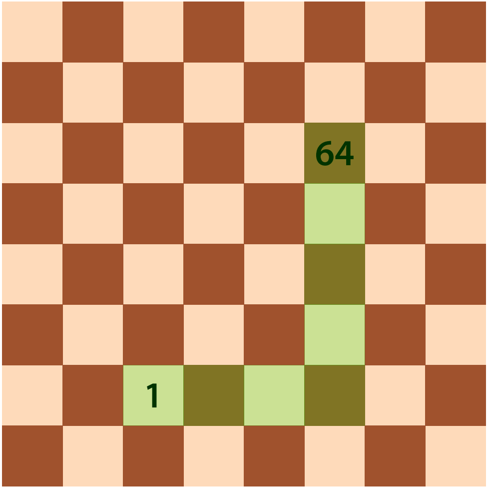

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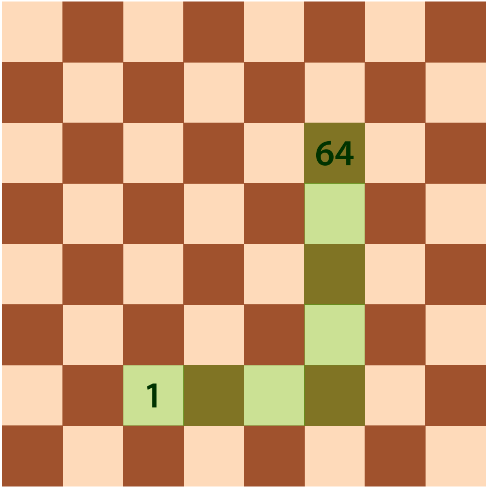

We need 7 steps to get from 1 to 64 in this example

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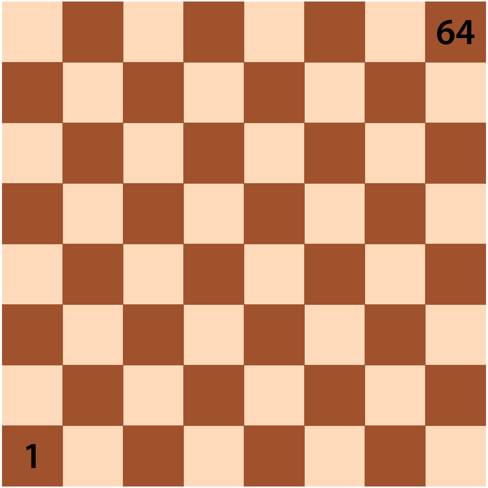

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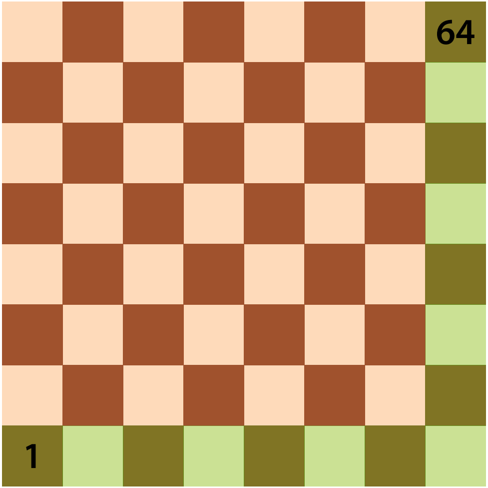

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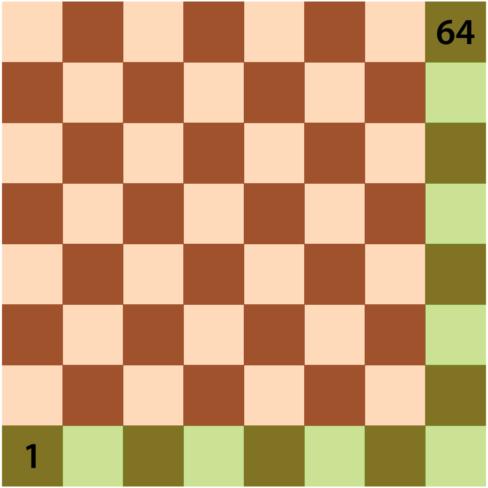

We need 14 steps to get from 1 to 64 in the worst case

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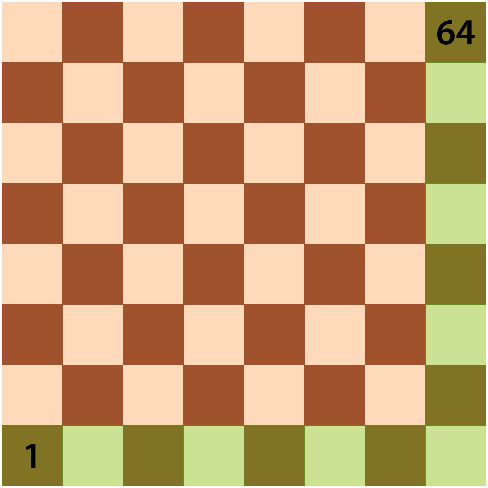

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the number can grow by at most
$4 \times 14 = 56$,
which is not enough

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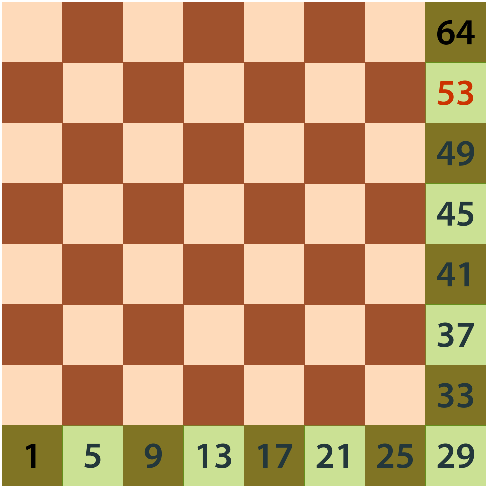

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Pigeonhole Principle

An (-1,0,1) Antimagic Square

Handshakes

Same Number of Hairs

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- Wikipedia says 8,537,673 (as of 2016)
- How many hairs does a person have?
- Wikipedia says 150,000 at most
- There are way more people in NYC than possible numbers of hairs!
- Thus there should be people with the same number of hairs

Pigeonhole Principle

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- Proof by contradiction: if there is at most one pigeon in each hole, then summing up we have at most n pigeons in all holes in total

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- Simple, but very useful principle
- Proof by contradiction: if there is at most one pigeon in each hole, then summing up we have at most n pigeons in all holes in total
- In the previous example people of NYC are "pigeons" and possible numbers of hairs are "holes"

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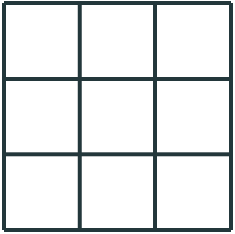

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- There are 3 rows, 3 columns and 2 diagonals, 8 in total

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- We are experienced: let's count

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- Note: 0 and 29 handshakes are impossible simultaneously! Now we have 29 pigeonholes!

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- But sometimes can help a lot on its own
- Pigeonhole principle: one of the most basic proof ideas
- Basically amounts to counting