Matrix Transformations

- If f is a function with domain R^n and codomain R^m, then f is a transformation from R^n to R^m

$$f : R^n \longrightarrow R^m$$

If $\boxed{m = n}$ $\rightarrow$ operators

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$$w_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$w_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots$$
$$w_m = a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n$$

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{w = Ax}$$

$T_A : R^n \to R^m$ — matrix transformation

$$\boxed{w = T_A(x)}$$

$$\boxed{x \xrightarrow{T_A} w}$$

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[4] Transformation from R^4 to R^3

$$w_1 = 2x_1 - 3x_2 + x_3 - 5x_4$$

$$w_2 = 4x_1 + x_2 - 2x_3 + x_4$$

$$w_3 = 5x_1 - x_2 + 4x_3$$

Solⁿ:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}$$

Let, $x = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = T_A(x) = Ax = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}$$

Zero Transformations,

If 0 is the $m \times n$ zero matrix, then,

$$T_0(x) = 0x = 0$$

multiplication by zero, maps every vector in R^n into the zero vector in R^m.

Identity Operators,

$$T_I(x) = Ix = x$$

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[4] For every matrix A the matrix transformation $T_A : R^n \to R^m$

a] $T_A(0) = 0$.

b] $T_A(ku) = kT_A(u)$

c] $T_A(u+v) = T_A(u) + T_A(v)$

d] $T_A(u-v) = T_A(u) - T_A(v)$

[5]

$$T_A(k_1u_1 + k_2u_2 + \cdots + k_ru_r) = k_1T_A(u_1) + k_2T_A(u_2) + \cdots + k_rT_A(u_r)$$

Every linear transformation from R^n to R^m is a matrix transformation and vice-versa.

If $T_A : R^n \to R^m$

and $T_B : R^n \to R^m$

and $T_A(x) = T_B(x)$

Then,

$$A = B$$

Finding the Standard Matrix for a Matrix Transformation

<u>step-1:</u> Find the images of the standard basis vectors $e_1, e_2, \dots e_n$ for R^n.

<u>step-2:</u> construct the matrix that has the images obtained in step 1 as its successive columns. This matrix is the standard matrix for the transformation

Find the standard matrix A for the linear transformation $T: R^2 \to R^2$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Soln:

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

If,

$$T\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \\ 3 \end{bmatrix}$$

$$T(x, y) = (x, -y)$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$T(x, y) = (-x, y)$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$T(x, y) = (y, x)$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$T(x, y, z) = (x, y, -z)$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$T(x,y,z) = (x,-y,z)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(x,y,z) = (-x,y,z)$

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(x,y) = (x,0)$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$T(x,y) = (0,y)$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

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$T(x,y,z) = (x,y,0)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

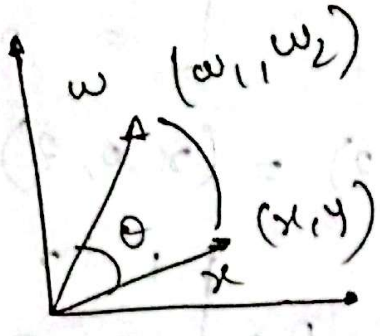
$T(x,y,z) = (x,0,z)$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$T(x,y,z) = (0,y,z)$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

☐ counterclockwise rotation about the origin through an angle θ.



$$w_1 = x\cos\theta - y\sin\theta$$

$$w_2 = x\sin\theta + y\cos\theta$$

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Ex Find the image of $x=(1,1)$ under a rotation of $\pi/6$ radius ~~or~~ about the origin.

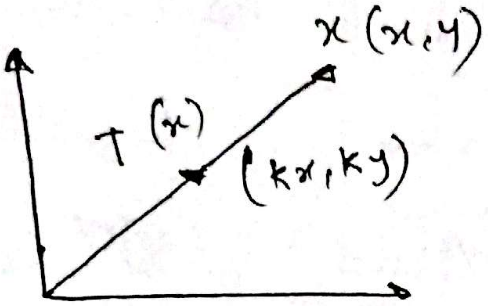
Sol^n:

$$R_{\pi/6}x = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

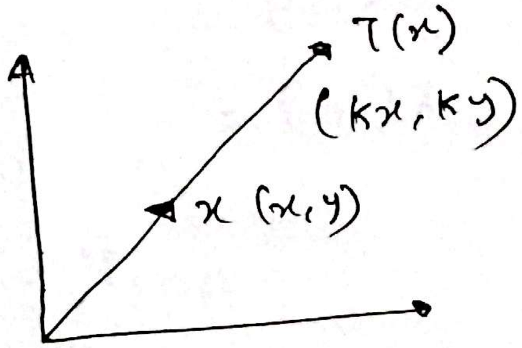
$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}+1}{2} \end{bmatrix}$$

Contraction with factor k in R^2 $(0 \le k < 1)$



Dilation with factor k in R^2 $(k > 1)$



$$A = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

<u>For R^3:</u>

$$A = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$T_A : R^n \to R^k$

$T_B : R^k \to R^m$

$$(T_B \circ T_A)(x) = T_B(T_A(x))$$
$$= B(T_A(x))$$
$$= B(Ax)$$
$$= (BA)x \qquad = T_{BA}$$

$T_C \circ T_B \circ T_A = T_{CBA}$

$T_A \circ T_B \neq T_B \circ T_A$

[E] A matrix transformation $T_A : R^n \to R^m$ is said to be one-to-one if T_A maps distinct vectors (points) in R^n into distinct vectors (points) in R^m.

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□: If $T_A : R^n \to R^m$ is a matrix transformation, then the set of all vectors in R^n that T_A maps into 0, is called the kernel of T_A and denoted by $\ker(T_A)$. The set of all vectors in R^m that are images under this transformation of at least one vector in R^n is called the range of T_A denoted $R(T_A)$.

$$\ker(T_A) = \text{null space of } A$$
$$R(T_A) = \text{column space of } A$$

If A is an $m \times n$ matrix, there are three ways of viewing the same subspace of R^n:

- Matrix view → the null space of A
- System view → the solution space of $Ax=0$
- Transformation view → the kernel of T_A

Three ways of viewing the same subspace of R^m:

- Matrix view → the column space of A
- System view → all b in R^m for which $Ax=b$ cosistent
- Transformation view → the range of T_A

General Linear Transformations

If $T: V \to W$ is a mapping from a vector space V to a vector space W, the T is called a linear transformation from V to W if the following two properties hold for all vectors u and v in V and for all scalars k:

i) $T(ku) = kT(u)$

ii) $T(u+v) = T(u) + T(v)$.

If $T: V \to W$

a) $T(0) = 0$

b) $T(u - v) = T(u) - T(v)$

Zero Transformation,

$T(u+v) = 0$ $\quad$ $T(kU) = 0$

$T(u) = 0$

$T(v) = 0$

$T(u+v) = 0 = 0 + 0 = T(u) + T(v)$

$T(kU) = 0 = kT(u)$

Let $T: V \to W$ be linear transformation where V is finite-dimensional. If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then the imge of any vector v in V can be expressed as

$$T(v) = c_1 T(v_1) + c_2 T(v_2) + \cdots + c_n T(v_n)$$

where $c_1, c_2 \ldots c_n$ are the coefficient required to express v as a linear combination of the vector in the basis V.

Q] Cosider the basis $S = \{v_1, v_2, v_3\}$ for R^3, where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$

Let $T: R^3 \to R^2$ be the linear transformation for which $T(v_1) = (1,0)$, $T(v_2) = (2,-1)$,

$T(v_3) = (4,3)$

Find a formula for $T(x_1, x_2, x_3)$ and then use that formula to compute $T(2,-3,5)$.

Soln:

$$\boxed{T(v) = c_1 T(v_1) + c_2 T(v_2) + c_3 T(v_3)}$$

$\Rightarrow$ ~~$T(x_1, x_2, x_3) =$~~ $c_1 ($

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\Rightarrow (x_1, x_2, x_3) = c_1 (1,1,1) + c_2 (1,1,0) + c_3 (1,0,0)$$

$c_1 + c_2 + c_3 = x_1$

$c_1 + c_2 = x_2$

$c_1 = x_3$

$c_2 = x_2 - x_3$

$c_3 = x_1 - x_2$

$$(x_1, x_2, x_3) = x_3(1,1,1) + (x_2 - x_3)(1,1,0) + (x_1 - x_2)(1,0,0)$$

$$(x_1, x_2, x_3) = x_3 v_1 + (x_2 - x_3) v_2 + (x_1 - x_2) v_3$$

$$T(x_1, x_2, x_3) = x_3 T(v_1) + (x_2 - x_3) T(v_2) + (x_1 - x_2) T(v_3)$$

$$= x_3(1,0) + (x_2 - x_3)(2,-1) + (x_1 - x_2)(4,3)$$

$$= (4x_1 - 2x_2 - x_3,\ 3x_1 - 4x_2 + x_3)$$

$$T(2,-3,5) = (9, 23)$$

☐ If $T: V \to W$ is a linear transformation, then:

a] The kernel of T is a subspace of V

b] The range of T is a subspace of W

☐ Let $T: V \to W$ be a linear transformation. If the range of T is finite-dimensional, then its dimension is called the rank of T; and if the kernel of T is finite-dimensional, then it's dimension is called the nullity of T. The rank of T is denoted by rank(T) and the nullity of T by nullity(T)

☐ If $T: V \to W$ is a linear transformation from a finite-dimensional vector space V to vector space W, then the range of T is finite-dimensional, and

$$\boxed{\text{rank}(T) + \text{nullity}(T) = \dim(V)}$$