Induction and Reccursion

Principal of Mathematical Induction:

Suppose P is a proposition defined on the positive integers N and suppose P has the following two properties:

1. P(1) is true.
2. P(K+1) is true when wherever P(K) is true. Then P is true for every positive integers.

1. P(1) → true
2. P(K) → P(K+1) → true → inductive step

$(P(1) \wedge \forall_K P(K)) \rightarrow P(K+1) \rightarrow \forall_n P(n)$

[i] To complete the iductive step of a proof using the principle of mathematical induction, we assume that p(k) is true, then p(k+1) must be true. The assumption that p(k) is true is called the inductive hypothesis.

4 Show that if n is a positive integer, then

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

Soln:

$$p(n) = \frac{n(n+1)}{2}$$

$$\therefore p(1) = \frac{1(1+1)}{2} = 1$$

assume: $1+2+3+\cdots+k = \frac{k(k+1)}{2}$

Then, $1+2+3+\cdots+k+k+1 = \frac{(k+1)((k+1)+1)}{2}$

$$= \frac{(k+1)(k+2)}{2}$$

we can

$1+2+3+\cdots+k+k+1$

$$= \frac{k(k+1)}{2} + k+1$$

$$= \frac{k^2+k+2k+2}{2}$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

so, it is true, $1+2+\cdots+n = \frac{n(n+1)}{2}$

☐ Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Soln:

$\cancel{n = 1, 2, 3},$

$1 = 1, \quad 1+3 = 4 \quad 1+3+5 = 9$

$1+3+5+7 = 16 \quad 1+3+5+7+9 = 25$

$1+3+5+7+9+ \cdots\cdots + (2n-1) = n^2$

$P(1) = 1^2 = 1$

$P(k) = 1+3+5+ \cdots\cdots + (2k-1) = k^2$ (assume)

$P(k+1) = 1+3+5+ \cdots\cdots + (2k-1) + (2k+1) = (k+1)^2$

$\therefore 1+3+5+ \cdots\cdots + (2k-1) + (2k+1)$

$= k^2 + 2k + 1$

$= (k+1)^2$

$\therefore$ so, it's true, $1+3+5+7+ \text{———} + (2n-1) = n^2$

Q. Use mathematical induction to show that,

$$1+2+2^2+\cdots+2^n = 2^{n+1}-1$$

$\rightarrow (n+1)$ terms

S.ln,

$P(0) = 2^{1+0}-1 = 1$

assume

$P(k) = 1+2+2^2+\cdots+2^k = 2^{k+1}-1$

$\therefore P(k+1) = 1+2+2^2+\cdots+2^k+2^{k+1} = 2^{(k+1)+1}-1 = 2^{k+2}-1$

$\therefore 1+2+2^2+\cdots+2^k+2^{k+1}$

$= 2^{k+1}-1+2^{k+1}$

$= 2\cdot 2^{k+1}-1$

$= 2^{k+2}-1$

Q Use mathematical induction to show that

$$\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1} - a}{r-1}$$

terms $(n+1)$

Sol^n:

$$P(0) = \frac{ar^{1+0} - a}{r-1}$$

$$= \frac{a(r-1)}{(r-1)} = a$$

assume,

$$P(k) = a + ar + ar^2 + \cdots ar^k = \frac{ar^{k+1} - a}{r-1}$$

$$P(k+1) = a + ar + ar^2 + \cdots ar^k + ar^{k+1} = \frac{ar^{k+1+1} - a}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1}$$

$$= \frac{ar^{k+1} - a}{r-1} + ar^{k+1}$$

$$= \frac{ar^{k+1} - a + ar^{k+1}(r-1)}{r-1}$$

$$= \frac{\cancel{ar^{k+1}} - a + ar^{k+2} - \cancel{ar^{k+1}}}{r-1}$$

$$= \frac{ar^{k+2} - a}{r-1}$$ so, equation is true.

Use mathematical induction to prove the inequality,

$$n < 2^n$$

Sol^n:

$P(1) : \quad 1 < 2^1$

assume,

$P(k) : \quad k < 2^k$

$P(k+1) : \quad k+1 < 2^{k+1}$

$k+1 < 2^k + 1 \qquad [1 \leq 2^k]$

$\Rightarrow 2^{k+1} \leq 2^k + 2^k$

$\Rightarrow 2^{k+1} \leq 2^{k+1}$

$\therefore k+1 \leq 2^{k+1}$

Use mathematical induction to prove that $2^n < n!$ $[n \geq 4]$.

Sol^n:

$p(4)$: $2^4 = 16$

$\therefore 16 < 24$

$16 < 4!$

assume,

$p(k)$: $2^k < k!$ $(k \geq 4)$

$p(k+1)$: $2^{k+1} = 2 \cdot 2^k$

$2 \cdot 2^k < 2 \cdot k!$ $[2 < k+1]$

$\therefore 2 \cdot 2^k < (k+1) \cdot k!$

$\therefore 2 \cdot 2^k < (k+1)!$

$\therefore 2^{k+1} < (k+1)!$

Use mathematical induction to show that,

Harmonic series,

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{j}$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

$$H_{2^n} \geq 1 + \frac{n}{2} \quad \text{[equation]}$$

Sol^n:

$P(0)$; $H_{2^0} = H_1 = 1$

$1 + \frac{n}{2} = 1 + \frac{0}{2} = 1$

$\therefore H_1 \geq 1$

assume,

$P(k)$; $H_{2^k} \geq 1 + \frac{k}{2}$

$P(k+1)$: $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$

$$H_{2^{k+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots \frac{1}{2^{k+1}}$$

$$= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}}$$

$$\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}}$$ [there are 2^k terms and each terms are $\geq \frac{1}{2^{k+1}}$]

$$\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2}$$

$$\geq 1 + \frac{k+1}{2}$$

☐ use mathematical induction to prove ~~than~~ that $n^3 - n$ is divisible by 3 whenever n is a positive integer. [Fermat's little theorem]

Sol^n:

$$n^3 - n$$

$P(1)$: $1^3 - 1 = 0$, that is divisible by 3.

assume,

$$P(k) : k^3 - k.$$

$$P(k+1) : (k+1)^3 - (k+1)$$

$$= k^3 + 3k^2 + 3k + \cancel{1} - k - \cancel{1}$$

$$= (k^3 - k) + 3(k^2 + k)$$

$\uparrow$ divisible by 3

$\downarrow$ $3 \times \square$ $\downarrow$ divisible by 3

Use mathematical induction to prove De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

Sol^n:

Basic step: P(2): $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

P(K): $\overline{\bigcap_{j=1}^{K} A_j} = \bigcup_{j=1}^{K} \overline{A_j}$ $[K \geq 2]$

P(K+1):

$$\overline{\bigcap_{j=1}^{K+1} A_j} = \overline{\left(\bigcap_{j=1}^{K} A_j\right) \cap A_{K+1}}$$

$$= \overline{\left(\bigcap_{j=1}^{K} A_j\right)} \cup \overline{A_{K+1}}$$

$$= \bigcup_{j=1}^{K} \overline{A_j} \cup \overline{A_{K+1}}$$

$$= \bigcup_{j=1}^{K+1} \overline{A_j}$$

Use mathematical statement to prove that $1^2+3^2+5^2+ \cdots + (2n+1)^2 = \frac{(n+1)(2n+1)(2n+3)}{3}$

for non-negative integeres.

Solⁿ:

$$P(0) = \frac{(0+1)(2\times 0+1)(2\times 0+3)}{3}$$

$$= \frac{1\times 1\times 3}{3}$$

$$= 1 \quad \text{true}$$

[1st term of n=0]

$$P(k) = 1^2+3^2+5^2+ \cdots + (2k+1)^2 = \frac{(k+1)(2k+1)(2k+3)}{3}$$

is true

$$P(k+1) = 1^2+3^2+ \cdots + (2k+3)^2 = \frac{(k+2)(2k+3)(2k+5)}{3}$$

$\therefore 1^2+3^2+ \cdots + (2k+3)^2$

$$= 1^2+3^2+ \cdots + (2k+1)^2 + (2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3} + (2k+3)^2$$

$$= \frac{(k+1)(2k+1)(2k+3) + 3(2k+3)^2}{3}$$

$$= \frac{(2k+3)\{(k+1)(2k+1) + 3(2k+3)\}}{3}$$

$$= \frac{(2k+3)\{2k^2 + k + 2k + 1 + 6k + 9)}{3}$$

$$= \frac{(2k+3)\ (2k^2 + 9k + 10)}{3}$$

$$= \frac{(2k+3)\ (2k^2 + 5k + 4k + 10)}{3}$$

$$= \frac{(2k+3)\{k(2k+5) + 2(2k+5)\}}{3}$$

$$= \frac{(2k+3)\ (2k+5)(k+2)}{3}$$

$$= \frac{(k+2)(2k+3)(2k+5)}{3}$$ (proof)

Strong Induction

- when $P(k)$ is not enough for proving $P(k+1)$.
- There can be exist multiple base cases.
- some times we can use $P(k-n)$ instead of $P(k)$.
- we can use $P(k+l)$ instead of $P(k+1)$
- Inductive step, $P(j)$, last base case $\leq j \leq k$

Ex: Define $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$ we have $f_n \geq \left(\frac{3}{2}\right)^{n-3}$ for all $n \in N$. Prove it using strong induction.

Sol^n:

Base cases:

$$n = 1, \quad f_1 = 1 > \frac{4}{9} = \left(\frac{3}{2}\right)^{-2} = \left(\frac{3}{2}\right)^{1-3}$$

$$n = 2, \quad f_2 = 1 > \frac{2}{3} = \left(\frac{3}{2}\right)^{-1} = \left(\frac{3}{2}\right)^{2-3}$$

Inductive step:

assume, for $k \geq 1$ $\quad f_m \geq \left(\frac{3}{2}\right)^{m-3}$ for $m \leq k$

consider,

$$f_{k+1} = f_k + f_{k-1} \geq \left(\frac{3}{2}\right)^{k-3} + \left(\frac{3}{2}\right)^{k-4}$$

$$\geq \left(\frac{3}{2}\right)^{k-4}\left(\frac{3}{2} + 1\right)$$

$$\geq \left(\frac{3}{2}\right)^{k-4} \cdot \left(\frac{5}{2}\right) \geq \frac{9}{4}\left(\frac{3}{2}\right)^{k-4}$$

$$= \left(\frac{3}{2}\right)^{k-2}$$

$$= \left(\frac{3}{2}\right)^{(k+1)-3}$$

proof

Ex: Proof ; for all $n \in N$, we have $6 | n^3 - n$.

S.M.

Base case,

$n=1$, $1^3 - 1 = 6.0$

$n=2$, $2^3 - 1 = 6.1$

$n=3$, $3^3 - 1 = 6.4$

Inductive step: assume for. $6 | m^3 - m$ for $3 \le m \le k$

Consider,

$$(k+3)^3 - (k+3) = k^3 + 9k^2 + 27k + 27 - k - 3$$

$$= (k^3 - k) + (9k^2 + 27k + 24)$$

$$= (k^3 - k) + 3(3k^2 + 9k + 8)$$

$(k^3 - k) \downarrow$ multiple of 6

$3(3k^2+9k+8) \downarrow 3(3(k+1)(k+2) + 2)$

$3(k+1)(k+2) \rightarrow$ even

$+2$

$\downarrow$

even

$\downarrow \times 3$

multiple of 6

$\therefore (k+3)^3 - (k+3)$ is a multiple of 6.

$\therefore 6 | n^3 - n \rightarrow$ proof.

Q. What different kinds of postage can I make with 3 and 5 cents stamps. Proof this number by strong induction.

Solⁿ:

3 → cents , 5 → cents

Amount of postage: 3, 5, 6, 8, 9, 10, 11, 12, 13, 14 ---

↓

3 and 5 এর combination

↓

$(3a+5b)$ → a, b integer

So, we can make any amount of postage greater than 8, cause after 8 every number possible by combinations of 3 and 5.

proof:

Base case:

P(8) : 5 cents and 3 cents stamp.

P(9) : 3 3-cents stamp.

P(10) : 2 5-cents stamp.

Inductive: Assume $P(j)$ is true for

$$8 \leq j \leq k.$$

so, $P(k-2)$ is true. that means

$k-2$ cents of stamp postage can be made with 3 and 5 cents stamp.

Then, by adding one 3 stamps, we can make $k-2+3 = k+1$ cents of postage.

Hence, $P(k+1)$ is true.

so, we can make any number of stamp postage by 3 and 5 cents greater than or equal 8.