AMATH 271 Final Project

Analysis of the Equations of Motion of Driven Damped Pendulums and their Chaotic Motion

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Abstract

In this paper we present an analysis of the equations governing the motion of a driven damped pendulum (DDP). A DDP is a pendulum with an element of damping via air resistance or other dissipative forces, and a sinusoidal driving force which allows the system to continue its motion. Not only does this system have non-linear equations of motion, but it is known to exhibit chaotic behaviour for certain parameters, meaning that its motion is not fully predictable. Chaotic systems are highly sensitive to initial conditions, and can produce entirely different results with only small differences. We will discuss techniques used to illustrate chaotic motion with the use of computer simulations, while introducing terminology and graphical representations, such as bifurcation diagrams, state space orbits and time-series plots.

Contents

1 Introduction

The driven damped pendulum (DDP) is an example of a nonlinear mechanical system. What makes this particularly interesting is that it exhibits **chaos**, meaning that very small changes in initial conditions may result in big changes down the line. Despite being simple to describe with only Newton's Laws, introducing a dissipative force and a driving force oscillating different from the natural frequency can create aperiodic pseudo-random motion that one might not expect. The DDP is just one example, and many such chaotic systems exist in nature which can offer vast insight into currently unsolved problems.
Although chaotic systems have been known for a long time, such as the general three body problem [1], it was not until the advent of modern computers that chaos theory was finally able to be studied properly. meaning that this is still a new field that is also quite complex, therefore much of the content in this paper will not be very rigorous. Instead, we offer a brief description of the equations that govern the system and proceed to solve it numerically using a computer, then analyze the results and describe key ideas using figures and at times, animations.

For a system to exhibit chaos it must be nonlinear, meaning that it can be modelled by a nonlinear differential equation (DE). In comparison, a linear differential equation is one written in the form:

$$0 = q(x) + p_1(x)y(x) + p_2(x)y'(x) + \cdot + p_n(x)y^{(n)}(x) \tag{1}$$

On the other hand, nonlinear DEs include some nonlinear function of $y^{(i)}(x)$ in their terms. For example, the equation of motion for a simple pendulum of mass m and length L is given by

$$mL^2\ddot{\phi} = -mgL\sin\phi \tag{2}$$

where $\phi(t)$ is a function of time. Since it contains the term $\sin\phi$, it is nonlinear. If we make the small angle approximation $\sin\phi \approx \phi$ it becomes

$$mL^2\ddot{\phi} = -mgL\phi \tag{3}$$

which is easy to solve, $\phi = A\cos((\sqrt{g/L})t + \theta)$. But this solution clearly does not satisfy the original case when ϕ in not sufficiently close to 0.

Although many linear DEs are analytically solvable, virtually all nonlinear DEs **cannot** be solved analytically using current methods. It should also be noted that not all nonlinear systems are chaotic, the example of a simple pendulum is still periodic even with the $\sin(\phi)$ term. However, just by adding in a damping force $-bL^2\dot{\phi}$ and a driving force $F(t)$, the system can exhibit chaos for some parameters.

$$mL^2\ddot{\phi} = -bL^2\dot{\phi} - mgL\sin\phi - F(t) \tag{4}$$

Before looking at how we might approach modelling the motion of a driven, damped pendulum, there is an important fact about nonlinear DEs to remember. The superposition principle which states that if $y_1(x), \dots, y_n(x)$ are solutions to an nth order homogeneous linear differential equation, then the linear combination

$$y(x) = a_1 y_1(x), \dots, a_n y_n(x) \tag{5}$$

must also be a solution, for some constants $a_1, \dots, a_n$ is not necessarily followed nonlinear DEs since the chain rule applies when taking the derivative of the new function. This means that even if a particular solution is found for a nonlinear DE, it may not provide any helpful information about other possible solutions.

Since we have established that the equation of the DDP cannot be solved analytically, the only way to understand the solutions is through numerical methods, i.e. computer simulation. First, we will have to fix our choice of parameters and set initial conditions. Let us assume that the driving force is sinusoidal and of the form $F(t) = F_\circ \cos(\omega t)$, where $F_\circ$ is the drive amplitude and ω is the drive frequency. Dividing (4) by mL^2 and rearranging, we see the system can be modelled by

$$\ddot{\phi} + 2\beta\dot{\phi} + \omega_\circ^2 \sin\phi = \gamma\omega_\circ^2 \cos\omega t \tag{6}$$

where $2\beta = b/m$ the damping constant, as in linear air resistance, and $\omega_\circ^2 = \sqrt{g/L}^2$ is the square of the natural frequency of the system. These will be fixed parameters in our simulation, as their effects on the system are not very important for the topic at hand (though we must at least have $\omega_\circ \neq \omega$). Instead, consider the effect of changing the **drive strength**, which we denote with the dimensionless parameter $\gamma = F_\circ/mg$. This expresses the ratio of the driving force to the force of gravity, which are the main forces that affect motion.

Now that we have established some preliminaries, it is time to start constructing a framework to interpret the system by looking at time series solutions the DE.

2 Numerical Solutions to Nonlinear ODEs

Let us first consider a simple case of very weak drive strength $\gamma \ll 1$. Pick the initial conditions as starting from rest at equilibrium. A weak drive strength means that the angle ϕ will remain small as well, and so the solution is well-approximated by $\sin\phi \approx \phi$ as seen in lecture. we expect the initial behaviour to not be sinusoidal as the natural frequency adapts to the driving force, but to then quickly stabilize. The initial differences from sinusoidal behaviour are called "transients", which die out at an exponential rate and result in the periodic motion of a unique "attractor". The computed results are shown in Figure 1.

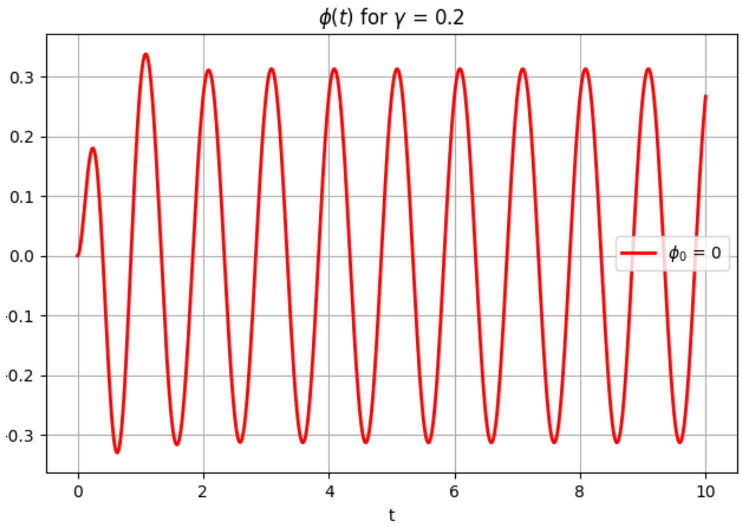

Figure 1: The motion of a DDP with relatively weak drive strength. It clearly settles down to sinusoidal motion after only a few cycles. The motion is also close to that of the simple harmonic oscillator which can be seen in this animation: https://imgur.com/a/JF7Cpl3

In order to compute the behaviour, and all subsequent plots, unless otherwise specified, we utilize an 8th order Runge-Kutta method as defined in the SciPy library [2].

This case demonstrates that our model behaves as expected for known parameters. In fact, if we increase γ to the point where the small angle approximation is no longer satisfactory, the attractor of the system is still the same as a simple pendulum with semi-sinusoidal motion (For a more detailed treatment consult [3], or any other source). Now we will focus on the case where this type of behaviour starts to break down. Increasing the drive strength to 1, the motion still appears to converge to a sinusoidal attractor. However, at a value slightly above 1, $\gamma = 1.073$ produces some interesting results, shown in Figure 2. At first, this seems to be the same as previous cases, albeit the transients last for much longer, but upon closer inspection reveals the attractor is not sinusoidal at all!

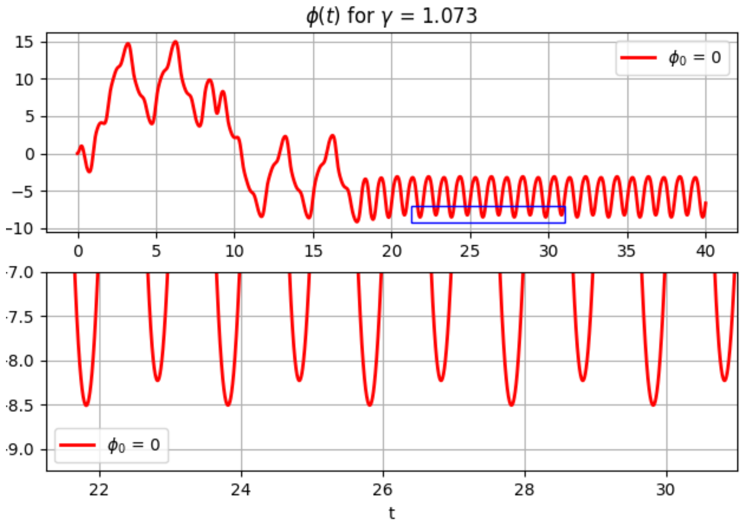

Figure 2: The motion of a DDP that shows period doubling. Transients die out after around 20 cycles and then it settles down to *nearly* sinusoidal motion. Upon closer inspection there are actually two distinct maxima every *two* periods. We can actually see the transients die out in the this animation https://imgur.com/a/muudf0n

The attractor clearly has two distinct troughs, occurring at a period twice that of the driving frequency. At first, one might assume this is simply another transient, but comparing the result from $20 \leq t \leq 30$ shows exactly the same graph as $990 \leq t \leq 1000$. So we can assume that this "period doubling" is the expected motion (according to the numerical method of approximation). While this is closer to chaos, it is certainly not chaotic.

Figure 3 shows the solution for successive values of γ. Looking at this, we can start to see a pattern here: the initial "period doubling" begins to double further—and at a faster rate with respect to the increase in γ. Past a certain point, as in Figure 4, the plot begins to look chaotic. But how do we know that it is actually chaos and not just extremely high period doubling? This is too complex a discussion for this paper, but it was shown by Feigenbaum that when periods double it is non-decreasing and occurs at specific threshold values. The interval of consecutive thresholds is a convergent geometric sequence determined by the **Feigenbaum constant** (delta). The thresholds converge to $\gamma_c = 1.0829$, after which point chaos sets in. (This argument is true for all chaotic systems, not just the DDP, due to universality [4]).

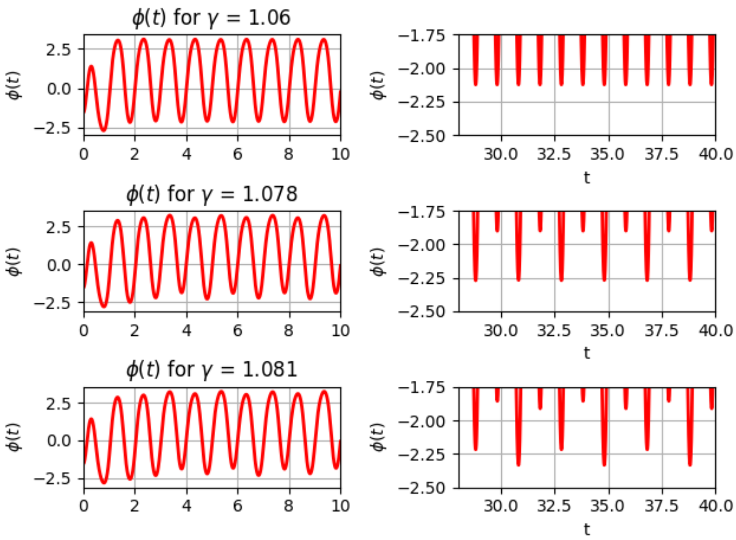

Figure 3: A period-doubling cascade. Each of the plots shares the exact same initial conditions, only the drive strength changes.

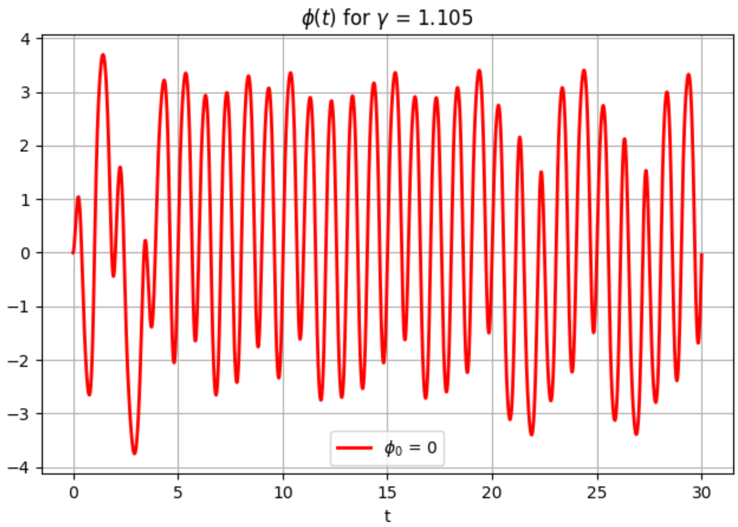

Figure 4: This is Chaos. The first 30 cycles are non periodic, and in fact, all cycles are. It never settles down to a fixed attractor.

Finally, we have reached chaos. Chaos has many interesting and unknown aspects, but perhaps the most important is called "sensitivity to initial conditions". This basically means that two pendulums with only a small difference in initial conditions can have wildly different chaotic solutions, as shown in Figure 5. To make this notion more precise, consider the motion of $\phi_1(t)$ and $\phi_2(t)$ which both satisfy the same equation of motion but with slightly different initial conditions (ICs). Denote the difference between the solutions as

$$\Delta\phi(t) = \phi_2(t) - \phi_1(t) \tag{7}$$

We now ask: what does the evolution of $\Delta\phi(t)$ look like? Does it stay the same? Decrease? Increase?

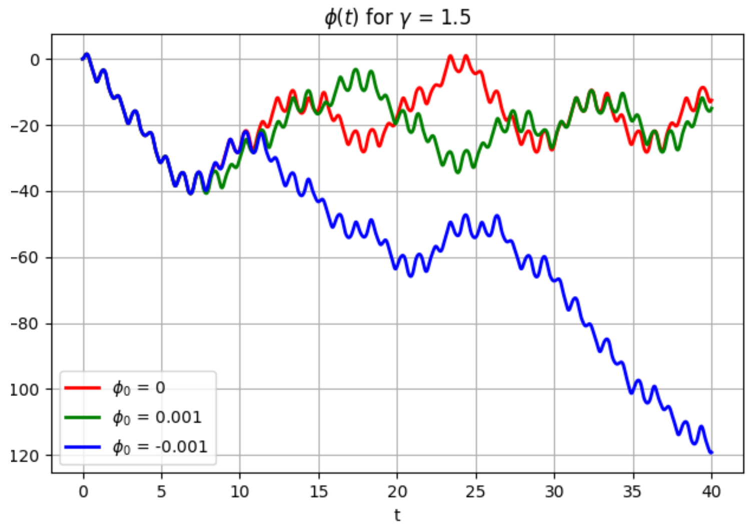

Figure 5: A plot of identical DDPs with a separation of 0.001 rad. For the first few cycles, the motion is indistinguishable, after which the sensitivity to ICs becomes very obvious. We can see this motion more clearly in this animation: https://imgur.com/a/o2Whpr0

We can plot $\Delta\phi(t)$ for different values of γ and see what it tells us about the behaviour of the system—see Figure 6. For the non-chaotic cases, $\Delta\phi(t)$ decays exponentially, as expected. This means that if two ICs are close to one another, they approach the same attractor as the transients die out. However, in the chaotic example, $\Delta\phi(t)$ starts to **grow** exponentially. That is, there are no transients dying out; since the chaotic motion itself has no attractors, it is "all transients". We can further generalize this notion to other chaotic systems by realizing $\Delta\phi(t) \sim Ke^{\lambda t}$ for some positive constant K and a parameter λ, called a **Liapunov exponent**. If λ is negative, then the system settles down to periodic oscillation and is non-chaotic. Conversely, if it is positive then we get a chaotic system. To conclude this section, note that further increases to γ will alternate between regions of chaos and periodic motion, but we will explore that later in the paper.

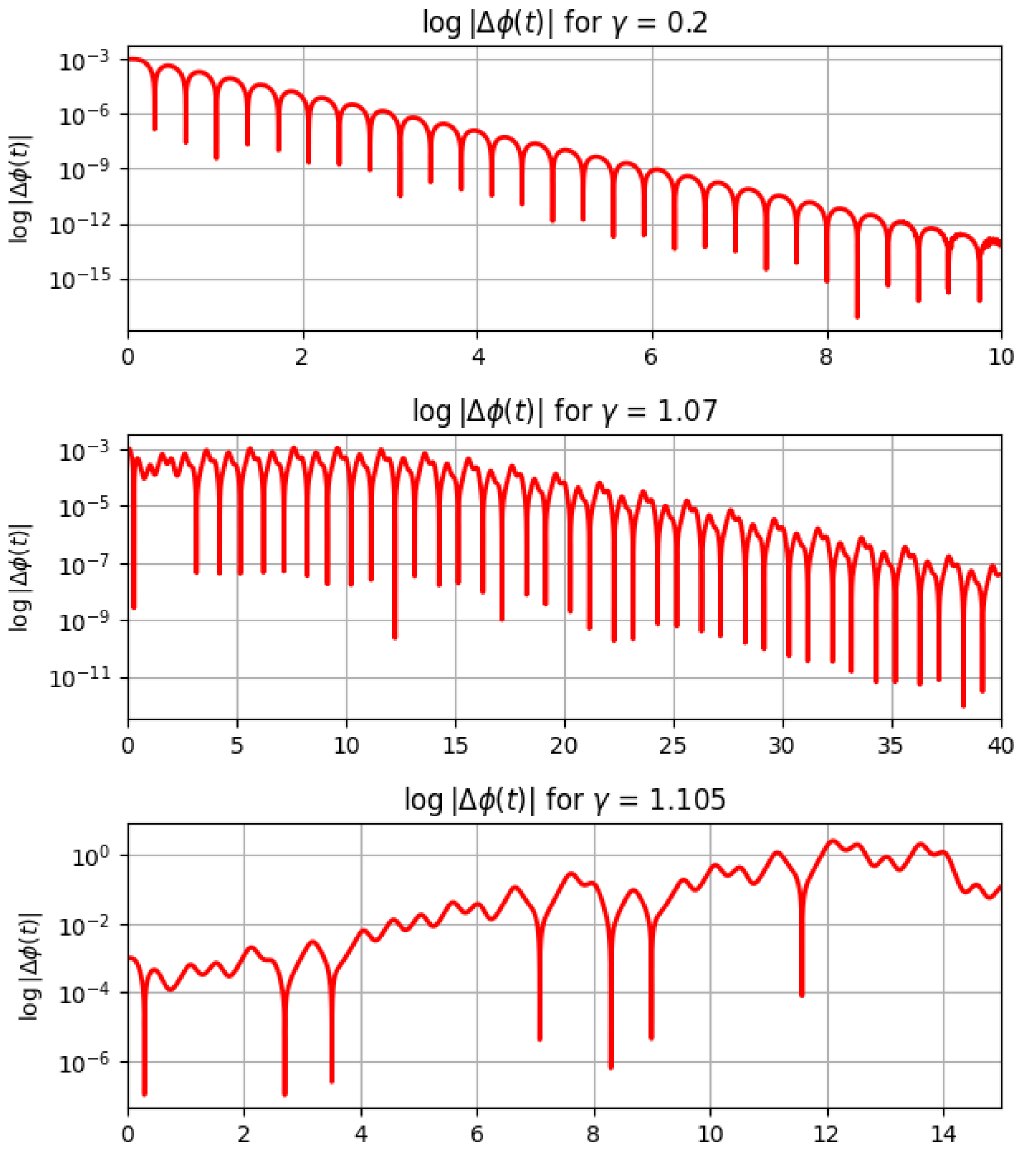

Figure 6: Logarithmic plot of the separation of two identical DDPs whose ICs differ by 0.001 radians. The top image depicts sinusoidal motion where the transients decay exponentially. The middle one is after period doubling, where transients are initially unpredictable but eventually the motion converges to a fixed attractor. The bottom image is of chaotic motion, where the difference diverges exponentially up to the order of 1 (completely uncorrelated).

3 State-Space Orbits

A state-space orbit is a way to visualize the chaotic behavior of these pendula using the angular position, $\phi(t)$, and angular velocity $\dot{\phi}(t)$. To begin, we will start by introducing some terminology, starting with the term state.

A **state**, short for state of motion, is a specific motion at some time t_0 from which we can decipher all subsequent motion of the pendulum. In this regard, we can think of them as initial conditions needed to specify a unique solution for the pendulum's equation of motion. The **state-space** is the space of all possible states of the pendulum, $(\phi, \dot{\phi})$, and a **state-space orbit** is the path formed by states of the pendulum. For any state at time t_i, there is only one orbit passing through $(\phi, \dot{\phi})$.

For example, a positive angular velocity, $(\dot{\phi}(t) > 0)$ will be above the x axis which indicates that the angular position is increasing, or heading towards the right on our state-space orbit. Conversely, negative angular velocity, $(\dot{\phi}(t) < 0)$ will be below the x axis, and this is a result of decreasing angular position, or heading towards the left.

State-space orbits also demonstrate tremendous sensitivity to initial conditions, as we will demonstrate in this upcoming figure. This is heavily dependent on the magnitude of γ, the **driving force**. The more we increase γ, the more chaotic the system's behavior becomes. We will compare and contrast using the same initial conditions for $\phi(t), \dot{\phi(t)}$, with varying γ to illustrate this in Figure (7).

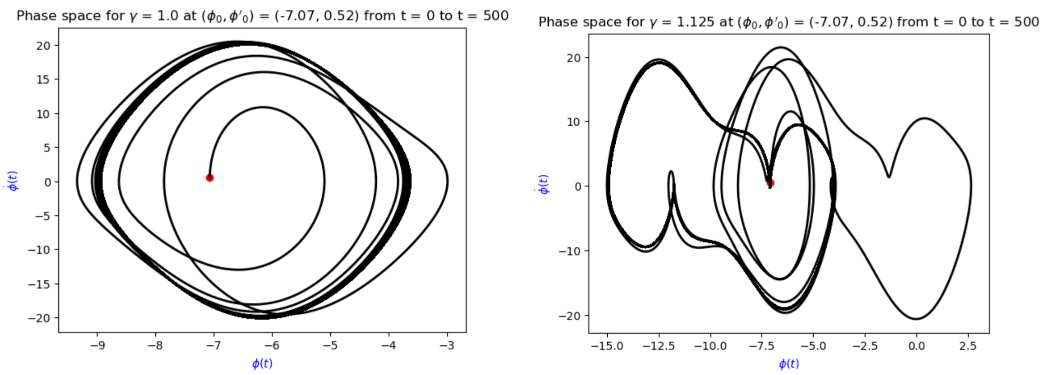

Figure 7: Right: $\gamma = 1.0$, Left: $\gamma = 1.125$

As we can see, increasing γ significantly impacts the orbit, even if we add as little as 0.1. On the other hand, we can also investigate what happens if we hold gamma constant while changing the initial conditions slightly. In the orbits below, we will shift our ICs for $\phi(t)$ by $\pm$ 0.1, with the original being in the center.

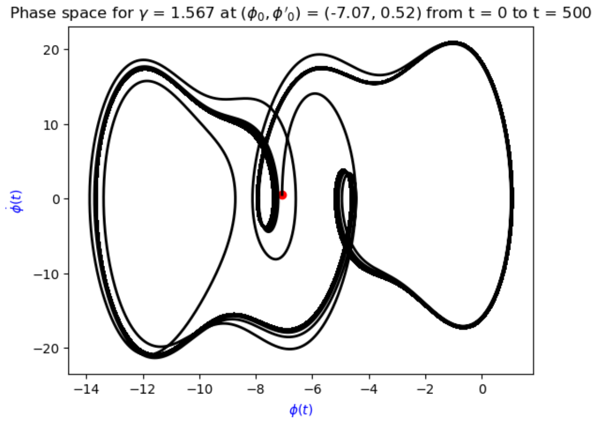

Figure 8: This is our "Base Case" with no varying initial conditions.

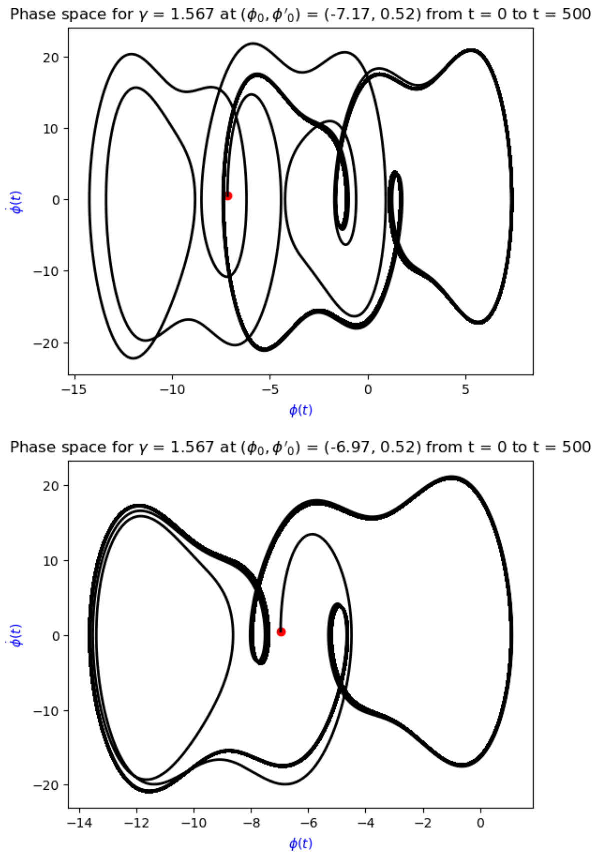

Figure 9: The middle is when we subtract .1, and the bottom is when we add .1, despite these small changes, we can see a lot of difference in the resulting orbit.

At this point, the natural question that arises is that if $\phi, \dot{\phi}$ are functions of time, why not use a 3 dimensional plot? However this 3rd dimension does not offer as much valuable input as a 2 dimensional state-space orbit [3]. To illustrate this, we offer a statespace orbit in 3 dimensions, Figure (10), plots of $\phi(t), \dot{\phi}(t)$, and t.

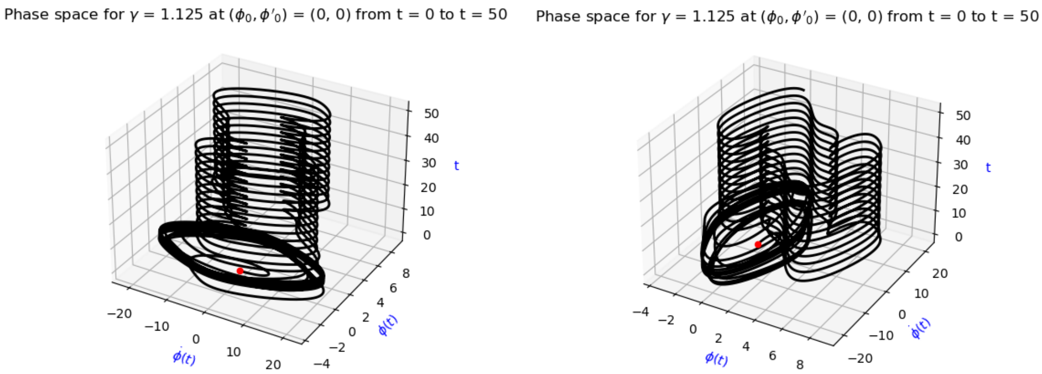

Figure 10: We can see that neither orientation provides input that we can find valuable, and often just comes off as cluttered. So we do not plot the time evolution of the system.

As such, what can we gain from state-space orbits? We can aim to get a clearer idea of the systems motion at specified time intervals. As the reader has surely noted above, at some time intervals, the orbits appear darker. Depending on the shape of the orbit, we can try to approximate solutions for the motion of the pendulum. For example, in the following Figure (11)

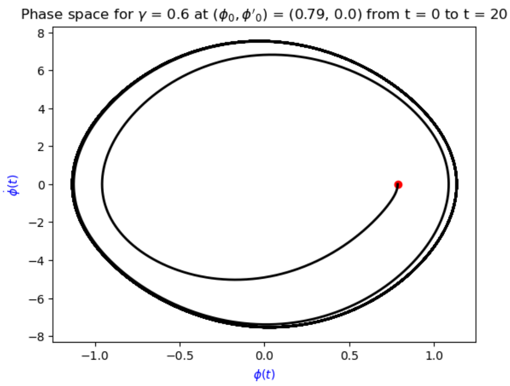

Figure 11: We note that there a lot of orbits in the elipsoid shape, meaning we can approximate this orbit with an elipitical function.

This is due to the location of the **attractors** of the system, which lead the orbit towards the outside, where the orbit can "orbit" around the attractor, leading to the darker lines on the orbit.

While this may just be an approximation, it takes us many steps closer to being able to predict the system's behavior. Suppose we were to consider the state-space orbits in varying time intervals, or **Periods**, where we can approximate the functions to describe the orbits. We could try and approximate the system's behavior as a whole over the time scale we are interested in, and aim to find some predictability in the chaos.

4 Bifurcation Diagrams

All of the solutions we have looked at so far for the driven dampened pendulum have been determined by a single value of of γ. If we want to observe how the motion of the pendulum changes as γ changes we can look at what is called a bifurcation diagram. To make a bifurcation diagram we can consider a range of values for γ and then we can solve the DE numerically for each value ranging from $t = 0$ to a time t_{max} such that all transients die out. We then consider every one of our solutions in a certain time interval and we take a point every period and we graph the point's value on the y-axis with the particular value for gamma as the x value.

For example, let's make a bifurcation diagram for the DDP given by (6), with initial conditions $\dot{\phi}(0) = 0,\ \phi(0) = \frac{\pi}{2},\ \omega = 2\pi,\ \omega_0 = 1.5\omega,\ \beta = \frac{\omega_0}{4}$.

We can choose 570 values for gamma spaced 0.00005 apart for $1.060 \leq \gamma \leq 1.0885$ and we solve the DE numerically for each value from $t = 0$ to $t = 500$. We then look at the following values of t:

$$t = 400, 401, 402, ..., 500$$

since every value of t is one period away from the previous point. For every value of t (lets call it t_j), we look at the corresponding value of $\phi(t_j)$ and we graph it on the coordinate $(\gamma_i,\ \phi(t_j))$ where γ_i is the value of gamma for our particular DE. We do this for every value of t, for every possible solution, for all values of γ—which adds up to $570 \cdot 100 = 5700$ calculations. This gives us the graph in Figure 12.

From this diagram, we can see that the pendulum has a constant period until $\gamma_1 = 1.0633$, at which the period doubles. The period doubles again at $\gamma_2 = 1.0793$ and again at $\gamma_3 = 1.0821$. At the left of the critical gamma value $\gamma_c = 1.0829$, the motion is mostly chaotic, although at $\gamma = 1.0845$ there is a brief island of stability where the period alternates between 6 different values.

Figure 12 shows only a small range of values for γ this is because of the fact that as γ increases, the pendulum can start "rolling" which means that rather than swaying back and forth. it could start doing complete rotations which adds a 2π to the angle that is permanent unless the pendulum does a counter rotation. In some cases, the pendulum may even continue this kind of behaviour indefinitely so that $\phi(t)$ approaches $\pm\infty$ and even if this kind of motion is perfectly periodic, we find that the values:

$$\phi(t_0), \phi(t_0 + 1), \phi(t_0 + 2), ...$$

will never repeat themselves which renders the plot above essentially useless as the values will increase by 2π every cycle. A way we can get around this is by noting that the angular velocity of the pendulum is unaffected by the amount of rotations it has done meaning it is immune to the ambiguity created by the extra 2π from

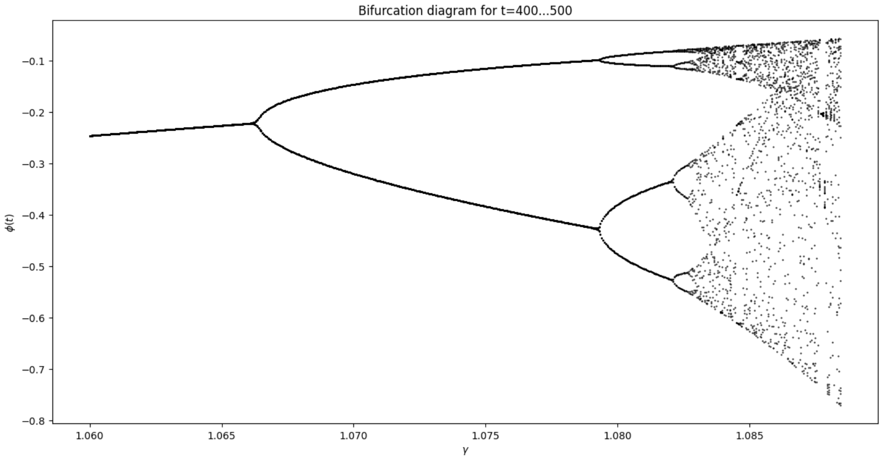

Figure 12: Bifurcation diagram for DDP with drive strengths $1.060 \leq \gamma \leq 1.087$. The period doubling cascade and descent into chaos is made very visible by this presentation.

rotations. This kind of rolling motion can be observed in the following animation: https://imgur.com/a/uqeWEVU

Thus, if the motion is periodic with period n, the values of $\dot{\phi}$ will always repeat themselves correspondingly, meaning that bifurcation diagrams drawn with $\dot{\phi}$ will work perfectly no matter how many rotations the pendulum does.

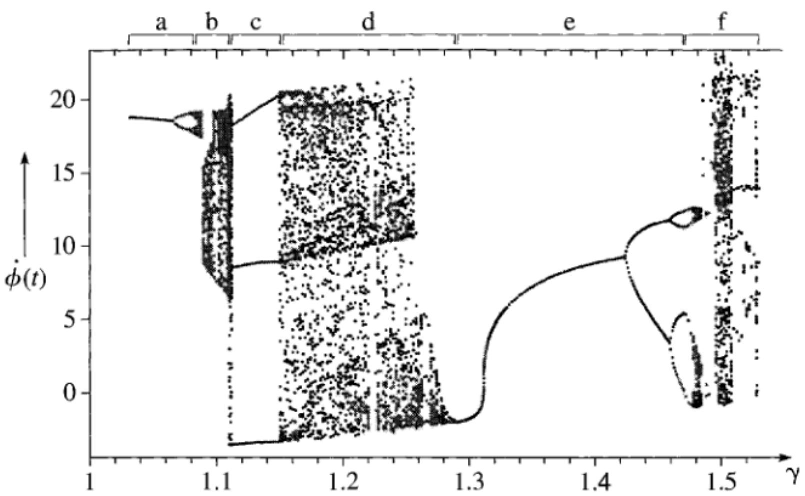

Figure 13: A very remarkable feature of this graph is the long stretch of single period motion that starts around 1.3 And if we actually make the graph for the pendulum at one of these points (say 1.4) we can see that the pendulum is actually rolling as its angle decreases by 2π every cycle implying that it is rotating. (Reprinted from [3])

So, we repeat the same process as above, but this time with a much larger value of γ where $1.03 \leq \gamma \leq 1.53$, giving us Figure 13. At the very far left we can see a version of the previous diagram, but shown in much less detail, where the pendulum starts with one period which then divides over and over until it descends into chaos. The pendulum continues moving chaotically, with a few islands of stability (not necessarily visible at this scale), until it suddenly switches to period 3 in part c. After that point, it continues acting chaotically until it gets to around $\gamma = 1.3$ in part d, where it again reaches a stable solution with a single period. However, this soon experiences a period doubling cascade and descends into chaos once more…

5 Numerical Methods and Discrete Difference Equations

One last idea we would like to finish up with is the connection between numerical methods of solving DEs and discrete difference equations. As you may recall, differential equations involve the instantaneous rate of change (derivative) of a continuous variable. When it is not possible to find the derivative analytically (in the case of nonlinear systems), the alternative is to take discrete samples and evaluate a **difference equation** instead. However, difference equations are not always as straightforward as they seem. For motivation, let us consider the **logistic map**.

Iterated functions are a big topic in chaos theory, and relate physical systems with constraints to mathematical idealizations like fractals. We will not be covering the logistic map explicitly here (it is thoroughly analyzed in this [5] seminal paper on the topic). Instead, we are only going to make a remark on the chaotic behaviour that arises—specifically, how it relates to continuous differential equations.

The equation for the logistic map is given by the recursively defined function

$$n_{t+1} = rn_t(1 - n_t) \tag{8}$$

for some growth rate r and a carrying capacity (maximum value) of 1. This is known to result in chaotic behaviour for certain values of r. However, there is a similar function known as the **logistic equation** which is defined

$$\dot{n} = rn(1 - n) \tag{9}$$

Although this is a nonlinear ODE, it is of a special type known as a Riccati equation that is analytically solvable. The general solution is [6]

$$n(t) = \frac{1}{1 + \left(\frac{1}{n_0} - 1\right) e^{-rt}} \tag{10}$$

for some initial condition n_0 (verify that this is a solution). The important thing to note is that, unlike the previous difference equation, this function is never chaotic.

However, it doesn't end here. In fact, there is a separate discrete difference equation that results in the closed form expression of (10) [7]

$$n_{t+1} = \frac{n_t}{mn_t + b} \tag{11}$$

At first glance, this does not look like the continuous function (9), despite being its discrete analog. Although the logistic map may seem more natural, we can analytically determine that (11) is the better choice for approximating discrete solutions. Furthermore, the wrong choice of difference equation can lead to unpredictable and unsolvable chaotic behaviour.

Now, remember that the numerical methods used to solve the equation of the DDP are themselves iterated difference equations. Is it possible that the chaotic motion observed in the DDP is simply an artifact of choosing the "wrong" method? This is not likely, as experiments show that chaos **is** an observable phenomenon. But the question still lingers as to how far off from the actual solution, if there is one, our approximations turn out to be—especially factoring in the extreme sensitivity to initial conditions. This is exactly motivation for the notion of "stiff" DEs that display numerical instability for certain iterative methods [2]. The topic is beyond the scope of this project, but it is nonetheless interesting to know that it exists. In any case, here is one final Figure 14 to ponder on.

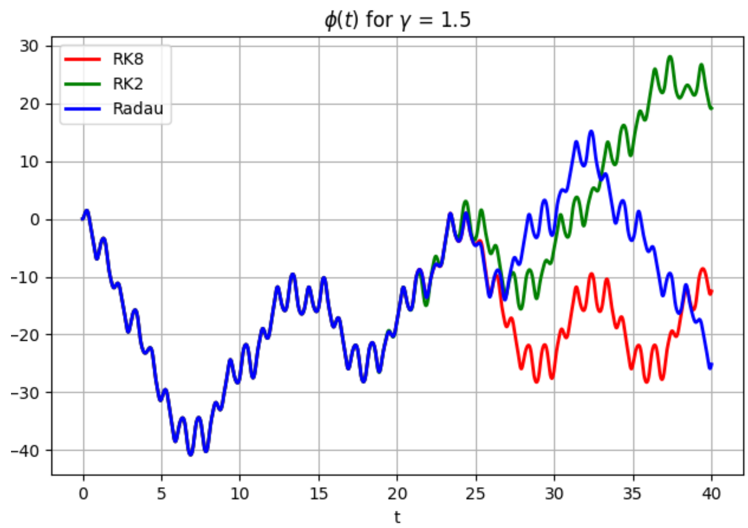

Figure 14: A plot of identical DDPs with the same ICs, solved by different numerical methods. The methods are a Runge-Kutta second-order, Runge-Kutta eighth-order, and the fifth-order implicit Runge-Kutta method of the Radau IIA family.

6 Conclusion

In summary, we first reviewed the nature of differential equations, and the conditions that give rise to chaotic motion. Specifically, the fact that non-linear DEs do not obey the superposition principle implies that we cannot find a general solution. We derived and classified the equations of motion of a Driven Damped Pendulum to fit within this model. Then, we explored possible interpretations of the non-linear system and determined the bounds of the chaotic regime. This is especially important considering the **universality** of chaotic systems.

Afterward, the focus shifted to different ways of interpreting numerical solutions of the DE. This was accomplished through tools such as the time-series plot, state-space orbit and bifurcation diagram, all of which gave us useful insights into the evolution of the system. These visualizations were especially helpful in understanding sensitivity to slight changes in initial conditions and parameters. Another key takeaway is how the Liapunov exponent describes the rate of divergence for such solutions.

Some topics for further exploration that we did not cover include, but are not limited to: the fractal nature of bifurcation diagrams and state-space orbits, behaviour of 3-dimensional attractors, the Poincaré recurrence theorem, and connections between chaos theory and ergodic theory. Of course, we cannot hope to even begin these topics within the confines of this paper, but it is nice to have some sense of how this problem fits in with other fields of research and dynamical systems.

We will point out, however, that the equations discussed are not exclusive to the driven damped pendulum. For example, if you use the small angle approximation, the resulting equations can be used to model the charge in a circuit with a resistor, inductor, and capacitor. This goes to show that a specific differential equation often can model many different types of systems. Taking this one step further, by introducing nonlinear elements into such a system, chaotic effects can still arise—see Chua's Circuit [8] for a simple extension to a circuit that exhibits chaos.

There are many ways to explore chaos theory, and it is a field that has only very recently started becoming more accessible due to the use of computer simulations. In a similar vein, we invite you to take a look at the interactive DDP simulator applet that we have made for the purpose of this project. Hopefully it can help reveal the more creative side of the yet undiscovered mathematics behind these systems...

7 Appendices

Credits

Mehdi: Bifurcation Diagrams, Conclusion, Editing, Code (figures and animations)
Khanjan: Abstract, State-Space Orbits, Conclusion (extension: Circuits), Code (state-space orbits)
Kyle: Introduction, Numerical Solutions to Nonlinear ODEs, Numerical Methods and Discrete Difference Equations, Code (simulator applet)

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