Bipartite Graphs

Alexander Golovnev

Outline

Job Assignment

Bipartite Graphs

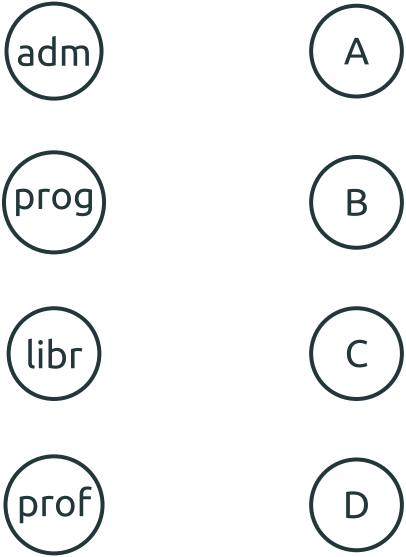
Matchings

Hall's Theorem

Job Assignment

	Alice	Ben	Chris	Diana
Administrator	+		+	
Programmer		+	+	
Librarian	+	+		
Professor				+

Job Assignment



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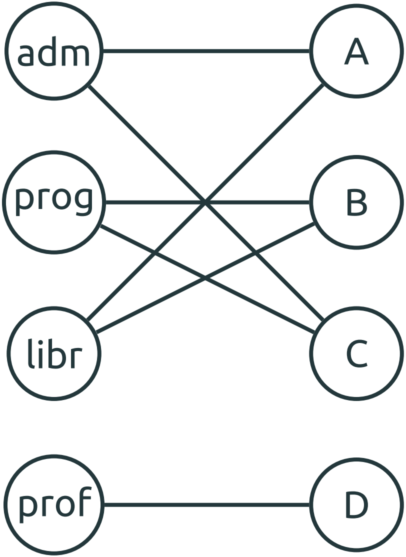

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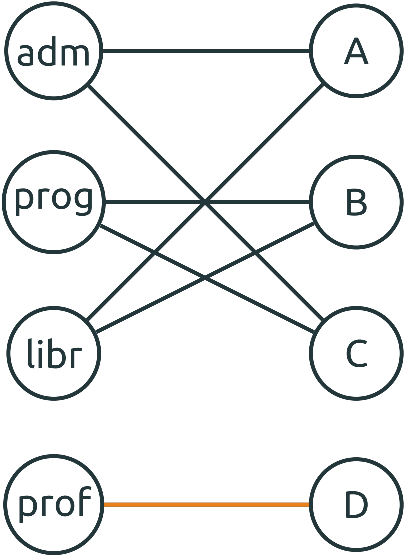

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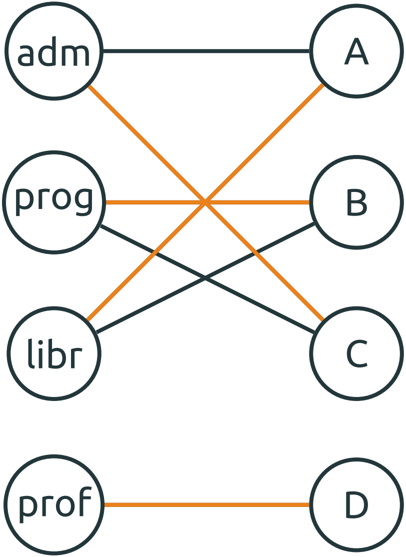

Room Assignment

	R# 1	R# 2	R# 3	R# 4	R# 5	R# 6
Aaron	+	+				
Bianca	+	+	+			
Carol				+	+	
Dana		+	+	+		+
Emma				+	+	
Francis				+	+	

Room Assignment

A	1
B	2
C	3
D	4
E	5
F	6

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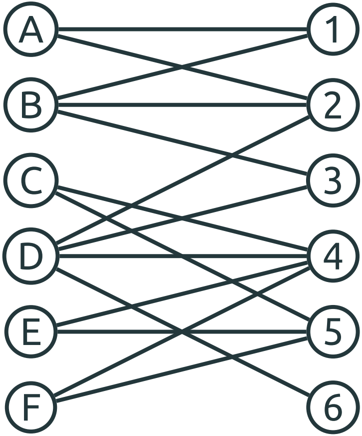

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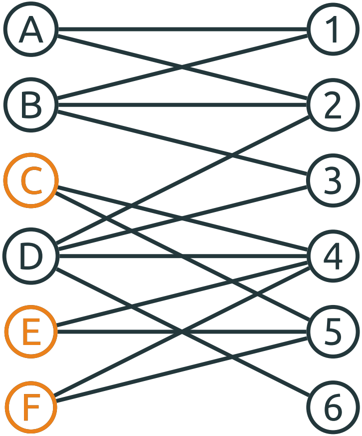

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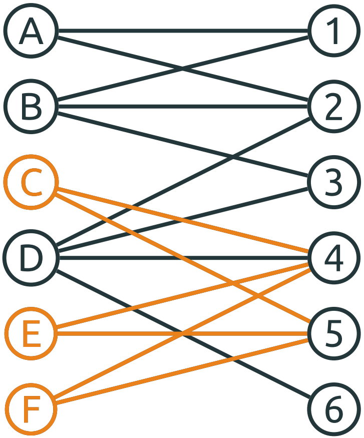

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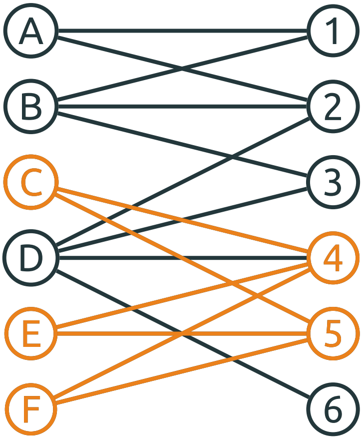

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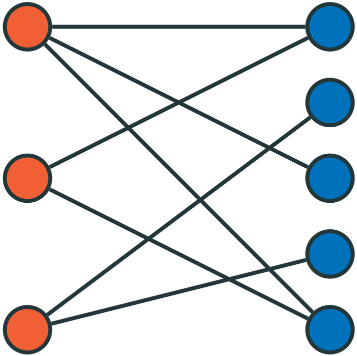
  - Every edge of G connects a vertex in L to a vertex in R
  - I.e., no edge connects two vertices from the same part
- L and R are called the parts of G

Bipartite Graphs: Examples



Bipartite Graphs: Characterization

Theorem

A graph is Bipartite if and only if it has no cycles of odd length.

Proof:

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Proof:

- Let $G = (L \cup R, E)$ be bipartite. Every edge goes from L to R (or from R to L)
- To end up in the original vertex, one has to make an even number of steps

Characterization: Proof

- Let's prove the other directions: if there are no cycles of odd length in *G*, then *G* is bipartite

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- If G has several connected components, fix one: C_1, and a vertex $v \in C_1$, color v red

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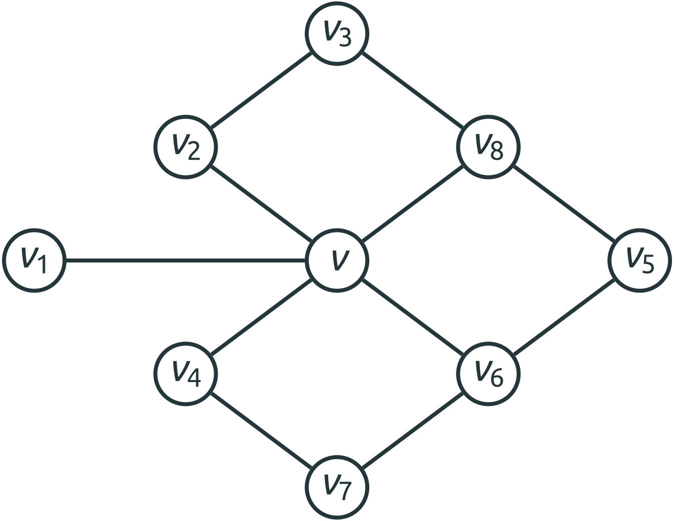
- If G has several connected components, fix one: C_1, and a vertex $v \in C_1$, color v red
- If there is a path from v to u of odd length, color u blue, otherwise: red

Characterization: Proof



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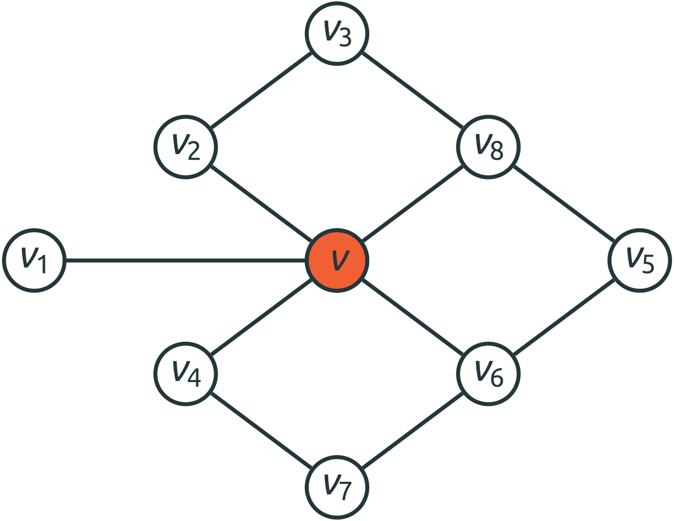

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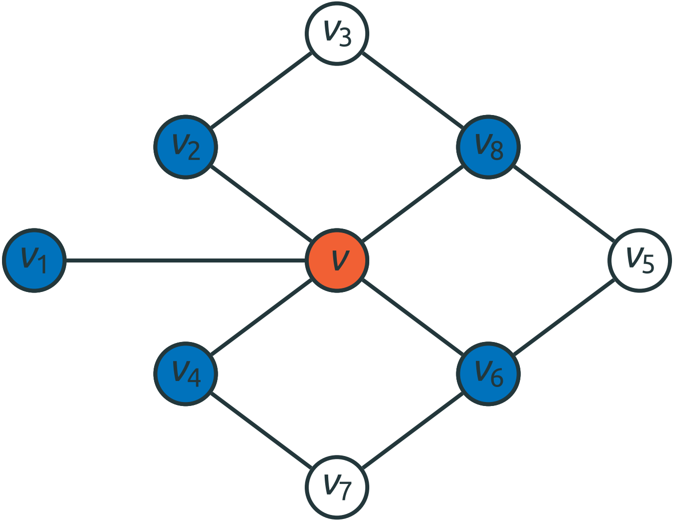

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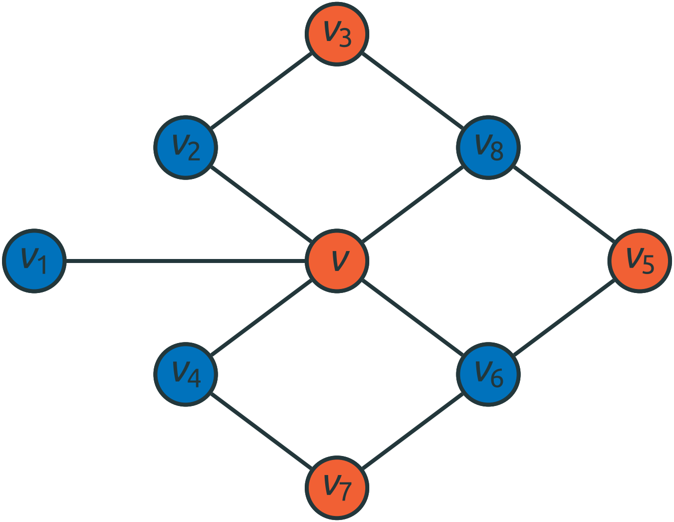

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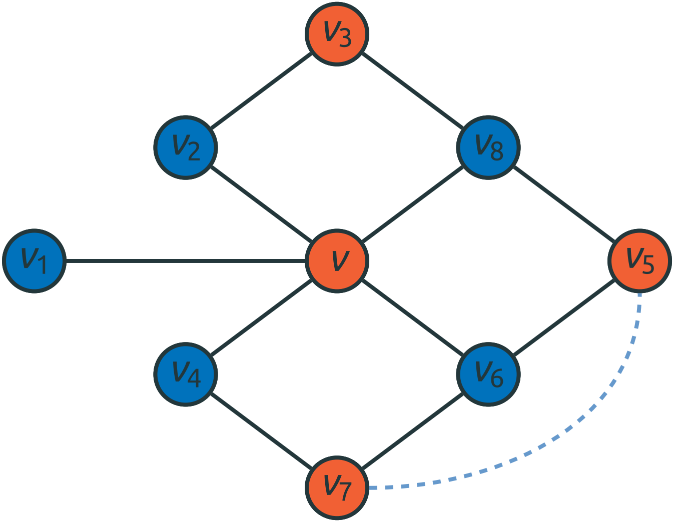

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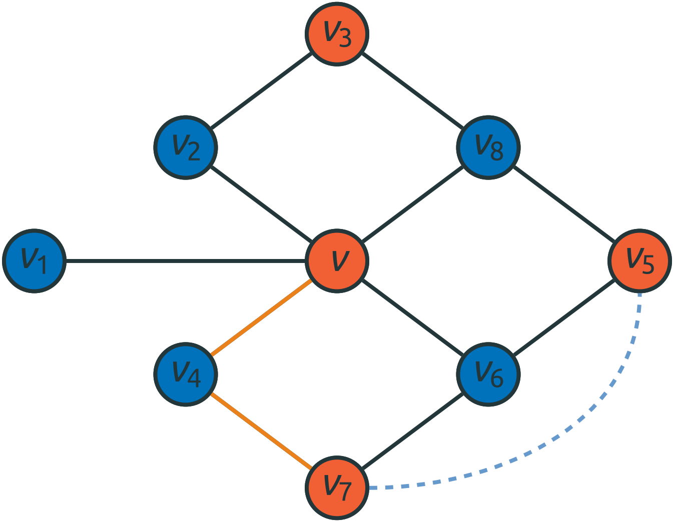

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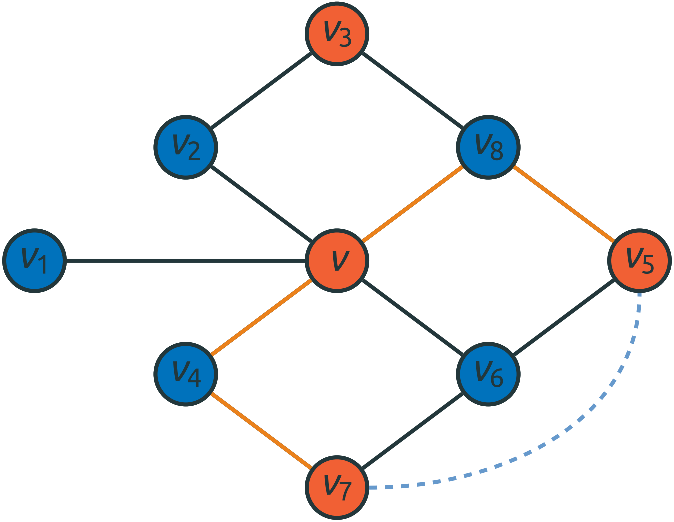

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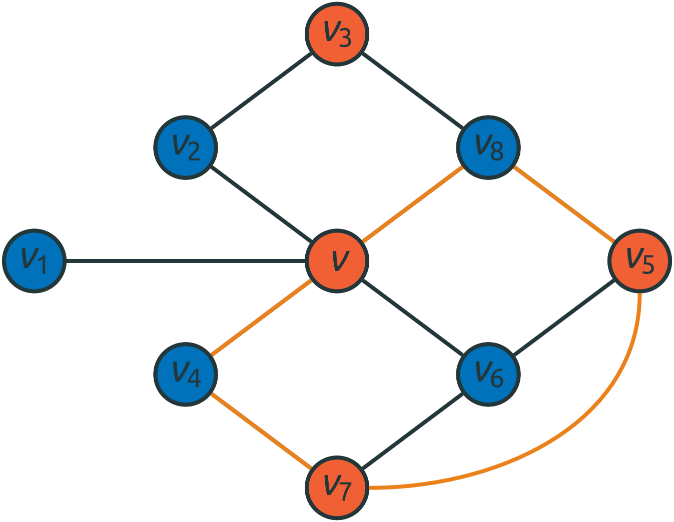

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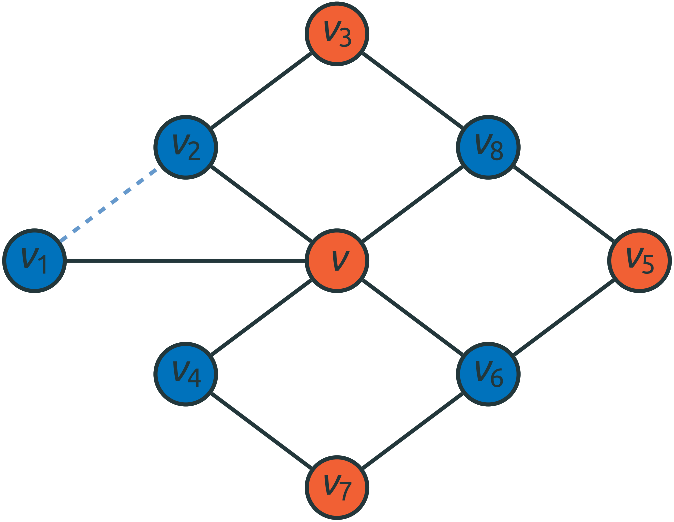

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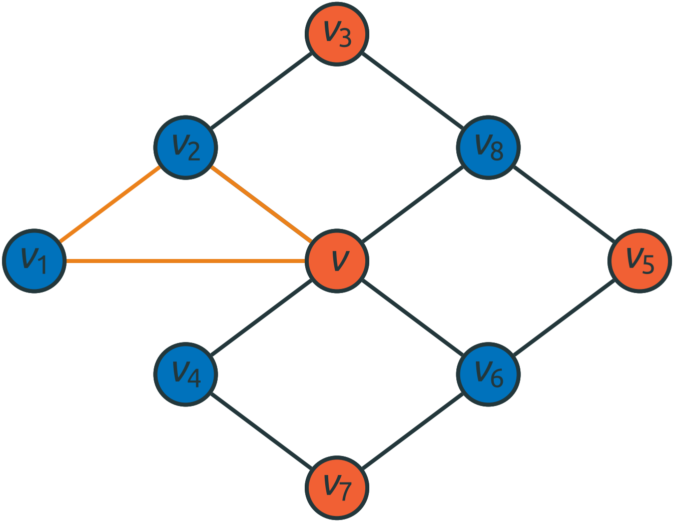

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- Repeat for other connected components

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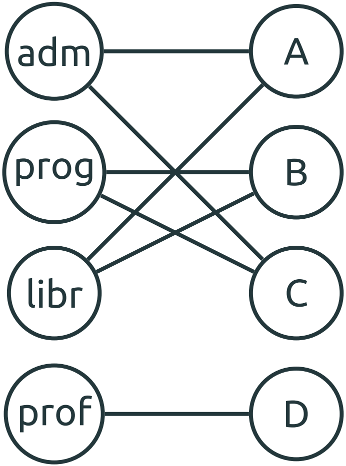
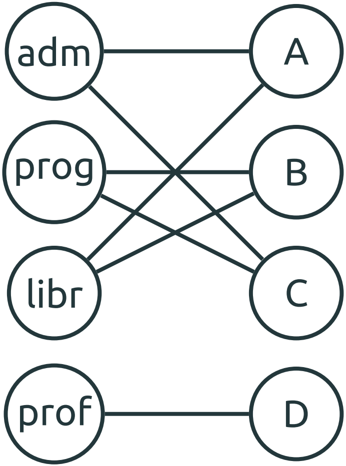
- A Maximum Matching is a matching of the largest size
- We often want to find a matching in a bipartite graph which covers all vertices of one side

Matchings in Bipartite Graphs



Matchings in Bipartite Graphs

We want a matching which covers all jobs



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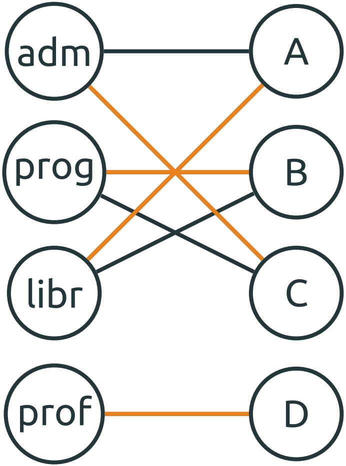

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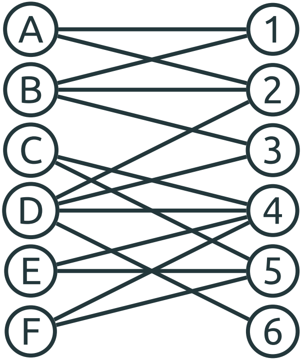

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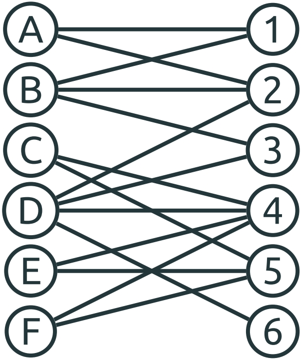
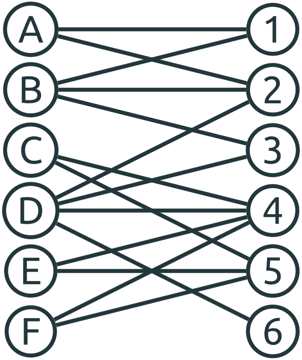


Matchings in Bipartite Graphs

We want a matching which covers all people

But it does not exist



Hall's Theorem

Definition

Let $G = (V, E)$ be a graph, and $S \subseteq V$ be a subset of vertices. The Neighborhood $N(S)$ of S is the set of all vertices connected to at least one vertex in S

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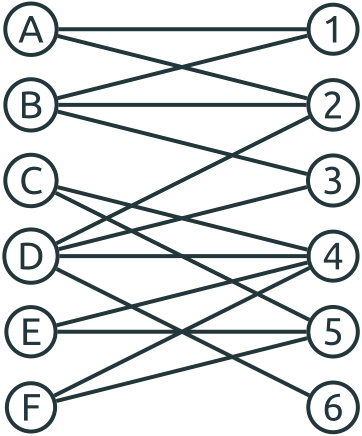
Let $G = (V, E)$ be a graph, and $S \subseteq V$ be a subset of vertices. The Neighborhood $N(S)$ of S is the set of all vertices connected to at least one vertex in S

Theorem (Hall, 1935)

In a bipartite graph $G = (L \cup R, E)$, there is a matching which covers all vertices from L if and only if for every subset of vertices $S \subseteq L$,

$$|S| \leq |N(S)| .$$

Hall's Theorem: Examples



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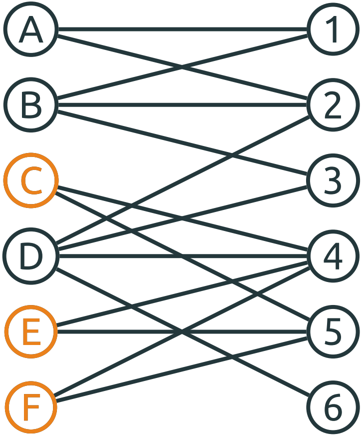

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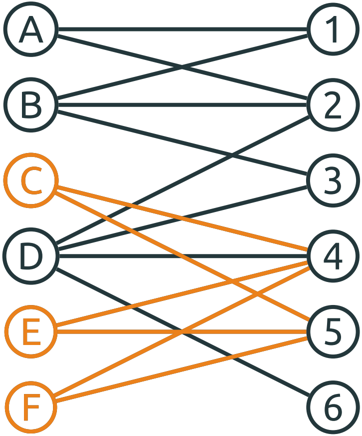

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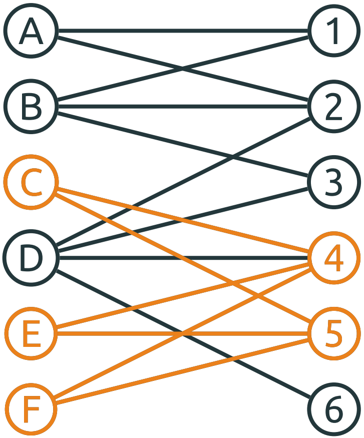

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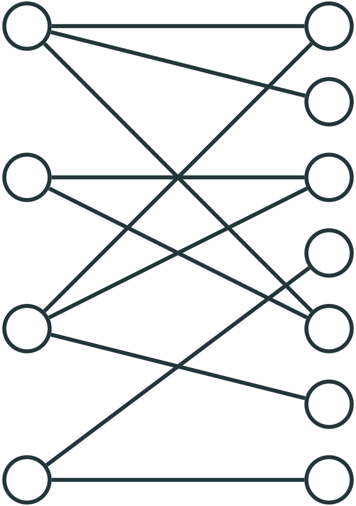
- There are at least $|S|$ of them, thus, $|N(S)| \geq |S|$

Hall's Theorem



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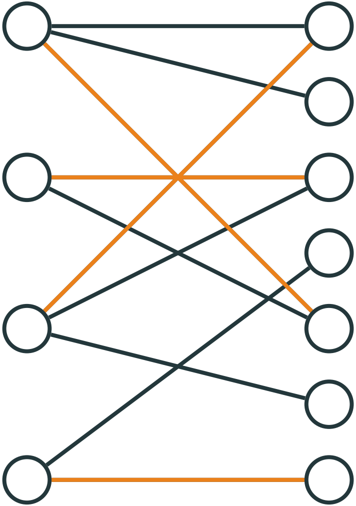

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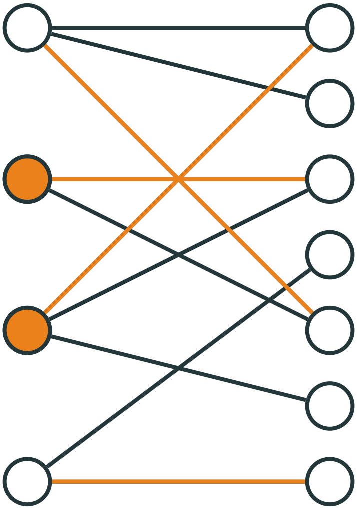

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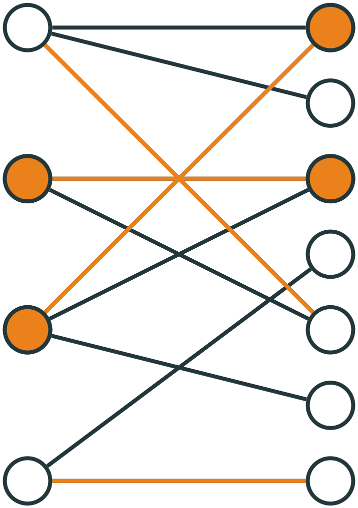

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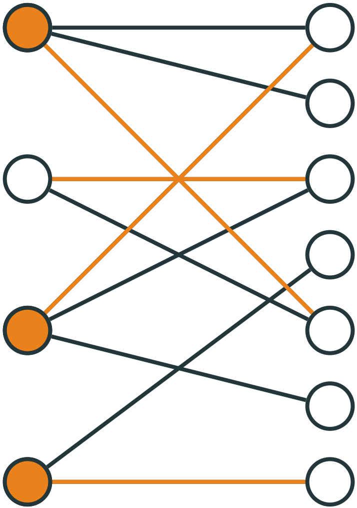

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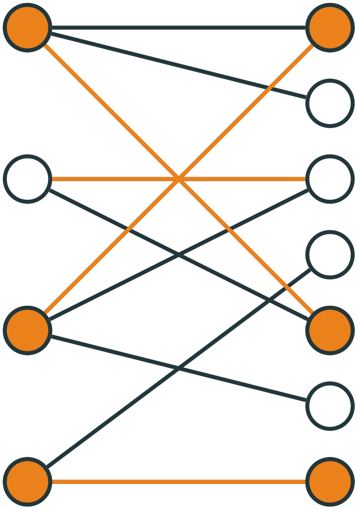

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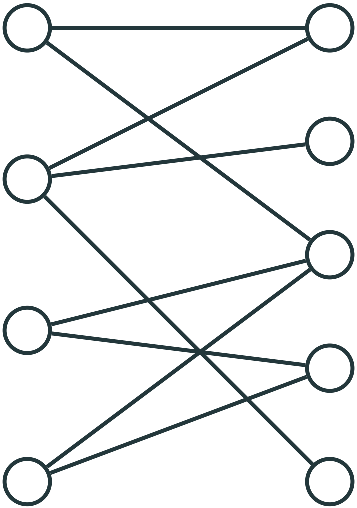

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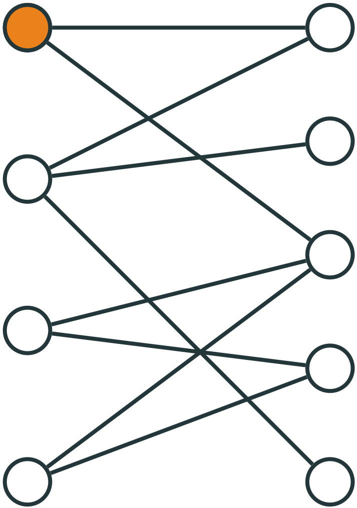

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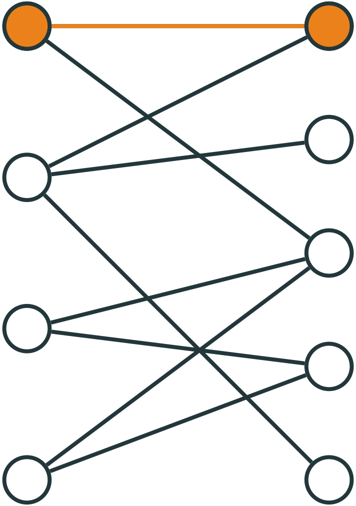

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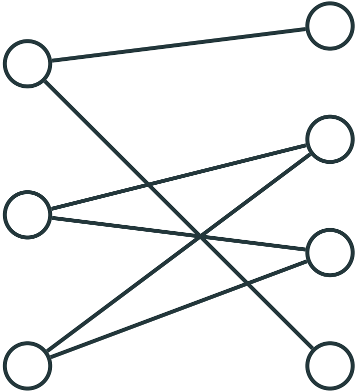

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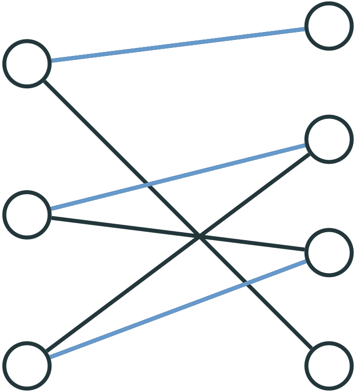

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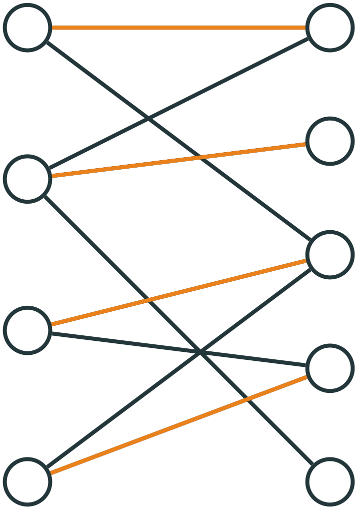

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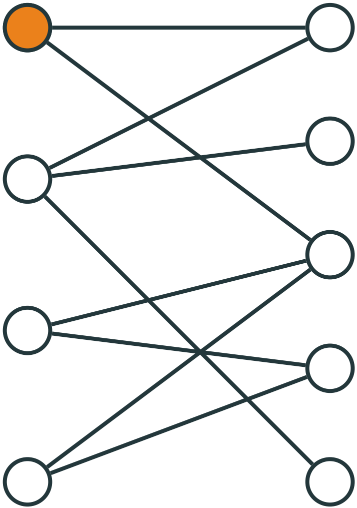

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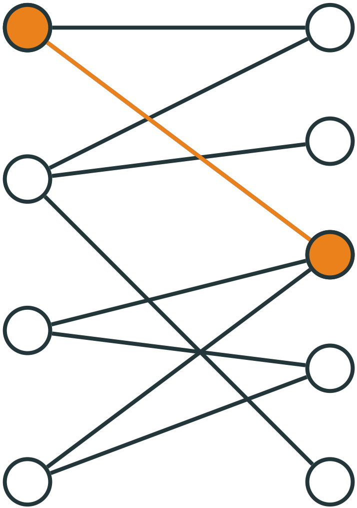

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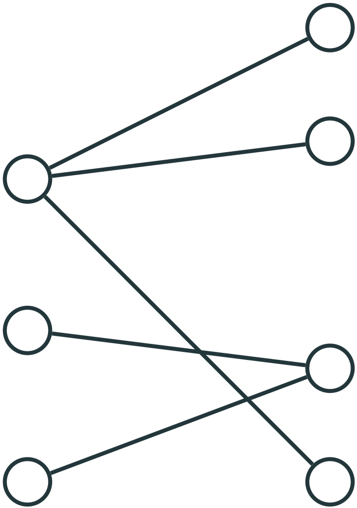

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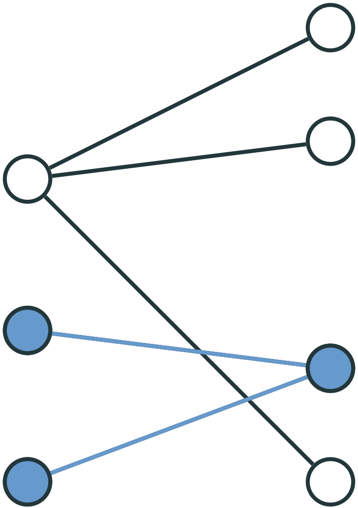

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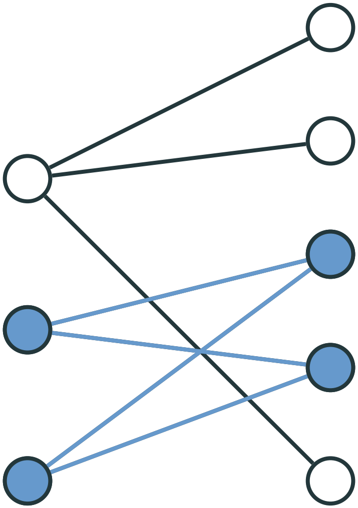

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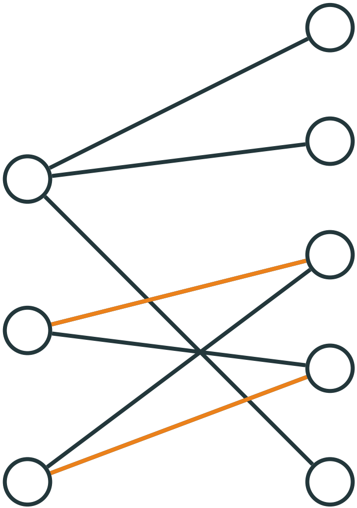

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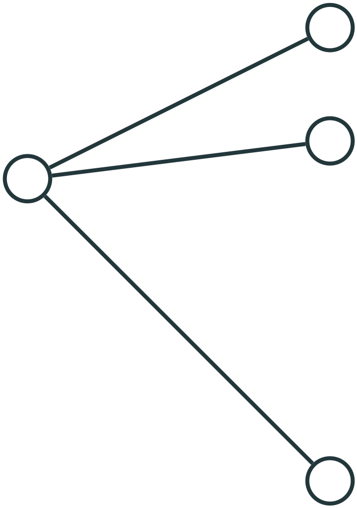

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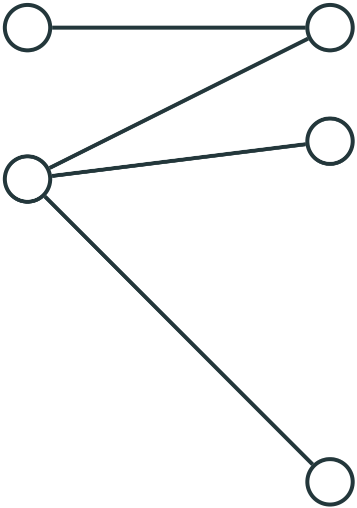

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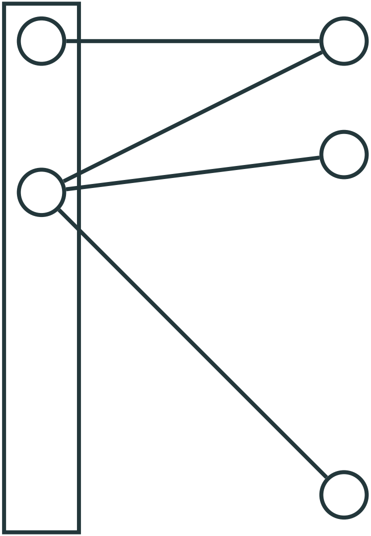

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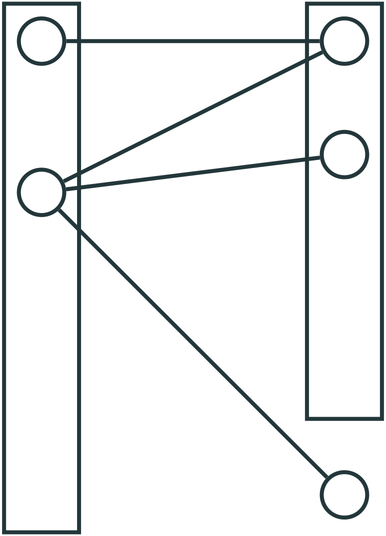

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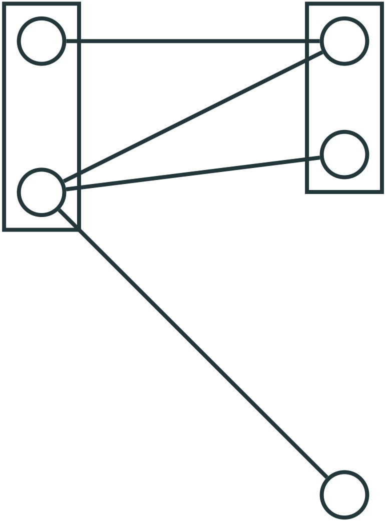

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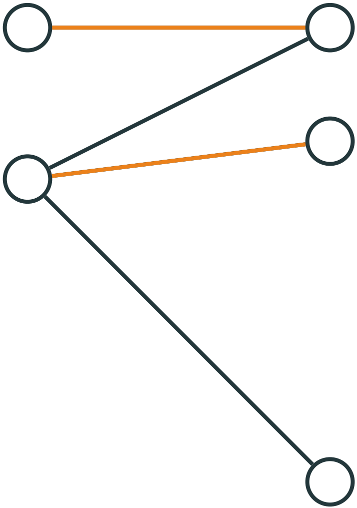

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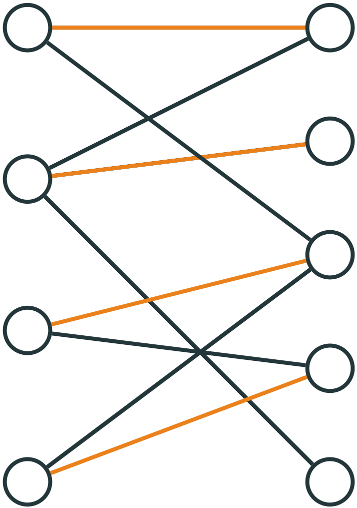

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- Pick a vertex $v \in L$ and its neighbor $u \in R$

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- Induction on $|L|$. Base Case: $|L| = 1$, $|N(L)| \geq |L| = 1$, so there is a matching of size 1
- Induction Hypothesis: The statement holds for all graphs with smaller $|L| \leq k$
- Induction Step: Prove the statement for $|L| = k + 1$
- Pick a vertex $v \in L$ and its neighbor $u \in R$
- If there is a matching on $L \setminus \{v\}$ and $R \setminus \{u\}$, then we're done!

Hall's Theorem: The Other Direction

- If there is no matching on $L \setminus \{v\}$ and $R \setminus \{u\}$, then there is a set $S_1 \subseteq L \setminus \{v\}$ s.t. its neigborhood in $R \setminus \{u\}$ is $< |S_1|$

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- In the remaining graph, every set $S \subseteq L$ has at least $|S| + |S_1| - |T_1| = |S|$ neighbors, there is a matching!