Lineare Equations

- 2D → $2x + 3y = 5$
- 3D → $2x + 3y + 4z = 6$

General form :

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

x → variable
a → coefficient

- systems → collection

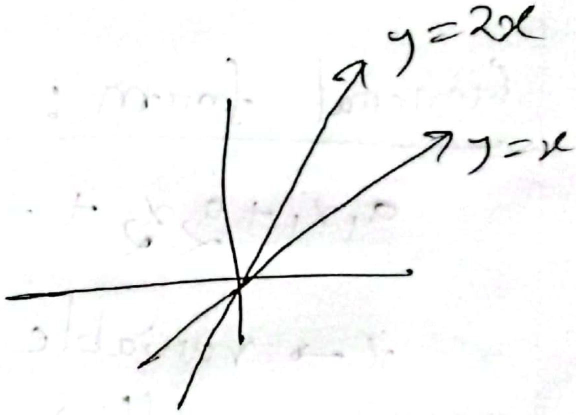
- If $b = 0$,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$$

homogeneous lineare equation

☐ System of linear Equations → collection of linear equations.

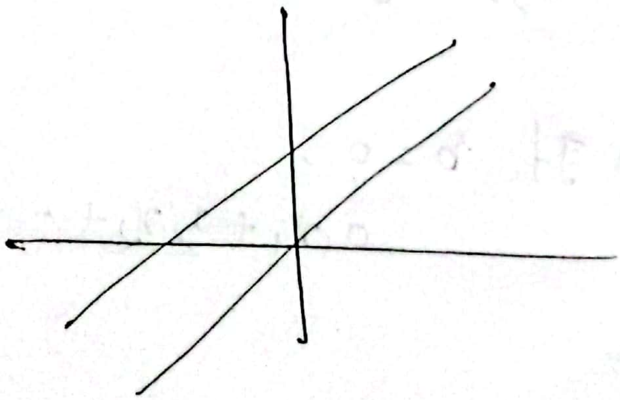
☐ $y = x$, $y = 2x$



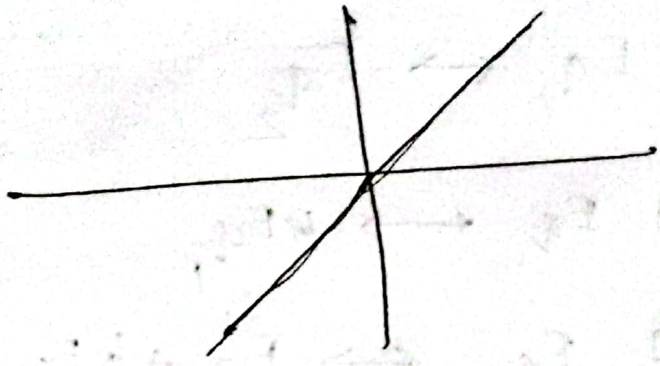
here, $\boxed{x = 0, y = 0}$

Solution possible
(consistent)

☐



parallel
↓
No solution
(Inconsistent)



Infinitely many
Solution possible

So, Three Case

No solution → Inconsistent System

One solution
Infinitely many Solutions } → Consistent System

We can replace Eq_1 by $k*Eq_1$ and $k \in R - \{0\}$

- $Eq_1 \leftrightarrow Eq_2$
- $Eq_1 \leftrightarrow kEq_1$
- $Eq_1 \leftrightarrow Eq_1 + kEq_1$

- No solution → lines may be parallel and distinct
- One solution → lines may intersect at only one point
- Infinitely many solutions → lines may coincide

- Every system of linear equations has zero, one or infinitely many solutions. There are no other possibilities.

Linear Equations with one soln:

$$x - y = 1$$

$$2x + y = 6$$

$$x = \frac{7}{3}, \quad y = \frac{4}{3}$$

A linear Systems with No Solutions:

$$x + y = 4 \quad \text{--- (i)}$$

$$3x + 3y = 6 \quad \text{--- (ii)}$$

$$\text{(i)} \times 3 - \text{(ii)}$$

$$0 = -6$$

Linear System with infinitely many sol^n:

$$4x - 2y = 1 \quad \text{--- (i)}$$
$$16x - 8y = 4 \quad \text{--- (ii)}$$

(i) $\times 4$ $-$ (ii)

$$0 = 0$$

so, x has no restriction on it's values,

Let's,

$$x = 1, \quad y = \frac{3}{2}$$
$$x = 2, \quad y = \frac{7}{2}$$

General Eq^n: (Parametric Sol^n)

$$x = t, \quad y = \frac{4t - 1}{2}$$

$$x-y+2z=5 \quad \text{— (i)}$$

$$2x-2y+4z=10 \quad \text{— (ii)}$$

$$3x-3y+6z=15 \quad \text{— (iii)}$$

(i) $\times 12 - ($(ii)$\times 3 +$ (iii) $\times 2)$

$$12x-12y+24z=60$$

$$\underset{(-)}{6x}\ \underset{(+)}{-6y}\ \underset{(-)}{+12z} = \underset{(-)}{30}$$

$$\underset{(-)}{6x}\ \underset{(+)}{-6y}\ \underset{(-)}{+12z} = \underset{(-)}{30}$$

$$0=0$$

General Sol^n:

$$x=t, \; y=p$$

$$z=\frac{5-t+p}{2}$$

Augmented Matrix

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

This called the augmented matrix.

To Solve a linear System

1. Multiply an equation through by a nonzero constant $(Eq_i \leftarrow kEq_i)$
2. Interchange two equations $(Eq_i \leftrightarrow Eq_j)$
3. Add a constant times one equation to another $(Eq_i \leftarrow Eq_i + kEq_j)$

For Augmented Matrix

1. Multiply a row through by a nonzero constant.
2. Interchange two rows
3. Add a constant times one row to another

→ These are called elementary row operations.

target,

$$\begin{matrix} 1 & \square & \\ 0 & 1 & \square \\ 0 & 0 & 1 \end{matrix}$$

$$x+y+2z=9$$

$$2x+4y-3z=1$$

$$3x+6y-5z=0$$

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

Row:

$II \leftarrow II + I(-2)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 3 & 6 & -5 & 0 \end{array}\right]$$

$III \leftarrow III + I(-3)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right]$$

$II \leftarrow II \times \frac{1}{2}$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{bmatrix}$$

$III \leftarrow III + II \times (-3)$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -\frac{1}{2} & -3/2 \end{bmatrix}$$

$III \leftarrow III \times (-2)$

$$\begin{bmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

CamScanner

$\therefore \quad x + y + 2z = 9$

$y - \frac{7}{2} z = -\frac{17}{2}$

$z = 3$

$\therefore \; y = 2$

$x = 1$

Gaussian Elimination

Echelon forms:

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$x = 4, \ y = 5, \ z = 7$

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\downarrow$ Row echelon form

$\downarrow$ Reduced Row Echelon form

properties :

1. If a row does not consist entirely of zeros, then the first non zero numbers in the row is a 1. (leading 1).

2. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.

3. In any two successive rows that do not consist entirely of zeros, the leading 1 in the lower row occurs further to the right that the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeros everywhere else in that column.

If A matrix follow (1 to 3) then it's a row echelon form

If A matrix follow (1 to 4) then it's a reduced row echelon form

□ Reduce any matrix to reduced row echelon form:-

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

<u>Step-I:</u> Locate the leftmost column that doesn't consist entirely of zeros

$$\begin{bmatrix} 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -10 & 6 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

↳ Leftmost nonzero column

<u>Step-2:</u> Interchange the top row with another row, if necessary; to bring a nonzero entry to the of the column found in step 1.

$$\begin{bmatrix} 2 & 4 & -10 & 6 & 12 & 28 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix}$$

Step-3: If the entry that is now at the top of the column in Step 1 is a, multiply the first row by 1/a in order to introduce a leading 1.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 2 & 4 & -5 & 6 & -5 & -1 \end{bmatrix} \quad I \leftarrow I \times \frac{1}{2}$$

Step-4: Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeros.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad III \leftarrow III + I \times (-2)$$

<u>Step-5:</u> Now cover the top row in the matrix and begin again with step 1 applied to the submatrix that remains. Continue in this way until the entire matrix is in row echelon form.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & -2 & 0 & 7 & 12 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix}$$

↳ Leftmost nonzero column in the submatrix

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 5 & 0 & -17 & -29 \end{bmatrix} \quad II \leftarrow II \times \frac{-1}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1/2 & 1 \end{bmatrix} \quad III \leftarrow III + II \times (-5)$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}$$

Leftmost nonzero column.

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad III \leftarrow III \times 2$$

Step 6: Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to introduce zeros above the leadings 1's.

0

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & -7/2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 6 & 14 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad II \leftarrow II + III \times \frac{7}{2}$$

$$\begin{bmatrix} 1 & 2 & -5 & 3 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad I \leftarrow I + III \times (-6)$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} \quad I \leftarrow I + II \times 5$$

Here, the last matrix is in reduced row echelon form.

- The procedure for reducing a matrix to reduced row echelon form is called Gauss-Jordan elimination.

- This process consists two phase

  forward phase → zeros are introduced below the leading 1's.

  backward phase → zeros are introduced above the leading 1's.

- If there is only forward phase then the procedure called Gaussian elimination

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = -1$$

$$5x_3 + 10x_4 + 15x_6 = 5$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 6$$

Sol^n:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix} \quad \begin{array}{l} II \leftarrow II + I(-2) \\ IV \leftarrow IV + I(-2) \end{array}$$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{array}\right]$$

$III \leftarrow III - 5\{II \times (-1)\}$

$IV \leftarrow IV - 4\{II \times (-1)\}$

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Interchange III and IV, then, $III \leftarrow III \times \frac{1}{6}$

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$II \leftarrow II + III(-3)$

$I \leftarrow I + II(2)$

$$\therefore \quad x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = \frac{1}{3}$$

$x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = \frac{1}{3}$

$x_5 = t,\ x_4 = s,\ x_3 = -2s,\ x_2 = r$

$\therefore x_1 = -3r - 4s - 2t$

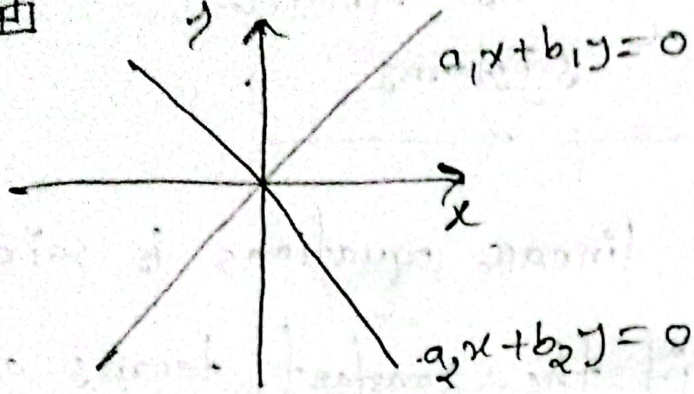
Leading 1's variables are called the Leading variables and the remaining variables ~~ca~~ are called free variables.

Homogeneous Linear Systems

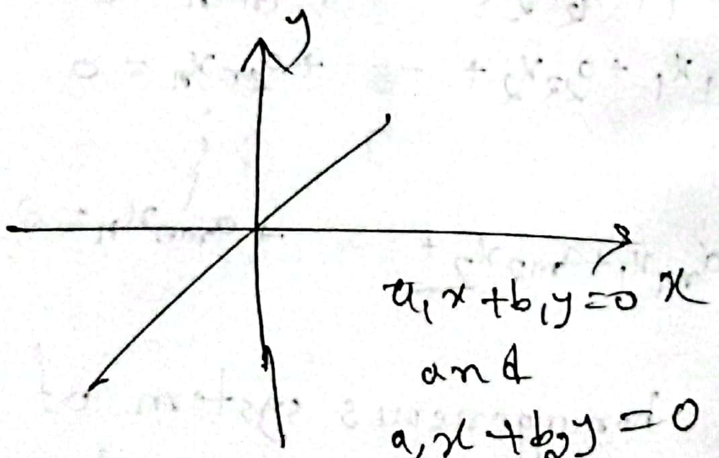
A system of linear equations is said to be homogenous if the constant terms are all zero.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots \qquad\qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

- Every homogeneous system of linear equations is consistent because all such systems have $x_1 = 0, x_2 = 0, \cdots, x_n = 0$ as a solution.
- These solution is called the trivial solution.
- If there are other solutions then it's called non-trivial solutions.



only the trivial solution



Infinitely many solutions

Example:

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 = 0$$

$$5x_3 + 10x_4 + 15x_6 = 0$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix}$$

same as previous example, we get

$$\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 4x_4 + 2x_5 = 0$$
$$x_3 + 2x_4 = 0$$
$$x_6 = 0$$

$x_3 = -2x_4$, $x_1 = -3x_2 - 4x_4 - 2x_5$

$x_6 = 0$, $x_5 = t$, $x_4 = s$, $x_3 = -2s$, $x_2 = r$

$$x_1 = -3r - 4s - 2t$$

trivial solⁿ: $\boxed{r = s = t = 0}$

In augmented matrix elementary row operations do not alter columns of zeros in a matrix.

so, If a matrix is homogeneous then reduced row echelon form is also homogeneous.

□ In homogeneous system, if there are n unknowns and reduced row echelon form of it's augmented matrix has r non-zero rows, then the system has $n-r$ free variables.

here,

free variables, $x_5 = t$, $x_4 = s$, $x_2\ r$

□ If a homogeneous system has more unknowns than equation, then this system has infinitely many solutions.

Facts:

1. Every matrix has a unique reduced row echelon form.

2. Row echelon forms are not unique

3. Despite this, reduced row echelon and row echelon has same number of zeros and leading 1's at the same position. Those are called pivot positions of A.

Matrix

Def^n. A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

Notations $A_{m \times n}$ $m \rightarrow$ rows

$n \rightarrow$ columns

$m \times 1 \longrightarrow$ row vector/matrix

$1 \times n \longrightarrow$ column vector/matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

① $[a_{ij}]_{m \times n}$ or a_{ij}

- If, $m = n$, then $A_{m \times n} \to$ square matrix

  $i = j$, $a_{11}, a_{22} --- a_{nn} \to$ main diagonal

- Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

- $A + B$, $A - B$ only possible when both A, B has same size.

cA means multiplying each entry of the matrix A by c. The matrix cA is said to be a scalare multiple of A.

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

$A_{m\times n}$, $B_{n\times n}$ then, multiplication possible, and result, $AB_{m\times n}$

Zerro matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Identify matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Triangularr matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 4 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 6 \end{bmatrix}$$

Diagonal matrix:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Matrix Multiplication

1. Row- column Rule:

$A = [a_{ij}]$ is an $m \times r$

$B = [b_{ij}]$ is an $r \times n$

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ \vdots & & & \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & & & \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

$$(AB)_{ij} = a_{i1} * b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

$$\boxed{AB_{ij} = \sum_{k=1}^{r} a_{ik} \times b_{kj}}$$

row col row col

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 40 & 33 \\ 62 & 97 & 90 \\ 114 & 154 & 147 \end{bmatrix}$$

Matrix multiplication by columns and by Rows

$$AB = A[b_1 \quad b_2 \; \cdots \; b_n] = [Ab_1 \quad Ab_2 \; \cdots \; Ab_n]$$ AB computed column by column

$$AB = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} B = \begin{bmatrix} a_1 B \\ a_2 B \\ \vdots \\ a_m B \end{bmatrix}$$ → AB computed row by row

So,

jth column vector of AB = A[jth column of B]

ith row vector of AB = [ith row of A] B

Ex:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 30 & 40 & 33 \\ 72 & 97 & 90 \\ 114 & 154 & 147 \end{bmatrix}$$

2nd column vector of $AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} * \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} 40 \\ 97 \\ 154 \end{bmatrix}$$

2nd row vector of $AB = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} * \begin{bmatrix} 2 & 4 & 7 \\ 8 & 9 & 10 \\ 4 & 6 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 72 & 97 & 90 \end{bmatrix}$$

Matrix Products as Linear Combinations

If $A_1, A_2, \ldots, A_r$ are matrices of the same size and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a linear combination of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, c_2, \ldots c_r$.

Let's, $A_{m\times n}$ and $x_{n\times 1}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$\boxed{5}$ If A is $m \times n$ matrix, and if x is an $n \times 1$ column vector, then the product Ax can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of x.

Ex:

$$A = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$Ax = 2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + -1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} 6 \\ -9 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+6 \\ 2-2-9 \\ 4-1-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 4\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}2\\6\end{bmatrix} + 2\begin{bmatrix}4\\0\end{bmatrix} & 1\times\begin{bmatrix}1\\2\end{bmatrix} - 1\begin{bmatrix}2\\6\end{bmatrix} + 7\begin{bmatrix}4\\0\end{bmatrix} & 4\begin{bmatrix}1\\2\end{bmatrix} + 3\begin{bmatrix}2\\6\end{bmatrix} + 5\begin{bmatrix}4\\0\end{bmatrix} & 3\begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\6\end{bmatrix} + 2\times\begin{bmatrix}4\\0\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix}4\\8\end{bmatrix} + \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}8\\0\end{bmatrix} & \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}-2\\-6\end{bmatrix} + \begin{bmatrix}28\\0\end{bmatrix} & \begin{bmatrix}4\\8\end{bmatrix} + \begin{bmatrix}6\\18\end{bmatrix} + \begin{bmatrix}20\\0\end{bmatrix} & \begin{bmatrix}3\\6\end{bmatrix} + \begin{bmatrix}2\\6\end{bmatrix} + \begin{bmatrix}8\\0\end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Column - Row Expansion

$A_{m \times r}$ matrix partitioned into its r column vectors $c_1, c_2, \ldots, c_r$ (each of size $m \times 1$)

$B_{r \times n}$ matrix partitioned into its r row vectors $r_1, r_2, \ldots, r_r$ (each of size $1 \times n$).

$$A = \begin{bmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_r \\ | & | & & | \end{bmatrix}_{m \times r}$$

$$B = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ & \vdots & \\ - & r_r & - \end{bmatrix}_{r \times n}$$

$$\therefore AB = c_1 r_1 + c_2 r_2 + \cdots + c_r r_r$$

Ex:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$r_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}, \quad r_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}\begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

Matrix Form of a Linear System:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$[A|b] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

↳ Augmented Matrix

Some Transposes :

$A_{m \times n}$ then, $A^T = A_{n \times m}$

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 7 & 0 \\ -5 & 8 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 & -5 \\ -2 & 7 & 8 \\ 4 & 0 & 6 \end{bmatrix}$$

Trace → some of main diagonal

$$tr(A) = 1 + 7 + 6 = 14$$

Properties :

1. $A + B = B + A$ (Commutative - add)

2. $A + (B + C) = (A + B) + C$ (Distributive - add)

3. $A(BC) = (AB)C$ (Associative - mul)

$ab(bC) = (ab)C$

$a(BC) = (aB)C$ $\qquad \boxed{AB \neq BA}$

4. $A(B + C) = AB + AC$

$(B + C)A = BA + CA$

$A(B - C) = AB - AC$

$(B - C)A = BA - CA$

$a(B \pm C) = aB \pm aC$

$(a \pm b)C = aC \pm bC$