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1- a)

$$P(\omega_1) = P(\omega_2) \quad , \quad x = \frac{a_1 + a_2}{2}$$

$$P(\omega_1 | x) = P(x|\omega_1) P(\omega_1) = \frac{1}{\pi b} \frac{1}{1 + (\frac{x-a_1}{b})^2} P(\omega_1)$$

$$\Rightarrow \frac{1}{\pi b} \frac{1}{1+(\frac{a_2-a_1}{2b})^2} P(\omega_2) = \frac{1}{\pi b} \frac{1}{1+(\frac{x-a_2}{b})^2} P(\omega_2) = P(\omega_2|x)$$

b)

$$P(\text{error}) = 1 - \sum_{c=1}^{2} \int_{R_c} f(x|\omega_c) P(\omega_c) dx = 1 - \int_{-\infty}^{\frac{a_1+a_2}{2}} f(x|\omega_1) P(\omega_1) dx - \int_{\frac{a_1+a_2}{2}}^{\infty} f(x|\omega_2) P(\omega_2) dx$$

$$= 1 - \frac{1}{2\pi} \left(tg^{-1}\left(\frac{a_2-a_1}{2b}\right) + \frac{\pi}{2} - tg^{-1}\left(\frac{a_1-a_2}{2b}\right) + \frac{\pi}{2} \right) = \frac{1}{2} - \frac{1}{\pi} tg^{-1}\left(\frac{a_2-a_1}{2b}\right)$$

C)

$$P_{max}(\text{error}) = \frac{\partial f(x)}{\partial x} = 0 \Rightarrow \left(\frac{x-a_i}{b}\right)\left(\frac{1}{1+(\frac{a-a_i}{2})^2}\right) = 0 \Rightarrow x = a_i$$

$$\Rightarrow \begin{cases} max\ P(\text{error})\Big|_{x=a_1} = \frac{1}{2} + \frac{1}{2} tg^{-1}\left(\frac{a_1-a_2}{b}\right) \\ max\ P(\text{error})\Big|_{x=a_2} = \frac{1}{2} - \frac{1}{2\pi} tg^{-1}\left(\frac{a_2-a_1}{b}\right) \end{cases}$$

$$\text{marginal} \Rightarrow max\ P(\text{error})\Big|_{x=\infty} = 1 - \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \frac{1}{2}$$

• دو حالت اول کم‌تر از $\frac{1}{2}$ بیش حالت marginal حداکثر احتمال خطا را دارد.

d)

$$f(x|\omega_1) P(\omega_1) \underset{R_1}{\overset{R_2}{\lessgtr}} f(x|\omega_2) P(\omega_2) \quad , \quad x = \frac{a_1+a_2}{2} \ , \ P(e) = 1 - \sum_{c=1}^{2} \int f(x|\omega_c) P(\omega_c) dx$$

e)

$$i^* = \arg\min \sum_{c=1}^{2} \lambda_{ic} P(\omega_c) f(x|\omega_c)$$

$$\Rightarrow \begin{cases} \lambda_{12} P(\omega_2) f(x|\omega_2) = \frac{1}{2\pi b} \frac{1}{1+(\frac{x-a_2}{b})^2} \\ \lambda_{21} P(\omega_1) f(x|\omega_1) = \frac{1}{\pi b} \frac{1}{1+(\frac{x-a_1}{b})^2} \end{cases}$$

kian $\Rightarrow \frac{1}{2\pi b} \frac{1}{1+(\frac{x-a_2}{b})^2} = \frac{1}{\pi b} \frac{1}{1+(\frac{x-a_1}{b})^2}$ assume $\begin{cases} a_1 = 3 \\ a_2 = 5 \\ b = 1 \end{cases}$ $\Rightarrow x^2 + 10 - 6x = 2x^2 + 52 - 200$

$$\Rightarrow x = 4.35$$

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2.a)

$$\begin{cases} \lambda_{11} P(\omega_1|x) + \lambda_{12} P(\omega_2|x) = R(\omega_1|x) \\ \lambda_{21} P(\omega_1|x) + \lambda_{22} P(\omega_1|x) = R(\omega_2|x) \end{cases}$$

$$\Rightarrow (\lambda_{21}-\lambda_{11}) P(x|\omega_1) P(\omega_1) \underset{R_1}{\overset{R_2}{\lessgtr}} (\lambda_{12}-\lambda_{22}) P(x|\omega_2) P(\omega_2)$$

$$\Rightarrow \frac{P(x|\omega_1)}{P(x|\omega_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} \times \frac{P(\omega_2)}{P(\omega_1)}$$

b)

$$\frac{P(x|\omega_2)}{P(x|\omega_1)} < \frac{\lambda_{21}-\lambda_{11}}{\lambda_{2}-\lambda_{22}} \frac{P(\omega_1)}{P(\omega_2)}$$ based on part a

3.

$$\min(P(error)) = \max\left(\sum_{c=1}^{2} \int_{R_c} P(x|\omega_c) P(\omega_c) dx\right)$$

$$\Rightarrow P(\omega_2) P(x|\omega_2) = P(\omega_1) P(x|\omega_1) \xRightarrow{P(\omega_1)=P(\omega_2)} P(x|\omega_2) = P(x|\omega_1)$$

$$\Rightarrow \frac{x}{\sigma_2^2} \exp\left(\frac{-x^2}{2\sigma_2^2}\right) = \frac{x}{\sigma_1^2} \exp\left(\frac{-x^2}{2\sigma_1^2}\right) \Rightarrow \ln(\sigma_1^2) + \frac{-x^2}{2\sigma_2^2} = \ln(\sigma_2^2) + \frac{-x^2}{2\sigma_1^2}$$

$$\Rightarrow x = \sqrt{(\ln(\sigma_2^2) - \ln(\sigma_1^2)) \frac{1}{\frac{-1}{2\sigma_2^2} + \frac{1}{2\sigma_1^2}}}$$

4. دو کلاس w_1 و w_2 را در نظر بگیرید که میانگین ها μ_1 و μ_2 دارند.
عدم وجود مرز تصمیم بین دو میانگین یعنی خط متصل کننده μ_1 و μ_2 عمود بر مرز تصمیم است.

خط متصل کننده μ_1 و μ_2 برابر است با $w^T(x-m)=0$ که $w=\mu_2-\mu_1$ و $m=\frac{\mu_1+\mu_2}{2}$
مرز تصمیم بر w عمود است پس بردار نرمال آن (n) باید با w موازی باشد.

$n = kw$, k is Constant $\Rightarrow n = k(\mu_2-\mu_1)$

پس شرط این است که $k(\mu_2-\mu_1)$ بین میانگین ها نباشد.

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5. a)

$$\mu_{red} = [1.33 \ , 1.61] \ , \mu_b = [-0.15 \ , 0.15]$$

$$\Sigma_{red} = \begin{pmatrix} 0.62 & 0.2 \\ 0.2 & 1.11 \end{pmatrix} , \quad \Sigma_b = \begin{pmatrix} 1.72 & 0.002 \\ 0.002 & 0.55 \end{pmatrix}$$

b)

$$P(\omega_1) P(x|\omega_1) = P(\omega_2) P(x|\omega_2)$$

$$\Rightarrow \frac{1}{\sqrt{2\pi \Sigma_1}} \exp\left(-\frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1)\right) = \frac{1}{\sqrt{2\pi \Sigma_2}} \exp\left(-\frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)\right)$$

$$\Rightarrow \ln\left(\frac{1}{\sqrt{\Sigma_1}}\right) - \frac{1}{2}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1) = \ln\left(\frac{1}{\sqrt{\Sigma_2}}\right) - \frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)$$

$$\Rightarrow \ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = \frac{1}{2}\left((x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1) - (x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)\right)$$

c)

$$risk = \sum_{c=1}^{2} \int_{R_c} \lambda_{c_i} P(x|\omega_c) P(\omega_c)\, dx = \int_{R_1} \lambda_{12} P(x|\omega_1) P(\omega_1)\, dx + \int_{R_2} \lambda_{21} P(x|\omega_2) P(\omega_2)$$

$$\Rightarrow \lambda_{12} P(\omega_1 x) - \lambda_{21} P(\omega_2|x) = 0 \Rightarrow 2a P(\omega_1|x) - a P(\omega_2|x) = 0 \Rightarrow$$

$$\Rightarrow 2P(\omega_1|x) = P(\omega_2|x)$$

$$\Rightarrow 2P(x|\omega_1) P(\omega_1) = P(x|\omega_2) P(\omega_2) \Rightarrow \ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = (x-\mu_1)^T \Sigma_1^{-1} (x-\mu_2) - \frac{1}{2}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)$$

d)

$$\ln\left(\sqrt{\frac{\Sigma_2}{\Sigma_1}}\right) = \frac{1}{3}(x-\mu_1)^T \Sigma_1^{-1} (x-\mu_1) - \frac{2}{3}(x-\mu_2)^T \Sigma_2^{-1} (x-\mu_2)$$

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6.

a) $\log p(D|\lambda) = \log \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \sum_{i=1}^{n} \left(x_i \log\lambda - \lambda - \log x_i! \right)$

$= \log\lambda \sum_{i=1}^{n} x_i - n\lambda - \sum_{i=1}^{n} \log x_i! \Rightarrow \frac{\partial \log p(D|\lambda)}{\partial\lambda} = \frac{1}{\lambda}\sum_{i=1}^{n} x_i - n = 0 \Rightarrow \boxed{\hat{\lambda}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i}$

b)

$$p(\lambda|D) = \frac{p(D|\lambda)\,p(\lambda)}{p(D)} \propto p(D|\lambda)\,p(\lambda)$$

$$= \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}\, c\,\lambda^{\alpha-1} e^{-\beta\lambda} = c' \cdot \lambda^{\sum x_i + \alpha - 1} e^{-(n+\beta)\lambda} = c'\,\lambda^{\alpha'-1} e^{-\beta'\lambda}$$

$$= \text{Gamma}(\lambda|\alpha', \beta'), \quad \alpha' = \sum x_i + \alpha, \quad \beta' = n + \beta$$

c) بله زیرا توزیع پسین و پیشین از یک نوع شده

d) $\lambda_{MAP} = \arg\max_{\lambda} p(\lambda|D) = \frac{\alpha' - 1}{\beta'}$

e) بله زیرا: $\lim_{n\to\infty} \frac{\sum x_i + \alpha - 1}{n+\beta} = \frac{\sum x_i}{n} = \hat{\lambda}_{ML}$

f) در صورت داده برداری مناسب و هنگامی که داده به اندازه کافی داریم MLE روش مناسبی است
ولی هنگامی که دانش پیشین نسبت نحوه توزیع داده‌ها داریم و می‌خواهیم از یک سری فرض
پیشین استفاده کنیم MAP بهتره.

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