Vectors

A ⟶ B terminal point
initial

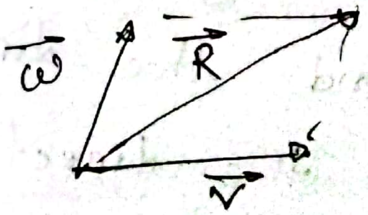
$v = \overrightarrow{AB}$

Equivalent vectors
{length and direction same}

Zero vector:

→ length zero.
→ no natural direction

Vector addition:

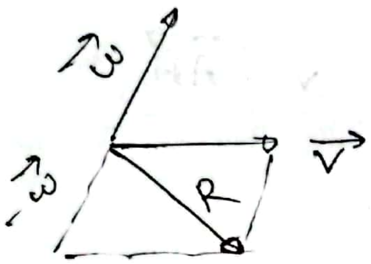


$\vec{R} = \vec{w} + \vec{v}$

$= \vec{v} + \vec{w}$

Vector Subtraction:

$\therefore R = \vec{v} - \vec{w}$



Scalar Multiplication:

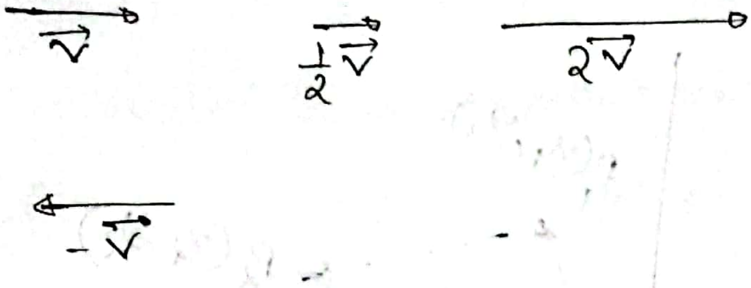
K is a non-zero scalar,

the scalar product of v by K to be the vector whose length is |K| times the length of v.

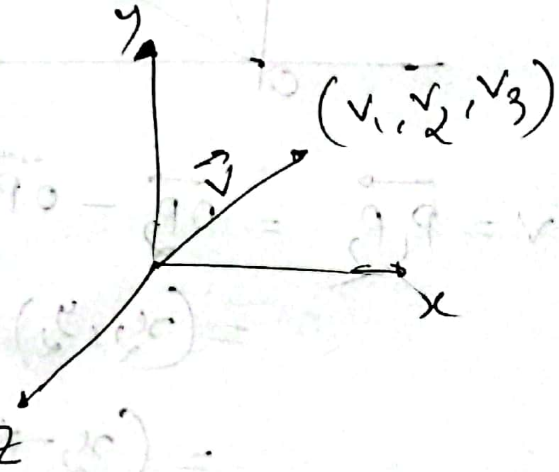
And the direction is same if (K>0)

the direction would be opposite if (K<0)

If K = 0 or v = 0 then Kv = 0



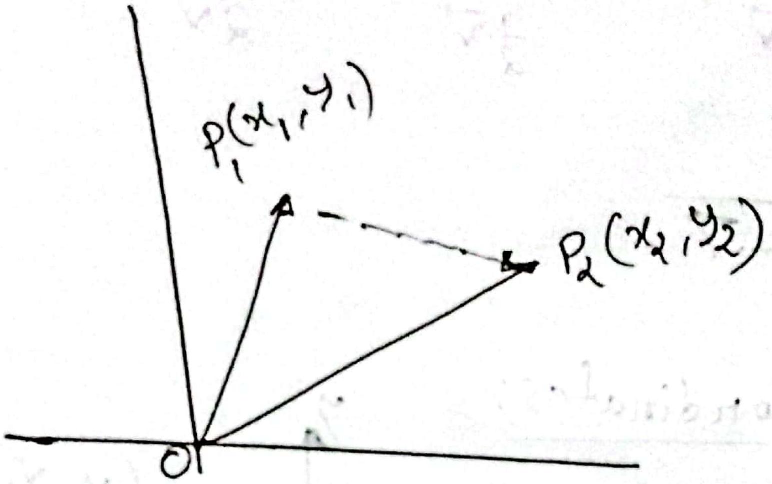
coordinates:



$v\,(v_1, v_2, v_3)$ $\quad w\,(w_1, w_2, w_3)$

v and w are equivalent if and only if,

$$v_1 = w_1$$
$$v_2 = w_2$$
$$v_3 = w_3$$



$$v = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$

$$= (x_2, y_2) - (x_1, y_1)$$

$$= (x_2 - x_1,\ y_2 - y_1)$$

If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(v_1, v_2 ---, v_n)$. The set of all ordered n-tuples is called n-space and is denoted by R^n.

$$v = (v_1, v_2 ---. v_n)$$

$$kv = (kv_1, kv_2 --- kv_n)$$

⊞ If u, v and w are vectors in R^n and if k and m are scalars, then:

a) $u + v = v + u$

b) $(u+v) + w = u + (v+w)$

c) $u + 0 = 0 + u = u$

d) $u + (-u) = 0$

e) $k(u+v) = ku + kv$

f) $(k+m)v = kv + mv$

g) $k(mu) = (km)u$

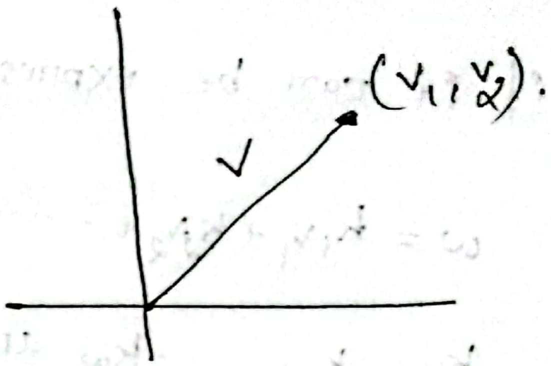
h) $1 \cdot u = u$

If w is a vector in R^n, then w is said to be a linear combination of the vectors $v_1, v_2 - - -, v_r$ if it can be expressed in the form

$$w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r$$

where, $k_1, k_2 - - - k_r$ are scalars.

Norm of a Vector



$$\boxed{\|v\| = \sqrt{v_1^2 + v_2^2}}$$

If, $v = (v_1, v_2 - \cdots v_n)$

$$\boxed{\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}}$$

If v is a vector in R^n and if k is any scalar then:

a) $\|v\| \geq 0$

b) $\|v\| = 0$ if and only if $v = 0$

c) $\|kv\| = |k| \|v\|$

Unit vector:

$$\vec{u} = \frac{1}{\|v\|} \vec{v}$$

$$\|u\| = 1$$

Standard Unit vectors:

i $(1, 0, 0)$

j $(0, 1, 0)$

k $(0, 0, 1)$

If standard unit vectors,

$$e_1 = (1, 0, 0, \text{----})$$

$$e_2 = (0, 1, 0, 0 \text{----})$$

$$e_n = (0, 0 \text{-----}, 1)$$

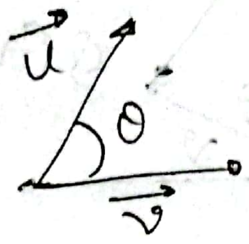
$$v = (v_1, v_2 \text{----} v_n)$$

$$= v_1 e_1 + v_2 e_2 + \text{--} + v_n e_n$$

If $u = (u_1, u_2 \text{----} u_n)$ and $v = (v_1, v_2 \text{----} v_n)$

$$\|u - v\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \text{----} (u_n - v_n)^2}$$

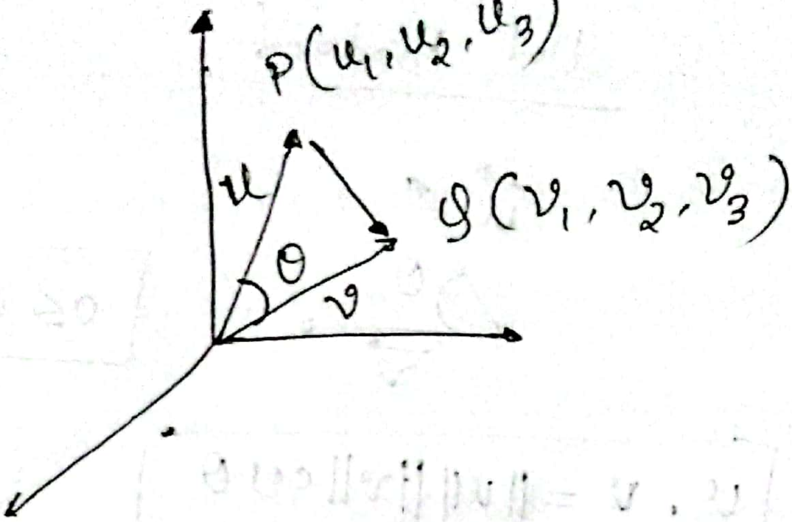
Dot Product



$$\boxed{0 \leq \theta \leq \pi}$$

$$\boxed{u \cdot v = \|u\| \, \|v\| \cos \theta}$$

If $u = 0$ or $v = 0$, then $u \cdot v = 0$



$$\|\overrightarrow{PQ}\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

$$\Rightarrow \|v-u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta$$

$$\Rightarrow \|u\|\|v\|\cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \|v-u\|^2\right)$$

$$\|u\|^2 = u_1^2 + u_2^2 + u_3^2$$

$$\|v\|^2 = v_1^2 + v_2^2 + v_3^2$$

$$\therefore\ u.v = \frac{1}{2}\left[u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 - \left\{(v_1-u_1)^2 + (v_2-u_2)^2 + (v_3-u_3)^2\right\}\right]$$

$$\Rightarrow \boxed{u.v = u_1v_1 + u_2v_2 + u_3v_3}$$

$\therefore u(u_1, u_2 - - - u_n), \quad v(v_1, v_2 - - - v_n)$

$$\therefore \boxed{u \cdot v = u_1 v_1 + u_2 v_2 + - - - + u_n v_n}$$

1. $u \cdot v = v \cdot u$

2. $u \cdot (v \pm w) = u \cdot v \pm u \cdot w$

3. $k(u \cdot v) = (ku) \cdot v$

4. $v \cdot v \geq 0$ and $v \cdot v = 0$ if and only if $v = 0$

CamScanner

Cauchy - Schwarz Inequality and Angles in R^n

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\|\|v\|}\right)$$

$$\therefore -1 \leq \frac{u \cdot v}{\|u\|\|v\|} \leq 1$$

$$\boxed{|u \cdot v| \leq \|u\|\|v\|}$$

$$\therefore |u_1 v_1 + u_2 v_2 + \cdots u_n v_n| \leq (u_1^2 + u_2^2 + \cdots u_n^2)^{1/2} (v_1^2 + v_2^2 + \cdots v_n^2)^{1/2}$$

$$\|u+v\| \le \|u\| + \|v\|$$

$$\begin{aligned}
\|u+v\|^2 &= (u+v)\cdot(u+v) \\
&= (u\cdot u) + 2(u\cdot v) + (v\cdot v) \\
&= \|u\|^2 + 2(u\cdot v) + \|v\|^2 \\
&\le \|u\|^2 + 2|u\cdot v| + \|v\|^2 \\
&\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \\
&= (\|u\| + \|v\|)^2.
\end{aligned}$$

$$\|u+v\|^2+\|u-v\|^2 = 2(\|u\|^2+\|v\|^2)$$

$$\|u+v\|^2+\|u-v\|^2 = (u+v)\cdot(u+v)+(u-v)\cdot(u-v)$$

$$= 2(u\cdot u)+2(v\cdot v)$$

$$= 2(\|u\|^2+\|v\|^2)$$

$$\cancel{u\cdot v = \frac{1}{4}\|u\|}$$

$$u\cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2$$

$$\|u+v\|^2 = (u+v)\cdot(u+v) = \|u\|^2+2(u\cdot v)+\|v\|^2$$

$$\|u-v\|^2 = (u-v)\cdot(u-v) = \|u\|^2-2(u\cdot v)+\|v\|^2$$

$$\|u+v\|^2-\|u-v\|^2 = 4(u\cdot v)$$

$$\therefore \boxed{\frac{1}{4}\|u+v\|^2-\frac{1}{4}\|u-v\|^2 = u\cdot v}$$

Dot Products as Matrix Multiplication

Case-1: u a column matrix and v a column matrix.

$$u \cdot v = u^T v = v^T u$$

$$u = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$u \cdot v = 1 \times 5 + (-3) \times 4 + 5 \times 0 = -7$$

$$u^T v = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = 5 - 12 = -7$$

$$v^T u = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = 5 - 12 = -7$$

Case-2: u a row matrix and v a column matrix

$$u \cdot v = uv = v^T u^T$$

$$u = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

$$u \cdot v = 5 - 12 = -7$$

$$uv = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = 5 - 12 = -7$$

$$v^T u^T = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = 5 - 12 = -7$$

Case-3: u a row matrix and v a column matrix

$$u \cdot v = vu = u^T v^T$$

$$u = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix}$$

$$u \cdot v = 5 - 12 = -7$$

$$vu = \begin{bmatrix} 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = 5 - 12 = -7$$

$$u^T v^T = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = 5 - 12 = -7$$

Case -4: u a row matrix and v a row matrix

$$u \cdot v = uv^T = vu^T$$

$u = [1 \quad -3 \quad 5]$ $\quad v = [5 \quad 4 \quad 0]$

$u \cdot v = 5 - 12 = -7$

$$uv^T = [1 \quad -3 \quad 5] \cdot \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} = 5 - 12 = -7$$

$$vu^T = [5 \quad 4 \quad 0] \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix} = 5 - 12 = -7$$

A is $n \times n$ matrix

u and v are $n \times 1$ matrix

$$Au \cdot v = v^T(Au) = (v^T A)u = (A^T v)^T u$$
$$= u \cdot A^T v$$

$$u \cdot Av = (Av)^T u = (v^T A^T)u = v^T(A^T u)$$
$$= A^T u \cdot v$$

$$\therefore \boxed{Au \cdot v = u \cdot A^T v}$$

$$\boxed{u \cdot Av = A^T u \cdot v}$$

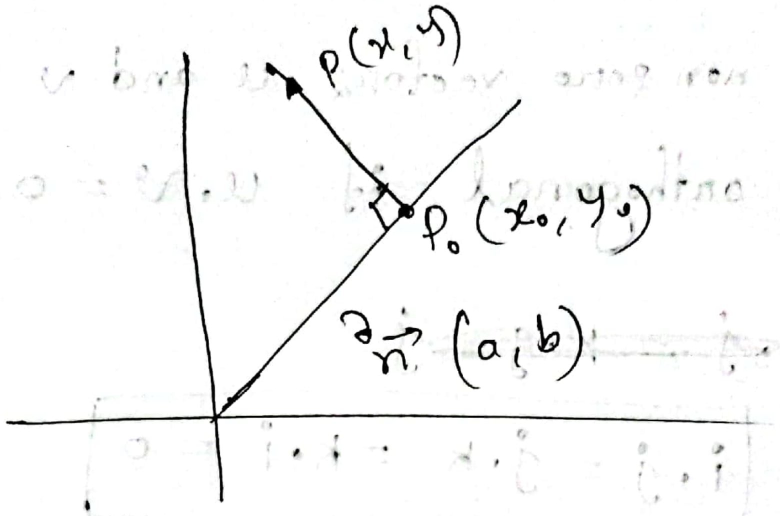
Orthogonality

Two non zero vectors u and v are said to be orthogonal if $u \cdot v = 0$. $[\theta = \pi/2]$

~~$i \cdot j = k \cdot j$~~ j

$$\boxed{i \cdot j = j \cdot k = k \cdot i = 0}$$

Point Normal Equations



$\therefore\ \vec{n} \cdot \vec{P_0P} = 0$

$(a, b) \cdot (x - x_0,\ y - y_0) = 0$

$\Rightarrow ax - ax_0 + by - by_0 = 0$

$\Rightarrow ax + by - ax_0 - by_0 = 0$

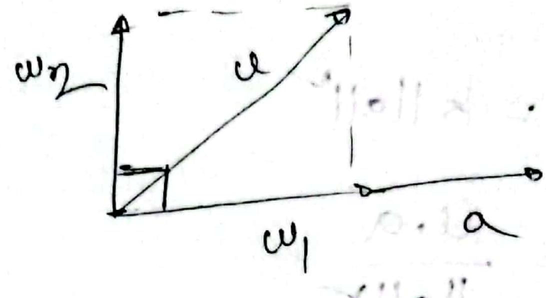
$\boxed{c = -ax_0 - by_0}$

$\boxed{ax + by + c = 0}$

$\hookrightarrow n\ (a, b)$

$\boxed{ax + by + cz + d = 0}$ $\hookrightarrow n\ (a, b, c)$

Orthogonal Projections



If u and a are vectors in R^n, and if $a \neq 0$, then u can be expressed in exactly one way in the form $u = w_1 + w_2$, where w_1 is a scalar multiple of a and w_2 is orthogonal to a.

proof:

$$w_1 = ka$$

$$u = w_1 + w_2$$

$$u = ka + w_2$$

$$u \cdot a = (ka + w_2) \cdot a$$

$$u \cdot a = k\|a\|^2 + w_2 \cdot a$$

$\boxed{w_2 \cdot a = 0}$

w_2 and a orthogonal

$$u \cdot a = k\|a\|^2$$

$$k = \frac{u \cdot a}{\|a\|^2}$$

$$w_2 = u - w_1$$

$$= u - ka$$

$$= u - \frac{u \cdot a}{\|a\|^2} a$$

$w_1 \rightarrow \text{proj}_a u$

$$\boxed{\text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a}$$

$\rightarrow$ vector component of u along a

$$w_2 = u - \text{proj}_a u$$

$$\boxed{w_2 = u - \frac{u \cdot a}{\|a\|^2} a}$$

$\rightarrow$ vector component of u orthogonal to a

Ex: Let $u = (2, -1, 3)$ and $a = (4, -1, 2)$. Find the vector component of u along a and the vector component of u orthogonal to a.

Sol^n:

$$u \cdot a = 2 \times 4 - 1 \times -1 + 3 \times 2$$

$$= 8 + 1 + 6$$

$$= 15$$

$$\|a\|^2 = 4^2 + (-1)^2 + (2)^2 = 21$$

$$\text{proj}_a u = \frac{u \cdot a}{\|a\|^2} a = \frac{15}{21}(4, -1, 2)$$

$$= \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

$$u - \text{proj}_a u = (2, -1, 3) - \left(\frac{20}{7}, -\frac{5}{7}, \frac{10}{7}\right)$$

$$= \left(-\frac{6}{7}, -\frac{2}{7}, \frac{11}{7}\right)$$

$$\|proj_a u\| = \left\| \frac{u \cdot a}{\|a\|^2} a \right\| = \left| \frac{u \cdot a}{\|a\|^2} \right| \|a\|$$

$$= \frac{|u \cdot a|}{\|a\|^2} \cdot \|a\|$$

$$\boxed{\|proj_a u\| = \frac{|u \cdot a|}{\|a\|}}$$

$$\boxed{\|proj_a u\| = \|u\| |\cos\theta|}$$

If u and v are orthogonal vectors in R^n with the Euclidean inner product, then

$$\boxed{\|u+v\|^2 = \|u\| + \|v\|^2}$$

Distance

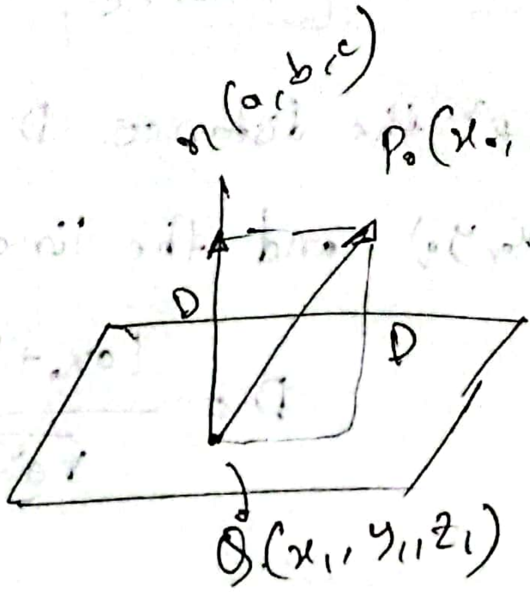
a) In R^2 the distance D between the point $P_0(x_0, y_0)$ and the line $ax+by+c=0$ is

$$D = \frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$$

b) In R^3 the distance D between the point $P_0(x_0, y_0, z_0)$ and the plane $ax+by+cz+d=0$ is.

$$D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

proof:



$$D = \|\text{proj}_n \overrightarrow{QP_0}\| \qquad \|n\| = \sqrt{a^2+b^2+c^2}$$

$$= \frac{|\overrightarrow{QP_0} \cdot n|}{\|n\|}$$

$$\overrightarrow{QP_0} = (x_0 - x_1, y_0 - y_1, z_0 - z_1)$$

$$\overrightarrow{QP_0} \cdot n = a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)$$

$$= ax_0 + by_0 + cz_0 - ax_1 - by_1 - cz_1$$

$$= ax_0 + by_0 + cz_0 + d \qquad [d = -ax_1 - by_1 - cz_1]$$

$$\therefore D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$