Probability: Dos and Don’ts

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Outline

Not Equiprobable Outcomes

More About Finite Spaces

Not All Questions Make Sense

World is Not Perfect

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  more exercises than RL

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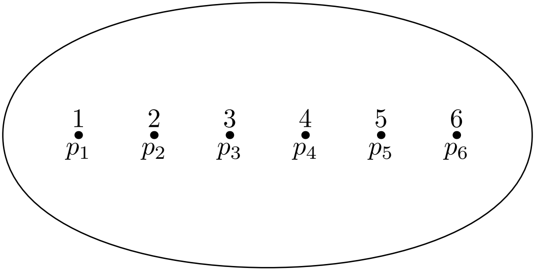
  but these outcomes do not matter

An Example



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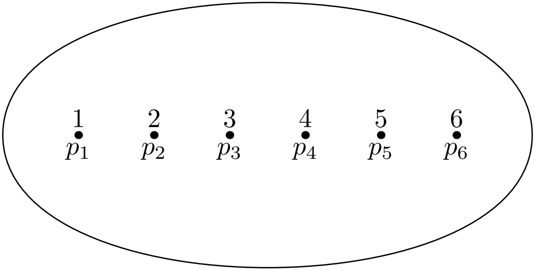

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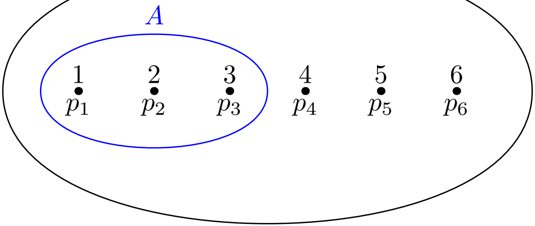

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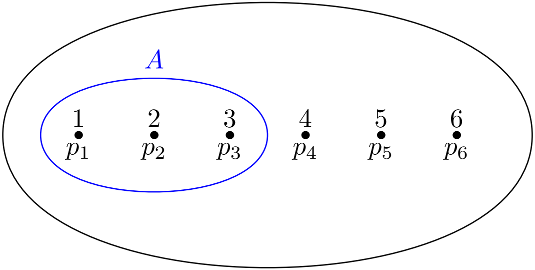

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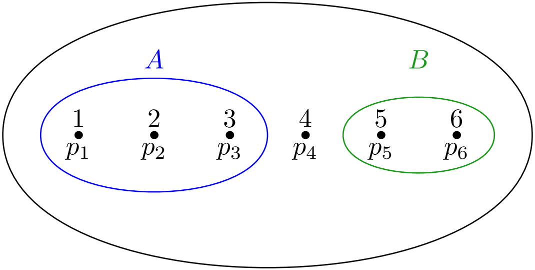

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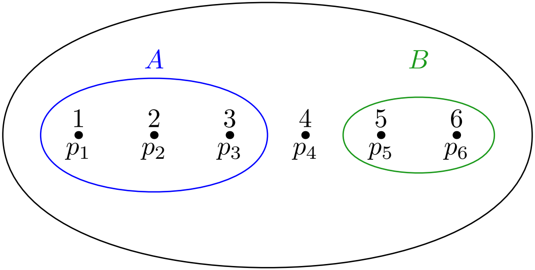

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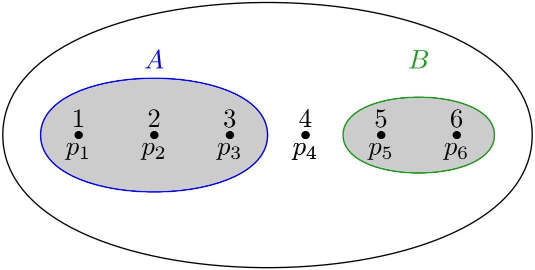

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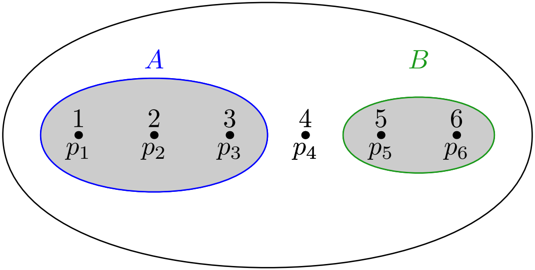

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$\Pr[A \text{ or } B] = \Pr[A] + \Pr[B] = p_1 + p_2 + p_3 + p_5 + p_6$

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  $\Pr[A \text{ or } B] <> \Pr[A] + \Pr[B]$?

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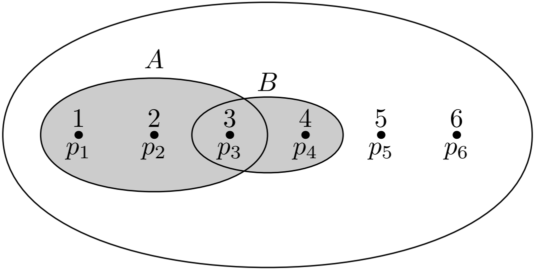

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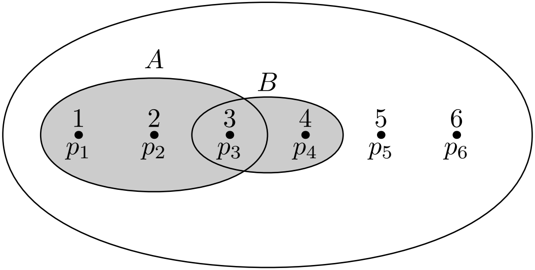

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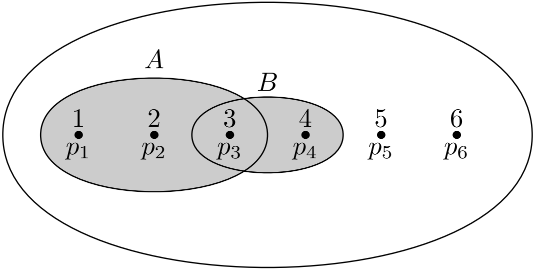

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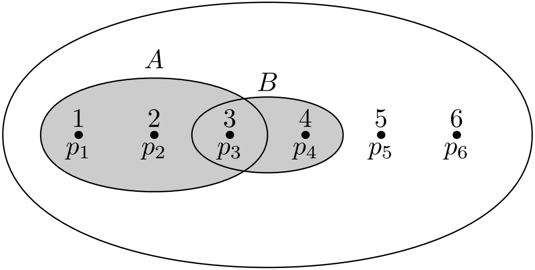

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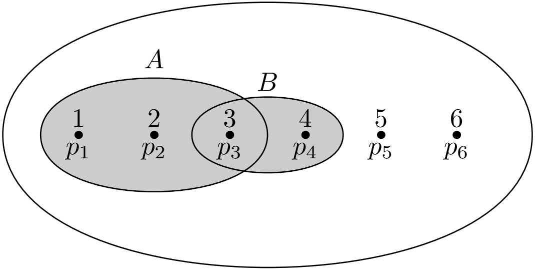

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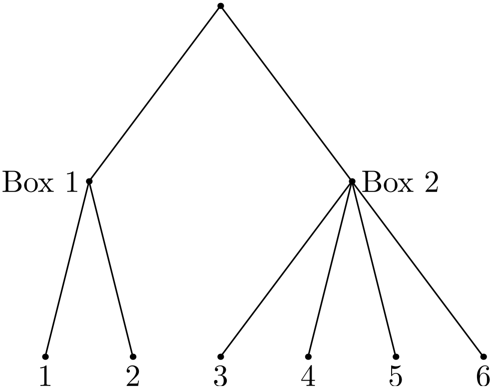

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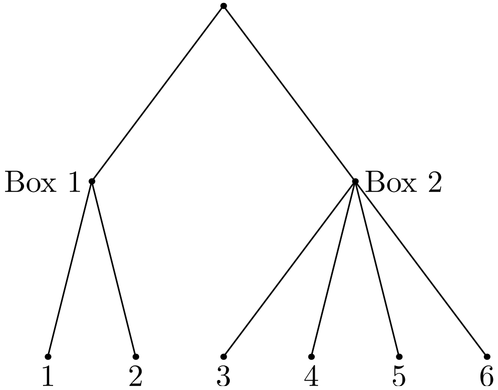

at each point all choices are equiprobable

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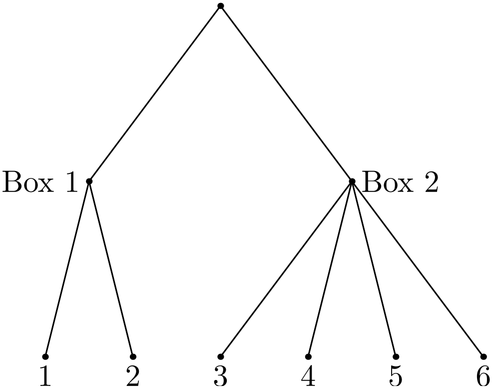

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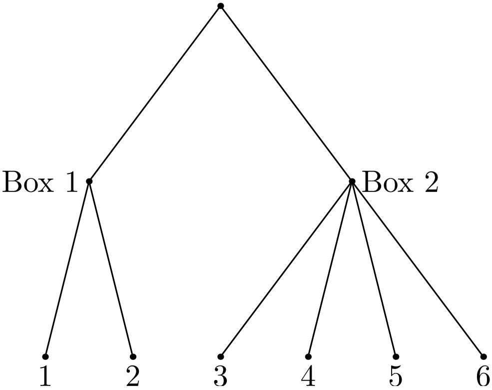

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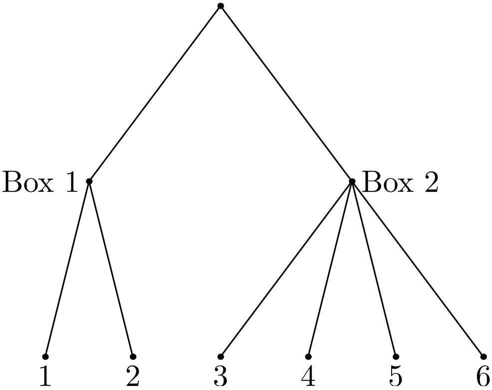

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- [case of boxes of equal size]

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  - can you "prove" the sum rule for mutual exclusive events?