Linear algebra IV
Positive semidefinite matrices

Topics we'll cover

1. Positive semidefinite matrices
2. Properties of PSD matrices
3. Checking if a matrix is PSD
4. A hierarchy of square matrices

When is a square matrix "positive"?

- A superficial notion: when all its entries are positive
- A deeper notion: **when the quadratic function defined by it is always positive**

Example: $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

Positive semidefinite matrices

Recall: every **square** matrix M encodes a **quadratic function**:

$$x \;\mapsto\; x^T M x \;=\; \sum_{i,j=1}^{d} M_{ij} x_i x_j$$

(M is a $d \times d$ matrix and x is a vector in $\mathbb{R}^d$)

A symmetric matrix M is **positive semidefinite (psd)** if:

$$x^T M x \geq 0 \;\text{ for all vectors } x$$

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We saw that $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is PSD. What about $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$?

A symmetric matrix M is **positive semidefinite (psd)** if:

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When is a diagonal matrix PSD?

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If M is PSD, must cM be PSD for a constant c?

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If M, N are of the same size and PSD, must $M + N$ be PSD?

Checking if a matrix is PSD

A matrix M is PSD if and only if it can be written as $M = UU^T$ for some matrix U.

Quick check: say $U \in \mathbb{R}^{r \times d}$ and $M = UU^T$.

1. M is square.
2. M is symmetric.
3. Pick any $x \in \mathbb{R}^r$. Then

$$\begin{aligned} x^T M x \;=\; x^T U U^T x \;&=\; (x^T U)(U^T x) \\ &=\; (U^T x)^T (U^T x) \\ &=\; \|U^T x\|^2 \;\geq\; 0. \end{aligned}$$

Another useful fact: any covariance matrix is PSD.

A hierarchy of square matrices

Square — $M \in \mathbb{R}^{d \times d}$

$\bigcup$

Symmetric — $M = M^T$

$\bigcup$

Positive semidefinite — $x^T M x \geq 0$ for all $x \in \mathbb{R}^d$