Inner Products

■ Inner Products space is a function that takes two vectors (of a real vector space) and gives a scalar, And have to satisfies four conditions,

1| $\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$

2| $\langle \bar{u}+\bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$

3| $\langle k\bar{u}, \bar{v} \rangle = k \langle \bar{u}, \bar{v} \rangle$

4| $\langle \bar{v}, \bar{v} \rangle \geq 0$ and $\langle \bar{v}, \bar{v} \rangle = 0$ if and only if $v = 0$

proof:

$$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + u_2 v_2$$

$$\bar{u} = (u_1 ; u_2)$$
$$\bar{v} = (v_1 , v_2)$$

1]

$$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + u_2 v_2$$
$$= v_1 u_1 + v_2 u_2$$
$$= \langle \bar{v}, \bar{u} \rangle$$

2]

$$\cancel{\langle (\bar{u}+\bar{v}), \bar{w} \rangle}$$

$$\bar{u} + \bar{v} = (u_1 + v_1 , u_2 + v_2)$$

$$\langle (\bar{u}+\bar{v}), \bar{w} \rangle = (u_1 + v_1) w_1 + (u_2 + v_2) w_2$$
$$= u_1 w_1 + v_1 w_1 + u_2 w_2 + v_2 w_2$$
$$= u_1 w_1 + u_2 w_2 + v_1 w_1 + v_2 w_2$$
$$= \langle u, w \rangle + \langle v, w \rangle$$

3)

$$k\bar{u} = (ku_1, ku_2)$$

$$\langle k\bar{u}, \bar{v}\rangle = ku_1v_1 + ku_2v_2$$

$$= k(u_1v_1 + u_2v_2)$$

$$= k\langle u, v\rangle$$

$\langle u, v \rangle = u \cdot v = u_1 v_1 + \cdots + u_n v_n$

↳ Euclidean inner Product

Norm, $\|v\| = \sqrt{\langle v, v \rangle}$

Distance, $d(u,v) = \|u - v\| = \sqrt{\langle u - v, u - v \rangle}$

$\langle u, v \rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \cdots + w_n u_n v_n$

↳ weighted Euclidean inner product

If $\|u\| = 1$ → unit sphere/unit circle

$\boxed{1}$ Let $u=(u_1, u_2)$ and $v=(v_1, v_2)$

verify $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$

satisfies four inner product axioms.

1] $\langle u, v\rangle = 3u_1v_1 + 2u_2v_2$

$= 3v_1u_1 + 2v_2u_2$

$= \langle v, u\rangle$

2] $\langle u+v, w\rangle = 3(u_1+v_1)w_1 + 2(u_2+v_2)w_2$

$= 3u_1w_1 + 3v_1w_1 + 2u_2w_2 + 2v_2w_2$

$= 3u_1w_1 + 2u_2w_2 + 3v_1w_1 + 2v_2w_2$

$= \langle u, w\rangle + \langle v, w\rangle$

3| $\langle ku, v\rangle = 3ku_1v_1 + 2ku_2v_2$

$= k(3u_1v_1 + 2u_2v_2)$

$= k\langle u, v\rangle$

4| $\langle v, v\rangle = 3v_1 \cdot v_1 + 2v_2v_2$

$= 3v_1^2 + 2v_2^2 \geq 0$ if and only if $v_1 = v_2 = 0$

$v = 0$

Matrix Inner Products:

$$\langle u, v\rangle = \underset{n\times n}{A}\,\underset{n\times 1}{u} \cdot A\,\underset{n\times 1}{v}$$

$$\langle u, v\rangle = (Av)^T \cdot Au$$

$$= v^T A^T \cdot Au$$

$$\langle u, v\rangle = \text{trace}(u^T v)$$

$\hookrightarrow$ standard inner product

$$= u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\|u\| = \sqrt{\langle u, u\rangle} = \sqrt{\text{tr}(u^T u)} = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

Calculate inner products, $\langle u-2v, 3u+4v\rangle$

Sol^n:

$\langle u-2v, 3u+4v\rangle$

$= \langle u, 3u+4v\rangle - \langle 2v, 3u+4v\rangle$

$= \langle u, 3u\rangle + \langle u, 4v\rangle - \langle 2v, 3u\rangle - \langle 2v, 4v\rangle$

$= 3\langle u, u\rangle + 4\langle u, v\rangle - 6\langle v, u\rangle - 8\langle v, v\rangle$

$= 3\|u\|^2 + 4\langle u, v\rangle - 6\langle u, v\rangle - 8\|v\|^2$

$= 3\|u\|^2 - 2\langle u, v\rangle - 8\|v\|^2$

Angle :

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|}\right)$$

$$-1 \leq \frac{u \cdot v}{\|u\| \|v\|} \leq 1$$

Cauchy - Schwarz Inequality

$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

$$\langle u, v\rangle^2 \leq \langle u, u\rangle \langle v, v\rangle$$

$$\langle u, v\rangle^2 \leq \|u\|^2 \|v\|^2$$

$$\|u + v\| \leq \|u\| + \|v\|$$

$$d(u, v) \leq d(u, w) + d(w, v)$$

4. Find angle between them,

$$u = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad v = \begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$$

Sol^n:

$$\langle u, v \rangle = tr(u^T v) = 1(-1) + 2(0) + 3(3) + 4(2)$$

$$= 16$$

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{tr(u^T u)}$$

$$= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2}$$

$$= \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

$$\|v\| = \sqrt{(-1)^2 + 0^2 + (3)^2 + (2)^2} = \sqrt{14}$$

$$\cos\theta = \frac{\langle u . v \rangle}{\|u\| \|v\|} = \frac{16}{\sqrt{30}\sqrt{14}} = 0.78$$

$$\theta = 38.67$$

☐ Two vectors u and v in an inner product space V called orthogonal if $\langle u, v \rangle = 0$

☐ If u and v are orthogonal then, $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

[10] Find that $u = (1,1)$ and $v = (1,-1)$ are onthogonal on not with respect to Euclidean inner product.

Sol^n:

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2$$

$$= 1 \times 1 + 1(-1)$$

$$= 0$$, onthogonal

☐ If W is a subspace of a real inner product space V, then the set of all vectors in V that are orthogonal to every vector in W is called orthogonal complement of W and is denoted by the symbol $W^{\perp}$.

- $W^{\perp}$ is a subspace of V
- $W \cap W^{\perp} = \{0\}$
- $(W^{\perp})^{\perp} = W$

☐ Let W be the subspace of R^6 spanned by the vectors :

$$w_1 = (1, 3, -2, 0, 2, 0), \quad w_2 = (2, 6, -5, -2, 4, -3)$$
$$w_3 = (0, 0, 5, 10, 0, 15), \quad w_4 = (2, 6, 0, 8, 4, 18)$$

A set of two or more vectors in a real inner product space is said to be orthogonal if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vector has norm 1 is said to be orthonormal.

If $S = \{v_1, v_2, \dots, v_n\}$ is an orthogonal set of nonzero vectors in an inner product space, then S is linearly independent.

□ If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal basis,

any vector in V

$$u = \frac{(u, v_1)}{\|v_1\|^2} v_1 + \frac{(u, v_2)}{\|v_2\|^2} v_2 + \cdots + \frac{(u, v_n)}{\|v_n\|^2} v_n$$

onthonormall,

$$u = (u, v_1) v_1 + (u, v_2) v_2 + \cdots + (u, v_n) v_n$$

□ $$(u)_S = \left(\frac{(u, v_1)}{\|v_1\|^2}, \frac{(u, v_2)}{\|v_2\|^2} \cdots \frac{(u, v_n)}{\|v_n\|^2} \right)$$

relative to an onthogonal basis

$$(u)_S = ((u, v_1), (u, v_2) \cdots (u, v_n))$$

relative to an onthonormal basis

☐ Normalizing,

$$u = \frac{1}{\|v\|} v$$

☐ Let $v_1 = (0,1,0)$, $v_2 = (1,0,1)$, $v_3 = (1,0,-1)$

normalize this,

Soln:

$\|v_1\| = 1$, $\|v_2\| = \sqrt{2}$, $\|v_3\| = \sqrt{2}$

Euclidean norms

$$u_1 = \frac{v_1}{\|v_1\|} = (0,1,0)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

Orthonormal,

$(u_1, u_2) = (u_1, u_3) = (u_2, u_3) = 0$

$\|u_1\| = \|u_2\| = \|u_3\| = 1$

$\boxed{+}$ $v_1 = (0,1,0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$,

$v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$

Express $u = (1,1,1)$ as a linear combination of these vectors (v_1, v_2, v_3); and Find coordinate vector $(u)_S$

Sol^n^:

$$u_1 = \frac{v_1}{\|v_1\|} = (0,1,0)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

$$(u_1, u_2) = (u_2, u_3) = (u_1, u_3) = 0$$

$$\|u_1\| = \|u_2\| = \|u_3\| = 1$$

orthonormal basis

now,

$$\langle u, v_1\rangle = 1$$

$$\langle u, v_2\rangle = -\frac{1}{5}$$

$$\langle u, v_3\rangle = \frac{7}{5}$$

For orthonormal,

$$u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3$$

$$u = v_1 - \frac{1}{5} v_2 + \frac{7}{5} v_3$$

$$(1,1,1) = (0,1,0) - \frac{1}{5}\left(\frac{-4}{5}, 0, \frac{3}{5}\right) + \frac{7}{5}\left(\frac{3}{5}, 0, \frac{4}{5}\right)$$

coordinate vector,

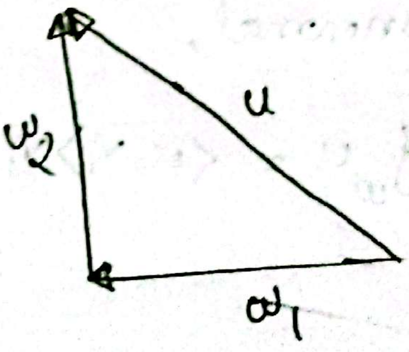
$$(u)_S = (\langle u, v_1\rangle, \langle u, v_2\rangle, \langle u, v_3\rangle)$$

$$= \left(1, -\frac{1}{5}, \frac{7}{5}\right)$$

☐ If W is a finite-dimensional subspace of an inner product space V, then every vector u in V can be expressed in exactly one way as,

$$\boxed{u = w_1 + w_2}$$

where w_1 is in W and w_2 is in $W^{\perp}$



$w_1 = \text{proj}_W u$

$w_2 = \text{proj}_{W^{\perp}} u$

$$u = \text{proj}_W u + \text{proj}_{W^{\perp}} u$$

$$= \text{proj}_W u + (u - \text{proj}_W u)$$

If $\{v_1, v_2, \cdots, v_r\}$ is orthogonal basis for W and u is any vector in V, then

$$\text{proj}_W u = \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2\rangle}{\|v_2\|^2} v_2 + \cdots + \frac{\langle u, v_r\rangle}{\|v_r\|^2} \cdot v_r$$

for orthonormal,

$$\text{proj}_W u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \cdots + \langle u, v_r\rangle v_r$$

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$v_1 = (0,1,0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$ → orthonormal vectors

then orthogonal projection of $u = (1,1,1)$ on W is -

Soln:

$$\text{proj}_W u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2$$
$$= 1\,(0,1,0) + \left(-\frac{1}{5}\right)\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$
$$= \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

$$\text{proj}_{W^\perp} u = u - \text{proj}\, u$$
$$= (1,1,1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$
$$= \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

⊞ Every non-zero finite dimensional inner product space has an orthonormal basis.

⊞ **Gram - Schimidt Process:**

To convert a basis $\{u_1, u_2, \ldots, u_n\}$ into an orthogonal basis $\{v_1, v_2, \ldots, v_n\}$ perform the following computations:

Step-1: $v_1 = u_1$

Step-2: $v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Step-3: $v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

$\vdots$

continue for n steps

optional: To convert orthogonal basis into an orthonormal basis normallize the orthogonal basis vectors.

[4] Apply the Gram Schmidt process to transform the basis vectors $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$ to orthogonal $\{v_1, v_2, v_3\}$ and orthonormal $\{q_1, q_2, q_3\}$ basis.

Solⁿ:

$$v_1 = u_1 = (1,1,1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$$

$$= (0,1,1) - \frac{2}{3}(1,1,1)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\|v_1\| = \sqrt{3} \quad , \quad \|v_2\| = \frac{\sqrt{6}}{3} \quad , \quad \|v_3\| = \frac{1}{\sqrt{2}}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

□ If A is an $m \times n$ matrix with linearly independent column vectors, the A can be factored as,

$$\boxed{A = QR}$$

where Q is an $m \times n$ matrix with orthonormal column vectors, R is an $n \times n$ invertible upper triangular matrix.

Find a QR decomposition of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Sol^n:

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v_1 = u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1/3}{2/3} \cdot \begin{bmatrix} -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/6 \\ 1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} (1, 1, 1) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{3}{\sqrt{6}} \times \left(-2/3, 1/3, -1/3\right)$$

$$= \left(-2/\sqrt{6}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \sqrt{2}\left(0, -\tfrac{1}{2}, \tfrac{1}{2}\right)$$

$$= \left(0, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

$$q_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad q_2 = \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_3, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_3, q_2 \rangle \\ 0 & 0 & \langle u_3, q_3 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & \frac{2}{\sqrt{6}} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$A = Q \cdot R$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & -2/\sqrt{6} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -\frac{1}{\sqrt{2}} \\ 1/\sqrt{3} & 1/\sqrt{6} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3/\sqrt{3} & 2/\sqrt{3} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 1/\sqrt{2} \end{bmatrix}$$