

## Le hors d'œuvre

$$U_X = \begin{bmatrix} -\frac{\alpha}{2} & 0 & \frac{\alpha}{2} \\ -\frac{(1-\alpha)}{2} & 0 & (1-\alpha) \\ -\frac{\alpha}{2} & 0 & \frac{\alpha}{2} \end{bmatrix} \quad \alpha > 0$$

1/ Pour obtenir masque Sobel, Prewitt :

$$\text{Prewitt} \cdot U_X = \boxed{\begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}} \cdot U_X \quad \text{Sobel} \quad U_X = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & 1 \end{bmatrix} \times \frac{1}{3}$$

$$-\frac{\alpha}{2} = -\frac{1}{3} \Rightarrow \alpha = \frac{2}{3}$$

$$1-\alpha = \frac{1}{3} \Rightarrow \alpha = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\boxed{\alpha = \frac{2}{3}}$$

$$-\frac{\alpha}{2} = -\frac{1}{4} \Rightarrow \alpha = \frac{1}{2}$$

$$1-\alpha = \frac{1}{4} \Rightarrow \alpha = 1 - \frac{1}{4} = \frac{1}{2}$$

$$\boxed{\alpha = \frac{1}{2}}$$

$$2/ \quad I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

a- Produit de convolution :

$$I_X = I \cdot U_X$$

$$= -\frac{\alpha}{2} - (1-\alpha) - \frac{\alpha}{2}$$

$$I_X = -\alpha - 1 + \alpha = -1$$

$$U_Y = \begin{bmatrix} -\frac{\alpha}{2} & -(1-\alpha) & -\frac{\alpha}{2} \\ 0 & 0 & 0 \\ \frac{\alpha}{2} & 1-\alpha & \frac{\alpha}{2} \end{bmatrix}$$

$$I_Y = I \cdot U_Y$$

$$= -\frac{\alpha}{2} - (1-\alpha) + \frac{\alpha}{2} + 1 - \alpha$$

$$\boxed{I_Y = 0}$$

b- Module de gradient du bloc I au centre

$$G_I = \sqrt{(-1)^2 + 0^2} = 1$$

c- Module de gradient du bloc J

$$J_X = J \cdot U_X$$

$$= -\frac{\alpha}{2} - (1-\alpha) - \frac{\alpha}{2} + \frac{\alpha}{2}$$

$$= -\frac{\alpha}{2} - 1 + \alpha = -1 + \frac{\alpha}{2} =$$

$$J_X = \frac{\alpha}{2} - 1$$

$$J_Y = J \cdot U_Y$$

$$= -\frac{\alpha}{2} + \frac{\alpha}{2} + 1 - \alpha + \frac{\alpha}{2}$$

$$= 1 - \frac{\alpha}{2}$$

$$J_Y = 1 - \frac{\alpha}{2}$$

$$G_J = \sqrt{J_X^2 + J_Y^2}$$

$$= \sqrt{\left(\frac{\alpha}{2} - 1\right)^2 + \left(1 - \frac{\alpha}{2}\right)^2}$$

$$= \sqrt{\frac{\alpha^2}{4} - \alpha + 1 + 1 - \alpha + \frac{\alpha^2}{4}}$$

$$= \sqrt{\frac{\alpha^2}{2} - 2\alpha + 2}$$

④

d. Pour quelle  $\alpha$  les 2 modules du gradient sont identiques

$$G_x = G_y$$

$$\sqrt{\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}} = 1$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = 1$$

$$\begin{aligned} \frac{\partial^2}{\partial x^2} - 4\alpha + \alpha^2 &= 1 \\ \alpha^2 - 4\alpha + 1 &= 1 \\ \alpha^2 - 4\alpha &= 0 \end{aligned}$$

$$\Delta = b^2 - 4ac$$

$$= 16 - 4 \times 1 \times 1 = 8 = (\sqrt{2})^2$$

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{4 - \sqrt{2}}{2} = 2 - \sqrt{2}$$

$$x_2 = 2 + \sqrt{2}$$

$\Rightarrow$  On choisit la plus petite valeur de  $x \Rightarrow \alpha = 2 - \sqrt{2}$

Pq? Donc  $- (1-\alpha) = -\alpha \leq 0$  d'après  $x_1$  et  $x_2$

$$\begin{array}{c} 1-\alpha > 0 \\ \boxed{1 \quad \alpha \quad 1} \end{array}$$

$$\text{D'où } \boxed{\alpha = 2 - \sqrt{2}}$$

### Le plan principal

$$P(C_1) = \frac{1}{3} \quad P(C_2) = \frac{2}{3}$$

① Formule de Bayes

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

$$P(X|C_i) = \frac{P(X) \cdot P(C_i|X)}{P(C_i)} \quad \text{avec } i = \{1, 2\}$$

$$P(X|C_i) \cdot P(C_i) = P(X) \cdot P(C_i|X)$$

② Distribution de probabilité  $p(x)$

$$P(X) = \sum_{i=1}^2 P(X|C_i) \cdot P(C_i) = P(X|C_1) \cdot P(C_1) + P(X|C_2) \cdot P(C_2)$$

$$P(X|C_1) = ax + b \quad (\text{selon graph})$$

$$x=0 \Rightarrow b=1$$

$$a = \frac{0-1}{2-0} = -\frac{1}{2} \Rightarrow a = -\frac{1}{2}$$

$$P(X|C_1) = -\frac{x}{2} + 1 = \frac{2-x}{2}$$

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$$P(X|C_2) = ax + b$$

$$x=0 \Rightarrow b=0$$

$$a = \frac{1-0}{2-0} = \frac{1}{2}$$

$$\Rightarrow P(X|C_2) = \frac{x}{2}$$

$$P(X) = P(X|C_1) \cdot P(C_1) + P(X|C_2) \cdot P(C_2)$$

$$P(X) = \frac{2-x}{2} \times \frac{1}{3} + \frac{x}{2} \times \frac{1}{3} = \frac{2-x}{6} + \frac{x}{3} = \frac{2-x+2x}{6} = \frac{x+2}{6}$$

$$P(X) = \frac{x+2}{6}$$

3°/ Expression des proba à posteriori

$$P(C_1|X) = \frac{P(C_1) \cdot P(X|C_1)}{P(X)}$$

$$= \frac{\frac{1}{3} \cdot \frac{2-x}{2}}{\frac{x+2}{6}} = \frac{\frac{2-x}{6}}{\frac{x+2}{6}} = \frac{2-x}{x+2}$$

$$P(C_2|X) = \frac{P(C_2) \cdot P(X|C_2)}{P(X)}$$

$$= \frac{\frac{2}{3} \cdot \frac{x}{2}}{\frac{x+2}{6}} = \frac{\frac{x}{3}}{\frac{x+2}{6}} = \frac{2x}{x+2}$$

4°/ Règle de décision pour un pixel donné  $X$ :

$$P(C_1|X) \underset{C_1}{\geq} P(C_2|X)$$

$$\frac{2-x}{x+2} \underset{C_2}{\geq} \frac{2x}{x+2}$$

$$2-x \geq 2x \Rightarrow 0 \geq 3x-2$$

5°/ Valeur du seuil où il y a indécision

$$P(C_1|X) = P(C_2|X)$$

$$\Leftrightarrow 3x-2=0$$

$$x = \frac{2}{3}$$

6°-7°/ Les expressions du risque  $\pi(a_i|X)$  pour  $i=1,2,3$   $\{\pi(a_i|X) = \sum_{j=1}^2 P(a_i|C_j) \cdot P(C_j|X)\}$

$$\begin{aligned} \pi(a_1|X) &= \sum_{j=1}^2 P(a_1|C_j) \cdot P(C_j|X) = P(a_1|C_1) \cdot P(C_1|X) + P(a_1|C_2) \cdot P(C_2|X) \\ &= 0 + \frac{1}{2} \cdot \frac{2x}{x+2} \end{aligned}$$

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$$r(a_1/x) = P(a_1/G_1) \cdot P(G_1/x) + P(a_1/G_2) \cdot P(G_2/x)$$

$$= 1 \cdot \frac{q-x}{q+x} + 0 \Rightarrow r(a_1/x) = \frac{q-x}{q+x}$$

$$r(a_3/x) = P(a_3/G_1) \cdot P(G_1/x) + P(a_3/G_2) \cdot P(G_2/x)$$

$$= \frac{1}{4} - \frac{q-x}{q+x} + \frac{1}{4} \cdot \frac{q-x}{q+x}$$

$$= \frac{q-x+qx}{4(q+x)} = \frac{q+x}{4(q+x)} = \frac{1}{4}$$

8° Déterminer les domaines de valeurs de  $x$  vérifiant l'action optimale.

$$a^*(x) = \min \{ r(a_i/x) \}, i = \{1, 2, 3\}$$

$$a^*(x) = \min \left\{ \frac{q-x}{q+x}, \frac{q-x}{q+x}, \frac{1}{4} \right\}$$

Les intervalles :

$$\frac{q-x}{q+x} = \frac{1}{4}$$

$$\frac{q-x}{q+x} = \frac{1}{4}$$

$$q-x = q+x$$

$$8-4x = 8+4x$$

$$x = \frac{q}{2} \approx 0,3$$

$$x = \frac{6}{5} \approx 1,2$$

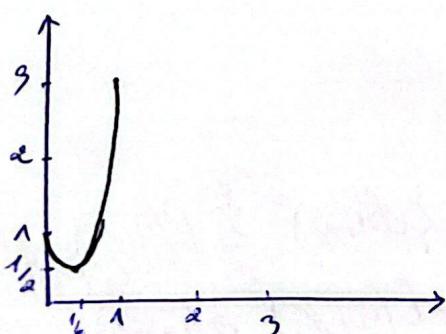
D'où  $[0, \frac{q}{2}], [\frac{q}{2}, \frac{6}{5}], [\frac{6}{5}, q]$

Le Risque total =  $R(a_i) = \int_0^q r(a_i/x) \cdot P(x) dx \approx 0,1714$

Le dessin - (7 pts)

$$R(r) = -4r + 6r^2 + 1, r \in [0, q]$$

9° Histogramme :



r	0	1	2
R(r)	1	3	17/8

b) Valeur de  $\lambda$  définissant le minimum de  $R'$

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$$\text{Minimum} \Rightarrow R'(\lambda) = 0$$

$$-4 + 12\lambda = 0$$

$$\lambda = \frac{4}{12} = \frac{1}{3} \Rightarrow \boxed{\lambda = \frac{1}{3}}$$

c) Densité de probabilité:  $\int_0^1 R(\lambda) d\lambda = 1$

$$\begin{aligned} \int_0^1 R(\lambda) d\lambda &= \int_0^1 -4\lambda + 6\lambda^2 + 1 d\lambda \\ &= -4 \left[ \frac{\lambda^2}{2} \right]_0^1 + 6 \left[ \frac{\lambda^3}{3} \right]_0^1 + [\lambda]_0^1 \\ &= -\frac{4}{2} + \frac{6}{3} + 1 = -2 + 2 + 1 = 1 \end{aligned}$$

d) La moyenne de l'image:

$$\begin{aligned} \int_0^m \lambda R(\lambda) d\lambda &= \int_0^m -4\lambda^2 + 6\lambda^3 + \lambda d\lambda \\ &= -4 \left[ \frac{\lambda^3}{3} \right]_0^m + 6 \left[ \frac{\lambda^4}{4} \right]_0^m + \left[ \frac{\lambda^2}{2} \right]_0^m \\ &= -\frac{4}{3} m^3 + \frac{6}{4} m^4 + \frac{1}{2} m^2 = \frac{2}{3} m^3 \end{aligned}$$

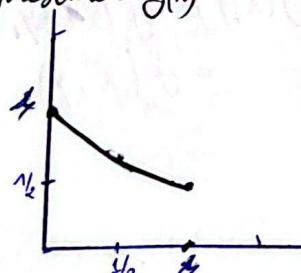
e) Transformation  $s = g(\lambda)$  permettant d'égaliser l'image:

$$\begin{aligned} g(\lambda) &= \int_0^m R(\lambda) d\lambda = (1-0) \int_0^m R(\lambda) d\lambda \\ &= \int_0^m -4\lambda^2 + 6\lambda^3 + \lambda d\lambda \\ &= -4 \left[ \frac{\lambda^3}{3} \right]_0^m + 6 \left[ \frac{\lambda^4}{4} \right]_0^m + [\lambda]_0^m \\ &= -\frac{4}{3} m^3 + \frac{6}{4} m^4 + m = -2m^3 + 2m^4 + m \end{aligned}$$

f)

$$m = g(\lambda) = \frac{1}{\lambda+1}$$

Représenter  $g(\lambda)$



$$\begin{array}{ccccccc} \lambda & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots \\ g(\lambda) & 1 & \frac{1}{2} & \frac{2}{3} & \frac{3}{4} & \dots \end{array}$$

g) Expression de l'histogramme  $f(m)$  obtenu après l'application de  $g$  sur  $R$

$$f(m) = \frac{R(\lambda)}{|g'(\lambda)|} = R(\lambda) \cdot \frac{d\lambda}{dm} \quad \underline{\lambda = g^{-1}(m)}$$

$$g(\lambda) = m$$

$$\frac{1}{\lambda+1} = m \Rightarrow \lambda = m(1-m) \Rightarrow m\lambda + m = 1 \Rightarrow m\lambda = 1-m \Rightarrow \lambda = \frac{1-m}{m}$$

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$$\lambda = \frac{1-m}{m}$$

$$\begin{aligned} R(\lambda) &= -4\left(\frac{1-m}{m}\right) + 6\left(\frac{1-m}{m}\right)^2 + 1 \\ &= \frac{-4(1-m)}{m} + \frac{6-12m+6m^2}{m^2} + \frac{m}{m^2} \\ &= \frac{-4m+4m^2+6-12m+6m^2+m^2}{m^2} = \frac{11m^2-16m+6}{m^2} \end{aligned}$$

$$g'(\lambda) = \left(\frac{1}{\lambda+1}\right)' = \frac{-1}{(\lambda+1)^2} = \frac{-1}{\left(\frac{1-m}{m}\right)^2} = -\left(\frac{m}{1-m}\right)^2$$

$$\text{d'après } \frac{f(m)}{g'(\lambda)} = \frac{R(\lambda)}{m^2} \times -\frac{m^2}{(1-m)^2} = -\frac{11m^2-16m+6}{(1-m)^2}$$

i) Supposons  $f(m) = 5m^2 - 10m + 6$

Peut-on considérer  $f$  comme une densité de proba.

$$\begin{aligned} \int_0^1 f(m) dm &= \int_0^1 5m^2 - 10m + 6 dm \\ &= 5 \left[ \frac{m^3}{3} \right]_0^1 - 10 \left[ \frac{m^2}{2} \right]_0^1 + 6 [m]_0^1 \\ &= \frac{5}{3} - \frac{10}{2} + 6 = \frac{8}{3} \neq 1 \end{aligned}$$

J'ai plus faim !

1% Définition de TF  $F(U, V)$  de  $f(x, y)$

$$F(U, V) = \frac{1}{HN} \sum_{x=0}^{N-1} \sum_{y=0}^{H-1} f(x, y) \exp\left(-q\pi_j\left(\frac{ux}{N} + \frac{vy}{H}\right)\right)$$

soit

$$\begin{aligned} \text{On pose } X' &= x - my \\ X &= X' + my \quad \text{et } X' = dX \end{aligned}$$

$$\begin{aligned} F(U, V) &= \frac{1}{HN} \sum \sum g(x, y) \exp\left(-q\pi_j\left(\frac{ux}{N} + \frac{vy}{H}\right)\right) \\ &= \frac{1}{HN} \sum \sum f(x, y) \exp\left(-q\pi_j\left(\frac{ux' + u_my}{N} + \frac{vy}{H}\right)\right) \\ &= \frac{1}{HN} \sum \sum f(x', y) \exp\left(-q\pi_j\left(\frac{ux'}{N} + \frac{vy}{H} + \frac{umy}{N}\right)\right) \end{aligned}$$

$$= \frac{1}{HN} \sum \sum f(x', y) \exp\left(-q\pi_j x'\left(\frac{u}{N}\right)\right) \cdot \exp\left(-q\pi_j y\left(\frac{v}{H} + \frac{um}{N}\right)\right)$$

$$\begin{aligned} g(x, y) &= f(x - my, y) \\ &= f(x', y) \end{aligned}$$