Relations

Def^n: Let A and B be two sets. A binary relation R from A to B is a subset of $A \times B$.

OR,

$$R \subseteq A \times B$$

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Notation: aRb to denote $(a,b) \in R$

$a \not{R} b$ to denote $(a,b) \notin R$

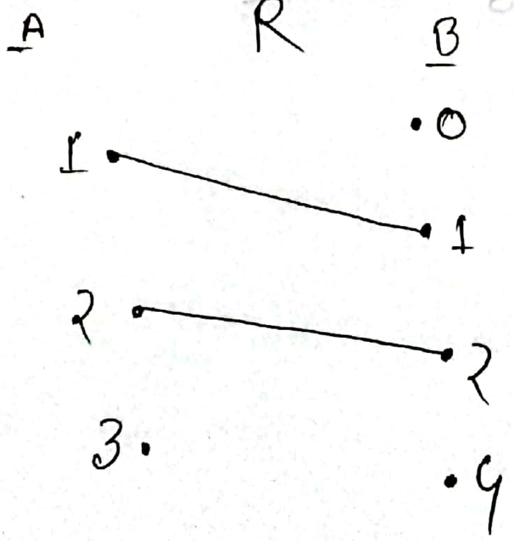
Ex: $A = \{1, 2, 3\}$, $B = \{0, 1, 2, 4\}$

$$A \times B = \{(1,0), (1,1), (1,2), (1,4), (2,0), (2,1), (2,2), (2,4), (3,0), (3,1), (3,2), (3,4)\}$$

Let say, R is the relation where $(a,b) \in R$ if and only if $a = b$ then,

$$R = \{(1,1), (2,2)\} \text{ and } R \subseteq A \times B$$

Graphical:



Arrows used to represent ordered pairs of relation R

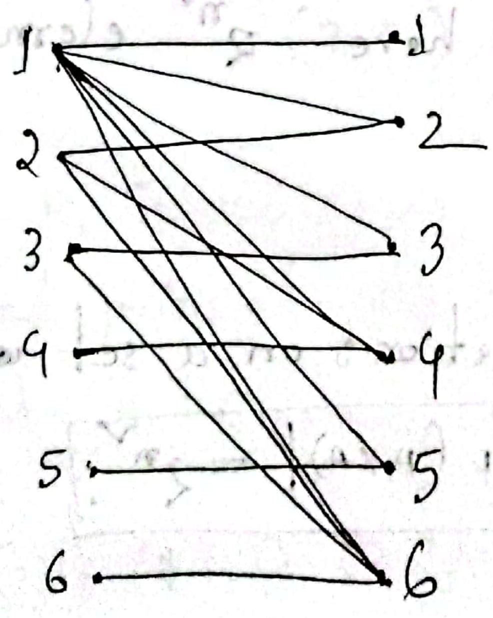
☐ A relation on a set A is a relation from A to A.

Ex:

Let $R = \{(a,b) \mid a \text{ divides } b\}$ $\quad A = \{1,3,4,5,6\}$

$\therefore R = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,2),$
$(2,4),(2,6),(3,3)\,(3,6),(4,4),(5,5),$
$(6,6)\}$

Graphical: (R)



Number of relations on a set with n elements :

- A relation on a set A is a subset of $A \times A$. Set A contains n elements and $A \times A$ contains n^2 elements.

- Let say, A has n elements. So, P(A) must have 2^n elements.

- $P(A \times A)$ must have 2^{n^2} elements.

So,

- number of relations on a set A with n elements $\boxed{= |P(A \times A)| = 2^{n^2}}$.

Reflexive :

* A relation R on a set A is called reflexive if $(a,a) \in R$ for every element $a \in A$.

* $\boxed{\forall a \, ((a,a) \in R)}$

Ex: $A = \{1,2,3,4\}$

$R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),$
$(3,3),(3,4),(4,4)\}$

Relation R_1 is reflexive because it contains all ordered pairs of the form (a,a) for every element $a \in A$. i.e. R_1 has $(1,1),(2,2),(3,3),(4,4)$

Sup: $R_2 = \{(1,1),(1,2),(2,1),(2,2),(3,1)(4,4)\}$

Relation R_2 is not reflexive because the ordered pair $(3,3)$ is not in R_2.

[d] Irreflexive Relation:

⊛ A relation R on a set A is called irreflexive if $\boxed{\forall a \in A ; (a,a) \notin R.}$

Ex: $A = \{1,2,3,4\}$

$R_3 = \{(1,2), (2,1), (3,3), (4,4)\}$ is not irreflexive because $(3,3)$ and $(4,4)$ is there in R_3.

$R_4 = \{(1,2), (2,1)\}$ is irreflexive because $\forall a \in A ; (a,a) \notin R_4$

Symmetric relation:

* A relation R on a set A is called symmetric if $(b,a) \in R$ holds when $(a,b) \in R$ for all $a,b \in A$.

* $\boxed{\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)}$

Ex: Relation $R_5 = \{(1,1),(1,2),(2,1),(2,2)\}$ is symmetric because for every $(a,b) \in R_5$ $(b,a) \in R_5$

like, $(1,2)$ $(2,1)$ is in R_5.

$R_6 = \{(1,1),(1,2),(1,3),(1,4)\}$ is not symmetry because for $(1,2)$ there is no $(2,1)$ in R_6. Same goes for $(1,3)$ and $(1,4)$.

Antisymmetric Relation:

A relation R on a set A is called antisymmetric if $\boxed{\forall a \forall b ((a,b) \in R \land (b,a) \in R \to (a=b))}$

* whenever we have (a,b) in R, we will never have (b,a) in R untill or unless $(a=b)$

Ex: $R_7 = \{(1,1), (2,1)\}$ on set A is antisymmetric because (2,1) is in R_7 but (1,2) is not in R_7.

Transitive Relation:

- A relation R on a set A is called transitive if $\boxed{\forall a \forall b \forall c\ (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)}$

- If relation has (a,b) and (b,c) it's must have (a,c)

Ex: $A = \{1, 2, 3, 4\}$

$R_8 = \{(2,1), (3,1), (3,2), (4,4)\}$ is transitive

because $(3,2)$, $(2,1)$, $(3,1)$ are there in R_8

$R_9 = \{(2,1), (1,3)\}$ is not transitive as $(2,1)$ and $(1,3)$ are there but $(2,3)$ is not in R_9.

4) Asymmetric Relation:

A relation R on a set A is called asymmetric if $\boxed{\forall a \forall b ((a,b) \in R \rightarrow (b,a) \notin R)}$

* so, if there (a,b), (b,a) can't be there.
* can't be $a=b$.

Ex:

$R_{10} = \{(1,1), (1,2), (1,3)\}$ is not an asymmetric because of $(1,1)$.

$R_{11} = \{(1,2), (1,3), (2,3)\}$ is an asymmetric relation.

Problem: Determine whether relation R on the set of all real numbers is reflexive, symmetric, antisymmetric, transitive, where $(x,y) \in R$ if and only if

a) $x+y=0$ b) $x-y$ is a rational number

c) $xy=0$ d) $x=1$ or $y=1$

Sol^n:

a) $R=\{(x,y) \mid x+y=0\}$

1. **Reflexive:** $\forall a \in R((a,a) \in R)$

so, not reflexive. any true for $(0,0)$. otherwise it's false

2. **Symmetric:** $\forall_a \forall_b \in R((a,b) \in R \rightarrow (b,a) \in R)$

If $a+b=0$. then $b+a=0$

$\therefore$ so, R is symmetric

3. **Antisymmetric:** $\forall_a \forall_b \in R((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b)$

so, it's not antisymmetric,

Ex: $\left.\begin{array}{l} 1+(-1)=0 \\ -1+1=0 \\ 1 \neq -1 \end{array}\right\}$

4. Transitive: $\forall_a \forall_b \forall_c (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$

Not Transitive:

Ex: $1 + (-1) = 0$, $-1 + 1 = 0$

$1 + 1 \neq 0$

b) $R = \{(x,y) \mid x-y$ is a rational no. $\}$

1. Reflexive: $a-a=0$ is a rational number

2. Symmetry:

If $a-b$ is a rational number

then, $b-a = -(a-b)$ is also a rational number.

3. Antisymmetry:

Not Antisymmetry.

cause

$3-2=1$ is rational num

$2-3=-1$ is rational num

$(3,2)$ থাকলে $(2,3)$ থাকতে পারবে না

4. Transitive:

$a-b$ ~~is~~ and $b-c$ is a rational number,

then, $a-c = (a-b)+(b-c)$ is a rational number

c) $R = \{(x,y) \mid xy = 0\}$

1. Reflexive:

Not reflexive.

cause, $a \cdot a = a^2 = 0$ only true for $a = 0$

2. Symmetry: $ab = 0$ then $ba = 0$

3. Antisymmetric:

Not antisymmetric,

$1 \times 0 = 0$ and $0 \times 1 = 0$

but $1 \neq 0$.

4. Transitive:

Not transitive,

$-1 \times 0 = 0$, $0 \times 2 = 0$

but $-1 \times 2 \neq 0$.

d) $R = \{(x, y) \mid x = 1 \text{ or } y = 1\}$

1. Reflexive:

not reflexive. cause $(0,0)$ not in R

2. Symmetry:

if $(a,b) \in R$, $a = 1$ or $b = 1$

This means $(b,a) \in R$

3. Antisymmetry:

not antisymmetry

if $a = 1$, $b = 0$

then $(a,b) \in R$, $(b,a) \in R$

but $a \neq b$.

4. Transitive: Not transitive

if $a = 2$, $b = 1$, $c = 2$

$(a,b) \in R$

$(b,c) \in R$

$(a,c) \notin R \rightarrow$ does n't have 1

Amongst the properties {reflexivity, symmetry, antisymmetry, transitivity}, the relation $R = \{(x,y) \in N^2 \mid x \neq y\}$ satisfies?

Soln:

1. Reflexivity: $\forall a\ (a,a) \in R$

$a \neq a$, Not possible

Therefore, R is not reflexive.

2. Symmetry: $\forall a \forall b\ ((a,b) \in R \rightarrow (b,a) \in R)$

$a \neq b$, then $b \neq a$

Therefore R is symmetric

3. Antisymmetry: $\forall a \forall b\ (((a,b) \in R \wedge (b,a) \in R) \rightarrow (a=b))$

$(a \neq b \wedge b \neq a) \rightarrow (a = b) \rightarrow$ false

so, not antisymmetric.

4. Transitivity: $a \neq b$, $b \neq c$

$a = 1$, $b = 2$, $c = 1$

$1 \neq 2$, $2 \neq 1$ but $1 = 1$

so, R is not transitive.

Ex The binary relation $S = \phi$ (empty set) on a set $A = \{1, 2, 3\}$ is

(A) reflexive, symmetry, Transitive check.

Soln:

$S = \phi$

1. There is no element in S.

so, $(a, a) \rightarrow$ not possible

not reflexive

2. Symmetry: $\forall a \forall b \in A((a, b) \in S \rightarrow (b, a) \in S)$

$(a, b) \in S$ is false

$\therefore$ Implication is true,

$\therefore$ It's symmetry

3. Transitivity: $\forall a \forall b \forall c \in A(((a, b) \in S \wedge (b, c) \in S) \rightarrow (a, c) \in S$

$(a, b) \in S \wedge (b, c) \in S$ is false

$\therefore$ Implication is true

$\therefore$ It's transitive

Operations on Relations

Let, considere two relations R_1 and R_2 from set A to B.

These relations can be combined in a way two sets can be combined i.e., $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$, $R_1 \oplus R_2$.

Ex: $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$

$R_1 = \{(1,3), (1,4), (3,3)\}$

$R_2 = \{(2,3), (2,4), (3,3), (2,5)\}$

$R_1 \cup R_2 = \{(1,3), (1,4), (2,3), (2,4), (3,3), (2,5)\}$

$R_1 \cap R_2 = \{(3,3)\}$

$R_1 - R_2 = \{(1,3), (1,4)\}$

$R_2 - R_1 = \{(2,3), (2,4), (2,5)\}$

$$\boxed{R_1 \oplus R_2 = (R_1 \cup R_2) - (R_1 \cap R_2)}$$

$$= \{(1,3), (1,4), (2,3), (2,4), (2,5)\}$$

Problem: Suppose that R and S are reflexive relations on a set A. Prove or disprove each of these statements.

a) $R \cup S$ is reflexive.

b) $R \cap S$ is reflexive.

c) $R \oplus S$ is irreflexive.

d) $R - S$ is irreflexive.

Soln:

a) R is reflexive. $\forall a \in A \ (a,a) \in R$

S is reflexive $\forall a \in A \ (a,a) \in S$

a). $R \cup S$ must contain all elements in either R or S. Therefore, $R \cup S$ is reflexive.

b) $R \cap S$ must contain all elements in either R or S. Therefore, $R \cap S$ is reflexive.

c)

c) $R \oplus S$ contains all ~~em~~ elements in R or S but not in both.

$\forall a \in A (a, a) \notin R \oplus S$

Therefore, $R \oplus S$ is irreflexive.

d) $R - S$ contains all elements in R that are not in S.

$\forall a \in A (a, a) \notin R - S$

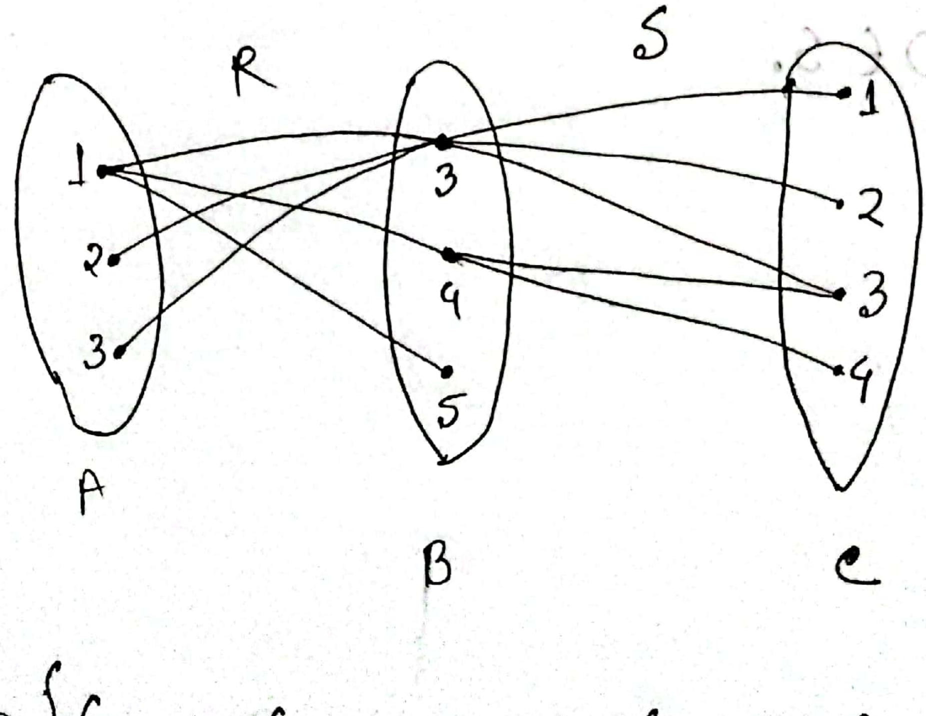
Therefore, $R - S$ is irreflexive

Composition of Relations

Let A, B and C three sets. Suppose, R is a relation from A to B, and S is a relation from B to C. The composite of R and S, denoted by $S \circ R$, is a binary relation from A to C consisting of ordered pairs (a,c) where $a \in A$ and $c \in C$. Also, $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$.

Ex: What is the composite of the relations R and S, where R is the relation from set $A = \{1,2,3\}$ to set $B = \{3,4,5\}$ with $R = \{(1,3), (1,4), (1,5), (2,3), (3,3)\}$ and S is the relation from set $B = \{3,4,5\}$ to set $C = \{1,2,3,4\}$ with $S = \{(3,1), (3,2), (3,3), (4,3), (4,4)\}$

Sol^n:



$$S \circ R = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Let R be the relation on the set $\{1,2,3,4,5\}$ containing the ordered pairs $(1,1),(1,2),(1,3),(2,3),(2,4),(3,1),(3,4),(3,5),(4,2),(4,5),(5,1),(5,2)$ and $(5,4)$. Find.

a) R^2 b) R^3 c) R^4

Soln:

$R=\{(1,1),(1,2),(1,3),(2,3),(2,4),(3,1),(3,4),(3,5),$
$(4,2),(4,5),(5,1),(5,2),(5,4)\}$

$R=\{(1,1),(1,2),(1,3),(2,3),(2,4),(3,1),(3,4),(3,5)$
$(4,2),(4,5),(5,1),(5,2),(5,4)\}$

a)

$R^2=R\circ R$

$R\circ R=\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,4),$
$(2,5),(2,2),(3,1),(3,2),(3,3),(3,5),$
$(3,4),(4,3),(4,4),(4,1),(4,2),(5,1),(5,2),$
$(5,3),(5,4),(5,5)\}$

$R^3 = R^2 \circ R$

$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (2,5), (3,1), (3,2), (3,3), (3,4), (3,5), (4,1), (4,2), (4,3), (4,4), (4,5), (5,1), (5,2), (5,3), (5,4), (5,5)\}$

$R^4 = R^3 \circ R$

R^3 already contains all possible ordered pairs in $A \times A$; R^4 will also contain all possible ordered pairs.

Therefore, $R^4 = R^3$.

Representation of Relations

Let say,
$$A = \{1, 2, 3\}$$
$$B = \{0, 1, 2, 4\}$$

Let, we want to represent a relation R which consists of all ordered pairs (x, y) where $x \in A$ and $y \in B$ and $x \leq y$.

There are multiple ways to represent a relation.

1. Listing Method: $R = \{(1,1), (1,2), (1,4), (2,2), (2,4), (3,4)\}$

2. Set Builder Method:
$$R_{A \text{ to } B} = \{(x,y) \mid x \leq y\} \text{ OR } R = \{(x,y) \mid x \in A \wedge x \in B \wedge x \leq y\}$$

3. Statement Representations:
$$\forall x \in A, \forall y \in B, \; xRy \text{ iff } x \leq y$$

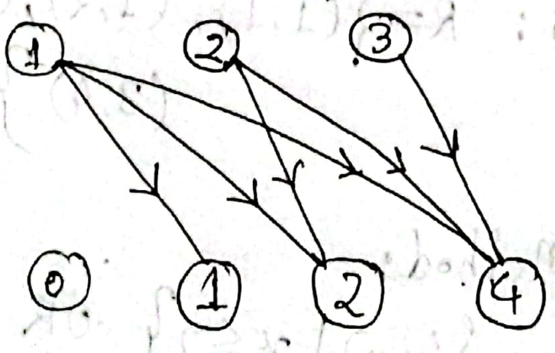
4. Matrix representations:

$|A| = m, \quad |B| = n$

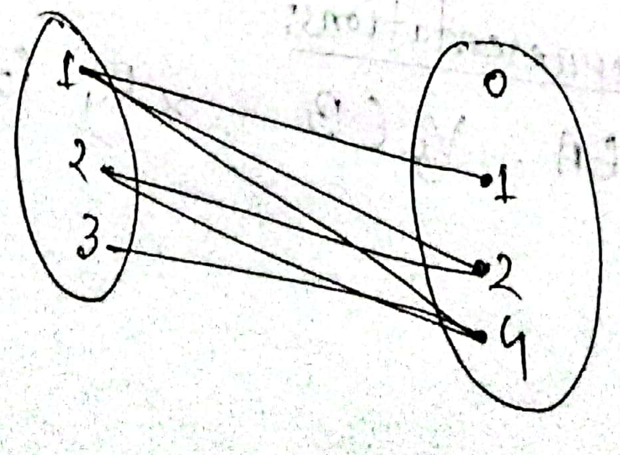
$$\begin{array}{c|cccc|l} & 0 & 1 & 2 & 4 & \\ \hline 1 & 0 & 1 & 1 & 1 & \\ 2 & 0 & 0 & 1 & 1 & \\ 3 & 0 & 0 & 0 & 1 & m \times n \end{array}$$

5. Graph Representation:

Directed graph:



6. Arrow diagram representation:



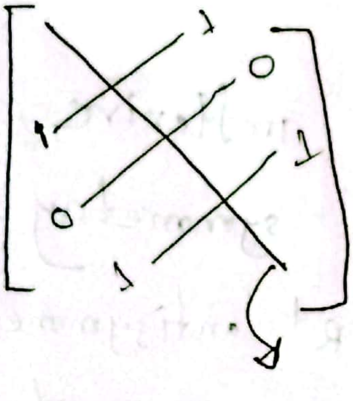
Matrix → মুখ্য কর্ণ এর elements গুলো 1 হলে reflexive

Matrix → প্রতিসম হলে symmetric

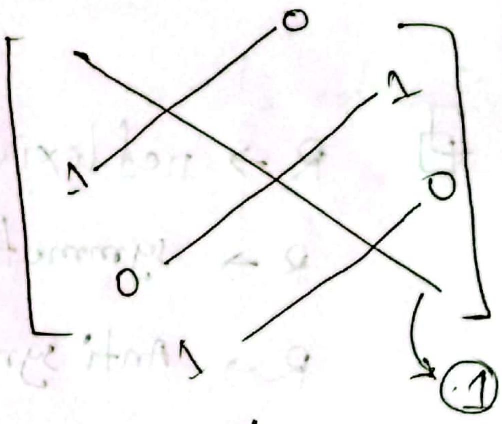
Matrix → * either, $m_{ij} = 1$ $(i \neq j)$. then $m_{ji} = 0$

* $m_{ij} = 0$ or $m_{ji} = 0$. when $i \neq j$

→ Antisymmetric



Symmetric



Antisymmetric

[illegible]

[illegible] relations,

[illegible]

[illegible] → Boolean Product

[illegible] R → reflexive শহলে R' reflexive

R → symmetric শহলে R' symmetric

R → anti symmetric শহলে R' antisymmetric

R → transitive শহলে R' transitive

[illegible] row, column interchange করলে

[illegible] পাওয়া যাবে।

[illegible]

[illegible]

④ $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is R reflexive, symmetric, and/or antisymmetric?

Sol^n:

Main diagonal all elements 1.

$\therefore$ R - reflexive.

$M_R = (M_R)^t$,
R - symmetric

as it is symmetric so its not antisymmetric.

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \quad M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_{S} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$M_{S \circ R} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$ then,

$$M_{R^{-1}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$M_{\overline{R}} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$M_{R^2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Closure of Relations

Reflexive Closure: Reflexive closure of a binary relation R on set A is the smallest reflexive relation of the set A that contains R.

(*) Reflexive clousure of R is denoted by R_r^*.

$$R_r^* = R \cup \{(a,a) \mid a \in A\}$$

Ex: $A = \{1, 2, 3\}$, $R = \{(1,1), (2,2), (2,3)\}$

$\therefore R_r^+ = \{(1,1), (2,2), (2,3), (3,3)\}$

Symmetric Closure: Symmetric closure of a binary relation R on a set A is the smallest symmetric relation on a set A that contains R.

$$R_s^{+} = R \cup \{(b,a) \mid (a,b) \in R\}$$

Ex: $A = \{0,1,2,3\}$, $R = \{(0,1),(1,1),(1,2),(2,0),(2,2),(3,0)\}$

Find symmetric closure of R.

Soln:

$$R_s^{+} = R \cup \{(b,a) \mid (a,b) \in R\}$$

$$= \{(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(1,0),(2,1),(0,2),(0,3)\}$$

Transitive closure: Transitive closure of a binary relation R on a set A is the smallest relation on a set A ~~for~~ that contain R.

$$R_t^+ = R \cup \{(a,c) \mid (a,b) \in R \wedge (b,c) \in R\}$$

Ex: $A = \{1, 2, 3\}$; $R = \{(1,1), (2,3), (3,1)\}$

Find transitive closure of R.

$$\therefore R_t^+ = \{(1,1), (2,3), (3,1), (2,1)\}$$

Ex: Let R be the relation $\{(a,b) \mid a \neq b\}$ on the set of integers. What is the reflexive closure of R?

Soln:

$$R = \{(a,b) \mid a \neq b\}$$

$$R_r^{+} = R \cup \{(a,a) \in A\}$$ [A is the set of all integers]

$$= \{(a,b) \mid a \neq b\} \cup \{(a,b) \mid a = b\}$$

This means all pairs of integers must be included in the reflexive closure of R

$$= \{(a,b) \mid a,b \in Z\}$$

$$= Z \times Z$$

Ex: Let R be the relation $\{(a,b) \mid a \text{ devides } b\}$ on the set of integers. What is the symmetric closure of R?

Soln:

$R = \{(a,b) \mid a \text{ derides } b\}$

A = set of integeres

$\therefore R_s^+ = R \cup \{(b,a) \mid (a,b) \in R\}$

$= \{(a,b) \mid a \text{ devides } b\} \cup \{(b,a) \mid a \text{ divides } b\}$

$= \{(a,b) \mid a \text{ devides } b\} \cup \{(a,b) \mid b \text{ devides } a\}$

$= \{(a,b) \mid a \text{ devides } b \text{ or } b \text{ devides } a\}$

Warshall's Algorithm :

* Warshall's algorithm is considered an efficient method in finding the transitive closure of a relation.

Ex: By using Warshall's algorithm, find the transitive closure of the relation.

$$R = \{(2,1), (2,3), (3,1), (3,4), (4,1), (4,3)\}$$

on set $A = \{1,2,3,4\}$

Sol^n: First, write R in matrix form,

$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times 4} \end{array}$$

There are, 4 elements in set A. Therefore, 4 steps are required in order to find the transitive closures of relation R [According to Warshall's Algorithm]

Step-1: we will consider 1st column and 1st row of the matrix as C_1 and R_1

* write all the position where 1 is present in column 1.

$$C_1 = \{2, 3, 4\}$$

* write all the position where 1 is present in row 1.

$$R_1 = \phi$$

$$C_1 \times R_1 = \phi$$

Therefore, no new ~~editions~~. additions.

Step-2: we will consider 2nd column and 2nd row of the matrix.

$$C_2 = \phi$$

$$R_2 = \{1, 3\}$$

$$C_2 \times R_2 = \phi$$

therefore, no new additions.

Step-3: 3rd column and 3rd rows

$$C_3 = \{2, 4\}$$

$$R_3 = \{1, 4\}$$

$$C_3 \times R_3 = \{(2,1), (2,4), (4,1), (4,4)\}$$

এখন এ pair গুলো পাওয়া গেছে main matrix এ ঐ pair গুলোতে 1 না থাকলে 1 বসাতে হবে।

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}_{4 \times 4} \rightarrow \text{new matrix.}$$

Step-4: 4th coloumn and 4th row

$$C_4 = \{2, 3, 4\}$$

$$R_4 = \{1, 3, 4\}$$

$$C_4 \times R_4 = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$$

∴ এ pair গুলো পাওয়া গেছে আগের matrix - এ যে খানে 1 না থাকলে 1 বসাতে হবে।

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 1 & 1 \\ 4 & 1 & 0 & 1 & 1 \end{array}_{4 \times 4} \rightarrow \text{New Matrix} \rightarrow R_t^+$$

$$\therefore R_t^+ = \{(2,1), (2,3), (2,4), (3,1), (3,3), (3,4), (4,1), (4,3), (4,4)\}$$

Equivalence Relation:

A relation R on a set A is an equivalence relation iff R is symmetric, reflexive and transitive.

Ex: $A = \{0, 1, 2, 3\}$

$R_1 = \{(0,0), (1,1), (2,2), (3,3)\}$

R_1 is reflexive,

R_1 is symmetric, cause, (a,b) and (b,a) included

R_1 is transitive, cause $(a=b)$ → for all elements

∴ R_1 is an equivalence relation.

$R_2 = \{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3)\}$

R_2 → is reflexive

→ is symmetric

→ is transitive

∴ R_2 is an equivalence relation.

$R_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3)\}$

$R_3 \rightarrow$ is reflexive
$\rightarrow$ isn't symmetric

So, not an equivalence Relation

$R_4 = \phi$

$R_4 \rightarrow$ isn't reflexive

So, not an equivalence Relation.

$R_5 = A \times A$

$R_5 \rightarrow$ is reflexive
$\rightarrow$ is symmetric
$\rightarrow$ is transitive

$\therefore R_5$ is an equivalence relation.

Q. Let us assume that R is a relation on the set of integers defined by aRb if and only if a-b is an integer. Prove that R is an equivalence relation.

Sol^n:

$A = \{\ldots -2, -1, 0, 1, 2 \ldots\}$ [set of all int]

R is defined on set A

aRb iff a-b is a integer.

(i) **Reflexivity:** $\forall a \in A \; (a,a) \in R$

$a - a = 0$, ~~is a n~~ 0 is a integer

so, it's reflexive.

(ii) **Symmetry:** $\forall a \forall b \in A \, [(a,b) \in R \rightarrow (b,a) \in R]$

if, a-b is an integer

then, b-a is also an integer.

so, it's symmetric.

$\forall a \forall b \forall c [(a,b) \in R \wedge (b,c) \in R) \to (a,c) \in R]$

(ii) Transitivity: If, $a-b$ is an ~~iteg~~ integer

$b-c$ is an integer

then, $a-c = (a-b)+(b-c)$ also an integer, so, it's transitive

so, R is a equivalence relation.

Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.

Soln: $R = \{((a,b),(c,d)) \mid ad = bc\}$, A = positive integers

(i) Reflexivity: $\forall a \in A, ((a,a) \in R)$

$\forall (a,b) \in A (((a,b),(a,b)) \in R)$

$\therefore (a,b),(a,b) \in R$ means $ab = ba$ which is true,

$\therefore$ R is reflexive.

(ii) Symmetry: $\forall a, b \in A \, ((a,b) \in R \rightarrow (b,a) \in R)$

$\forall (a,b),(c,d) \in A \, (((a,b),(c,d)) \in R \rightarrow ((c,d),(a,b)) \in R)$

$((a,b),(c,d)) \in R$ means,

$$ad = bc \equiv da = cb$$
$$\equiv c\cancel{b} = da$$
$$\downarrow$$
$$(c,d),(a,b)$$

So, R is symmetric.

(iii) Transitivity: ~~$\forall a \forall b \forall c \in A \, ((a,b) \in R \Rightarrow$~~

$\forall_a \forall_b \forall_c \in A \, (((a,b) \in R \wedge (b,c) \in R) \rightarrow (a,c) \in R)$

$\forall (a,b),(c,d),(e,f) \in A \, ((((a,b),(c,d)) \in R) \wedge ((c,d),(e,f)) \in R)$

$\rightarrow ((a,b),(e,f)) \in R)$

$((a,b),(c,d)) \in R \rightarrow$ means $ad = bc \Rightarrow a = \dfrac{bc}{d}$

$((c,d),(e,f)) \in R \rightarrow$ mens $cf = de \Rightarrow f = \dfrac{de}{c}$

$\therefore af = \dfrac{b\cancel{c}}{\cancel{d}} \times \dfrac{\cancel{d}e}{\cancel{c}} = be$

$\therefore (a,b),(e,f)$

$\therefore$ it's transitive

$\therefore$ R is an equivalence relation.

Equivalence class

Equivalence class is the name given to a subset of some equivalence relation R which includes all elements that are equivalent to each other.

Let R be an equivalence relation on a set A. The set of all elements which are related to an element x of set A is called the equivalence class of x.

$$\underbrace{[x]}_{\text{Equivalence class of } x \in A} = \{ y \mid (x, y) \in R \}$$

Ex: let $A = \{1, 2, 3, 4, 5\}$

$R = \{(a,b) \mid a+b \text{ is even}\}$

First, check equivalence,

1. Reflexive, $a+a=2a$ is even

2. Symmetric; If $a+b$ → is even, then $b+a$ is also even

3. Transitive:

If $a+b$ is even. and $b+c$ is even

$a+c$ must be even.

So, R is an equivalence relation.

$[1] = \{1, 3, 5\}$ → cause

$\left.\begin{array}{l} 1+1=2 \\ 1+3=4 \\ 1+5=6 \end{array}\right\}$ even

$[2] = \{2, 4\}$

$[3] = \{1, 3, 5\}$

$[4] = \{2, 4\}$

$[5] = \{1, 3, 5\}$

Equivalence ~~eal~~ class of elements 1, 3 and 5 are same and equivalence class of elements 2, 4 are same.

* So, any elements out of 1, 3, 5 can be chosen as a representive of the ~~e~~ equivalence class $\{1,3,5\}$

* Also, any elements out of 2, 4 can be choosen for $\{2,4\}$

$\therefore$

$[1] = \{1, 3, 5\}$

$[2] = \{2, 4\}$

$\therefore$ Relation R has two equivalce class.

Q. $a \equiv b \pmod n$ is an equivalence relation.

~~proof~~ : $R = \{(a,b) \mid a \equiv b \pmod n\}$

proof:

Reflexivity: $\forall a \in A((a,a) \in R)$

$\therefore$ $a \equiv a \pmod n$

$\therefore$ $a - a = 0$ divisible by n.

$\therefore$ so, it's reflexive.

Symmetry: $\forall a,b \in A((a,b) \in R \rightarrow (a,b) \in R)$

$(a,b) \in R$, $a \equiv b \pmod n$

$(b,a) \in R$, $b \equiv a \pmod n$

if, $a-b$ is divisible by n

then, $b-a$ is also divisible by n

$\therefore$ so, it's symmetry.

<u>Transitivity:</u> $\forall a,b,c \in A[((a,b)\in R \wedge (b,c)\in R) \rightarrow (a,c)\in R]$

$(a,b)\in R \rightarrow \quad a\equiv b \pmod n \rightarrow a-b=nk$

$(b,c)\in R \rightarrow \quad b\equiv c \pmod n \longrightarrow b-c=nl$

$\therefore (a-b)+(b-c) = n.k+nl$

$\Rightarrow a-c=n(k+l)=nk'$

$\Rightarrow a-c=nk'$

$\therefore a\equiv c \pmod n$

$\downarrow$

(a,c)

$\therefore$ so, it's transitive.

$\therefore$ Congruent modulo n is an equivalence relation.

Ex: What is the equivalence class of 2 with respect to congruence modulo 5?

Soln:

$$x \equiv 2 \pmod 5$$

$$x = \{\ldots -8, -3, 2, 7, 12, \ldots\}$$

$$\therefore [2] = \{\ldots, -8, -3, 2, 7, 12 \ldots\}$$

The equivalence classes of congruence modulo n are called congruent classes modulo n. It's denoted by $[X]_m$.

$$\therefore [X]_m = \{\ldots, x-2m, x-m, x, x+m, x+2m \ldots\}$$

Equivalence Classes and Partitions

Let R be an equivalence relation on a set A.
Following statements for elements a and b of set A are equivalent.

(i) aRb (ii) $[a]=[b]$. (iii) $[a]\cap[b]\neq\phi$

proof:

method: if (i) implies (ii), (ii) implies (iii), (iii) implies (i), then (i), (ii), and (iii) are equivalent.

(a) (i) $\rightarrow$ (ii)

if aRb then $[a]=[b]$

assume, aRb

$\therefore$ we have to show $[a]\subseteq[b]$ and $[b]\subseteq[a]$

$\therefore$ Let, $x\in[a]$

$x\in[a]\rightarrow aRx$

aRb means bRa [equivalence]

$\therefore$ $aRx\rightarrow bRx$

$\therefore$ $x\in[b]$

$\therefore$ $[a]\subseteq[b]$

$y\in[b]$

$y\in[b]\rightarrow bRy$

$\therefore$ $bRy\rightarrow aRy$ [transitive]

$\therefore$ $y\in[a]$

$\therefore [b]\subseteq[a]$

So, $[a]=[b]$

b] (ii) $\rightarrow$ (iii)

If $[a] = [b]$ then $[a] \cap [b] \neq \phi$

assume, $[a] = [b]$

$$[a] \cap [b] = [a] \cap [a]$$
$$\neq \phi$$

c] (iii) $\rightarrow$ (i)

If, $[a] \cap [b] \neq \phi$ then, $[a] = [b]$

assume, $[a] \cap [b] \neq \phi$

so, $x \in [a]$ and $x \in [b]$; for some $x \in A$

$x R a$ and $x R b$

$\therefore x R a \rightarrow a R x$ (symmetry)

$a R x$ and $x R b \rightarrow a R b$ (transitivity)

$\therefore$ (i), (ii), (iii) are equivalent.

<u>conl:</u> Equivalent classes of two elements of set A are either identical or disjoint.

* $[a]_R \cap [b]_R \neq \phi$ when $[a]_R \neq [b]_R$

* $\bigcup_{a \in R} [a] = A$

because no matter what, every element exist in its own equivalence class because of reflexivity

$$[\forall a \in A \ (a,a) \in R]$$

* Therefore, equivalence classes of set A form partition of A.

* Partition of set A is a collection of all disjoint non-empty subsets A_i of A where $i \in I$ (I is the index set)

* $A_i \neq \phi \quad \forall i \in I$

* $A_i \cap A_j = \phi$ when $i \neq j$

* $\bigcup_{i \in I} A_i = A$

Ex: $A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(a, b) \mid a+b \text{ is even}\}$

R is an equivalence relation

$A_1 = \{1, 3, 5\} = [1]$

$A_2 = \{2, 4, 6\} = [2]$

$A_1 \cap A_2 = \phi$

$A_1 \neq \phi, A_2 \neq \phi$

$A_1 \cup A_2 = \{1, 2, 3, 4, 5, 6\} = A$

therefore, $[1]$ and $[2]$ forms the partition of A.

Ex: Show that the relation R on the set of all bit strings such that sRt iff s and t contain the same number of 1s is an equivalence relation. Also, determine the equivalence class of the bit strings 011 for the equivalence relation R.

Sol^n: a) A = set of all bit strings

$R = \{(s,t) \mid s \text{ and } t \text{ have the same number of 1s}\}$

(i) **Reflexivity:** $\forall s \in A\ (s,s) \in R$

$(s,s) \in R$ means s and s have same number of 1s

$\therefore$ R is reflexive.

(ii) **Symmetry:** $\forall s,t \in A\ [(s,t) \in R \rightarrow (t,s) \in R]$

$(s,t) \in R$ means s and t have same no. of 1s

$(t,s) \in R$ means, s and t have same no. of 1s.

$\therefore$ R is symmetric.

(iii) Transitivity: $\forall s,t,u \in A\,[((s,t) \in R \wedge (t,u) \in R) \rightarrow (s,u) \in R]$

$(s,t) \in R \rightarrow$ s and t have same no. 1s

$(t,u) \in R \rightarrow$ t and u have same no. 1s

$\therefore$ (s,u)

s and u have same no. 1s

$\therefore$ R is transitive

$\therefore$ R is equivalence relation.

b)

[011] = set of all bit strings having exactly two 1s