Linear Equations

- 2D → $2x + 3y = 5$
- 3D → $2x + 3y + 4z = 6$

General form :

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

x → variable

a → coefficient

- systems → collection
- If $b = 0$,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

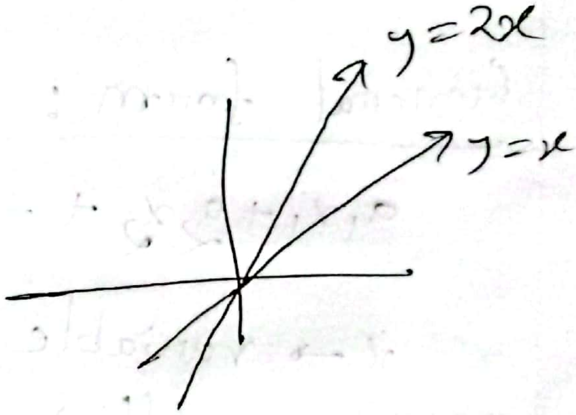
homogeneous linear equation

System of linear Equations → collection of linear equations.

$y = x$, $y = 2x$

here, $\boxed{x = 0, y = 0}$

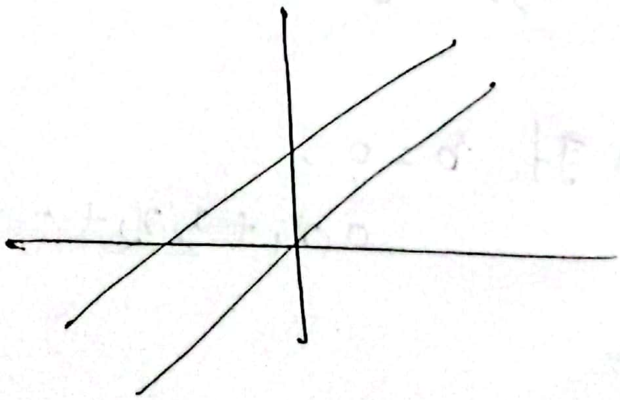
Solution possible (consistent)



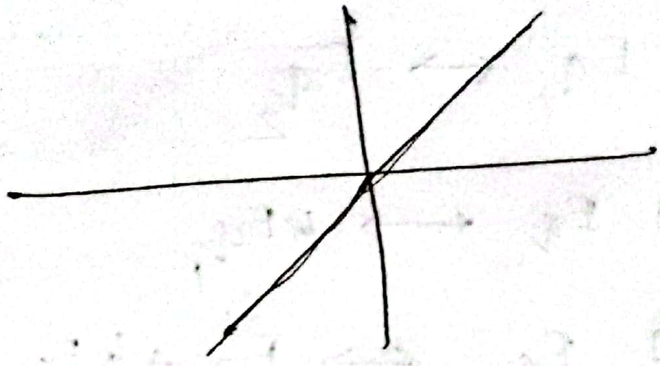
parallel

↓

No solution (Inconsistent)



Infinitely many
Solution possible



So, Three Case

No solution $\longrightarrow$ Inconsistent System

One solution
Infinitely many Solutions
$\Big\}\longrightarrow$ Consistent System

we can replace Eq_1 by $k*Eq_1$ and $k \in R - \{0\}$

- $Eq_1 \leftrightarrow Eq_2$
- $Eq_1 \leftrightarrow kEq_1$
- $Eq_1 \leftrightarrow Eq_1 + kEq_1$

- No Solution → lines may be parallel and distinct
- One solution → lines may intersect at only one point
- Infinitely many solutions → lines may coincide

- Every ~~linear~~ system of linear equations has zero, one or ~~two~~ infinitely many solutions. There are no other possibilities.

□ Linear Equations with one solⁿ:

$$x - y = 1$$
$$2x + y = 6$$
$$x = \frac{7}{3}, \quad y = \frac{4}{3}$$

□ A linear Systems with No Solutions:

$$x + y = 4 \quad \text{—(i)}$$
$$3x + 3y = 6 \quad \text{—(ii)}$$

$$\text{(i)} \times 3 - \text{(ii)}$$
$$0 = -6$$

Linear System with infinitely many sol^n:

$$4x - 2y = 1 \quad \text{— (i)}$$
$$16x - 8y = 4 \quad \text{— (ii)}$$

$(i) \times 4 - (ii)$,

$$0 = 0$$

so, x has no restriction on it's values,

Let's,

$$x = 1, \quad y = \frac{3}{2}$$
$$x = 2, \quad y = \frac{7}{2}$$

General Eq^n: (Parametric Sol^n)

$$x = t, \quad y = \frac{4t - 1}{2}$$

$$x - y + 2z = 5 \quad \text{— (i)}$$

$$2x - 2y + 4z = 10 \quad \text{— (ii)}$$

$$3x - 3y + 6z = 15 \quad \text{— (iii)}$$

$$\text{(i)} \times 12 - \left(\text{(ii)} \times 3 + \text{(iii)} \times 2\right)$$

$$\begin{array}{rrrcl}
12x & - 12y & + 24z & = & 60 \\
6x & - 6y & + 12z & = & 30 \\
(-) & (+) & (-) & & (-) \\
6x & - 6y & + 12z & = & 30 \\
(-) & (+) & (-) & & (-) \\
\hline
 & & 0 & = & 0
\end{array}$$

General Sol^{n}:

$$x = t, \quad y = p$$

$$z = \frac{5 - t + p}{2}$$

Augmented Matrix

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This called the augmented matrix.

To Solve a linear System

1. Multiply an equation through by a nonzero constant $(Eq_i \leftarrow kEq_i)$
2. Interchange two equations $(Eq_i \leftrightarrow Eq_j)$
3. Add a constant times one equation to another $(Eq_i \leftarrow Eq_i + kEq_j)$

For Augmented Matrix

1. Multiply a row through by a nonzero constant.
2. Interchange two rows
3. Add a constant times one row to another

→ These are called elementary row operations.

$\boxed{\text{H}}$

$$x+y+2z=9$$

$$2x+4y-3z=1$$

$$3x+6y-5z=0$$

target,

$$\begin{matrix} 1 & \square & \square \\ 0 & 1 & \square \\ 0 & 0 & 1 \end{matrix}$$

<u>Augmented Matrix:</u>

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

<u>Row:</u>

$$II \leftarrow II + I(-2)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$$III \leftarrow III + I(-3)$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$$II \leftarrow II \times \frac{1}{2}$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$$III \leftarrow III + II \times (-3)$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -\frac{1}{2} & -3/2 \end{bmatrix}$$

$$III \leftarrow III \times (-2)$$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\therefore \quad x + y + 2z = 9$

$y - \frac{7}{2} z = -\frac{17}{2}$

$z = 3$

$\therefore \; y = 2$

$x = 1$

Gaussian Elimination

Echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$x = 4, \quad y = 5, \quad z = 7$$

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\downarrow$ Row echelon form

$\downarrow$ Reduced Row Echelon form

properties :

1. If a row does not consist entirely of zeros, then the first non zero numbers in the row is a 1. (leading 1).

2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs further to the right that the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else in that column.

If A matrix follow (1 to 3) then it's a row echelon form

If A matrix follow (1 to 4) then it's a reduced row echelon form

☐ Reduce any matrix to reduced row echelon form:-

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step-I: Locate the leftmost column that doesn't consist entirely of zeros.

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

↳ Leftmost nonzero column

Step-2: Interchange the top row with another row, if necessary; to bring a nonzero entry to the of the column found in step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step-3: If the entry that is now at the top of the column in Step 1 is a, multiply the first row by 1/a in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad I \leftarrow I \times \frac{1}{2}$$

Step-4: Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad II \leftarrow II + I \times (-2)$$

Step-5: Now covers the top row in the matrix and begin again with step 1 applied to the submatrix that remains. Continue the submatrix in this way until the entire matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

↳ Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad II \leftarrow II \times \frac{-1}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \quad III \leftarrow III + II \times (-5)$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Leftmost nonzero column.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad III \leftarrow III \times 2$$

Step 6: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leadings 1's.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad II \leftarrow II + III \times \frac{7}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad I \leftarrow I + III \times (-6)$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad I \leftarrow I + II \times 5$$

Here, the last matrix is in reduced row echelon form.

- The procedure for reducing a matrix to reduced row echelon form is called Gauss-Jordan elimination.

- This process consists two phase

  forward phase → zeros are introduced below the leading 1's.

  backward phase → zeros are introduced above the leading 1's.

- If there is only forward phase then the procedure called Gaussian elimination

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

Soln:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \quad \begin{matrix} II \leftarrow II + I(-2) \\ IV \leftarrow IV + I(-2) \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix} \quad \begin{array}{l} III \leftarrow III - 5\{II \times (-1)\} \\ IV \leftarrow IV - 4\{II \times (-1)\} \end{array}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Interchange III and IV, then, $III \leftarrow III \times \frac{1}{6}$

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} II \leftarrow II + III(-3) \\ I \leftarrow I + II \times (2) \end{array}$$

$$\therefore \quad x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$

$$x_3 + 2x_4 = 0$$

$$x_6 = \frac{1}{3}$$

$x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = \frac{1}{3}$

$x_5 = t, \quad x_4 = s, \quad x_3 = -2s, \quad x_2 = r$

$\therefore x_1 = -3r - 4s - 2t$

Leading 1's variables are called the Leading variables and the remaining variables are called free variables.

Homogeneous Linear Systems

A system of linear equations is said to be homogenous if the constant terms are all zero.

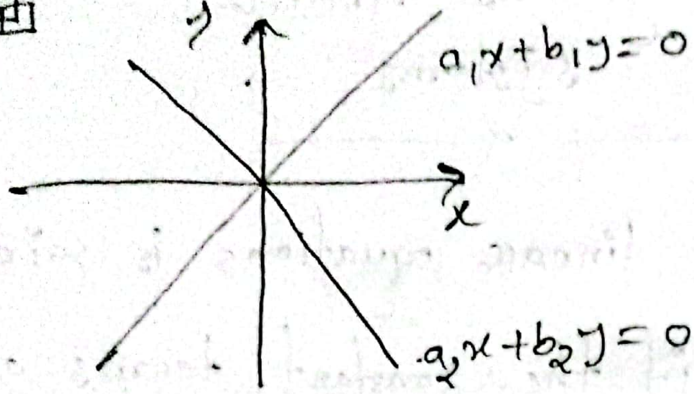
$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

* Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution.

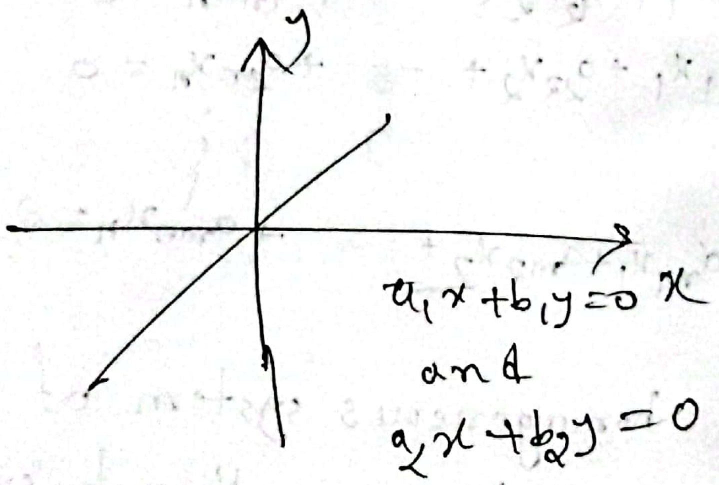
* These solution is called the trivial solution.

* If there are other solutions then it's called non-trivial solutions.

(ii)



only the trivial solution



Infinitely many solutions

Example:

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$

$$5x_3 + 10x_4 + 15x_6 = 0$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

same as previous example, we get

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

$\therefore$

$x_3 = -2x_4$, $x_1 = -3x_2 - 4x_4 - 2x_5$

$x_6 = 0$, $x_5 = t$, $x_4 = s$, $x_3 = -2s$, $x_2 = r$

$$x_1 = -3r - 4s - 2t$$

trivial solⁿ: $\boxed{r = s = t = 0}$

In augmented matrix elementary row operations do not alter columns of zeros in a matrix.

so, If a matrix is homogeneous then reduced row echelon form is also homogeneous.

◻ In homogeneous ~~of~~ system, if there are n unknowns and reduced row echelon form of it's augmented matrix has r non-zero rows, then the system has $n-r$ free variables.

here,

free variables, $x_5 = t$, $x_4 = s$, $x_2 = r$

◻ If a homogeneous system has more unknowns than equation, then this system has infinitely many solutions.

Facts:

1. Every matrix has a unique reduced row echelon form.

2. Row echelon forms are not unique

3. Despite this, reduced row echelon and row echelon has same number of zeros and leading 1's at the same position. Those are called pivot positions of A.

Matrix

Defn. A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

Notations $A_{m \times n}$ $m \rightarrow$ rows

$n \rightarrow$ columns

$m \times 1 \longrightarrow$ row vector / matrix

$1 \times n \longrightarrow$ column vector / matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

① $[a_{ij}]_{m \times n}$ or a_{ij}

- If, $m = n$, then $A_{m \times n} \to$ square matrix

  $i = j$, $a_{11}, a_{22} \cdots a_{nn} \to$ main diagonal

- Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

- $A+B$, $A-B$ only possible when both A, B has same size.

cA means multiplying each entry of the matrix A by c. The matrix cA is said to be a scalare multiple of A.

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

$A_{m\times r}$, $B_{r\times n}$ then, multiplication possible, and result, $AB_{m\times n}$

Zero matrix:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangular matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix}$$

Diagonal matrix:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Matrix Multiplication

1. Row- column Rule:

$A = [a_{ij}]$ is an $m \times r$

$B = [b_{ij}]$ is an $r \times n$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & & & \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1} * b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

$$\boxed{AB_{ij} = \sum_{k=1}^{r} a_{ik} * b_{kj}}$$

row col row col

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 40 & 33 \\ 62 & 97 & 90 \\ 114 & 154 & 147 \end{bmatrix}$$

Matrix multiplication by columns and by Rows

$$AB = A[b_1 \; b_2 \; \cdots \; b_n] = [Ab_1 \; Ab_2 \; \cdots \; Ab_n]$$ AB computed column by column

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$ → AB computed row by row

So,

jth column vector of $AB = A[j^{th}$ column of $B]$

i^{th} row vector of $AB = [i^{th}$ row of $A]B$

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 40 & 33 \\ 72 & 97 & 90 \\ 114 & 154 & 147 \end{bmatrix}$$

2nd column vector of $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} * \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 40 \\ 97 \\ 154 \end{bmatrix}$$

2nd row vector of $AB = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 72 & 97 & 90 \end{bmatrix}$$

Matrix Products as Linear Combinations

If $A_1, A_2, \dots, A_r$ are matrices of the same size and if $c_1, c_2, \dots, c_r$ are scalars, then an expression of the form,

$$c_1A_1 + c_2A_2 + \dots + c_rA_r$$

is called a linear combination of $A_1, A_2, \dots, A_r$ with coefficients $c_1, c_2, \dots, c_r$.

Let's, $A_{m\times n}$ and $x_{n\times 1}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

If A is $m \times n$ matrix, and if x is an $n \times 1$ column vector; then the product Ax can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of x.

Ex:

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \qquad x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$Ax = 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + -1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ -9 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+6 \\ 2-2-9 \\ 4-1-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Ex:

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 0 \end{bmatrix} & 1\times\begin{bmatrix} 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 3 \\ 6 \end{bmatrix} + 7\begin{bmatrix} 4 \\ 0 \end{bmatrix} & 4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 4 \\ 0 \end{bmatrix} & 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} + 2\times\begin{bmatrix} 4 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -6 \end{bmatrix} + \begin{bmatrix} 28 \\ 0 \end{bmatrix} & \begin{bmatrix} 4 \\ 8 \end{bmatrix} + \begin{bmatrix} 6 \\ 18 \end{bmatrix} + \begin{bmatrix} 20 \\ 0 \end{bmatrix} & \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} 8 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Column - Row Expansion

$A_{m \times r}$ matrix partitioned into its r column vectors $c_1, c_2, \ldots, c_r$ (each of size $m \times 1$)

$B_{r \times n}$ matrix partitioned into its r row vectors $r_1, r_2, \ldots, r_r$ (each of size $1 \times n$).

$$A = \begin{bmatrix} | & | & \cdots & | \\ c_1 & c_2 & \cdots & c_r \\ | & | & \cdots & | \end{bmatrix}_{m \times r}$$

$$B = \begin{bmatrix} \cdots & r_1 & \cdots \\ \cdots & r_2 & \cdots \\ \cdots & r_r & \cdots \end{bmatrix}_{r \times n}$$

$$\therefore AB = c_1 r_1 + c_2 r_2 + \cdots + c_r r_r$$

Ex:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}, \quad r_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

Matrix Form of a Linear System:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad \vdots \qquad\quad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{bmatrix}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{bmatrix}
$$

$$
[A|b] =
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}\right]
$$

↳ Augmented Matrix

Some Transposes :

$A_{m \times n}$ then, $A^T = A_{n \times m}$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Trace → some of main diagonal

$$tr(A) = 1 + 7 + 6 = 14$$

Properties :

1. $A+B = B+A$ (Commutative - add)

2. $A+(B+C) = (A+B)+C$ (Distributive - add)

3. $A(BC) = (AB)C$ (Associative - mul)

$a(bC) = (ab)C$

$a(BC) = (aB)C$ $\boxed{AB \neq BA}$

4. $A(B+C) = AB+AC$

$(B+C)A = BA+CA$

$A(B-C) = AB-AC$

$(B-C)A = BA-CA$

$a(B \pm C) = aB \pm aC$

$(a \pm b)C = aC \pm bC$

Rank of Matrix

A matrix is said to be of rank r, when

i. There is at least one minor of A of order r which does not vanish.

ii. Every minor of A of order $(r+1)$ or higher vanishes.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$|A| = 0 \qquad \rho(A) < 3$$

* Rank of matrix no. of different rows. (linearly independent)

At least one minor non zero,

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}_{2\times 2}$$

$$\therefore \rho(A) = 2$$

Rank = Number of non-zero row in upper triangular matrix.

the rank of a matrix is the largest order of any non-vanishing minor of the matrix.

i. If a matrix has a non-zero minor of order r, its rank is $\geq r$.

ii. If all minors of a matrix of order $r+1$ are zero, its rank is $\leq r$.

The rank of a matrix A shall be denoted by $\rho(A)$.

⊞ Elementary transformations do not alter the rank of a matrix i.e., equivalent matrices have the same rank.

⊞ The rank of the transpose of a matrix is the same as than of the original matrix.

⊞ The rank of a product of two matrices can not be exceed the rank of either matrix i.e.,

$$\rho(AB) \le \rho(A) \text{ and } \rho(AB) \le \rho(B)$$

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Rank of Matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$\det(A) = 0$$

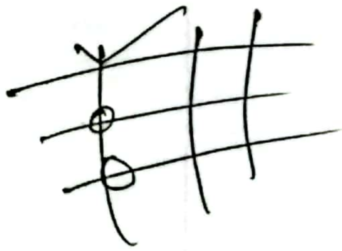
$$\therefore \rho(A) < 3.$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} = 4 - 2 = 2 \neq 0$$

$$\therefore \rho(A) = 2$$

Transformation: $R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$

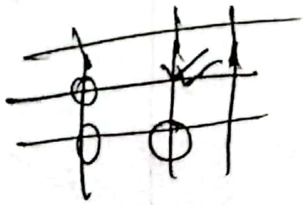
$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 2 & -1 \end{bmatrix}$$



$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \rho(A) = 2$$

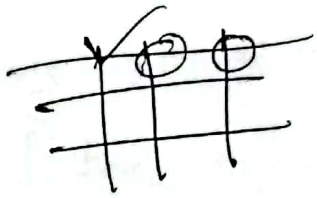


Normal Form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

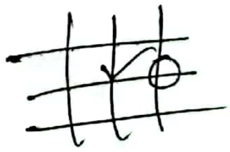
$C_2 \rightarrow C_2 - 2C_1, \quad C_3 \rightarrow C_3 - 3C_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$



$C_3 \rightarrow 2C_3 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$



$\rho(A) = 2$

Significance :

1. Rank of matrix = number of var,
the system has a unique solⁿ

2. Rank of matrix < number of vars,
it indicates infinite solⁿ

3. Rank of matrix < number of eqⁿ,
the system is inconsistent (no solⁿ)

4. If rank < dimension, matrix
is not inventible

5.

Properties of Zero Matrix

a) $A + O = O + A = A$

b) $A - O = A$

c) $A - A = O$

d) $O \cdot A = O$

e) if $cA = O$ then $c = 0$ or $A = O$

Identity Matrices:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & \cdots & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \cdots 0 \\ 0 & 0 & 0 & \cdots 1 \end{bmatrix}_{n \times n}$$

If A is $m \times n$ matrix, then,

$$A I_n = A$$

$$I_m A = A$$

If A is $n \times n$ then,

$$A I_n = A$$

$$I_n A = A$$

☐ If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

$\boxed{R \text{ is a row of zeros or } R = I_n}$

Inverse Matrix :

If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be invertible (or nonsingular) and B is called inverse of A. If no such matrix B can be found, then A is said to be singular.

□ If square matrix with a row or column of zeroes is singular.

If B and C are both inverses of the matrix A, then $B = C$

$$AB = I = BA$$
$$AC = I = CA$$

$$\therefore AB = I$$
$$\Rightarrow CAB = CI$$
$$\Rightarrow I.B = C$$
$$\therefore \boxed{B = C}$$

$$\boxed{AA^{-1} = A^{-1}A = I}$$

The matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad-bc \neq 0$,

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□ If A and B are invertible matrices with the same size, then AB is invertible.

$$\boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$\boxed{(A_1A_2 \cdots A_K)^{-1} = A_K^{-1}A_{K-1}^{-1} \cdots A_2^{-1}A_1^{-1}}$$

$AB\,B^{-1}A^{-1}$

$= A(BB^{-1})A^{-1}$

$= A.I.A^{-1}$

$= AA^{-1}$

$= I$

$B^{-1}A^{-1}AB$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}.I.B$

$= B^{-1}B$

$= I$

□ $(A^{-1})^{-1} = A$

$A^n = I$

$A^n = A.A \cdots A$ (n factors)

$A^{-n} = (A^{-1})^n = A^{-1}.A^{-1} \cdots A^{-1}$ (n factors)

$$A^r A^s = A^{r+s}$$

$$(A^r)^s = A^{rs}$$

If A is invertible and n is a non-negative integer, then,

a) A^{-1} is invertible and $(A^{-1})^{-1} = A$

b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$

c) kA is invertible for any non-zero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$

Properties of the Transpose

a) $(A^T)^T = A$

b) $(A \pm B)^T = A^T \pm B^T$

c) $(K.A)^T = K A^T$

d) $(AB)^T = B^T A^T$

e) $(A^{-1})^T = (A^T)^{-1}$

Elementary Matrices

Row operations :

1. Multiply a row by a non-zero constant c.
2. Interchange two rows
3. Add a constant c times one row to other.

If we obtain B from A by row operations, then,

1. Multiply the same row by $1/c$.
2. Interchange two rows.
3. If B resulted by adding c times r_i of A to row r_j, then add $-c$ times r_j to r_i.

applying above operations to B we can obtain A.

☐ Matrices A and B are said to be row equivalent if either can be obtain from the other by a sequence of elementary row operations.

☐ A matrix E is called an elementary matrix if it can be obtained from an identity matrix by performing a single elementary row operation

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 5 & 6 \end{bmatrix}$$

$A \rightarrow R_1 = R_1 + R_2$

B

$$A = \begin{bmatrix} 7 & 9 & 11 \\ 6 & 7 & 8 \\ 0 & 5 & 6 \end{bmatrix}$$

$A \rightarrow R_2 \leftrightarrow R_3$ (interchange)

$$A = \begin{bmatrix} 7 & 9 & 11 \\ 0 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix}$$

$A \rightarrow R_3 = 2 \times R_3$

$$A = \begin{bmatrix} 7 & 9 & 11 \\ 0 & 5 & 6 \\ 12 & 14 & 16 \end{bmatrix} = B$$

then, A, B said to be row equivalent.

Ex

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ (R_2, R_3 interchange)

$E_1 \to$ is an elementary matrix

Row Operations by Matrix Multiplication

If the elementary matrix E results from performing a certain row operation on I_m and if A is an $m \times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Example:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$I \rightarrow R_3 \leftarrow R_3 + R_1 * 3$$

$A \to R_3 \leftarrow R_3 + R_1 * 3$

$$A' = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

$$E\,A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix} = A'$$

$\boxplus$ Row Operation on I that produces E

1. Multiply row i by $c \neq 0$.
2. Interchange rows i and j.
3. Add c times row i to row j

$\boxplus$ Row Operation on E that produces I

1. Multiply row i by $1/c$
2. Interchange rows i and j
3. Add $-c$ times row i to row j

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7

Multiply the second row by $1/7$

☐ Every elementary matrix is invertible, and the inverse is also an elementary matrix.

if E is an elementary matrix, then E results by performing some row operation on I.

Let E_0 be the matrix that results when the inverse of this operation is performed on I.

then,
$$E E_0 = I$$
$$E_0 E = I$$

so, we can say that elementary matrix E_0 is the inverse of E.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \quad [R_3 \leftarrow R_3 + 3 * R_1]$$

$$E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \quad [R_3 \leftarrow R_3 - 3 * R_1]$$

$$E E_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

$$E_0 E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Equivalent Statements

If A is an $n \times n$ matrix, then the following statements are equivalent, that is all true or all false.

1. A is invertible.
2. $Ax = 0$ has only trivial solution
3. The reduced row echelon form of A is I_n
4. A is expressible as a product of elementary matrices.
5. $Ax = b$ is consistent for every $n \times 1$ matrix b.
6. $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

proof:

$\boxed{1 \Rightarrow 2}$

Assume A is invertible and X_0 be any sol^n of $Ax=0$

$$\therefore \quad Ax_0 = 0$$

$$\Rightarrow A^{-1}Ax_0 = 0$$

$$\Rightarrow Ix_0 = 0$$

$$\Rightarrow x_0 = 0$$

, so $Ax=0$ has only one sol^n.

$\boxed{2 \Rightarrow 3}$ Let $Ax=0$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

Assume that, system has only trivial sol^n.

If we solve by Gauss-Jordan elimination then the system of equations corresponding to the reduced row echelon form of the augmented matrix will be,

$$\begin{aligned} x_1 \quad\quad\quad\quad &= 0 \\ x_2 \quad\quad &= 0 \\ \ddots \quad & \\ x_n &= 0 \end{aligned}$$

Thus the augmented matrix,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & a_{nn} & 0 \\ a_{n1} & a_{n2} & \cdots & & \end{bmatrix}$$

will become

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

It's an Identity matrix

$\boxed{3 \Rightarrow 4}$ Assume that the reduced row echelon form of A is I_n. So, A can be reduced to I_n, by a finite sequence of row operations.

Thus we find elementary matrices, $E_1, E_2 \cdots E_k$ such that

$$\boxed{E_k E_{k-1} \cdots E_2 E_1 A = I_n}$$

and, we know, $E_1, E_2 \cdots E_k$ are invertible,

$\therefore$ multiplying both sides by $E_k^{-1}, E_{k-1}^{-1} \cdots E_2^{-1}, E_1^{-1}$

$$A = E_k^{-1} . E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} I_n$$

$$\boxed{\therefore A = E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1}}$$

$\boxed{\text{I}}$ To find the inverse of an inventible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$[A \mid I]$.

$$\left[\begin{matrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right]$$

$$\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & -2 & 5 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right] \quad \begin{matrix} R_2 \leftarrow R_2 + (-2) \times R_1 \\ R_3 \leftarrow R_3 + (-1) * R_1 \end{matrix}$$

$$\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & -1 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -5 & 2 & 1 \end{matrix}\right] \quad R_3 \leftarrow R_3 + 2 * R_2$$

$$\left[\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{matrix}\right] \left[\begin{matrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & -1 \end{matrix}\right] \quad R_3 \leftarrow (-1) * R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -14 & 6 & 3 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \leftarrow R_2 + 3 * R_3 \\ R_1 \leftarrow R_1 + (-3) * R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \quad R_1 \leftarrow R_1 + (-2) * R_2$$

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

A system of linear equations has zero, one or infinitely many solutions. There are no other possibilities.

Assume, $Ax = b$ has more than one sol^n,

Let, $x_0 = x_1 - x_2$, x_1, x_2 are any two distinct sol^n

$$Ax_0 = A(x_1 - x_2)$$
$$= Ax_1 - Ax_2$$
$$= b - b$$
$$= 0$$

For; $k \rightarrow$ be any scalar.

$$A(x_1 + kx_0) = Ax_1 + Akx_0$$
$$= b + k(Ax_0)$$
$$= b + 0$$
$$= b$$

$\therefore$ $x_1 + kx_0$ is sol^n. k has many values so, there are many solutions.

▣ If A is an invertible $n \times n$ matrix, then for each $n \times 1$ matrix b, the system of equations $Ax = b$, has exactly one solution. namely $x = A^{-1}b$.

$$Ax = b$$

$$\Rightarrow A^{-1}Ax = A^{-1}b$$

$$\Rightarrow \boxed{x = A^{-1}b}$$

Ex: $x_1 + 2x_2 + 3x_3 = 5$

$2x_1 + 5x_2 + 3x_3 = 3$

$x_1 + 8x_3 = 17$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$$x = A^{-1} b$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore \; x_1 = 1$

$x_2 = -1$

$x_3 = 2$

☐ Let A be a square matrix,

1. If B is a square matrix satisfying $BA = I$,
then $B = A^{-1}$

2. If B is a square matrix satisfying $AB = I$,
then $B = A^{-1}$

☐ Let A and B be square matrices of the same size. If AB is invertible then A and B must also be invertible.

Diagonal Matrices:

$$D = \begin{bmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}_{3\times 3}$$

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/d_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/d_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1/d_{nn} \end{bmatrix}$$

$$D^K = \begin{bmatrix} d_1^K & 0 & \cdots & 0 \\ 0 & d_2^K & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & d_n^K \end{bmatrix}$$

Upper and Lower Triangular Matrices

$$\begin{bmatrix} 4 & 7 & 8 \\ 0 & 9 & 4 \\ 0 & 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} 4 & 0 & 0 \\ 9 & 7 & 9 \\ 10 & 11 & 10 \end{bmatrix}$$

$\downarrow$ upper triangular

$\downarrow$ Lower triangular

- Transpos of upper is lower and vice versa.
- Product of lower is lower and upper is upper.
- Inverse of lower is lower and upper is upper.

- A square matrix A is said to be symmetric if $A = A^T$

- If A and B is symmetric, then $A+B$ and $A-B$ are symmetric

- A and B commute means, $AB = BA$

- The product of two symmetric matrices is symmetric if and only if the matrices commute

- If A is an invertible symmetric matrix then A^{-1} is symmetric.

- If A is invertible, then AA^T and A^TA also are invertible

Determinants

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after the ith row and jth column are deleted from A. The number $(-1)^{i+j} M_{ij}$ is denoted by C_{ij} and is called the cofactor of entry a_{ij}

Ex:

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

Minor, $M_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 5 \times 8 - 6 \times 4 = 16$

cofactor $C_{11} = (-1)^{1+1} \cdot M_{11} = 16$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & & \end{bmatrix}$$

If A is an $n\times n$ matrix, then the numbers obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansions of A. That is,

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

(cofactor expansion along the jth column)

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

(cofactor expansion along the ith row)

Ex:

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$\det(A) = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - 1(-11) + 0$$

$$= -1$$

Determinant of lower and upper triangular.

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix}$$

$$\det(A) = a_{11} \times a_{22} \times a_{33} \qquad \det(B) = b_{11} \times b_{22} \times b_{33}$$

☐ Let A be a square matrix. If A has a row of zeros or a column of zeros, then

$$\det(A) = 0$$

☐ Let A be square matrix. Then

$$\det(A) = \det(A^T)$$

☐ Let A be $n \times n$ matrix;

1. If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then

$$\det(B) = k \cdot \det(A).$$

2. If B is the matrix the results when two rows or two columns of A are interchanged, then

$$\det(B) = -\det(A)$$

3. If B is the matrix that results when a multiple of one row of A is added to another or when a multiple of one column is added to another, then

$$\det(B) = \det(A)$$

Let E be an $n \times n$ elementary matrix

1. If E results from multiplying a row of I_n by a non-zero number k,

   then $\det(E) = k$.

2. If E results from interchanging two rows of I_n then,

   $\det(E) = -1$

3. If E results from adding a multiple of one row of I_n to another,

   then, $\det(E) = 1$

[d] In square matrix A has two proportional rows or columns, then.

$$\det(A) = 0$$

- If A is $n \times n$ matrix then,

$$\boxed{\det(KA) = k^n \cdot \det(A)}$$

- $\det(A+B) \neq \det(A) + \det(B)$

- $\det(AB) = \det(A)\det(B)$

- If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\boxed{\det(EB) = \det(E)\det(B)}$$

Theorem: A square matrix A is inveatible if and only if $\det(A) \neq 0$.

Let, R is a reduced row echelon form of A.

$E_1, E_2, ----, E_r$ are elementary matrices.

So, we can say,

$$R = E_1 E_2 ----E_r A$$

$$\det(R) = \det(E_1)\det(E_2) ---- \det(E_r) \det(A)$$

We know that, determinant of elementary matrix is non-zero.

So, If both R and A is zero or non-zero.

Let's assume A is inventible, then we can say that

$R = I$

so, $\det(R) = 1 \neq 0$

so, $\det(A) \neq 0$

Again, let's $\det(A) \neq 0$, so, $\det(R) \neq 0$

so, R have to be I.

So, A is inventible if and only if $\det(A) \neq 0$.

Theorem: If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} \cdot adj(A)$$

Let's,

$$A\,adj(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{j1} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{j2} & & c_{n2} \\ \vdots & & & & & \\ c_{1n} & c_{2n} & \cdots & c_{jn} & \cdots & c_{nn} \end{bmatrix}$$

In the product entry of (ith row and jth column) is,

$$a_{i1}c_{j1} + a_{i2}c_{j2} + \cdots + a_{in}c_{jn}$$

we know, if $i=j$, then the entry of product would be $\det(A)$

if, $i \neq j$, then the entry of product would be 0.

$$A\,adj(A) = \begin{bmatrix} det(A) & 0 & \cdots & 0 \\ 0 & det(A) & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & det(A) \end{bmatrix}$$

$$= det(A) \cdot I$$

Since, $det(A) \neq 0$ [A is inventible]

$$\frac{A\,adj(A)}{det(A)} = I$$

$$\Rightarrow A\left[\frac{1}{det(A)} \cdot adj(A)\right] = I$$

$$\Rightarrow A^{-1}A\left[\frac{1}{det(A)} \cdot adj(A)\right] = A^{-1}I$$

$$\Rightarrow A^{-1} = \frac{1}{det(A)} \cdot adj(A)$$

$$\therefore \boxed{A^{-1} = \frac{1}{det(A)} \cdot adj(A)}$$

Theorem: Cramer's Rule

If $Ax = b$ is a system of n linear eqn in n unknowns such that $\det(A) \neq 0$, then the system has a unique soln. This solution is,

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)} \quad \cdots \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Let's,

A is invertibe $[\det(A) \neq 0]$

$$\therefore \quad x = A^{-1} b$$

$$\Rightarrow x = \frac{1}{\det(A)} \operatorname{adj}(A) \cdot b$$

$$x = \frac{1}{\det(A)} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & & \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{\det(A)} \begin{bmatrix} b_1 c_{11} + b_2 c_{21} + \cdots + b_n c_{n1} \\ b_1 c_{12} + b_2 c_{22} + \cdots + b_n c_{n2} \\ \vdots \\ b_1 c_{1n} + b_2 c_{2n} + \cdots + b_n c_{nn} \end{bmatrix}$$

The entry in the jth row of x is therefore,

$$x_j = \frac{b_1 c_{1j} + b_2 c_{2j} + \cdots + b_n c_{nj}}{\det(A)}$$

Now let,

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

A_j differs from A only in the j^{th} column.
it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are same as the cofactors of the corresponding entries in the jth column of A.

$$\det(A_j) := b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

$$\therefore \; x_j = \frac{\det(A_j)}{\det(A)}$$

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