Eigenvalues and Eigenvectors

If A is an $n \times n$ matrix, then a nonzero vector x in R^n is called and eigenvector of A (or of the matrix operator T_A) if Ax is a scalar multiple of x; that is,

$$\boxed{Ax = \lambda x}$$

$\lambda \rightarrow$ this scalar is called eigenvalue of A

$x \rightarrow$ is called eigenvector corresponding to λ.

☐ If A is an $n\times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation,

$$\boxed{\det(\lambda I - A) = 0}$$

☐ If A is an $n\times n$ triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A.

☐ If A is an $n\times n$ matrix, the following statements are equivalent.

a) λ is an eigenvalue of A.

b) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$

c) The system of equations $(\lambda I - A)x = 0$ has non trivial solution

d) There is a non zero vector x such that $Ax = \lambda x$.

Find the eigenvalues of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

Solⁿ:

$$\det(\lambda I - A)$$

$$\lambda I - A$$

$$= \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{bmatrix}$$

$$= \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0$$

$$\Rightarrow \lambda = 4,\ 2+\sqrt{3},\ 2-\sqrt{3}$$

[1] Find bases for the eigenspace of the matrix,

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

Sol^n:

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda(\lambda+1) - 6 = 0$$

$$\therefore \lambda = 2, -3$$

There are two eigen values, so, there are two eigenspaces of A.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

we know,

x is an eigenvector of A corresponding to an eigenvalue λ if and only if

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\boxed{\lambda = 2}$

$$\begin{bmatrix} 3 & -3 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x_1 - 3x_2 = 0$

$\Rightarrow x_1 = x_2$

$2x_1 - 2x_2 = 0$

$\Rightarrow x_1 = x_2$

$\therefore x_1 = x_2 = t$

$\therefore x = \begin{bmatrix} t \\ t \end{bmatrix}$

$= t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

base $\rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = -3$

$$\begin{bmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-2x_1 - 3x_2 = 0$

$x_1 = -\frac{3}{2}x_2$

$\Rightarrow x = \begin{bmatrix} -3/2\, t \\ t \end{bmatrix}$

$-2x_1 - 3x_2 = 0$

$\Rightarrow x_1 = -\frac{3}{2}x_2$

$= t \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

$x_2 = t$

$x_1 = -3/2\, t$

bases $= \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$

Th A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

Diagonalization

$A \rightarrow P^{-1}AP \rightarrow$ similarity transformation

$B = P^{-1}AP$

$\Rightarrow \det(B) = \det(P^{-1}AP)$

$= \det(P^{-1}) \cdot \det(A) \cdot \det(P)$

$= \frac{1}{\cancel{\det(P)}} \cdot \det(A) \cdot \cancel{\det(P)}$

$= \det(A)$

Similarity Invariants:

1. A and $P^{-1}AP$ have the same determinant.
2. A is invertible if and only if $P^{-1}AP$ is invertible
3. A and $P^{-1}AP$ have the same rank
4. A and $P^{-1}AP$ have the same nullity
5. A and $P^{-1}AP$ have the same trace.
6. A and $P^{-1}AP$ have the same characteristic polynomial.
7. A and $P^{-1}AP$ have the same eigenvalues.

☐ If A and B are square matrices, then we say that B is similar to A if there is an invertible matrix P such that $B = P^{-1}AP$

☐ A square matrix A is said to be diagonalizable if it is similar to some digonal matrix. that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A.

☐ If A is $n \times n$ matrix, the following statements are equivalent.

a) A is diagonalizable.

b) A has n linearly independent eigenvectors.

☐ If $\lambda_1, \lambda_2, \lambda_3 \cdots \lambda_k$ are distinct eigenvalues of a matrix A, and if $v_1, v_2, \ldots, v_k$ are corresponding eigenvectors, then $\{v_1, v_2, \ldots v_k\}$ is a linearly independent set.

An $n \times n$ matrix with n distinct eigenvalues ~~is diagonl~~ is diagonalizable.

Find a matrix P that diagonalizes,

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Sol^n:

$\det(\lambda I - A) = 0$

$$\Rightarrow \begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} = 0$$

$\lambda(\lambda-2)(\lambda-3) + 2(\lambda-2) = 0$

$\Rightarrow (\lambda-2)(\lambda^2 - 3\lambda + 2) = 0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-1) = 0$

$\therefore \lambda = 2, 1$

$\therefore \quad (\lambda I - A)x = 0$

$$\begin{bmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\lambda = 2}$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_3 = 0$

$-x_1 - x_3 = 0$

$-x_1 - x_3 = 0$

$x_3 = s, \; x_2 = t$

$\therefore x_1 = -s$

$$x = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so, bases, $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$; $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

$\boxed{\lambda = 1}$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_3 = 0$

$-x_1 - x_2 - x_3 = 0$

$-x_1 - 2x_3 = 0$

$x_3 = s$

$x_2 = s$

$x_1 = -2s$

$$\therefore x = \begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ bases $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

$$\therefore P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Show that the following matrix is not diagonalizable:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$$

Solⁿ:

$$\det(\lambda I - A) = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-1)(\lambda-2)^2 = 0$$

$$\lambda = 1, 2$$

$$(\lambda I - A)x = 0$$

$$\begin{bmatrix} \lambda-1 & 0 & 0 \\ -1 & \lambda-2 & 0 \\ 3 & -5 & \lambda-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\lambda = 1}$

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 - x_2 = 0$

$3x_1 - 5x_2 - x_3 = 0$

$\boxed{x_3 = s}$ $\quad x_2 = -\frac{1}{8}s \quad , \quad x_1 = \frac{1}{8}s$

$$x = \begin{bmatrix} \frac{1}{8}s \\ -\frac{1}{8}s \\ s \end{bmatrix} = s \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}$$

$$\text{bases} = \begin{bmatrix} 1/8 \\ -1/8 \\ 1 \end{bmatrix}$$

$\boxed{\lambda = 2}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$, $x_2 = 0$

$x_3 = s$

$$\therefore \; x = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{bases} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$A \to 3 \times 3$ and bases are two.

so, A is not diagonalizable.

$\boxdot$ If k is a positive integer, λ is an eigenvalue of a matrix A, and x is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and x is a corresponding eigenvector.

Q. Find A^{13} where,

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Soln:

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$D = P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A

$$D = P^{-1}AP$$

$$D^k = P^{-1}A^kP$$

$$\Rightarrow PD^kP^{-1} = PP^{-1}A^kPP^{-1}$$

$$= A^k$$

$$(P^{-1}AP)^2$$

$$= P^{-1}APP^{-1}AP$$

$$= P^{-1}A.I.AP$$

$$= P^{-1}A^2P$$

$$\therefore A^{13} = PD^{13}P^{-1}$$

$$A^{13} = P D^{13} P^{-1}$$

$$= \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -8190 & 0 & -16382 \\ 8191 & 8192 & 8191 \\ 8191 & 0 & 16383 \end{bmatrix}$$