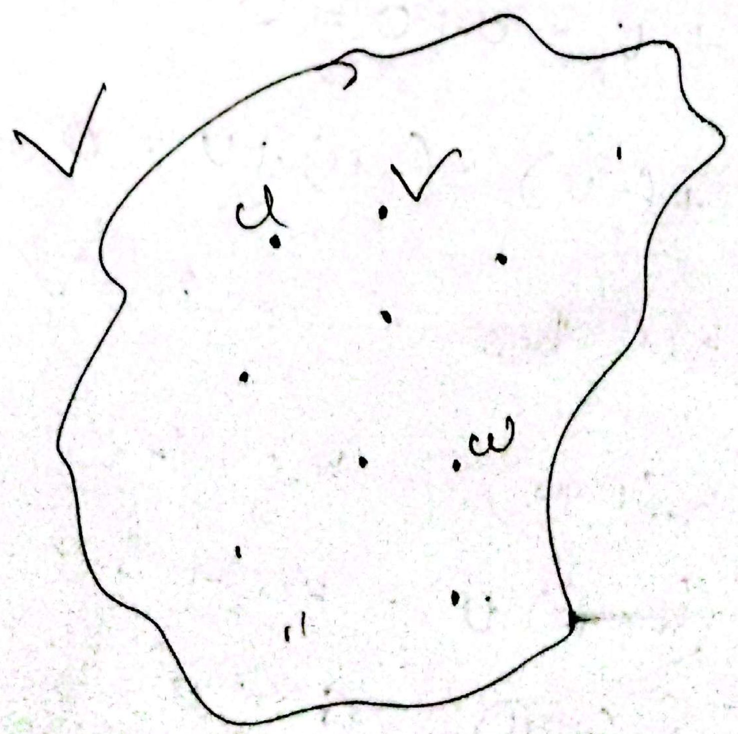
Vector Spaces

Let V be an arbitrary non empty set of objects.

We can call V a vector space if two operations are defined for V one is addition and other one is multiplication.

Let's, u, v, w are objects in V



If we want to call V a vector space then following 10 axioms has to be satisfied by v, u, w.

1. If u and v are in V, then

   $u+v$ has to be in V

2. $u+v = v+u$

3. $u+(v+w) = (u+v)+w$

4. $0+u = u+0 = u$

5. $u+(-u) = (-u)+u = 0$

6. ku has to be in V

7. $k(u+v) = ku+kv$

8. $(k+m)u = ku+mu$

9. $k(mu) = (km)u$

10. $1.u = u$

☒ If axioms 1 and 6 hold. Remaining axioms automatically holds.

Ex: Zero vectors space.

Soln:

Axioms 1: $0+0=0$

Axioms 6: $k.0=0$

Ex: R^n is a vector space

Soln:

$V=R^n$,

$u+v=(u_1,u_2 \ldots u_n)+(v_1+v_2+\ldots+v_n)$

$=(u_1+v_1, u_2+v_2, \ldots u_n+v_n)$

$ku=(ku_1, ku_2 \ldots ku_n)$

Ex: R^2 is a vector space or not!

$V = R^2$

$u = (u_1, u_2)$, $v = (v_1, v_2)$

let's define addition, multiplication.

$u + v = (u_1 + v_1 , u_2 + v_2)$

$ku = (ku_1, 0)$ → [define]

From axioms we know,

$1u = u$

$\therefore 1 \cdot u$

$= 1(u_1, u_2)$

$= u_1, 0$

$\neq u$

so, R^2 is not a vector space

[i] If u and v are objects in V, then $u+v$ is in V

↳ closure under addition

[ii] If k is any scalar and u is any object in V, then ku is in V.

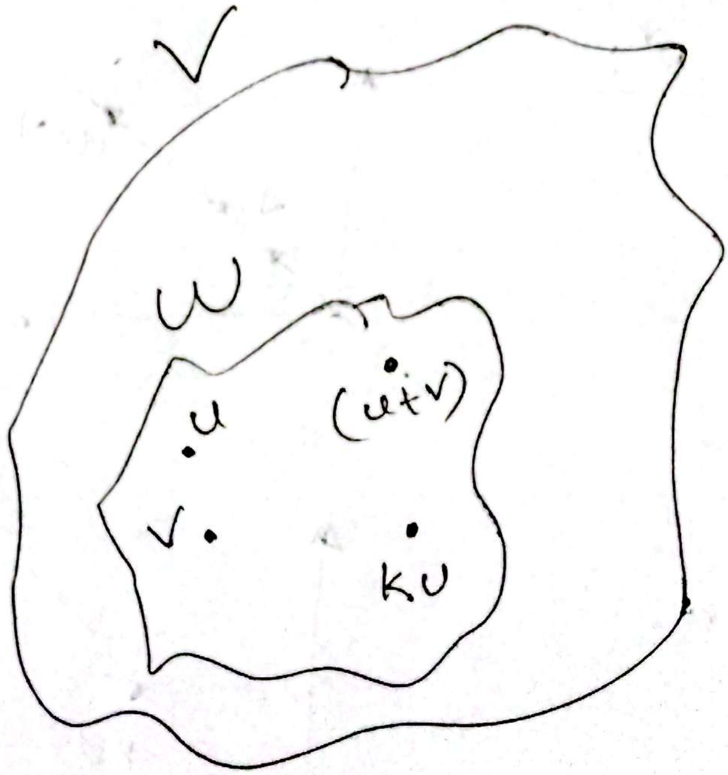
↳ closure under scalar Multiplication

Subspace

[d] W is a subspace of V, if and only if:

a] If u and v are in W and u+v also in W

b] ku also in W.



Ex: The zero subspace

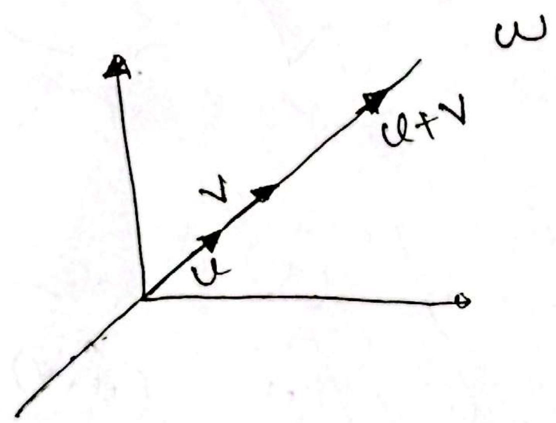
$$W = \{0\}$$

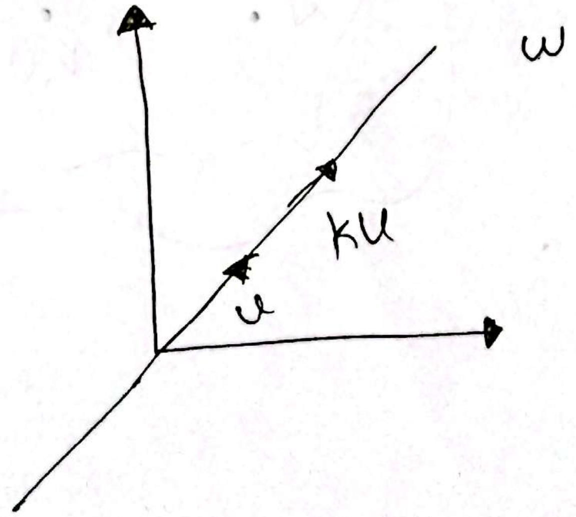
$$0 + 0 = 0$$

$$k0 = 0$$

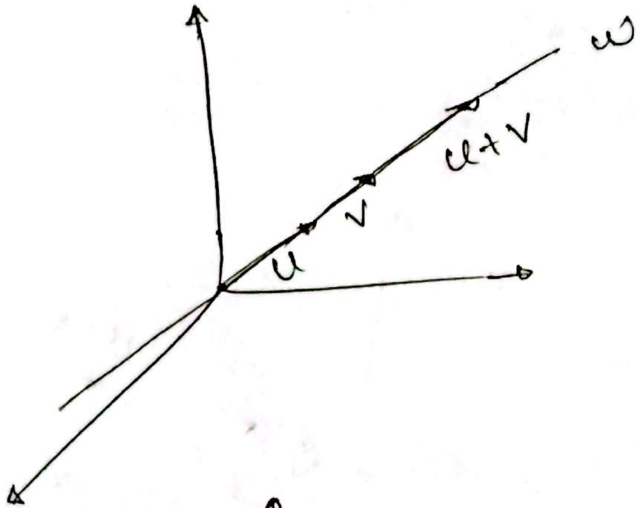
Ex: Lines through origin.

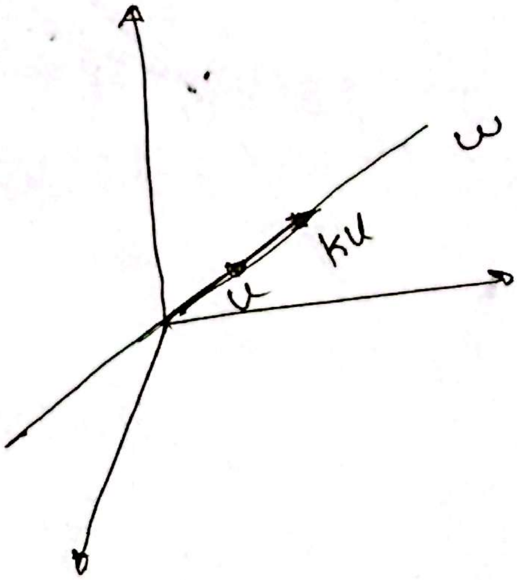
For R^2:



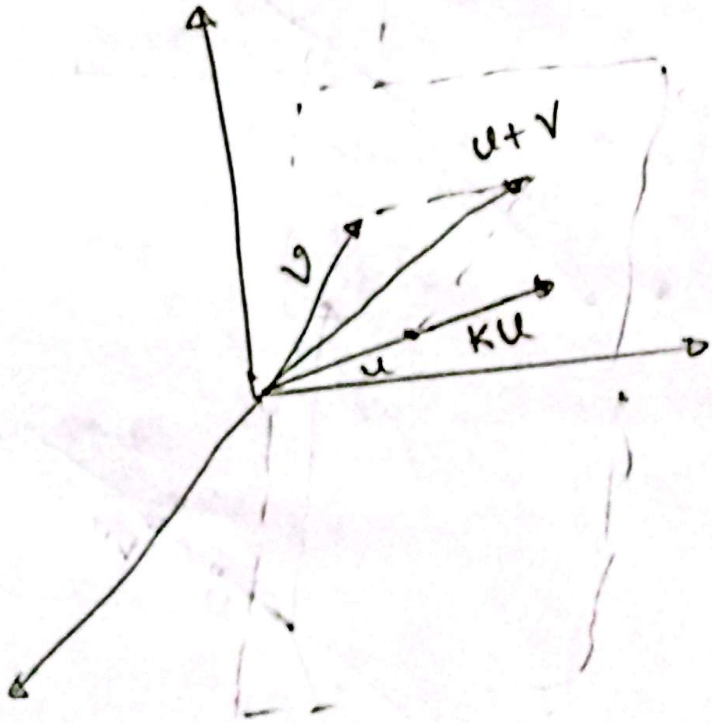


For R^3:

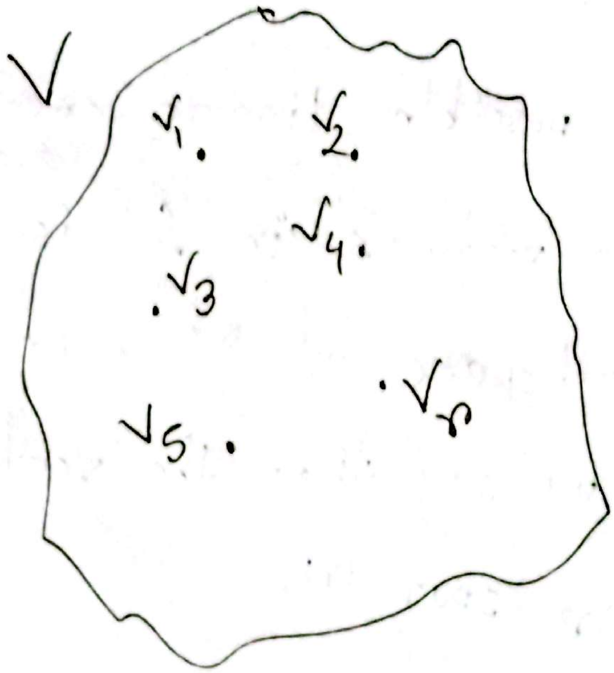




Ex: Planes through the origin in R^3.



If w is a vector in a vector space V, then w is said to be a linear combination of the vectors $v_1, v_2, \ldots, v_r$ in V if w can be expressed,



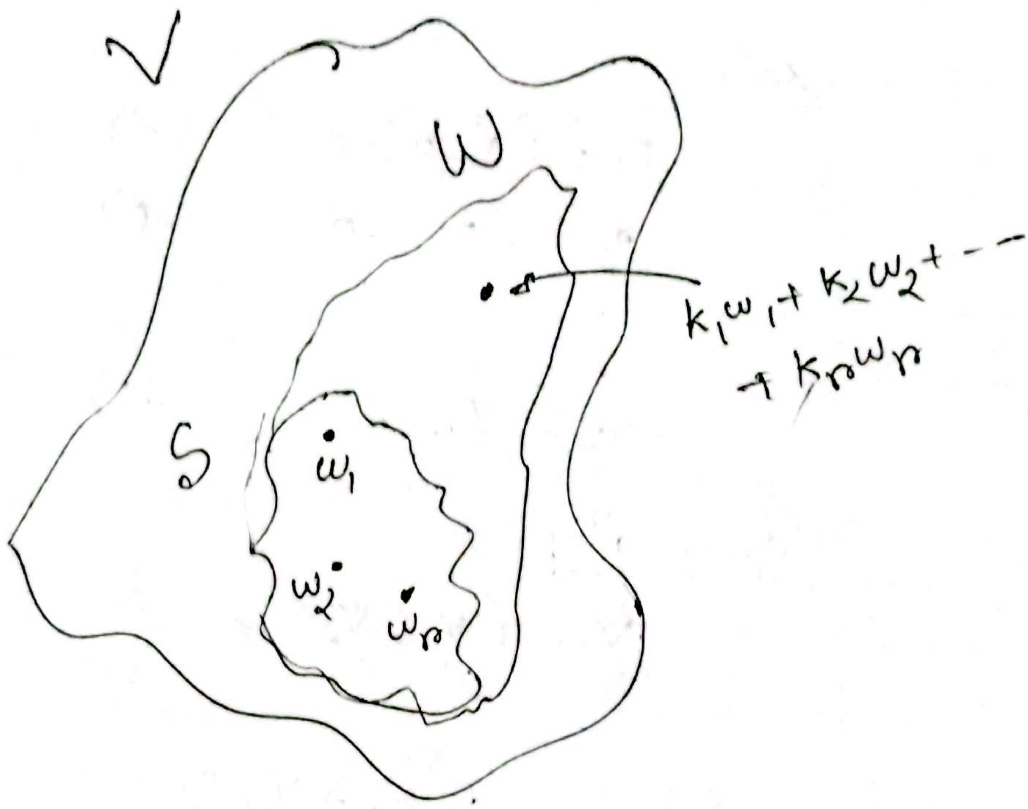
$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

☐ If $S = \{w_1, w_2 \cdots\cdots, w_r\}$ is a nonempty set of vectores in a vectore space V, then:

the subspace W of V that cosists of all possible linearc combinations of of the vectores in S is called the subspace of V generated by S, and we say that the vectores $w_1, w_2 \cdots$ $\cdots w_r$ span W.

$$W = \operatorname{span}\{w_1, w_2 \cdots\cdots w_r\}$$

$$W = \operatorname{span}(S).$$


V
W
S
w_1
w_2
w_r
$k_1w_1 + k_2w_2 + \cdots + k_rw_r$

☐ The standard unit vectors span R^n

Sol^n:

$$e_1 = (1, 0, 0 \cdots 0)$$
$$e_2 = (0, 1, 0 \cdots 0)$$
$$\vdots$$
$$e_n = (0, 0 \cdots 1)$$

Span R^n:

$$v = (v_1, v_2 \cdots v_n) \text{ in } R^n$$

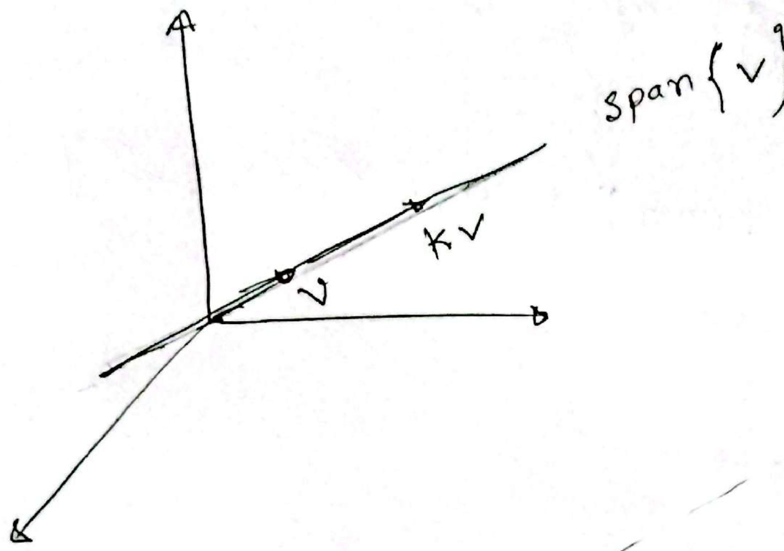
$$v = v_1 e_1 + v_2 e_2 + \cdots v_n e_n$$

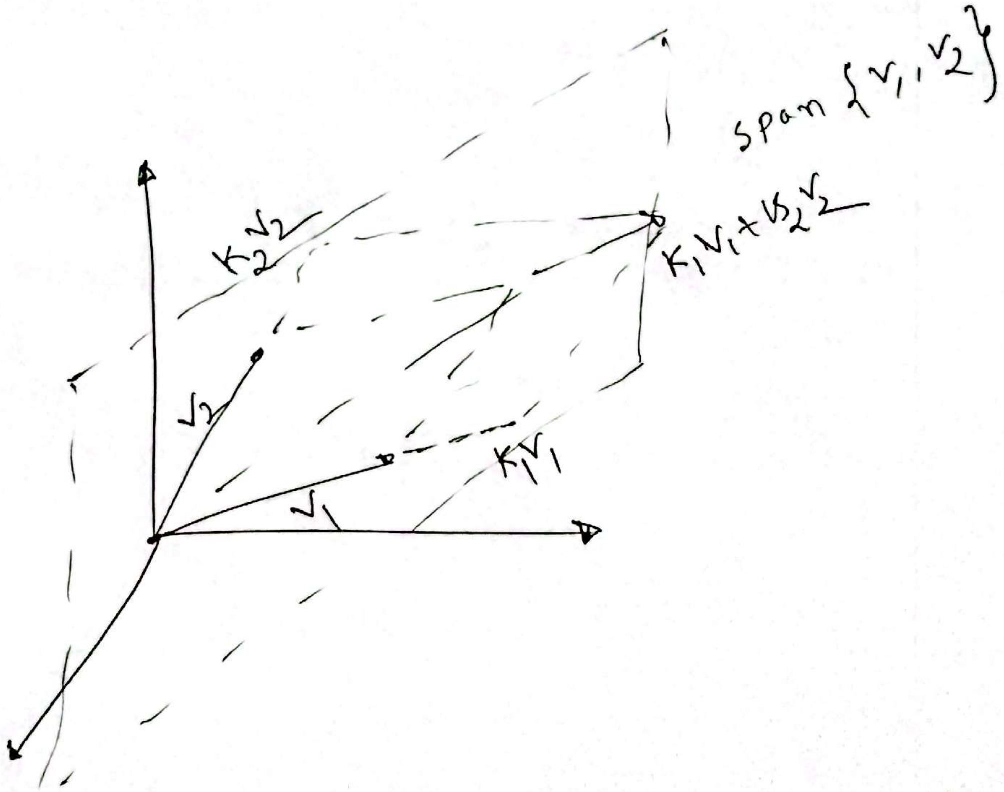
Span R^3:

$$v = (a, b, c) \text{ in } R^3$$

$$v = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1)$$

$$v = a\hat{i} + b\hat{j} + c\hat{k}$$


span{v}
kv
v


span{v1, v2}
k2v2
k1v1+k2v2
v2
k1v1
v1

☐ spaning of P_n

$$P_n = \text{span}[1, x, x^2, \ldots, x^n]$$

Example: If $u = (1,2,-1)$, $v = (6,4,2)$ in R^3 show that $w = (9,2,7)$ is a linear combination and $w' = (4,-1,8)$ not

Sol^n:

$$w = k_1 u + k_2 v$$

$$\Rightarrow (9,2,7) = k_1(1,2,-1) + k_2(6,4,2)$$

$$\Rightarrow (9,2,7) = (k_1 + 6k_2,\ 2k_1 + 4k_2,\ -k_1 + 2k_2)$$

$\therefore$

$$k_1 + 6k_2 = 9$$
$$2k_1 + 4k_2 = 2$$
$$-k_1 + 2k_2 = 7$$

$k_1 = -3$

$k_2 = 2$

w is a linear combination

and,

$$(4,-1,8) = (k_1 + 6k_2,\ 2k_1 + 4k_2,\ -k_1 + 2k_2)$$

$$k_1 + 6k_2 = 4$$
$$2k_1 + 4k_2 = -1$$
$$-k_1 + 2k_2 = 8$$

It's inconsistent.

so, w' is not linear combinations.

Determine whether $v_1 = (1,1,2)$, $v_2 = (1,0,1)$

$v_3 = (2,1,3)$ span R^3.

Soln: If $b = (b_1, b_2, b_3)$ in R^3 then

it's has to ~~b~~ expressed ~~ex~~ as

linear combination,

$$b = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$\therefore (b_1, b_2, b_3) = k_1(1,1,2) + k_2(1,0,1) + k_3(2,1,3)$$

$$\Rightarrow (b_1, b_2, b_3) = (k_1 + k_2 + 2k_3,\ k_1 + k_3,\ 2k_1 + k_2 + 3k_3)$$

$$k_1 + k_2 + 2k_3 = b_1$$

$$k_1 + k_3 = b_2$$

$$2k_1 + k_2 + 3k_3 = b_3$$

CamScanner

$$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$\det(A) = 0$, so linear combinations not possible

that's why v_1, v_2, v_3 do not span R^3

☐ Solution set of a homogenous linear system $Ax = 0$ of m equations in n unknowns is a subspace of R^n.

Proof: Let W be the solution set of the system. The set W is not empty because it contains at least the trivial solution $x = 0$.

$Ax = 0$

x_1, x_2 is a solution of the system,

$\therefore Ax_1 = 0$

$Ax_2 = 0$

$A(x_1 + x_2) = Ax_1 + Ax_2$

$= 0 + 0$

$= 0$

$A(kx_1)$

$= k \cdot Ax_1$

$= k0$

$= 0$

$\therefore$ Solution set of a homogeneous system in n unknowns is a subspace of R^n.

Linear Independence

If $S = \{v_1, v_2 ---- v_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a linearly independent set if no vector in S can be expressed as a linear combination of the others.

A set that is not linearly independent is said to be linearly dependent.

A nonempty set $S = \{v_1, v_2 \text{ ---- } v_r\}$ in a vector space V is linearly independent if and only if the vector equation

$k_1 v_1 + k_2 v_2 + \text{----} + k_n v_n = 0$ has only trivial solution.

$$k_1 = k_2 \text{----} = 0$$

Ex: Linear Independence of the standard Unit Vectors in R^n

$e_1 = (1, 0, 0 \text{ ----}, 0)$

$e_2 = (0, 1, 0, \text{ ----}, 0)$

$\vdots$

$e_n = (0, 0, 0 \text{ ----- } 1)$

$i = (1, 0, 0)$

$j = (0, 1, 0)$

$k = (0, 0, 1)$

$$k_1 i + k_2 j + k_3 k = 0$$

$$k_1 (1, 0, 0) + k_2 (0, 1, 0) + k_3 (0, 0, 1) = 0$$

$$\Rightarrow (k_1, k_2, k_3) = (0, 0, 0)$$

$$k_1 = 0, \; k_2 = 0, \; k_3 = 0$$

Determine whether the vectors:

$v_1 = (1, -2, 3)$, $v_2 = (5, 6, -1)$, $v_3 = (3, 2, 1)$

are linearly independent or linearly dependent in $\mathbb{R}^3$.

Soln:

For, linear independence,

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$k_1 (1, -2, 3) + k_2 (5, 6, -1) + k_3 (3, 2, 1) = (0, 0, 0)$$

$$k_1 + 5k_2 + 3k_3 = 0$$
$$-2k_1 + 6k_2 + 2k_3 = 0$$
$$3k_1 - k_2 + k_3 = 0$$

$$A = \begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix}$$ iii ← iii − i×3

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix}$$ ii ← i×2 + ii

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ iii ← ii + iii

Let's, $\boxed{k_3 = t}$

$16k_2 + 8k_3 = 0$

$\therefore \boxed{k_2 = -\frac{1}{2}t}$

$k_1 + 5k_2 + 3k_3 = 0$

$\Rightarrow k_1 - \frac{5}{2}t + 3t = 0$

$k_1 = -\frac{1}{2}t$

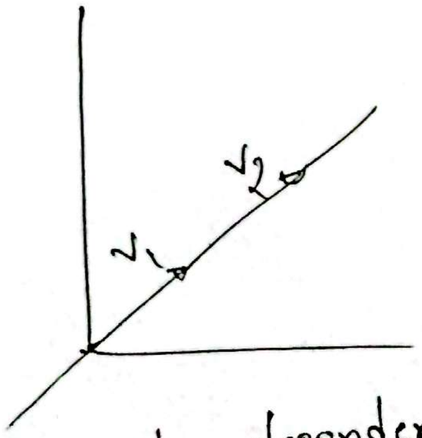
So, Infinit solution, linearly dependent

☐ A finite set that contains 0 is linearly dependent.

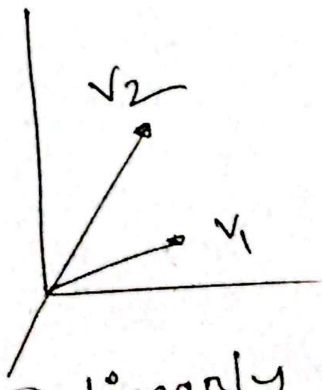
→ A set with only one vector is linearly independent if that vector is not 0.

→ A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

☐



(i) linearly dependent (same line)



(ii) linearly independent

☐ Let $S = \{v_1, v_2, \dots, v_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent.

☐ Determine whether the polynomials,

$$P_1 = 1-x, \; P_2 = 5+3x-2x^2, \; P_3 = 1+3x-x^2$$

are linearly dependent or linearly independent in P_2.

Solⁿ:

$$k_1P_1 + k_2P_2 + k_3P_3 = 0$$

$$\Rightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) = 0$$

$$\Rightarrow (k_1+5k_2+k_3) + (-k_1+3k_2+3k_3)x + (-2k_2-k_3)x^2 = 0$$

$$k_1+5k_2+k_3 = 0$$
$$-k_1+3k_2+3k_3 = 0$$
$$-2k_2-k_3 = 0$$

$$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$$

$\det(A) = 0$

$\therefore$ so, linearly dependent

Coordinates and Basis

If $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a basis for V if:

a) S spans V

b) S is linearly independent

□ Standard unit vectors are basis of R^n :

$e_1 = (1, 0, 0, \ldots, 0)$

$e_2 = (0, 1, 0, \ldots, 0)$

$\vdots$

$e_n = (0, 0, 0, \ldots, 1)$

Let's ~~v in R^n~~ v in R^n

$v = (v_1, v_2, \ldots v_n)$

$v = v_1 e_1 + v_2 e_2 + \ldots + v_n e_n$ can possible so, $e_1, e_2, \ldots e_n$ are basis of R^n.

next, $k_1 e_1 + k e_2 + \ldots + k_n e_n = 0$

$\Rightarrow$ ~~$k_1 +$~~

$\therefore k_1 = k_2 = \ldots = k_n = 0$

so, linearly independent

$\therefore$ Standard unit vectors are basis of R^n.

Show that the vectors $v_1 = (1,2,1)$, $v_2 = (2,9,0)$ and $v_3 = (3,3,4)$ form a basis for R^3.

Soln: we have to show the vectors are linearly independant and span R^3.

For linear Independence, we have to show that, $c_1v_1 + c_2v_2 + c_3v_3 = 0$ has only trivial solution.

For, span R^3, we must show that every vector $b = (b_1, b_2, b_3)$ in R^3 can be expressed as,
$$c_1v_1 + c_2v_2 + c_3v_3 = b$$

$\therefore c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) = 0$

$\therefore c_1 + 2c_2 + 3c_3 = 0$

$2c_1 + 9c_2 + 3c_3 = 0$

$c_1 + 4c_3 = 0$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

From, $c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4) = (b_1, b_2, b_3)$

$c_1 + 2c_2 + 3c_3 = b_1$

$2c_1 + 9c_2 + 3c_3 = b_2$

$c_1 + 4c_3 = b_3$

$\det.(A) = -1$

$\det(A) \neq 0$ From equivalent statements,

$Ax = 0$ has only trivial sol^n,

$Ax = b$ is consistent

$\therefore v_1, v_2, v_3$ form a basis for R^3.

☐ If $S = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed in the form

$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$ in exactly ~~the~~ one way

proof:

Let's two different way,

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n \quad \text{--- (i)}$$

$$v = k_1 v_1 + k_2 v_2 + \dots + k_n v_n \quad \text{--- (ii)}$$

(i) $-$ (ii)

$$0 = (c_1 - k_1) v_1 + (c_2 - k_2) v_2 + \dots + (c_n - k_n) v_n$$

$\therefore c_1 - k_1 = 0 \;\Rightarrow c_1 = k_1$ | $c_2 - k_2 = 0 \;\Rightarrow c_2 = k_2$ | $c_n - k_n = 0 \;\Rightarrow c_n = k_n$

☐ If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, and

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

then,

$c_1, c_2 \ldots c_n$ are called the coordinates of v relative to the basis S. The vector $(c_1, c_2 \ldots, c_n)$ in R^n constructed from these coordinates is called the coordinate vector of v relative to S: it is denoted by

$$(v)_S = (c_1, c_2 \ldots, c_n)$$

$\boxed{}$ $v_1 = (1,2,1)$, $v_2 = (2,9,0)$, $v_3 = (3,3,4)$ form a basis for R^3. Find the coordinate vector of $v = (5,-1,9)$ relative to the basis $S = \{v_1, v_2, v_3\}$

Soln:

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$$c_1 + 2c_2 + 3c_3 = 5$$
$$2c_1 + 9c_2 + 3c_3 = -1$$
$$c_1 + 4c_3 = 9$$

$$c_1 = 1, \; c_2 = -1, \; c_3 = 2$$

$$(v)_S = (1,-1,2)$$

Dimension

All bases for a finite-dimensional vector space have the same number of vectors.

- dimension of a finite-dimensional vector space V. $\rightarrow$ $\dim(V)$
- For zero vector space $\rightarrow$ $\dim(V) = 0$
- $\dim(R^n) = n$ [The standard basis has n vectors]
- $\dim(P_n) = n+1$ [The standard basis has $n+1$ vectors]
- $\dim(M_{mn}) = mn$ [The standard basis has mn vectors]

☐ If $S = \{v_1, v_2, \cdots v_n\}$ then every vector in span(S) is expressible as a linear combination of vectors in S. Thus, if the vectors in S are linearly independent, they automatically form a basis for span(S)

$$\dim[\text{span}\{v_1, v_2, \cdots, v_n\}] = n$$

☐ The dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

10. Find a basis for and the dimension of the solution space of the homogeneous system

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$
$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$
$$5x_3 + 10x_4 + 15x_6 = 0$$
$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

Solⁿ: $x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s$

$x_5 = t, \quad x_6 = 0$

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r-4s-2t, r, -2s, s, t, 0)$$

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3,1,0,0,0,0) + s(-4,0,-2,1,0,0) + t(-2,0,0,0,1,0)$$

$v_1 = (-3,1,0,0,0,0), \quad v_2 = (-4,0,-2,1,0,0)$

$v_3 = (-2,0,0,0,1,0)$

$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$

$\Rightarrow k_1(-3,1,0,0,0,0) + k_2(-4,0,-2,1,0,0) +$

$k_3(-2,0,0,0,1,0) = 0$

$-3k_1 - 4k_2 - 2k_3 = 0$

$k_1 = 0$

$-2k_2 = 0$

$\Rightarrow k_2 = 0, \quad k_1 = 0$

So, linearly independent

$\therefore$ The solution space has dimension 3

Plus/Minus Theorem

- If S is a linearly independent set, and if v is a vector in V that is outside of span(S), then the set $S \cup \{v\}$ that results by inserting v into S is still linearly independent.
- If v is a vector in S that is expressible as a linear combination of other vectors in S, ~~and if $S - \{v\}$~~

So, then,

$$\text{span}(S) = \text{span}(S - \{v\})$$

☐ Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V if and only if S space V or S is linearly independent.

If W is a subspace of a finite-dimensional vector space V, then:

a) W is finite-dimensional.

b) $\dim(W) \leq \dim(V)$

c) $W = V$ if and only if $\dim(W) = \dim(V)$

Change of Basis

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a finite-dimensional vector space V, and if

$$(v)_S = (c_1, c_2, \ldots, c_n)$$

is the coordinate vector of v relative to S.

$$v \longrightarrow (v)_S$$

$$V \ni v \xrightarrow{[\;]_S} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \in \mathbb{R}^n$$

coordinate map

Q. If v is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to a basis B', then how are coordinate vectors $[v]_B$ and $[v]_{B'}$ related?—

Let, $B = \{u_1, u_2\}$ $\quad B' = \{u_1', u_2'\}$

Let,

$$[u_1']_B = \begin{bmatrix} a \\ b \end{bmatrix} \quad [u_2']_B = \begin{bmatrix} c \\ d \end{bmatrix}$$

coordinate vector for new basis

$$u_1' = au_1 + bu_2$$

$$u_2' = cu_1 + du_2$$

Let v be any vector in V.

$$[v]_{B'} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$v = k_1 u_1' + k_2 u_2'$$

$$v = k_1 (a u_1 + b u_2) + k_2 (c u_1 + d u_2)$$

$$= (k_1 a + k_2 c) u_1 + (k_1 b + k_2 d) u_2$$

$$[v]_B = \begin{bmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \end{bmatrix}$$

$$[v]_{B'} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$[v]_B = \begin{bmatrix} a & c \\ b & d \end{bmatrix} [v]_{B'}$$

□ If we change the basis for a vector space V from an old basis $B = \{u_1, u_2, \ldots, u_n\}$ to a new basis $B' = \{u_1', u_2', \ldots, u_n'\}$ then for each vector v in V, the old coordinate vector $[v]_B$ is related to the new coordinate vector $[v]_{B'}$ by the equation,

$$\boxed{[v]_B = P\,[v]_{B'}}$$

P are the coordinate vectors of the new basis vectors relative to the old basis, the column vectors of P are,

$$[u_1']_B, [u_2']_B, \ldots [u_n']_B$$

$$P_{B' \to B} = \left[[u_1']_B , [u_2']_B \cdots [u_n']_B \right]$$

$$P_{B \to B'} = \left[[u_1]_{B'} , [u_2']_B \cdots [u_n]_{B'} \right]$$

Finding Transition Matrices:

Consider the bases $B = \{u_1, u_2\}$, $B' = \{u_1', u_2'\}$ for R^2, where

$u_1 = (1,0)$, $u_2 = (0,1)$, $u_1' = (1,1)$, $u_2' = (2,1)$

a) Find the transition matrix $P_{B' \to B}$ from B' to B

b) Find the transition matrix $P_{B \to B'}$ from B to B'

Sol^n:

a

$u_1' = u_1 + u_2$

$u_2' = 2u_1 + u_2$

$[u_1']_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[u_2']_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

CamScanner

b

$$u_1 = -u_1' + u_2'$$
$$u_2 = 2u_1' - u_2'$$

$$[u_1]_{B'} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad [u_2]_{B'} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

c Use an appropriate formula to find $[v]_B$ given that, $[v]_{B'} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$

<u>sol^n.</u>

$$[v]_B = P_{B' \to B} [v]_{B'}$$
$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix}$$
$$= \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\boxed{(P_{B' \to B})(P_{B \to B'}) = I}$$

If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.

$$[\text{new basis} \mid \text{old basis}] \xrightarrow{\text{row operations}} [I \mid \text{transition from old to new}]$$

previous problem,

a) $B' \to B$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

left matrix already identity matrix

so $P_{B' \to B} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

b) $P_{B \to B'}$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

$ii \leftarrow i - ii$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \end{array}\right] \quad i \leftarrow ii + i$$

$$i \leftarrow ii + i$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & -1 & -1 & 1 \end{array}\right] \quad ii \leftarrow ii \times -1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\therefore \quad P_{B \to B'} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

LU Factorization

L → Lower triangular

U → upper triangular

using LU to solve linear systems,

$$Ax = b$$

Let,

$$LUx = b \quad \text{—} ②$$

$$Ux = y \quad \text{—} ③$$

$\boxed{Ly = b}$ → solve y

using, y → solve $\boxed{Ux = y}$ → get(x)

$$E_k \cdots E_2 E_1 A = U$$

$$A = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1} U$$

$$A = LU$$

$$\therefore \boxed{L = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_k^{-1}}$$

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$$

Step 1: $i \times \frac{1}{2}$

$$\begin{bmatrix} 1 & 3 & 1 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \quad E_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_1^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step-2: $i \times 3 + ii$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 4 & 9 & 2 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ; \quad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step-3: $-4 \times i + iii$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

Step-4: $3 \times ii + iii$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdot E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Step-5: $iii \times \frac{1}{7}$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/7 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -3 & 7 \end{bmatrix}$$

☐ If A is an $m \times n$ matrix, then the subspace of R^n spanned by the row vectors of A is called the row space of A.

The subspace of R^m spanned by the column vectors of A is called the column space of A.

The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of R^n, is called the null space of A

☐ $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$r_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

$$r_2 = \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix}$$

$$\vdots$$

$$r_m = \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \quad c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

- A system of lineare equations $Ax = b$ is consistent if and only if b is in the column space of A.

- We can expressed,

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n$$

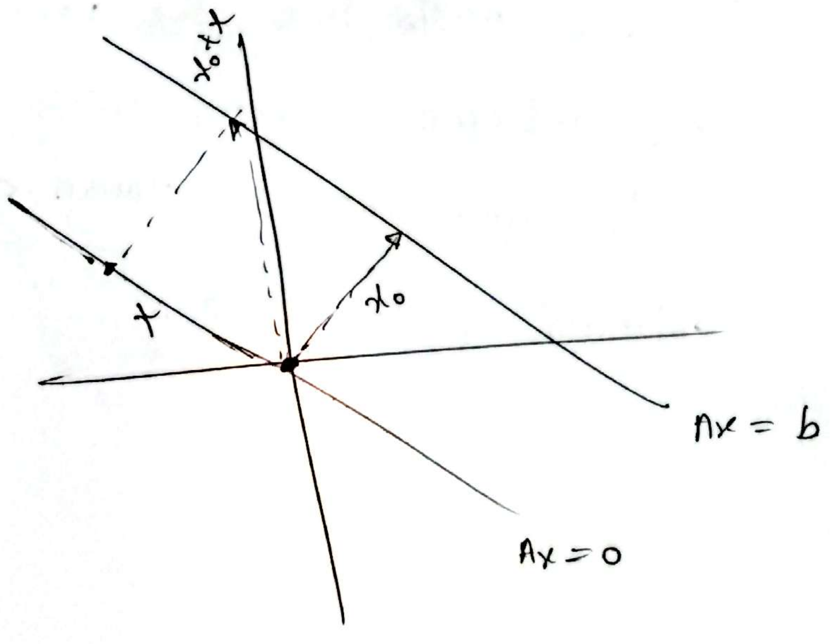
$$\therefore x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b$$

If x_0 is a solution of $Ax=b$ and x' is a solution of $Ax=0$ then the remaining solution of $Ax=b$ is

$$x = x_0 + x'$$

If $S = \{v_1, v_2, \ldots, v_k\}$ is a basis for the null space of A. Then,

$$x = x_0 + c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$



- Elementary row operations do not change the null space of a matrix.

- Elementary row operations do not change the row space of a matrix.

- If a matrix R is in row echelon form, then the row vectors with the leading 1's form a basis for the row space of R. The column vectors with the leading 1's of the row vectors form a basis for the column space of R.

☐ If A and B are row equivalent matrices, then:

a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.

b) A given set of column vectors of A forms a basis for the column vectors of B form a basis for the column space of B.

$\boxed{+}$ The row space and the column space of a matrix A have the same dimension.

$\boxed{+}$ The common dimension of the row space and column space of a matrix A is called the rank of A.

$\boxed{+}$ The dimension of the null space of A is called the nullity of A.

$\boxed{+}$ $A \to m \times n$ matrix

$$\text{rank}(A) \leq \min(m, n)$$

$n \to$ columns,

$$\text{rank}(A) + \text{nullity}(A) = n$$

If A is an $m \times n$ matrix, then,

rank(A) = the number of leading variables in the general solution of $Ax = 0$

nullity(A) = the number of parameters in the general solution of $Ax = 0$

If $Ax = b$ is a consistent linear system of m equations in n unknowns, and if A has rank r, then the general solution of the system contains $n - r$ parameters

The Fundamental Space of a Matrix.

row space of A
column space of A
null space of A
null space of A^T

CamScanner

■ If A is any matrix, then $\text{rank}(A) = \text{rank}(A^T)$

■ $A_{m \times n}$

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

$$\text{rank}(A) + \text{nullity}(A^T) = m$$

If $\text{rank}(A) = r$,

$\dim[\text{row}(A)] = r$, $\dim[\text{col}(A)] = r$

$\dim[\text{null}(A)] = n - r$, $\dim[\text{null}(A^T)] = m - r$

■ If W is a subspace of R^n, then the set of all vectors in R^n that are orthogonal to every vector in W is called the orthogonal component of W. Denoted by $W^\perp$

- The only common vector in W and $W^{\perp}$ is 0

- The null space of A and the row space of A are orthogonal complements in R^n.

- The null space of A^T and the column space of A are orthogonal complements in R^m.

CamScanner

Equivalent Statements:

If A is $n \times n$ matrix,

a) A is invertible.

b) $Ax = 0$ has only the trivial solution

c) The reduced row echelon form of A is I_n

d) A is expressible as a product of elementary matrices.

e) $Ax = b$ is consistent for every $n \times 1$ matrix b.

f) $Ax = b$ has exactly one solution for every $n \times 1$ matrix b.

g) $\det(A) \neq 0$.

h) The column vectors of A are linearly independent.

i) The row vectors of A are linearly independent.

j) The column vectors of A span R^n.

k] The row vectors of A span R^n.

l] The column vectors of A form a basis for R^n

m] The row vectors of A form a basis for R^n.

n] A has rank n.

o] A has nullity 0.

p] The orthogonal complement of the null space of A is R^n.

q] The orthogonal complement of the row space of A is $\{0\}$.

Finding a Basis for the Null Space of a Matrix,

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$$

Sol^n:

$$Ax = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -3r - 4s - 2t$, $x_2 = r$, $x_3 = -2s$, $x_4 = s$

$x_5 = t$, $x_6 = 0$

$v_1 = (-3, 1, 0, 0, 0, 0)$

$v_2 = (-4, 0, -2, 1, 0, 0)$

$v_3 = (-2, 0, 0, 0, 1, 0)$

☐ Find a basis for the row space and for the column space of the matrix.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solⁿ:

Row echelon form,

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A.

∴ These basis vectors are,

$$r_1 = [1 \ -3 \ 4 \ -2 \ 5 \ 4]$$

$$r_2 = [0 \ 0 \ 1 \ 3 \ -2 \ -6]$$

$$r_3 = [0 \ 0 \ 0 \ 0 \ 1 \ 5]$$

(4) A and R have different column space. so, we can not find a basis for the column space of A directly from the column vectors of R.

If we find ~~the~~ a set of vectors of R that forms a basis for the column space of R, then the corresponding column vectors of A will form a basis for the column space of A.

Since first, third, fifth columns of R contain the leading 1's of the row vectors, the vectors,

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

Thus, corresponding column vectors of A, which are,

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

CamScanner

Find a subset of the vectors;

$v_1 = (1, -2, 0, 3)$, $v_2 = (2, -5, -3, 6)$

$v_3 = (0, 1, 3, 0)$, $v_4 = (2, -1, 4, -7)$

$v_5 = (5, -8, 1, 2)$

that forms a basis for the subspace of R^4 spanned by these vectors.

Express each vector not in the basis as a linear combination of the basis vectors.

Soln;

$$A = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix} \\ \begin{matrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{matrix} \end{array}$$

$$R = \begin{array}{c} \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccccc} \;w_1 & w_2 & w_3 & w_4 & w_5 \end{array} \end{array}$$

$\{w_1, w_2, w_4\} \to$ basis for the column space of R

$\therefore \{v_1, v_2, v_4\} \to$ is a basis for the column space of A.

we can express,

$$w_3 = 2w_1 - w_2$$

$$w_5 = w_1 + w_2 + w_4$$

$$\therefore \quad v_3 = 2v_1 - v_2$$

$$v_5 = v_1 + v_2 + v_4$$

Find the rank and nullity of the matrix,

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

<u>Sol^n:</u>

row echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row and column spaces are two dimensional

$$\text{rank}(A) = 2.$$

solution,

$$x_1 = 4r + 28s + 37t - 13u$$

$$x_2 = 2r + 12s + 16t - 5u$$

$$x_3 = r \quad x_5 = t$$

$$x_4 = s \quad x_6 = u$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

nulity (A) = 4

A matrix has 6 columns.

rank(A) + nullity(A) = 2 + 4 = 6