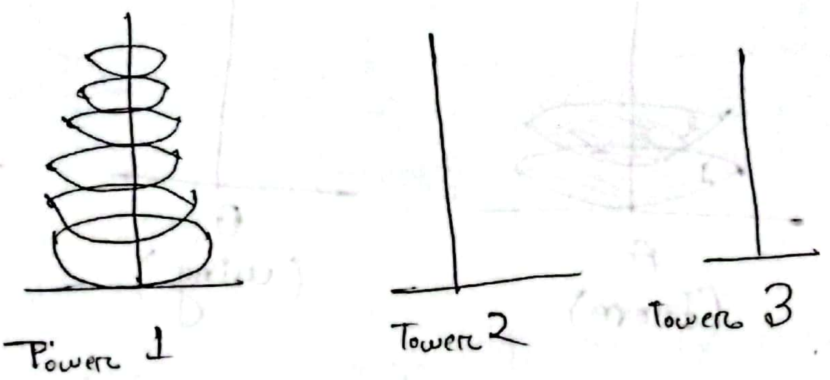
Tower of Hanoi Puzzle

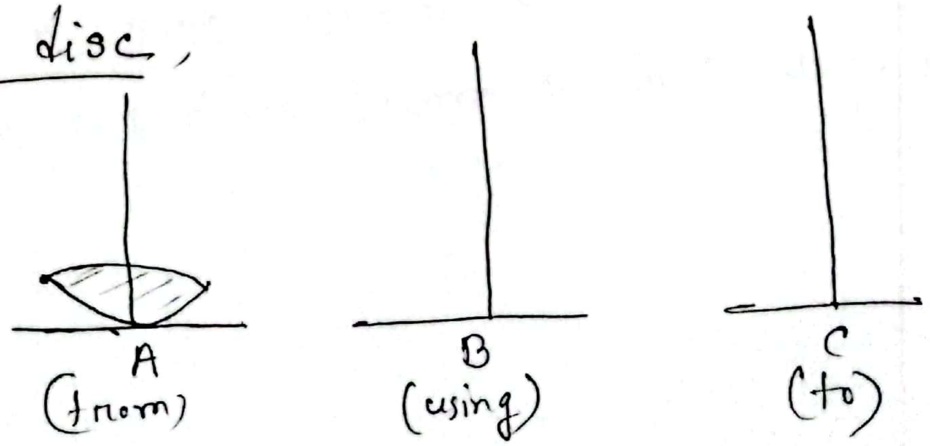


~~How How minin~~

How many minimum moves required ~~to~~ for transfering all disks from Tower 1 to Tower 3.

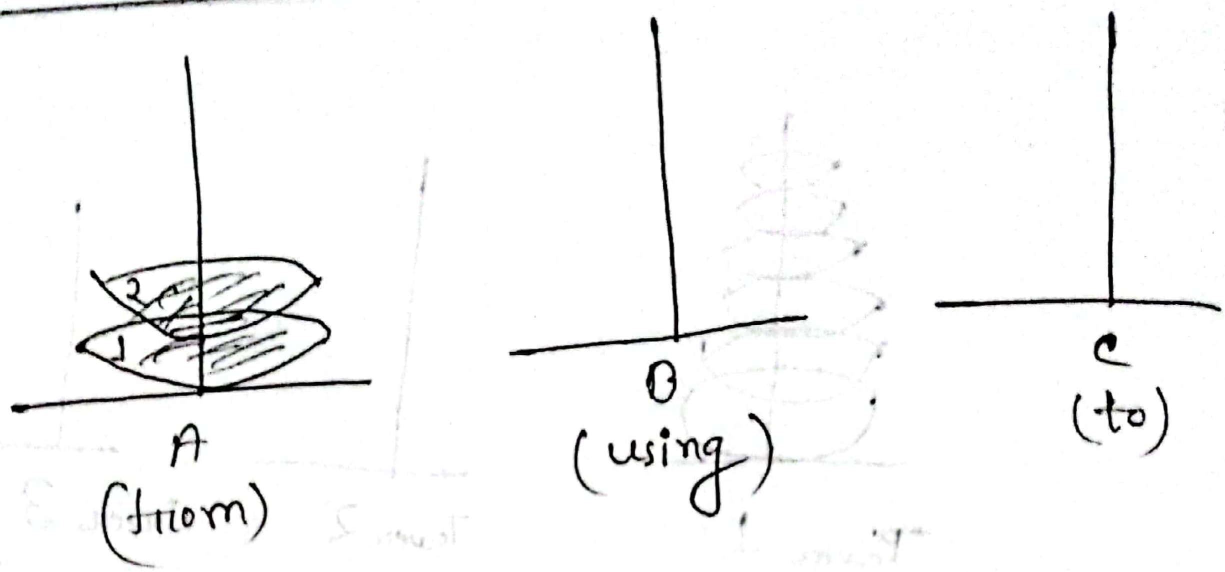
Sol^n:

For single disc,



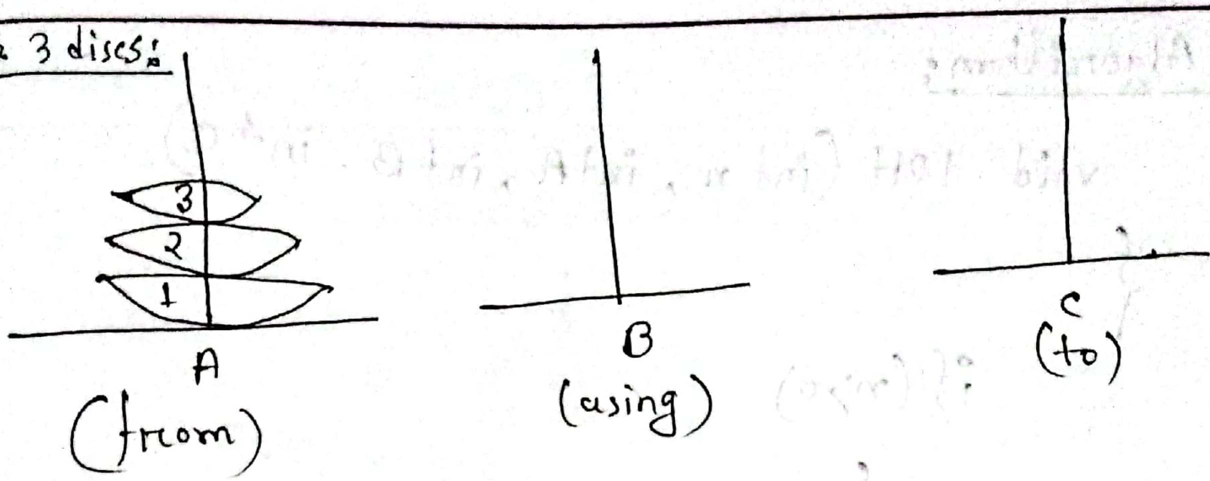
* Move a disc from A to C

For 2 disks:



* Move a disc from A to B using C,
* Move a disc from A to C.
* Move a disc from B to C using A.

For 3 dises:



- More 2 dises from A to B using C
- More a dise from A to C
- More 2 dises from B to C using A.

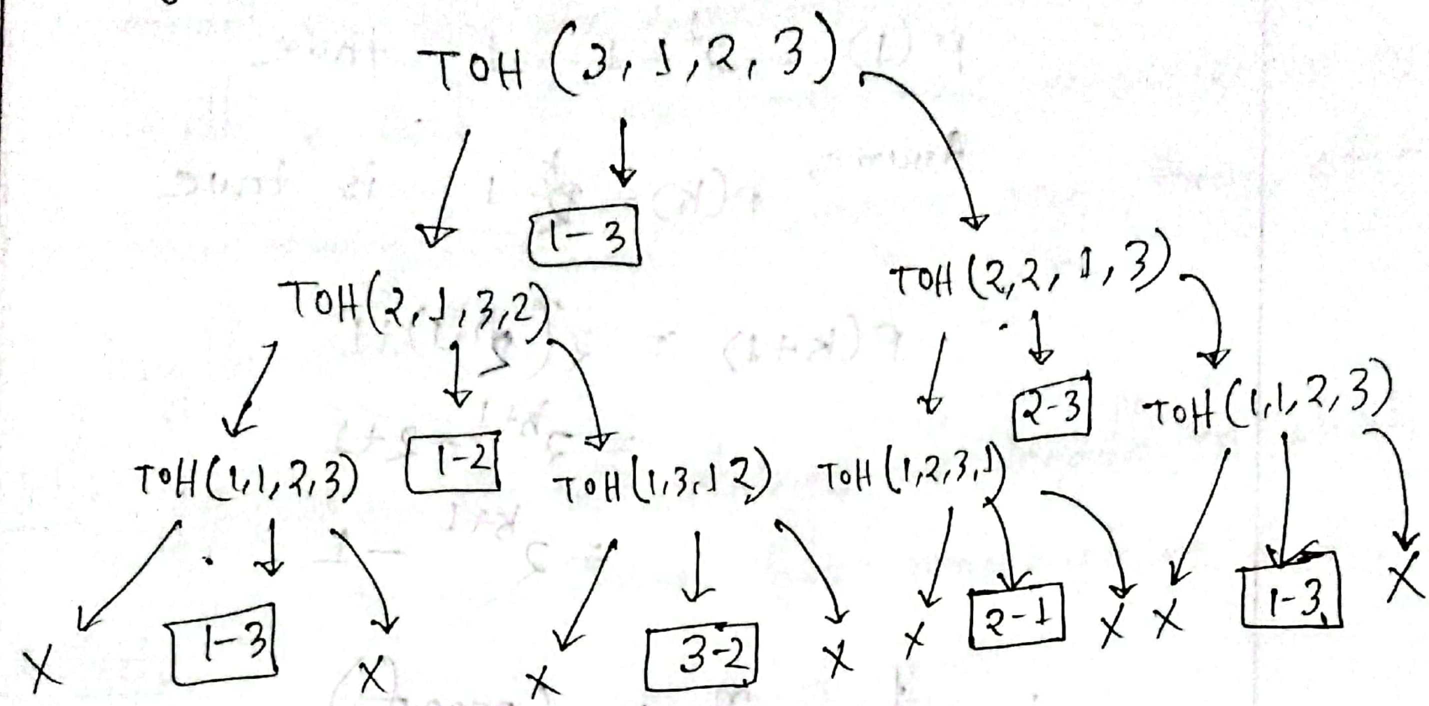
For n dises:

- More n-1 dises from A to B using C
- More a dise from A to C
- More n-1 dises from B to C using A.

Algorithm:

```
void TOH (int n, int A, int B, int C)
{
    if (n>0)
    {
        TOH (n-1, A, C, B);
        printf ("Move a disc from
                %d to %d," A, C);
        TOH(n-1, B, A, C);
    }
}
```

Tracing for 3 discs:



Moves: (1-3), (1-2), (3-2), (1-3), (2-1), (2-3), (1-3)

Number of moves: 7 $= 2^3 - 1$

$\therefore$ For n number of ~~moves~~ discs minimum moves required would be $2^n - 1$.

Recurrence relation, $\boxed{t_n = 2t_{n-1} + 1}$

Proof by induction: $p(n) = 2^n - 1$

$p(1) = 2^1 - 1 = 1$ true

Assume, $p(k) = 2^k - 1$, is true

$\therefore$ $p(k+1) = 2(2^k - 1) + 1$

$= 2^{k+1} - 2 + 1$

$= 2^{k+1} - 1$.

$\therefore$ $t_n = 2^n - 1$ (proof)

Ex: Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length five?

Sol^n:

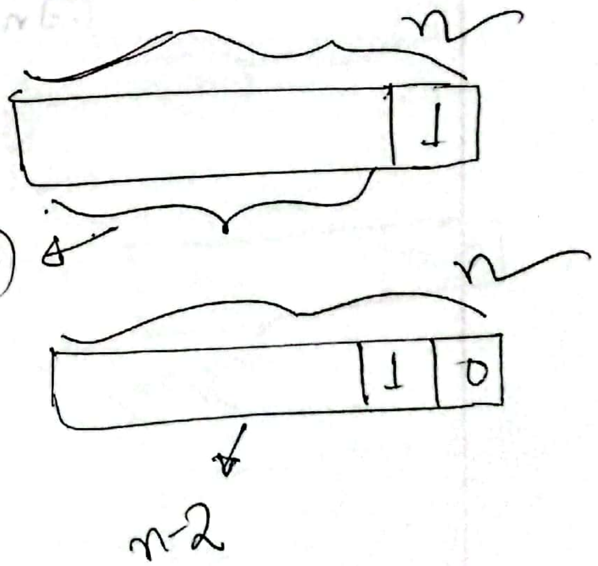
$f(n) \rightarrow$ no of bit strings of length n which do not have two consecutive 0's.

Initial;

$f(1) = 2 \longrightarrow 0, 1$

$f(2) = 3 \longrightarrow 01, 10, 11$

$$\boxed{f(n) = f(n-1) + f(n-2)}$$



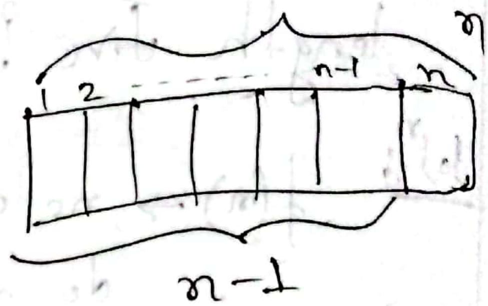
$f(3) = f(2) + f(1)$
$= 3 + 2 = 5$

$f(4) = f(3) + f(2)$
$= 5 + 3 = 8$

$f(5) = f(4) + f(3)$
$= 8 + 5 = 13$

Find a recurrence relation for the number of permutations of a set with n elements and find using iteration:

Soln: $f(n)$ = no of permutations of n elements.



$f_n = n \cdot f_{n-1}$

then, iteration.

$$f_n = n \cdot (n-1) \cdot (n-2) \cdots 1$$

$$= n!$$

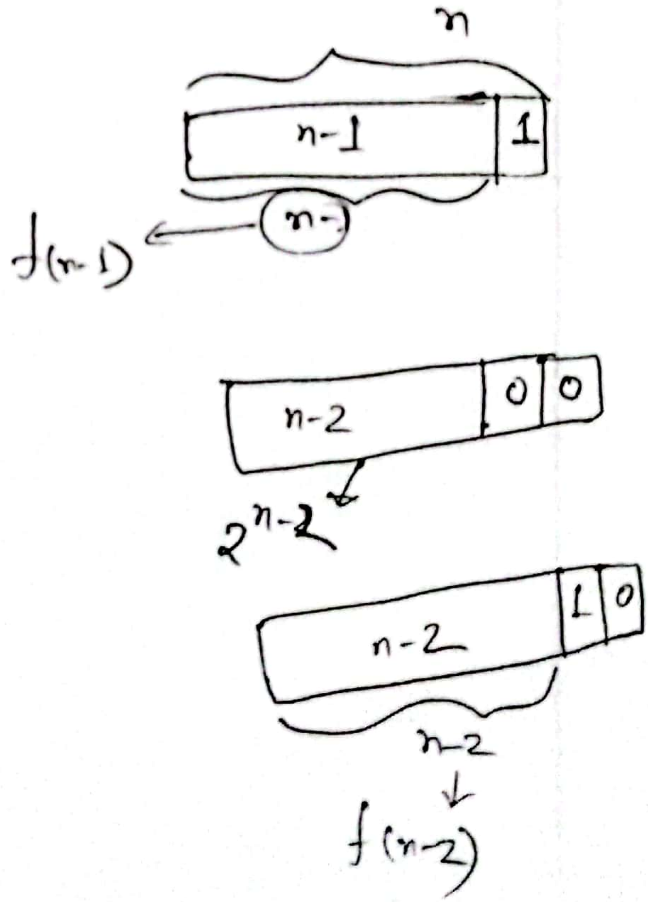
10) Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s.

Sol^n: $f(n)$ = the no. of bit strings of length n that contain a pair of consecutive 0s.

$f(1) = 0 \longrightarrow 0, 1$

$f(2) = 1 \longrightarrow 00, 01, 10, 11$

$$\boxed{f(n) = f(n-1) + f(n-2) + 2^{n-2}}$$

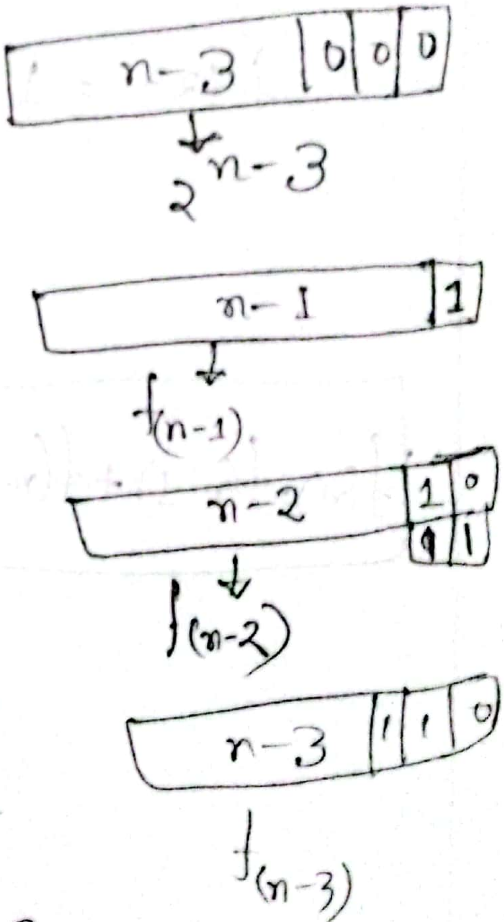


(8) a) Find a recurrence relation for the number of bit strings of length n that contain three consecutive 0s.

b) what are the initial conditions?

c) How many bit strings of length seven contain three consecutive 0s.

Sol^n:



a) $$\boxed{f_n = f_{n-1} + f_{n-2} + f_{n-3} + 2^{n-3}}$$

$n \geq 3$

b) $a_0 = 0$

$a_1 = 0$ } 0, 1

$a_2 = 0$ } 00, 10, 01, 11

c)

$a_3 = a_2 + a_1 + a_0 + 2^0 = 0+0+0+1$
$= 1$

$a_4 = a_3 + a_2 + a_1 + 2^1 = 1+0+0+2$
$= 3$

$a_5 = a_4 + a_3 + a_2 + 2^2 = 3+1+0+4$
$= 8$

$a_6 = a_5 + a_4 + a_3 + 2^3 = 8+3+1+8 = 20$

$a_7 = a_6 + a_5 + a_4 + 2^4 = 20+8+3+16 = 47$

[4] a) Find a recurrence relation for the number for bit strings of length n that contain the string 01.

b) What are the initial conditions.

c) How many bit strings of length seven contain the string 01.

Soln:

a)

$$\boxed{\therefore f_n = f_{(n-1)} + 2^{n-1} - 1}$$

b) $a_0 = a_1 = 0$

c)

$a_2 = a_1 + 2^{2-1} - 1 = 0 + 2 - 1 = 1$

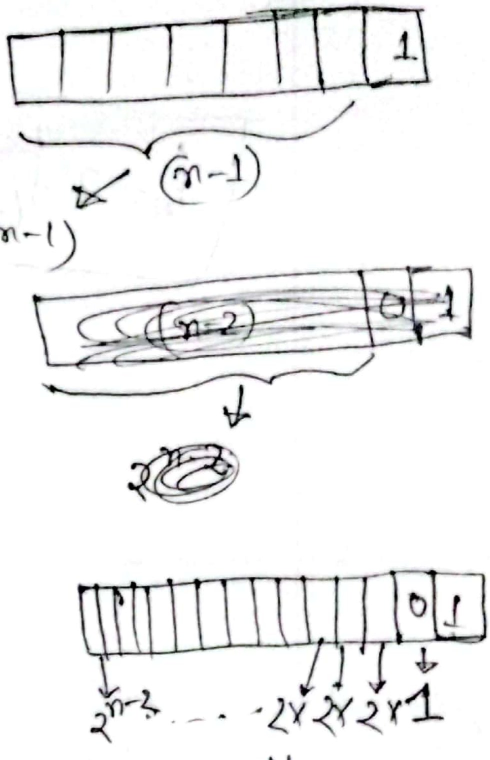
$a_3 = a_2 + 2^{3-1} - 1 = 1 + 2^2 - 1 = 4$

$a_4 = a_3 + 2^{4-1} - 1 = 4 + 2^3 - 1 = 11$

$a_5 = a_4 + 2^{5-1} - 1 = 11 + 2^4 - 1 = 26$

$a_6 = a_5 + 2^{6-1} - 1 = 26 + 2^5 - 1 = 57$

$a_7 = a_6 + 2^{7-1} - 1 = 57 + 2^6 - 1 = 120$



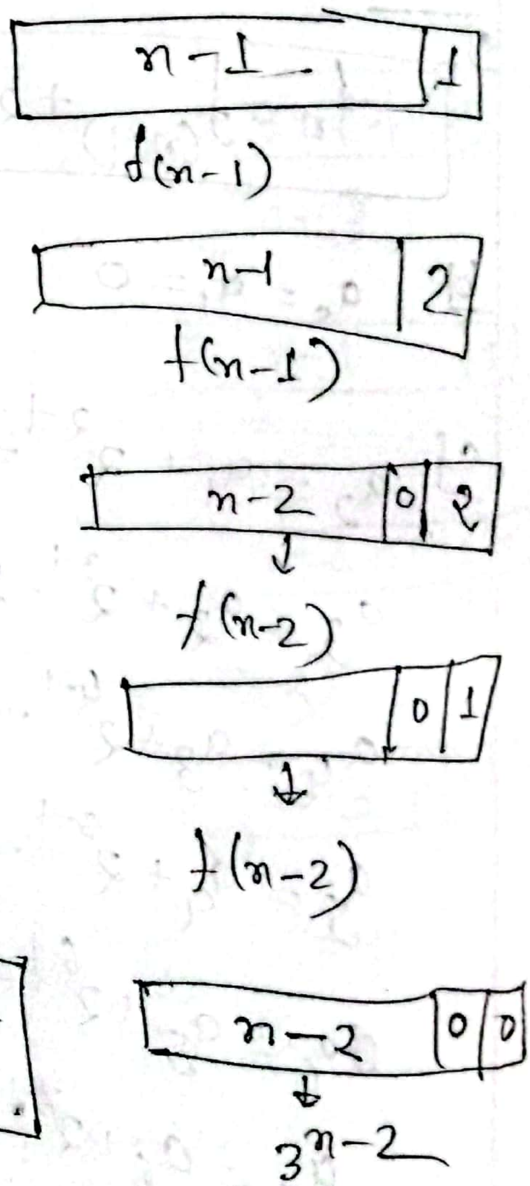
$2^{n-2+1} - 1$

$= 2^{n-1} - 1$

13| Find a recurrence relation for the number of ~~termo~~ ternary strings of length n that do not contain two consecutive 0s.

Sol^n: ternary strings = 0, 1, 2

$$\boxed{f_n = 2f_{(n-1)} + 2f_{(n-2)}}$$

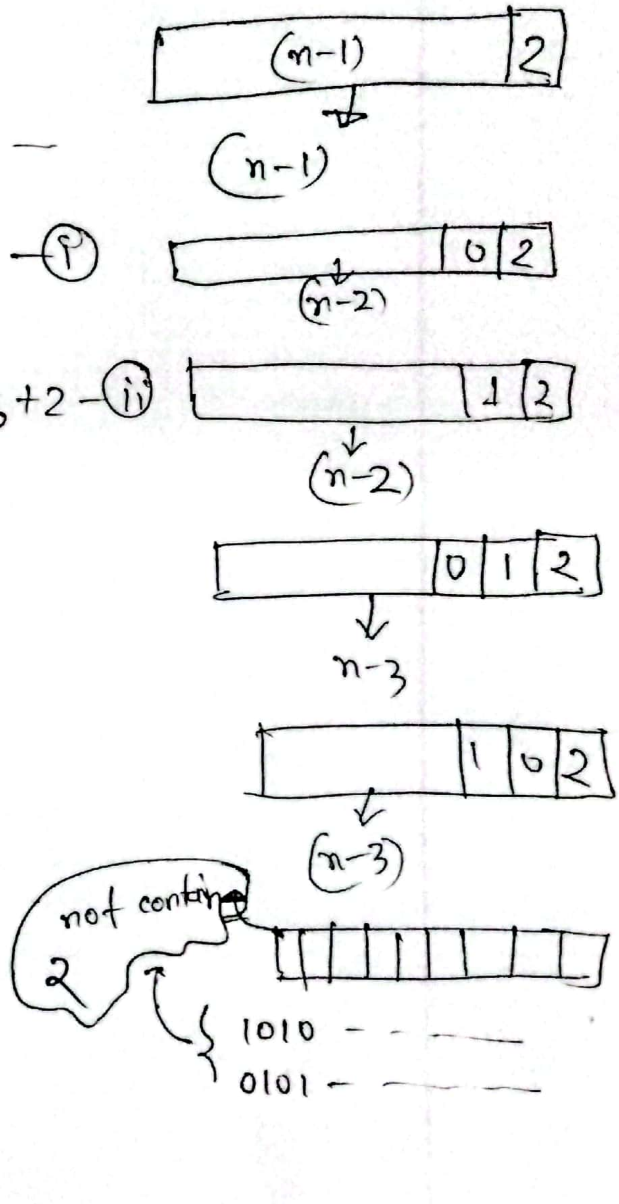


contain two consecutive 0s:

$$\boxed{f_n = 2f_{(n-1)} + 2f_{(n-2)} + 3^{n-2}}$$

Q) Find a reccurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s or two consecutive 1s.

Sol^n:



$$\therefore f_n = f_{n-1} + 2f_{n-2} + 2f_{n-3} + \cdots + 2f_0 + 2 \quad \text{--- (i)}$$

$$f_{n-1} = f_{n-2} + 2f_{n-3} + 2f_{n-4} + \cdots 2f_0 + 2 \quad \text{--- (ii)}$$

(i) − (ii)

$$f_n - f_{n-1} = f_{n-1} + f_{n-2}$$

$$\therefore \boxed{f_n = 2f_{n-1} + f_{n-2}}$$

⊛ contain two cosecce tive 0s or 1s -

$$\therefore \boxed{f_n = 2f_{n-1} + f_{n-2} + 2 \cdot 3^{n-2}}$$

$$\underbrace{\boxed{\quad n-2 \quad}}_{3^{n-2}}\boxed{0}\boxed{0}$$

$$\underbrace{\boxed{\quad n-2 \quad}}_{3^{n-2}}\boxed{1}\boxed{1}$$

The Principle of Inclusion - Exclusion

* $|A \cup B| = |A| + |B| - |A \cap B|$
* $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Ex: How many positive integers not exceeding 1000 are divisible by 7 or 11?

Soln:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \left\lfloor \frac{1000}{7} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{7 \cdot 11} \right\rfloor$$

$$= 142 + 90 - 12$$

$$= 220.$$

Ex: A total of 1232 students have taken a course in spanish, 879 have taken a course in French, and 114 have taken a course in Russian. 103 have taken both spanish and French, 23 have taken both Spanish and Russian, 14 have taken both French and Russian. If 2092 students have taken at least one of those course. How many students take all of those courses.

Sol^n:

$|S| = 1232$, $|F| = 879$, $|R| = 119$

$|S \cap F| = 103$, $|S \cap R| = 23$, $|F \cap R| = 14$

$|S \cup F \cup R| = 2092$

$\therefore |S \cup F \cup R| = |S| + |F| + |R| - |S \cap F| + |S \cap R| + |F \cap R|$
$+ |S \cap F \cap R|$

$\Rightarrow 2092 = 1232 + 879 + 119 - 103 - 23 - 14 +$
$|S \cap F \cap R|$

$\therefore \cancel{|S \cap R \cap A|}$

$|S \cap F \cap R| = 7$

General form:

$$|A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n| = \sum_{1 \le j \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| +$$

$$\sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - (-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n|$$

Term number $= 2^n - 1$.

$$|A_1 \cup A_2 \cup A_3 \cup A_4| = |A_1| + |A_2| + |A_3| + |A_4| - |A_1 \cap A_2| - |A_1 \cap A_3| -$$
$$|A_1 \cap A_4| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_3 \cap A_4|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4|$$
$$+ |A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Ex: Find the number of positive integers not exceeding 10000 ~~thats~~ that are not divisible by 3,4,7 or 11.

Sol^n:

divisible,

$$= \left\lfloor \frac{10,000}{3} \right\rfloor + \left\lfloor \frac{10,000}{4} \right\rfloor + \left\lfloor \frac{10000}{7} \right\rfloor + \left\lfloor \frac{10000}{11} \right\rfloor$$

$$- \left\lfloor \frac{10,000}{3.4} \right\rfloor + \left\lfloor \frac{10,000}{3.7} \right\rfloor - \left\lfloor \frac{10,000}{3.11} \right\rfloor - \left\lfloor \frac{10,000}{4.7} \right\rfloor$$

$$- \left\lfloor \frac{10,000}{4.11} \right\rfloor - \left\lfloor \frac{10,000}{7.11} \right\rfloor + \left\lfloor \frac{10,000}{3.4.7} \right\rfloor + \left\lfloor \frac{10000}{3.4.11} \right\rfloor$$

$$+ \left\lfloor \frac{10,000}{3.7.11} \right\rfloor + \left\lfloor \frac{10,000}{4.7.11} \right\rfloor - \left\lfloor \frac{10,000}{3.4.7.11} \right\rfloor$$

$$= 3333 + 2500 + 1428 + 909 - 833 - 476 -$$
$$303 - 357 - 227 - 129 + 119 + 75 +$$
$$43 + 32 - 10$$

$$= 6104$$

$\therefore$ Not divisible $= 10,000 - 6104$

$$= 3896$$