Introduction to Sets

Some symbol:

$N \to$ Natural Numbers $\{1, 2, 3, 4 \ldots\}$

$W \to$ whole Numbers $\{0, 1, 2, 3, \ldots\}$

$Z \to$ Integers $\{\ldots, -2, -1, 0, 1, \ldots\}$

$Q \to$ Rational $\to \frac{a}{b}$, $a, b \in Z$, $b \neq 0$,
$\frac{a}{b}$ is in lowest term

$R \to$ Real Numbers.

$C \to$ Complex Numbers $\to a+bi$

$Z^+ \to$ positive integers.

$Z^- \to$ Negative integers

Set Notation:

Rostere Notation: $S = \{1, 2, 3, 4, 5\}$

→ Discrete Sets

↓

(Countable)

* Not countable / continuous → $0 \leq x \leq 1$

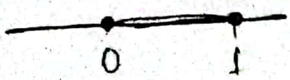
Set-Buildere Notation:

$$S = \{ x \mid x \in N,\ x \leq 5 \}$$

$\mid$ ↓ such that

Intereval Notation:

$$B = \{ x \mid 0 \leq x \leq 1 \}$$
$$= [0, 1]$$



$$M = \{ x \mid 0 < x \leq 1 \}$$
$$= (0, 1]$$

Universal Set: The set of all elements under consideration

Empty set: A set with no elements

ϕ, $\{ \}$

Set Equality:

Sets are equal if and only if or iff they have the same elements.

$$\boxed{\forall x\,(x \in A \leftrightarrow x \in B)}$$

Notation: $A = B$

$A = \{0, 1, 1, 3, 4, 4\}$

$B = \{0, 1, 3, 4\}$

$C = \{0, 1, 3, 4, 5\}$

$A = B$, $A \neq C$

* order matters করে না
* Same elements multiple times থাকতে পারে

Subsets: A set A is a subset of B iff every element of A is also an element of B.

$$\boxed{\forall x\,(x \in A \rightarrow x \in B)}$$

Notation: $A \subseteq B$

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4, 5\}$

Show $A \subseteq B$: Show every element of A belongs to B. i.e.

If $x \in A$ then $x \in B$.

Show $A \not\subseteq B$: Show x belongs to A but not set B.

$\exists x \in A \rightarrow x \notin B$

Show $A \subseteq B$ and $B \subseteq A$: Show $A = B$

Propere subsets:

If $A \subseteq B$ but $A \neq B$, meaning B contains an Element that not contained in A, then A is a propere subset of B.

$$\boxed{\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)}$$

Notation: $A \subset B$

$A = \{1, 2, 3\}$

$B = \{1, 2, 3, 4\}$

$A \subseteq B$

$B \nsubseteq A$

$A \neq B$

$\therefore A \subset B$

Cardinality : (size of a Set)

The numbers of distinct elements of a set.

Notation : $|A|$

$A = \{1, 2, 3, 3, 3, 5\}$

$\therefore |A| = 4$

$|\varphi| = 0$

Powers Sets :

The set of all subsets of a set.

Notation: $P(A)$

$A = \{0, 1, 2\}$

$P(A) = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\},$
$\{1, 2\}, \{0, 1, 2\}\}$

$|P(A)| = 8$

* cardinality of powers set of a set with n elements is 2^n.

Tuples:

An ordered n-tuple is an ordered collection that has a_1 as its first element, a_2 as its second and so on until a_n.

Notation: $(a_1, a_2, \cdots, a_n)$

Ordered pairs (a_1, a_2)

$(5,2) \neq (2,5)$

Cartesian Product:

The set of ordered pairs (a, b) where $a \in A$ and $b \in B$, resulting from $A \times B$

Notation: $\boxed{A \times B = \{(a,b) \mid a \in A \wedge b \in B\}}$

$A = \{0, 1\}$

$B = \{2, 3, 4\}$

$A \times B = \{(0,2), (0,3), (0,4), (1,2), (1,3), (1,4)\}$

$\therefore A^n = \{(a_1, a_2, a_3, \ldots) \mid a_i \in A \text{ for } i = 1, 2, \ldots n\}$

$A \longrightarrow m$ elements

$B \longrightarrow n$ elements

$\therefore A \times B \longrightarrow mn$ elements

$A^n \longrightarrow m^n$ elements

Truth Sets and Quantifiers:

A truth set of P is the set of elements x in D such that P(x) is true.

Notation: $\{x \in D \mid P(x)\}$

$D = \mathbb{Z}$

$P(x):\ |x| = 3$

$\{-3, 3\}$.

Union of Sets:

Let A, B be sets. The union of sets A and B is

$$\boxed{A \cup B = \{x \mid x \in A \vee x \in B\}}$$

$A = \{1, 4, 7\}$

$B = \{4, 5, 6\}$

$A \cup B = \{1, 4, 5, 6, 7\}$

Inclusion – Exclusion: $\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$

Intersection of sets:

Let A, B be sets. The intersection of A and B is —

$$\boxed{A \cap B = \{x \mid x \in A \wedge x \in B\}}$$

$A = \{1, 4, 7\}$

$B = \{4, 5, 6\}$

$\therefore A \cap B = \{4\}$

☐ If $A \cap B = \emptyset$, then A and B are said to be disjoint.

Complement :

Let A be a set. The complement of set A with respect to U is U − A.

$$\boxed{\overline{A} = A^c = \{x \in U \mid x \notin A\}}$$

$U = \{0, 1, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{1, 4, 7\}$, $B = \{4, 5, 6\}$

$\therefore \overline{A} = \{0, 3, 5, 6, 8, 9\}$

$\overline{B} = \{0, 1, 3, 7, 8, 9\}$

$\overline{A \cup B} = \{0, 3, 8, 9\}$

$\overline{A \cap B} = \{0, 1, 3, 5, 6, 7, 8, 9\}$

Difference of sets:

Let A and B be sets. The difference of A and B is the set containing elements of A that are not in B.

$$\boxed{\begin{aligned} A-B &= \{x \mid x \in A \wedge x \notin B\} \\ &= A \cap \overline{B} \end{aligned}}$$

$A = \{1, 4, 7\}$

$B = \{4, 5, 6\}$

$A - B = \{1, 7\}$

$U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$\overline{B} = \{0, 1, 2, 3, 7, 8, 9\}$

$\therefore A \cap \overline{B} = \{1, 7\}$

Set Identities:

* Identity Laws: $A \cup \emptyset = A$
$A \cap U = A$

* Domination laws: $A \cup U = U$
$A \cap \emptyset = \emptyset$

* Idempotent Laws: $A \cup A = A$
$A \cap A = A$

* Complementation law: $(\bar{\bar{A}}) = A$

* Commutative laws: $A \cup B = B \cup A$
$A \cap B = B \cap A$

* Associative laws: $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

* Distributive Laws: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C$

* De Morgan's laws: $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cup B} = \bar{A} \cap \bar{B}$

* Absorption Laws: $A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

* Complement Laws: $A \cup \bar{A} = U$

$A \cap \bar{A} = \phi$

Methods of Identity Proof:

1. Prove each set in the identity is a subset of the others.

2. Use propositional logic.

3. Use a membership table showing the same combination of sets do or don't belong to the identity.

☐ Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ by showing $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$ and $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$

proof idea
$A \subseteq B$
$B \subseteq A$
$A = B$

Solⁿ:

$x \in \overline{A \cap B}$ → By assumption

$= x \notin A \cap B$ → Defⁿ of complement

$= \neg((x \in A) \wedge (x \in B))$ → Defⁿ of Intersection

$= \neg(x \in A) \vee \neg(x \in B)$ → De Morgan for Propositional logic

$= (x \notin A) \vee (x \notin B)$ → Defⁿ of Negation

$= x \in \bar{A} \vee x \in \bar{B}$ → Defⁿ of comp.

$= x \in \bar{A} \cup \bar{B}$ → Defⁿ of union

$x \in \bar{A} \cup \bar{B}$ $\rightarrow$ By assumption

$= (x \in \bar{A}) \vee (x \in \bar{B})$ $\rightarrow$ Def^n of union

$= x \notin A \vee x \notin B$ $\rightarrow$ Def^n of comp.

$= \neg(x \in A) \vee \neg(x \in B)$ $\rightarrow$ Def^n of Negation

$= \neg((x \in A) \wedge (x \in B))$ $\rightarrow$ De Morgan's Law of Prop logic

$= \neg(x \in A \cap B)$ $\rightarrow$ Def^n of intersection

$= x \in \overline{A \cap B}$ $\rightarrow$ Def^n of comp.

$\therefore \overline{A \cap B} = \bar{A} \cup \bar{B}$

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ using set builder notation and propositional logic.

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$ → Def^n of comp.

$= \{x \mid \neg(x \in (A \cap B))\}$ → Def^n of $\notin$

$= \{x \mid \neg(x \in A \wedge x \in B)\}$ → def^n of Intersection

$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$ → De Morgan prop logic

$= \{x \mid x \notin A \vee x \notin B\}$ → Def^n of $\notin$

$= \{x \mid x \in \bar{A} \vee x \in \bar{B}\}$ → Def^n of comp

$= \{x \mid x \in \bar{A} \cup \bar{B}\}$ → Def^n of union

$= \bar{A} \cup \bar{B}$

Prove $\overline{A \cap B} = \bar{A} \cup \bar{B}$ using a membership table.

A	B	$A \cap B$	$\overline{A \cap B}$	$\bar{A}$	$\bar{B}$	$\bar{A} \cup \bar{B}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

$\therefore \quad \overline{A \cap B} = \bar{A} \cup \bar{B}$

Generalized Unions and Intersection:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

Computers Representation of Sets

* $\boxed{\text{Bit string}}$

↳ $x \in U$ then we put 1

$x \notin U$ the we put 0

$\therefore$ $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{\text{odd numbers}\}$

$\therefore A = \{1, 3, 5, 7, 9\}$

$\boxed{A \subset U}$

Bit string of $A = 1010101010$

$B \subset U$, $B = \{\text{even numbers}\}$

$= \{2, 4, 6, 8, 10\}$

$\therefore$ Bit string of $B = 0101010101$

$C \subset U$, $C = \{1,2,3,4,5\}$

Bit strings of $C = 1111100000$

Ex

(*) for Bit string complent set

we put 0 in place of 1
and 1 in place of 0

$\therefore$ $A = 1010101010$

$\bar{A} = 0101010101$

(*) $A = \{1,2,3,4,5\}$ are 1111100000

$B = \{1,3,5,7,9\}$ are 1010101010

union $1111100000 \vee 1010101010 = 1111101010$

$\{1,2,3,4,5,7,9\}$

intersection $1111100000 \wedge 1010101010 = 1010100000$

$\{1,3,5\}$

Multiset

Normal set, $A = \{1, 2, 3, 4\}$

multiset set, $B = \{1, 1, 1, 2, 2, 3, 3, 3, 4\}$

<u>Multiplicity:</u> How many times an element present in a set.

(Ex) $A = \{1, 1, 1, 2, 2, 3, 3, 3, 3, 4\}$

$Mul(1) \rightarrow 3$

$mul(2) \rightarrow 2$

$mul(3) \rightarrow 4$

$mul(4) \rightarrow 1$

Union: maximum occurrence of each element of set A and B.

Intersection: minimum occurrence of each element of set A and B.

Ex) $A = \{1, 1, 1, 2, 2, 3, 3, 3, 4\}$

$B = \{1, 1, 2, 3\}$

$A \cup B = \{1, 1, 1, 2, 2, 3, 3, 3, 4\}$

$A \cap B = \{1, 1, 2, 3\}$

$m_A(1) = 3$
$m_B(1) = 2$
$m_A(2) = 2$
$m_B(2) = 1$
$m_A(3) = 3$
$m_B(3) = 1$
$m_A(4) = 1$
$m_B(4) = 0$

④ **Summation:** $m_{(A)}(x) + m_B(x)$

$A = \{a, a, a, b, b, c\}$

$B = \{a, a, b, c\}$

$A + B = \{a, a, a, a, a, b, b, b, c, c\}$

ⓐ $3 + 2 = 5$

ⓑ $2 + 1 = 3$

ⓒ $1 + 1 = 2$

⑤ **Difference:** $\boxed{m_{(A)}(x) - m_B(x) \geq 1}$

$A - B = \{a, b\}$

ⓐ $3 - 2 = 1 \geq 1$

ⓑ $2 - 1 = 1 \geq 1$

ⓒ $1 - 1 = 0 \ngeq 1$

$A = \{1, 1, 1, 2, 3\}$

$B = \{1, 2, 2, 3\}$

$\therefore A - B = \{1, 1\}$

① $3 - 1 = 2 \geq 1$

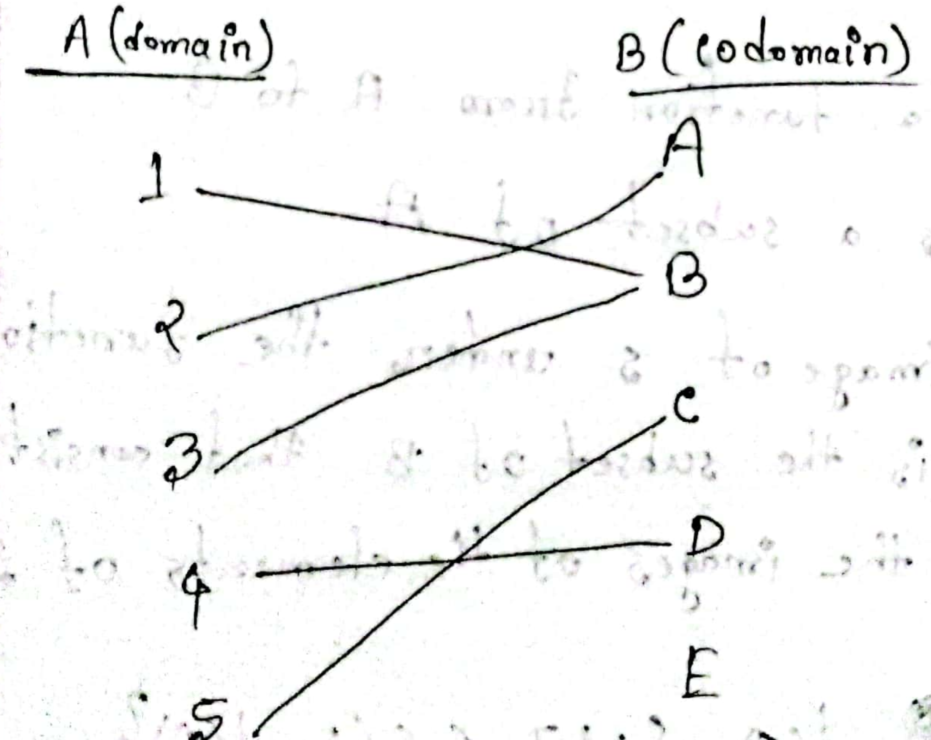
② $1 - 2 = -1 \ngeq 1$

③ $1 - 1 = 0 \ngeq 1$

Functions

A function f from A to B. denoted $f: A \to B$ assign each element of A to exactly one element of B. (একই x এর জন্য একাধিক y থাকতে পারবে না)

* Functions are sometimes also called mappings or transformations.



$f(1) = B$

image: B is the image of 1 under f

pre-image: $\{1, 3\}$ is the pre-image of B under f

Let f_1 and f_2 be functions from A to R. f_1+f_2 and f_1f_2 are also functions from A to R.

$\therefore x \in A$.

$$(f_1+f_2)(x) = f_1(x)+f_2(x)$$

$$(f_1f_2)(x) = f_1(x)\cdot f_2(x)$$

f is a function from A to B

S is a subset of A

The image of S under the function f is the subset of B that consists of the images of the elements of S.

$$f(S) = \{t \mid \exists s \in S\ (t = f(s))\}$$

Ex: $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$

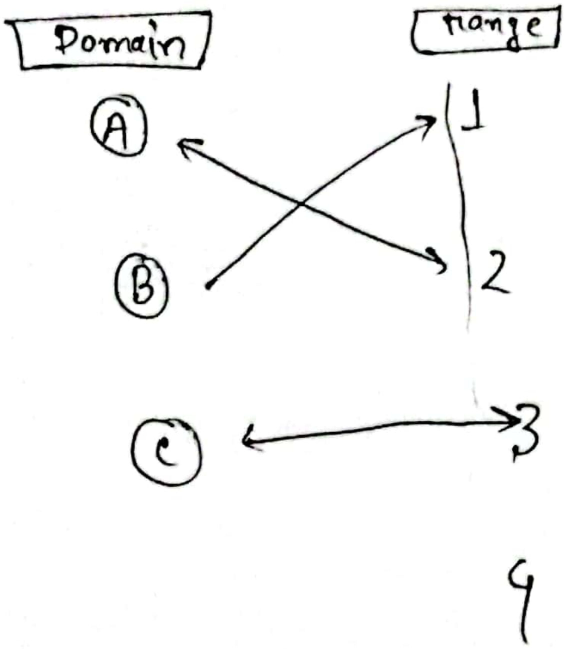
f defined by $f(1) = a$, $f(2) = b$, $f(3) = c$

Let, $S = \{1, 2\}$

Then the image of S under f is $f(S) = \{a, b\}$

(Injective) One-to-one functions (এক-এক)

Each value in the range corresponds to exactly one element in the domain.

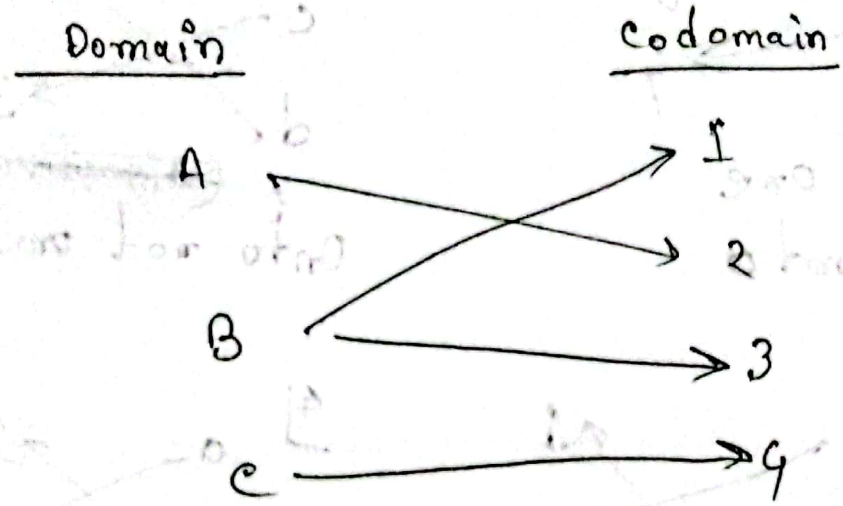


$$\forall a \forall b \left((a \neq b) \rightarrow (f(a) \neq f(b)) \right)$$

$$\forall a \forall b \left((f(a) = f(b)) \rightarrow (a = b) \right)$$

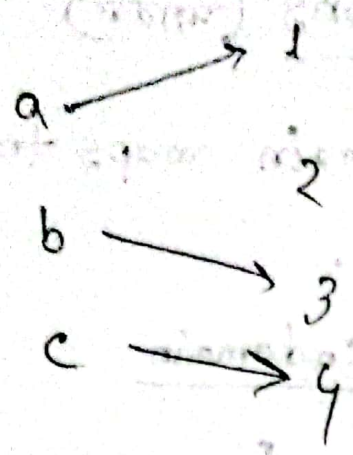
(Subjective) Onto functions ([illegible])

Every element in the codomain maps to at least one element in the domain.



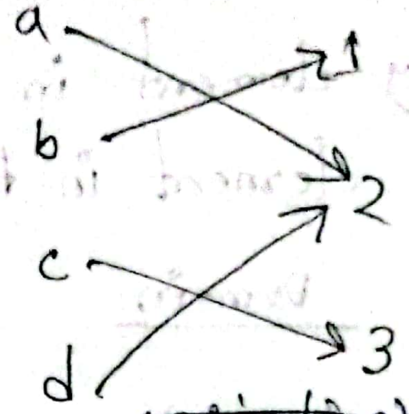
$$\forall y \, \exists x \, (f(x) = y)$$

1]



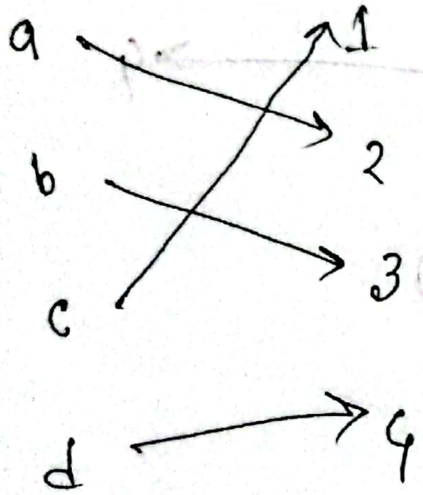
One to One
not onto

2]



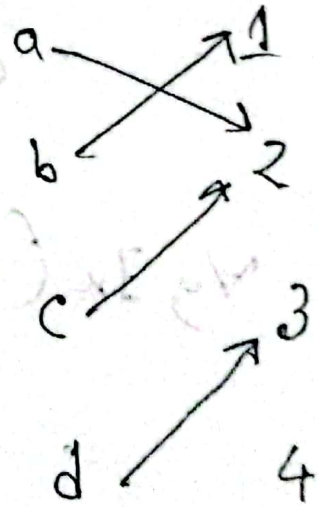
~~(Bijective)~~
Onto not one-to-one

3]



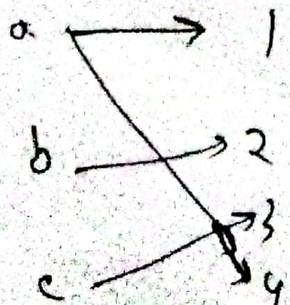
(Bijective)
One-to-One and
Onto

4]



Neither one to one
nor onto

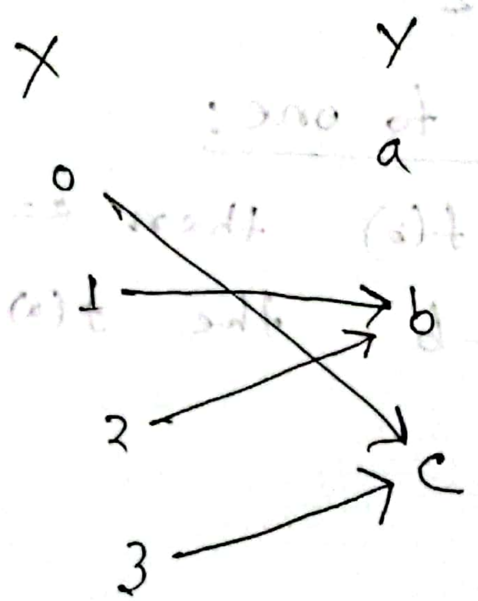
5]



Not a function

☐ Let f be a function from $X = \{0, 1, 2, 3\}$ to $Y = \{a, b, c\}$ defined by $f(0) = c$, $f(1) = b$, $f(2) = b$ and $f(3) = c$. Is $f: X \to Y$ either one-to-one or onto?

Soln:



This is not one-to-one and not onto.

Is the function $f(x) = x^2$ from the set of integers to the set of integers either one-to-one or onto?

S.M:

$f: Z \to Z$

for proof one to one:

If $f(a) = f(b)$ then $a = b$

or if $a \neq b$ the $f(a) \neq f(b)$

$a = 2$

$b = -2$

$\therefore f(a) = 4$

$f(b) = 4$

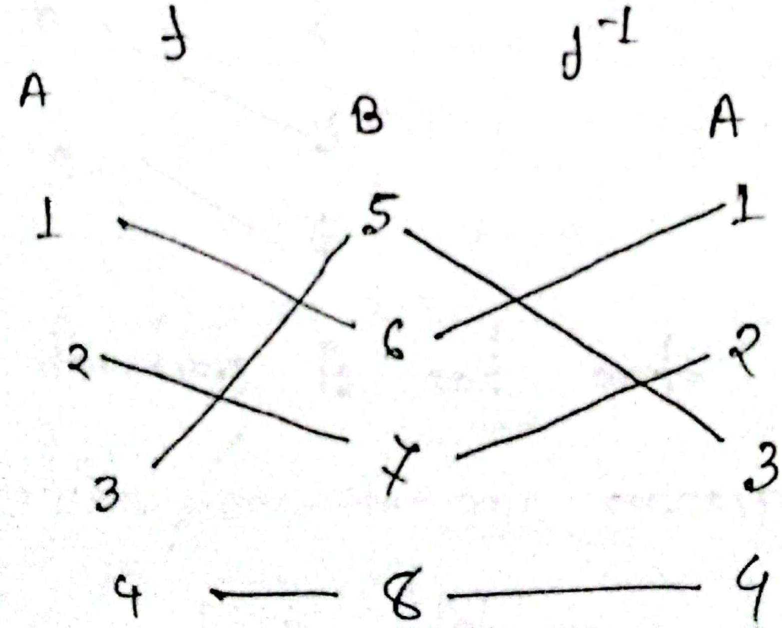
$\therefore a \neq b$ but $f(a) = f(b)$, so, it's not one-to-one.

Codomain $\in Z$, but $f(x) \neq -1$, so it's not onto.

Inverse Functions

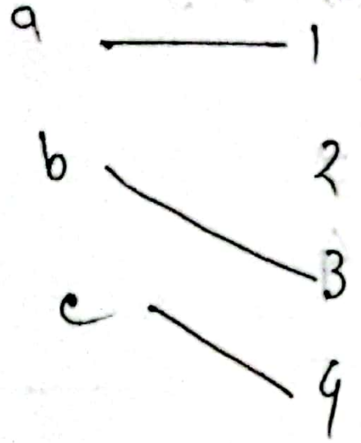
Let f be a bijection (both one-to-one and onto) from set A to B. The inverse of f, f^{-1}, is the function from B to A defined as

$f^{-1}(y) = x$ iff $f(x) = y$.



If f is the function from $\{a, b, c\}$ to $\{1, 2, 3, 4\}$ such that $f(a) = 1$, $f(b) = 3$ and $f(c) = 4$ is f invertible? If so, what is the inverse

Soln:



the function is not onto. so, it's not a bijection / one-to-one correspondence. so, it's not invertible.

Let $f: Z \rightarrow Z$ such that $f(x) = x+3$. Is f invertible? If, so, what is the inverse?

Soln:

one-to-one:

has to be $f(a) = f(b) \rightarrow a = b$

Let, $f(a) = 7$ | $f(b) = 7$

$\Rightarrow a+3 = 7$ | $\Rightarrow b+3 = 7$

$\Rightarrow a = 4$ | $\Rightarrow b = 4$

$\therefore a = b$

$\therefore$ so, f is one-to-one.

onto:

for every codomain the domain exists.

~~only one element~~

codomain

4 — 7

5 — 8

$\therefore$ so, it's onto

$y = f(x)$ | $y = x+3$

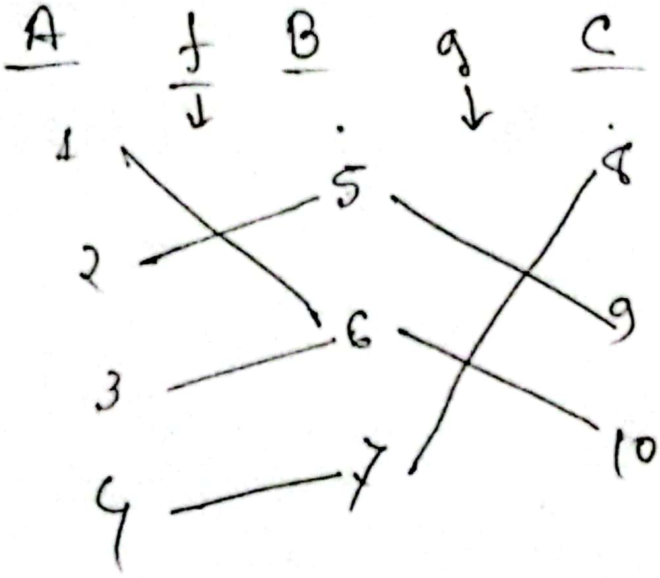
$\Rightarrow f^{-1}(y) = x$ | $\Rightarrow x = y-3$

$\therefore f^{-1}(y) = y-3$

$\therefore f^{-1}(x) = x-3$

Composition of Functions

Let $f: A \to B$ and $g: B \to C$. The composition of g with f, $g \circ f$, is the function from A to C defined by $g \circ f(x) = g(f(x))$



$\therefore\ g(f(1)) = 10$

$g \circ f(1) = 10$

[4] If $f(x) = x+3$ and $g(x) = x^2 - 2$ find:

$$fog(1) = f(1-2) = f(-1) = -1+3 = 2$$

$$fog(x) = f(x^2-2) = (x^2-2)+3 = x^2+1$$

$$gof(x) = g(x+3) = (x+3)^2 - 2 = x^2+6x+7$$

$$gof(1) = g(4) = 16-2 = 14$$

Useful Functions

Floor function, $f(x) = \lfloor x \rfloor$

largest integer less than or equal to x.

Ceiling Function, $f(x) = \lceil x \rceil$

Smallest integer greater than or equal to x.

$\lceil 2.2 \rceil = 3$ $\qquad$ $\lceil -3.7 \rceil = -3$

$\lfloor 2.2 \rfloor = 2$ $\qquad$ $\lfloor -3.7 \rfloor = -4$

Factorial Function

$f: \mathbb{N} \to \mathbb{Z}^{+}$ denoted by $f(n) = n!$ is the product of the first n positive integers when n is a non-negative integers.

$$f(n) = n \cdot (n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$$

stirolings formula : $\boxed{n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}$

Sequences

An ordered list of elements created by a function mapping the integers to a set S.

Notation: a_n represents the n^{th} term.

Ex: If $a_n = 2n$, find $\{a_n\}$

$$a_0 = 2(0) = 0$$
$$a_1 = 2(1) = 2$$
$$a_2 = 2(2) = 4$$
$$a_3 = 2(3) = 6$$

$$\therefore \{a_n\} = \{0, 2, 4, 6 \cdots\}$$

Arithmetic Sequence:

A sequence formed by adding the initial ~~term~~ term, a and the product of the common difference, d, and the term number, n.

$$\{a_n\} = a, a+d, a+2d, a+3d, \dots, a+nd$$

$$\therefore \boxed{a_n = a+nd}$$

$a_1, a_2, \dots a_n \rightarrow$ finite series $\rightarrow \boxed{\text{strings}}$

* String is also denoted by $a_1 a_2 \dots a_n$ (bit string $\rightarrow$ finite sequences of bit)

* Length of a string is the number of terms in this string.

* The empty string denoted by λ. In this string there is no terms so length is zero

Geometric Sequence

A sequence formed by multiplying the initial term, a, by the common ratio to the nth power, r^n.

$$\{a_n\} = \underset{a_0}{a},\ \underset{a_1}{ar},\ \underset{a_2}{ar^2},\ ar^3 \ldots\ldots ar^n$$

$$\boxed{a_n = ar^n}$$

Recurrence Relation

An equation that express a_n in terms of one or more of the previous terms of the sequence. Initial conditions are required to specify terms that precede the first term ~~to specify~~ where the relation takes effect.

Ex: $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 2$ whe $a_0 = 2$
$a_1 = 5$

$a_0 = 2,\ a_1 = 5$

$$\begin{aligned} a_2 &= a_{2-1} + 2a_{2-2} \\ &= a_1 + 2a_0 \\ &= 5 + 2 \times 2 \\ &= 9 \end{aligned}$$

$$\begin{aligned} a_3 &= a_{3-1} + 2a_{3-2} \\ &= a_2 + 2a_1 \\ &= 9 + 2 \times 5 \\ &= 19 \end{aligned}$$

$$\begin{aligned} a_4 &= a_{4-1} + 2a_{4-2} \\ &= a_3 + 2a_2 \\ &= 19 + 2 \times 9 \\ &= 37 \end{aligned}$$

Let a_n be the sequence that satisfies the recurrence relation $a_n = a_{n-1} + 6$ for $n \geq 1$

a) if $a_0 = 3$, find a_1, a_2, a_3.

b) if $a_0 = -7$, find a_1, a_2, a_3

soln:

a) $a_0 = 3$,

$a_1 = a_{1-1} + 6 = a_0 + 6 = 3 + 6 = 9$

$a_2 = a_{2-1} + 6 = a_1 + 6 = 9 + 6 = 15$

$a_3 = a_{3-1} + 6 = a_2 + 6 = 15 + 6 = 21$

$\therefore a_n = 3 + 6n$

b) $a_0 = -7$

$a_1 = a_{1-1} + 6 = a_0 + 6 = -7 + 6 = -1$

$a_2 = a_{2-1} + 6 = a_1 + 6 = -1 + 6 = 5$

$a_3 = a_{3-1} + 6 = a_2 + 6 = 5 + 6 = 11$

$\therefore a_n = -7 + 6n$

Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$f_0 = 0$

$f_1 = 1$

$f_2 = 1 = 1+0 = f_0 + f_1$

$= f_{2-1} + f_{2-2}$

$f_3 = 2 = 1+1$

$= f_1 + f_2$

$= f_{3-1} + f_{3-2}$

$f_4 = 3 = 1+2$

$= f_2 + f_3$

$= f_{4-1} + f_{4-2}$

$\therefore \boxed{f_n = f_{n-1} + f_{n-2}}$

Solving Recurrence Relations:

Finding a non-recursive formula to calculate a_n is called solving the recurrence relation. The solution is called a closed formula.

One method for solving is called iteration, involving substitution.

Let a_n be the sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n \geq 1$ with $a_0 = 2$

$a_0 = 2$

$a_1 = a_{1-1} + 3 = 2 + 3$

$a_2 = a_{2-1} + 3 = a_1 + 3 = 2 + 2 \times 3$

$a_3 = a_{3-1} + 3 = a_2 + 3 = 2 + 3 \times 3$

$\vdots$

$$\boxed{a_n = 2 + 3n}$$

$$\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 \\
&= a_{n-2} + 2 \times 3 \\
&= (a_{n-3} + 3) + 2 \times 3 \\
&= a_{n-3} + 3 \times 3 \\
&\vdots \\
&= a_{n-n} + n \times 3 \\
&= a_0 + 3n \\
&= 2 + 3n
\end{aligned}$$

Sigma Notation

To express the sum of the terms of the sequence

$a_n = \{a_m, a_{m+1}, \cdots a_n\}$, we write

$$\sum_{i=m}^{n} a_i = \Sigma_{i=m}^{n} a_i = \Sigma_{m \le i \le n} a_i$$

which represents, $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$

Ex: Use sigma notation to express the sum of the first 100 terms of the sequence a_i where $a_i = \frac{1}{i}$ for $i = 1, 2, 3, \cdots$

$a_1 = \frac{1}{1}$ $\qquad \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots \frac{1}{100}$

$a_2 = \frac{1}{2}$ $\qquad \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$

$a_3 = \frac{1}{3}$

$\therefore$ sigma notation: $\sum_{i=1}^{100} \frac{1}{i}$

Q. What is the value of $\sum_{i=1}^{5} i^2$?

$$\sum_{i=1}^{5} i^2 = 1^2+2^2+3^2+4^2+5^2$$
$$= 55$$

Q. what is the value of $\sum_{i=7}^{10} (-1)^i$?

$$\sum_{i=7}^{10} (-1)^i = (-1)^7+(-1)^8+(-1)^9+(-1)^{10}$$
$$= -1+1-1+1$$
$$= 0$$

Basic Properties

1. $$\sum_{k=1}^{n} c a_k = c\left(\sum_{k=1}^{n} a_k\right)$$

2. $$\sum_{k=1}^{n} c = nc$$

3. $$\sum_{k=1}^{n} (a_k \pm b_k) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

4. $$\sum_{k=1}^{n} a_k = \sum_{k=1}^{j} a_k + \sum_{k=j+1}^{n} a_k \quad \left[1 \le j < j+1 < n \text{ and } j \in N\right]$$

5. $$\sum_{k=1}^{n} a_k = \sum_{k=1+m}^{n+m} a_{k-m} \text{ for }$$

Summation Formula:

1. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

2. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

3. $$\sum_{i=1}^{n} i^3 = \left\{\frac{n(n+1)}{2}\right\}^2$$

4. $$\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1} \quad (r \neq 1)$$

5. $$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}, \quad |x|<1$$

6. $$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, \quad |x|<1$$

Matrices and Matrices Operations

Matrices: A matrix is a rectangular array of numbers.

$m \rightarrow$ rows

$n \rightarrow$ ~~colom~~ column

is called $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

□ Matrix Addition and Subtraction :-

□ ~~Matrix product.~~

$A = [a_{ij}]$, $B = [b_{ij}]$

$\therefore A+B = [a_{ij} + b_{ij}]$

□ Matrix product:

$A \to m \times k$

$B \to k \times n$

$\therefore AB \to mn$

if $AB = [c_{ij}]$, then,

$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + \cdots a_{ik} b_{kj}$

Identity Matrix:

$I_n = \delta_{ij}$, where, $\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ if $i \neq j$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$I_n A = A I_n = A$

Transpose Matrix:

$A \Rightarrow m \times n$

$A^T \rightarrow n \times m$

If $\rightarrow \boxed{A = A^T} \longrightarrow$ then A is symmetric.

Zero-One Matrices

A matrix where all entries are either 0 or 1.

Zero-One matrices are based on Boolean operations.

meet $\leftarrow b_1 \wedge b_2 = \begin{cases} 1 & \text{if } b_1 = b_2 = 1 \\ 0 & \text{otherwise} \end{cases}$

join $\leftarrow b_1 \vee b_2 = \begin{cases} 1 & \text{if } b_1 = 1 \text{ or } b_2 = 1 \\ 0, & \text{otherwise} \end{cases}$

alternative,

$\begin{cases} 1, & \text{otherwise} \\ 0, & \text{if } b_1 = b_2 = 0 \end{cases}$

Q) Find the join and meet of the zero one matrices:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$A = [a_{ij}] \rightarrow m \times k$

$B = [b_{ij}] \rightarrow k \times n$

} Zero-one matrix

~~A⊙→n~~

Boolean product $A \odot B \rightarrow m \times n \rightarrow (c_{ij})$

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj})$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A \odot B = \begin{bmatrix} (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \\ (0\wedge 1)\vee(1\wedge 0) & (0\wedge 1)\vee(1\wedge 1) & (0\wedge 0)\vee(1\wedge 1) \\ (1\wedge 1)\vee(0\wedge 0) & (1\wedge 1)\vee(0\wedge 1) & (1\wedge 0)\vee(0\wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$