Counting

The Product Rule: Suppose that a procedure can be broken down into a sequence of two tasks.

If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then are $n_1 n_2$ ways to the procedure.

Ex: There are 32 computers. Each computers has 24 ports. How many different computer ports are there?

Soln: First task → choosing computer → 32 ways

then → " ports → 24 ways

∴ There are $= 32 \cdot 24 = 768$ different ports.

Ex: How many ~~bit~~ different bit strings of length seven are there?

Soln:

Each of seven bits can be choosen in two ways, because it's either 0 or 1.

∴ Product rule, there are

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^7 = 128$ different bit strings.

Ex: How ~~can~~ many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even there ~~any~~ to are obscene?

Sol^n: 3 ~~digits~~ letters can be ~~sose~~ choose,

$26 \times 26 \times 26$ ways

3 digits can be choose,

$10 \times 10 \times 10$ ways

$\therefore$ possible license plates are $= 26 \times 26 \times 26 \times 10 \times 10 \times 10$

$= 17,576,000$.

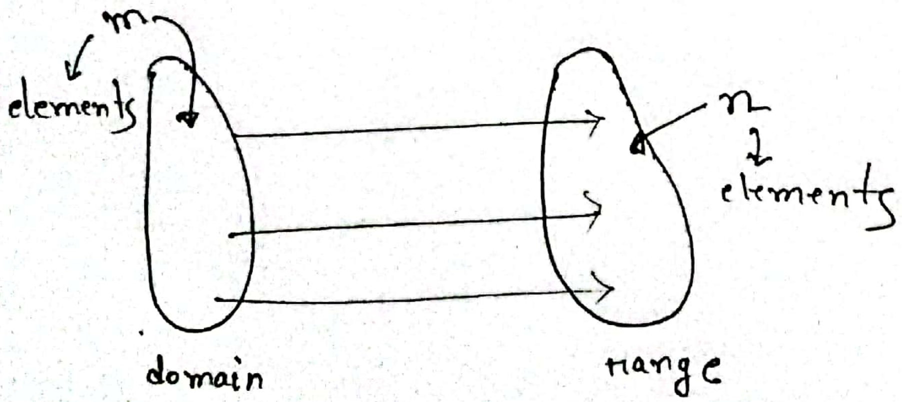
Counting Functions: How many functions are there from a set with m elements to a set with n elements?

Soln: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain.

So, using product rule,

~~no~~ number functions $= n \cdot n \cdot \text{-----}$ m সংখ্যকবার

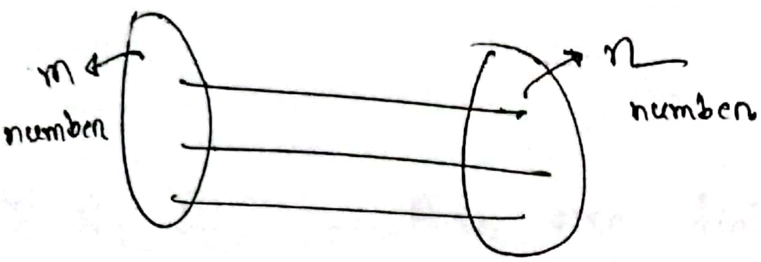
$$\boxed{\text{number functions} = n^m}$$



Counting One-to-One Functions:

How many one-to-one functions are there from a set with m elements to one with n elements?

Soln:



condition:

$$\boxed{m \leq n}$$

$\therefore$ Number of one-to one

$$\text{functions} = n(n-1)(n-2) \cdots (n-m+1).$$

Counting Subsets of a Finite Set:

$$\boxed{|P(S)| = 2^{|S|}}$$

$|S| \rightarrow$ length of bit string

If $A_1, A_2, \ldots, A_m$ are finite sets.

$$\boxed{|A_1 \times A_2 \times \cdots \times A_m| = |A_1| \cdot |A_2| \cdots |A_m|}$$

The sum rule: If a tasks can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are n_1+n_2 ways to do the task.

Generalized sum rule:

If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, And ~~none~~ no two of these tasks can be done at the same time, then there are $n_1+n_2+\cdots+n_m$ ways to do one of these tasks.

Ex: A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

Sol^n: $n_1 = 23$, $n_2 = 15$, $n_3 = 19$

$\therefore$ possible project $= n_1 + n_2 + n_3$

$= 23 + 15 + 19$

$= 57$

$$\boxed{|A_1 \cup A_2 \cup \cdots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|}$$

Ex: Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password contain at least one digit. How many possible passwords are there?

Soln: 6 characters $\longrightarrow P_6$

7 " $\longrightarrow P_7$

8 " $\longrightarrow P_8$

letter + digits $= 26 + 10 = 36$

letter $= 26$

letter + digits থেকে only letter -এর টা বিয়োগ করলে $\longrightarrow$ desire result পাবো।

$P_6 = 36^6 - 26^6 = 1,867,866,560$

$P_7 = 36^7 - 26^7 = 70,332,353,920$

$P_8 = 36^8 - 26^8 = 2,612,282,842,880$

$P = P_6 + P_7 + P_8 = 2,684,483,063,360$

Subtraction rule

If a task can be done in either n_1 ways or n_2 ways, then number of ways to do the task is n_1+n_2 minus the number of ways to do the task that are common to the different ways.

$$\boxed{|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|}$$

Ex: How many bit strings of length 8 either start with a 1 or end with 00?

Sol^n:

case 1: $\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_}_{7} \rightarrow 2^7$

$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{6}\ \underline{0}\ \underline{0} \rightarrow 2^6$

$\underline{1}\ \underbrace{\_\ \_\ \_\ \_\ \_}_{5}\ \underline{0}\ \underline{0} \rightarrow 2^5$

$\therefore$ Number of the ways $= 2^7 + 2^6 - 2^5$

$= 160$

Division rule

→ It task has n- ways to do it. But for a specific way w, has d ways for do it.

$\therefore$ $\boxed{\text{number of ways for this task} = \frac{n}{d}}$

যদি n সংখ্যক elements কে arrange করা যায় $n!$ way তে।

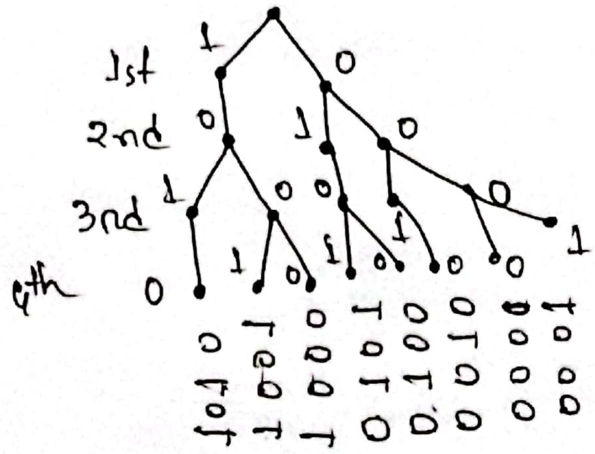
তার মধ্যে p সংখ্যক same হয় তাহলে,

$$\text{number way} = \frac{n!}{p!}$$

Tree Diagrams

How many bit strings of length four do not have two consecutive 1s?

Sol^n:



Piegon Hole Principle

If (N+1) or more objects are placed into N boxes then there is atleast one box containing two or more objects.

Generalized Pigeonhole Principle

If 'n' pigeon hole are occupied by kn+1 or more pigeons then atleast one pigeonhole is occupied by k+1 or more pigeon.

If N objects are placed into k boxes, then there is at least one box containing at least $\lceil \frac{N}{K} \rceil$ objects.

Ex: Find the minimum number of teachers in a college to be sure that four of them are born in the same month.

Soln:

$n = 12$ (month)

(min)

$k+1 = 4$

$\Rightarrow k = 3$

$\therefore$ minimum teachers $= kn+1$

$= 3 \times 12 + 1$

$= 37$

Ex: How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards for the same suit are selected?

Soln: Approach-1

box → 4

$n = 4$

$k+1 \geqslant 3$

$\therefore k = 3-1$

$= 2$

minimum, $kn+1$

$= 2 \times 4 + 1$

$= 9$

approach-2

$\left\lceil \frac{N}{4} \right\rceil \geq 3$

$\therefore N = 4 \cdot 2 + 1$

$= 9$

4 → box

Permutations

Theorem:

If n is a positive integer and r is an integer with $1 \leq r \leq n$.

$\therefore P(n, r) = {}^nP_r$

$$\therefore \boxed{{}^nP_r = n(n-1)(n-2) \cdots (n-r+1)}$$

$${}^nP_r = \frac{n!}{(n-r)!} \quad [0 \leq r \leq n]$$

Ex: How many ways are there to select a first-prize winner, a second prize winner, and a third prize winner from 100 different people who have entered a contest?

Soln:

$n = 100$, $r = 3$

$${}^{n}P_{r} = {}^{100}P_{3} = 970,200$$

Ex: How many permutation of the letters ABCDEFGH contain the string ABC?

Soln:

$$\underbrace{ABC}_{1} + \underbrace{D, E, F, G, H}_{5}$$

$\therefore$ $6! = 720$.

Combinations

The number of r- combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$\boxed{C(n,r) = \frac{n!}{r!(n-r)!}}$$

$\boxed{r \leq n}$

$$C(n,n-r) = \frac{n!}{(n-r)!\,(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!\,(\cancel{n}-\cancel{n}+r)!}$$

$$= \frac{n!}{(n-r)!\,r!}$$

$$\therefore \boxed{\begin{gathered} C(n,r) = C(n,n-r) \\ {}^{n}C_{r} = {}^{n}C_{n-r} \end{gathered}}$$

Ex: How many poker hands of five cards can be dealt from a standard deck of 52 cards?

Soln:

$$^{n}C_{r} = {}^{52}C_{5} = \frac{52!}{5!\,(52-5)!} = 2598960$$

Binomial Theorem

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

Ex: what is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Soln:

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} \cdot (-3y)^{j}$$

$\therefore$ $x^{12}y^{13}$ $\rightarrow$ coefficient,

$$^{25}C_{13} \cdot (2)^{12} \cdot (-3)^{13}$$

$$= -\frac{25!}{13!\,12!} \times 2^{12} \times 3^{13}$$

$\boxed{\square}$ $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $\quad n \rightarrow$ nonnegative integers

proof:

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^k \cdot 1^{n-k}$$

$n \rightarrow$ positive integers

$$= \sum_{k=0}^{n} \binom{n}{k}$$

$\boxed{\square}$ $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ $\quad n \rightarrow$ nonnegative integers

proof:

$$0 = 0^n = ((-1)+1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \cdot 1^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (-1)^k$$

Q) $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

<u>proof:</u> $3^n = (1+2)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot 1^{n-k} \cdot 2^k$

$$= \sum_{k=0}^{n} \binom{n}{k} \cdot 2^k$$

Pascal Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$n, k \to$ positive integers

$\boxed{n \geq k}$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

Vandermonde's Identity

Let, m, n and r be nonnegative integers and r not exceeding either m or n. Then,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$ $n \rightarrow$ nonnegative integers

Proof:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{n-k}\binom{n}{k} \qquad \binom{n}{n-k} = \binom{n}{k}$$

$$\cancel{\sum_{k=0}^{n} \binom{n+n}{n}}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2$$

Permutation with Repetition

Theorem: The number of r-permutations of a set of n objects with repetition ~~to~~ allowed is n^r.

Ex: How many strings of length 5 can be formed from the uppercase letters of English alphabet?

Sol^n:

$n = 26$

$r = 5$ because we repeat alphabet in this problem.

$\therefore$ strings possible $= 26^5$

Combinations with Repetition

Theorem: There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when repetition of elements is allowed.

$n \to$ elements (different)
$r \to$ combination

Ex: Suppose that a cookie shop has four different kinds of cookies. How many different ways can six cookies be chosen?

Sol^n:

$n = 4$, $r = 6$

$$\therefore \text{ways} = C(n+r-1, r)$$
$$= C(4+6-1, 6)$$
$$= C(9, 6)$$
$$= {}^{9}C_{6}$$
$$= 84$$

Ex: How many solutions does the equation,

$$x_1 + x_2 + x_3 = 11$$

Sol^n: $n = 3$, $r = 11$

no of solutions $= C(3+11-1, 11)$

$= C(13, 11)$

$= {}^{13}C_{11}$

$= 78$

Combinations and Permutations Formula

Type	Repetition	Formula
r-permutations	No	$^{n}P_{r} = \frac{n!}{(n-r)!}$
r-combinations	No	$^{n}C_{r} = \frac{n!}{r!\,(n-r)!}$
~~pe~~ r-permutations	Yes	n^{r}
r combinations	Yes	$^{n+r-1}C_{r} = \frac{(n+r-1)!}{r!\,(n-1)!}$

The number of different permutations of n objects, where n_1 is same type 1, n_2 is same type 2, ---- n_k is same type k. is

$$\boxed{{}^{n}C_{n_1} \times {}^{n-n_1}C_{n_2} \times \cdots \times {}^{n-n_1-n_2-\cdots-n_k}C_{n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}}$$

Ex: How many different strings can be made by ordering of the letters of the word SUCCESS.

Soln:

s → 3, c → 2

Total → 7

$$\therefore \text{strings no} = \frac{7!}{2!\,3!}$$

$$= 420.$$

Distinguishable objects and distinguishable boxes:

Distinguishable → different.

$n \rightarrow$ distinguishable objects

$k \rightarrow$ distinguishable boxes

$$\frac{n!}{n_1!\,n_2!\cdots n_k!}$$

[n_i objects places into box i.

$i = 1, 2, \cdots k$

Indistinguishable ~~boxes~~ objects and distinguishable boxes:

$r \rightarrow$ indistinguishable

$n \rightarrow$ distinguishable

ways $= C(n+r-1, n-1) = C(n+r-1, r)$