Coursework Quantitative Risk Management

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1 Basel Credit risk model

1. In this exercise we consider a homogeneous portfolio with 100 credits. Thus for each credit, the exposure at default, the expected LGD and the probability of default are set to 1 mn, 50% and 5%. We assume that the asset correlation $\rho = 10\%$. The homogeneity of the portfolio doesn't mean that the events of default on the loans are independent of each other, so we can't apply the CLT and say that the loss distribution converges to a normal distribution as the portfolio size increase. However, the distribution of the portfolio loss does converge to a limiting form and here we will plot such distribution, its quantile and density function.

In order to obtain a closed-form formula, we need a model of default times. Under the Merton-Vasicek model, one-factor approach is selected, where: Z_i be the normalized asset value of the entity i.

$$Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\epsilon_i, \tag{1}$$

- $\sqrt{\rho}X$ is the company's exposure to the common factor.
- $\sqrt{1-\rho}\epsilon_i$ represents the company's specific risk

Still, the default times are not independent, because they depend on the common risk factors X. However, conditionally to these factors, they become independent because idiosyncratic risk factors ϵ_i are not correlated. We now calculate the conditional default probability:

$$p_i(X) = Pr[Z_i < Bi|X] = Pr[\epsilon_i < \frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}] = \phi(\frac{B_i - \sqrt{\rho}X}{\sqrt{1-\rho}}) \tag{2}$$

Default occurs when Z_i is below the barrier B_i and it is equal to $\phi^{-1}(p_i)$ (p_i the unconditional default probability).

Understanding the conditional independence of default times was key to grasping its idea. Proving how, we plotted the quantile function, used **'bisect()'** function in python to inverse it and get the cumulative distribution. Finally, using the relationship between density and quantile functions we got the pdf. We can say that the portfolio loss distribution is highly skewed and higher correlation between assets means higher loss, which is obvious, but default is less likely to occur.

2. The risk contributions depend:
- on 3 credit parameters: EAD ω_i, E[LGDi], p_i;
- on 2 model parameters: α, ρ

The risk contribution is not a monotone function with respect to ρ. It depends on the sign of the term $\sqrt{\rho}\phi^{-1}(pi) + \phi^{-1}(\alpha)$. This implies that the risk contribution may decrease if the probability of default is very low and the confidence level is larger than 50%. We illustrate the behavior of the risk contribution with a predefined baseline: We verify that the risk contribution is an increasing function of $E[LGD_i]$ and α.
When p_i=10% and α=90% the risk contribution increases with ρ and reaches the value 35, which corresponds to the limiting case of ρ=1 where p_i = 1-α. While when p_i=5% p_i ¡ 1-α the credit risk vanishes at ρ=1. And this behavior changes when α=99.9%, we are in the case of p_i ¿ 1-α and the credit risk is equal to ω_i, E[LGDi].

2 Loss given default estimation

In this section we needed better statistical language like R. To our knowledge, the Maximum likelihood estimator in python doesn't exist or isn't performing well enough.
1. We present the assumption made by the stochastic modelling of LGD, considering it as a random variable, i.e. having a distribution function. Using a parametric distribution: generally the beta distribution is used $\mathcal{B}(\alpha, \beta)$.
Given a set of observed losses we estimated α and β using the method of moments and maximum likelihood. We report the estimated parameters:
α_{ML}=1.84 and β_{ML}=1.25
α_{MM}=1.37 and β_{MM}=0.92

2. a) To show the shortcoming of this assumption that LGDi are Beta distributed, when empirically it is bimodal: the recovery can be very high or very low, we use a non-parametric model.
We have reported the empirical distribution, and the corresponding (re-scaled) calibrated beta distribution. We notice that it is very far from the empirical distribution. Remember that the (re-scaled) variance:

$$V(X) = [\sum_{n=1}^{\kappa} p_i x_i^2] - [\sum_{n=1}^{\kappa} p_i x_i]^2 \tag{3}$$

b) Using Monte carlo methods, generating $N = 10^7$ scenarios, we compare the loss distribution when we consider the empirical distribution and the calibrated beta distribution for the loss given default. We also report the loss distribution when we replace the random variable LGDi by its expected value E [LGDi] = 50%.

We observe that the shape of L highly depends on the LGD model. And a more pronounced fat tail with the calibrated beta distribution.

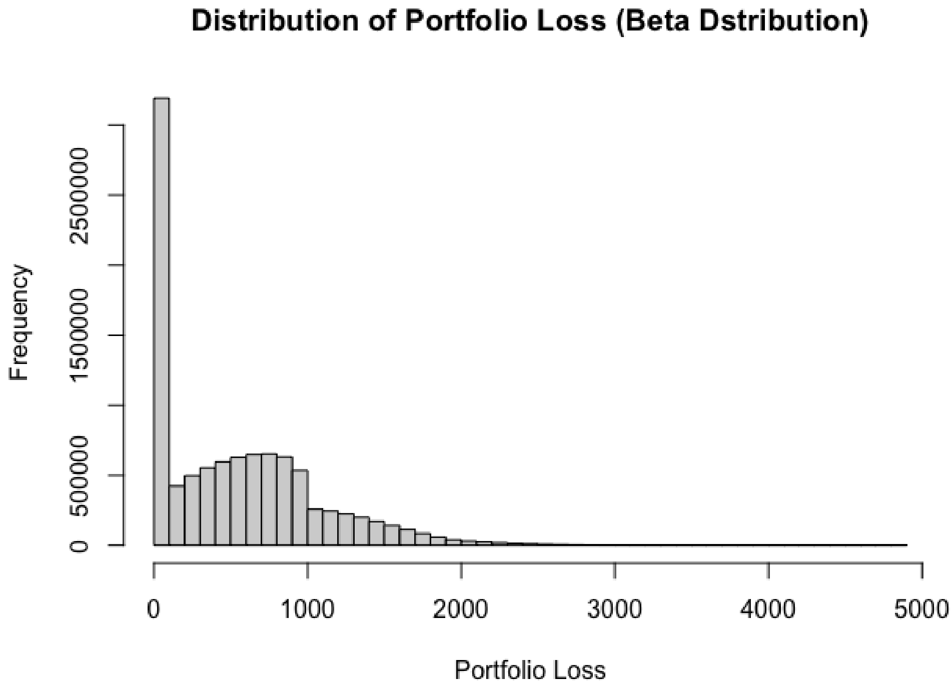
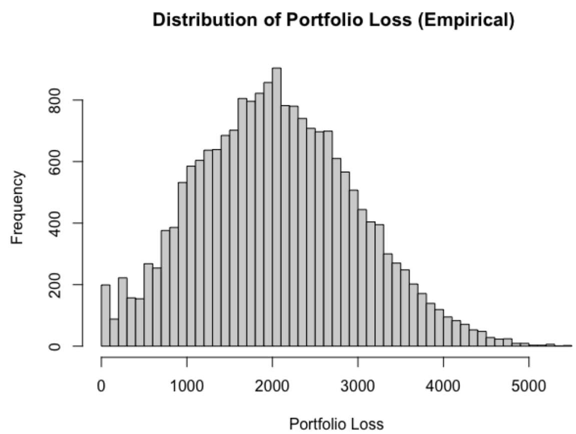


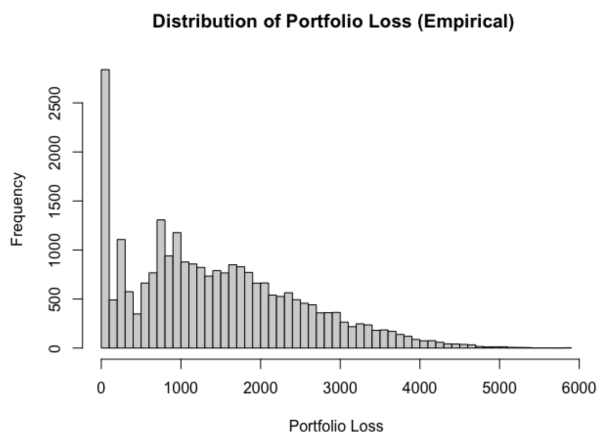
Figure 1:

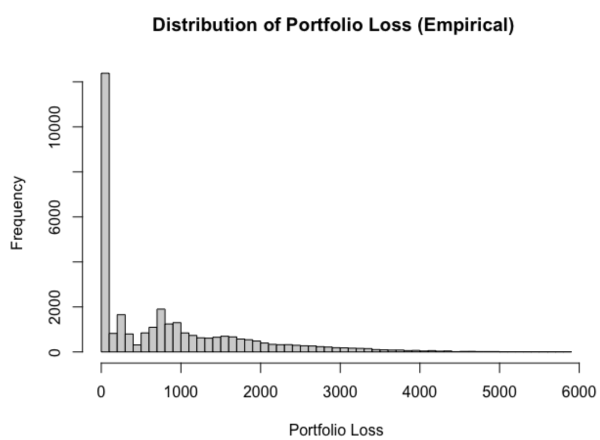
3 Default Probability

1. The survival function S(t) is the main tool to characterize the probability of default. And in survival analysis, the key concept is the hazard function λ(t), which is the instantaneous default rate given that the default has not occurred before t.
We will estimate and plot the hazard function under the exponential, the Gompertz and the piece-wise exponential assumptions. We deduce that the one-year default probability (PD=1-S(1)) is respectively equal to 1.366%, 1.211% and 1.200%.
2. a) Considering the credit migration matrix given we estimate the piece-wise exponential model over a period of 150 years.
We can see the for good initial ratings, hazard rates are low for short maturities and increase with time. For bad initial ratings, we obtain the opposite effect, because the firm can only improve its rating if it did not default. The hazard function of all the ratings converges to the same level, which is equal to 102.63 bps. Which means from these 1000 firms 1.02% default every year.
b) For continuous time we estimate Markov generators $\hat{\Lambda}$, which contain off-diagonal negative values: This implies that we can obtain transition probabilities which are negative for short maturities. To overcome that we used Israel et al. (2001) two proposed estimators. And by some matrix algebra we get $\bar{\Lambda}$ and $\tilde{\Lambda}$.
Finally we prove that the Markov generator $\tilde{\Lambda}$ is the estimator that minimizes the distance to $\hat{P}$ and use it to estimate and compare the probability density function of S&P ratings.

4 Default Correlation







By Monte Carlo methods we generate multiple scenarios where a random normal variable and a normal vector of length n (portfolio assets or in our example LGD empirical distribution) are simulated then the Basel copula each time to finally transform them by the inverse survival function (which is uniformly distributed) to get our defaults times τ_i. We plot for different ρ the loss distribution and calculate VaR and ES.

5 Credit Risk and Valuation Risk

1. Risk-free Mark-to-Market (MTM) Value: The Risk-free MTM value of the forward contract can be calculated using the Black-Scholes formula for a forward contract:

$$C = S_0 \cdot N(d_1) - X \cdot e^{-rT} \cdot N(d_2)$$

where:
C = Call option price
S_0 = Current price of the underlying asset
$N(\cdot)$ = Cumulative standard normal distribution function
X = Strike price of the option
r = Risk-free interest rate
T = Time to expiration
$d_1 = \frac{\ln(S_0/X)+(r+\sigma^2/2)T}{\sigma\sqrt{T}}$
$d_2 = d_1 - \sigma\sqrt{T}$

MTM = $S_0 * e^{r*T)} - K * e^{-r*T}$
with K = $e^{r*T} * S_0$= 108.3388 (strike)
C = 14,03162048

2. First-to-Default Credit Value Adjustment (CVA): To calculate the CVA, we'll consider the default probabilities of the Bank and the Counterparty, as well as the recovery rates:
CVA = max(0, (1 - RB) * PD_B * (1 - RC) * PD_C * EAD)
where
RB = 50% (Bank's recovery rate)
PD_B = 1 - e^{-CS_BT} (Bank's default probability)
RC = 60% (Counterparty's recovery rate)
PD_C = 1 - e^{-CS_CT} (Counterparty's default probability)
EAD = S_0 * (1 - RB) (exposure at default)

Plugging in the values:
PD_B = 1 - $e^{-0.005*2}$ = 0.009975
PD_C = 1 - $e^{-0.008*2}$ = 0.015872
EAD = 100 * (1 - 0.5) = 50

3. First-to-Default Debit Value Adjustment (DVA): Similar to CVA, DVA takes into account the default probabilities and recovery rates: DVA = max(0, (1 - RB) * PD_B * (1 - RC) * PD_C * EAD)

Plugging in the values: DVA = max(0, (1 - 0.5) * 0.009975 * (1 - 0.6) * 0.015872 * 50)

4. Total Fair Value (Bilateral Value Adjustment): The total fair value is calculated by subtracting the first-to-default CVA from the Risk-free MTM value: Total Fair Value = MTM - CVA