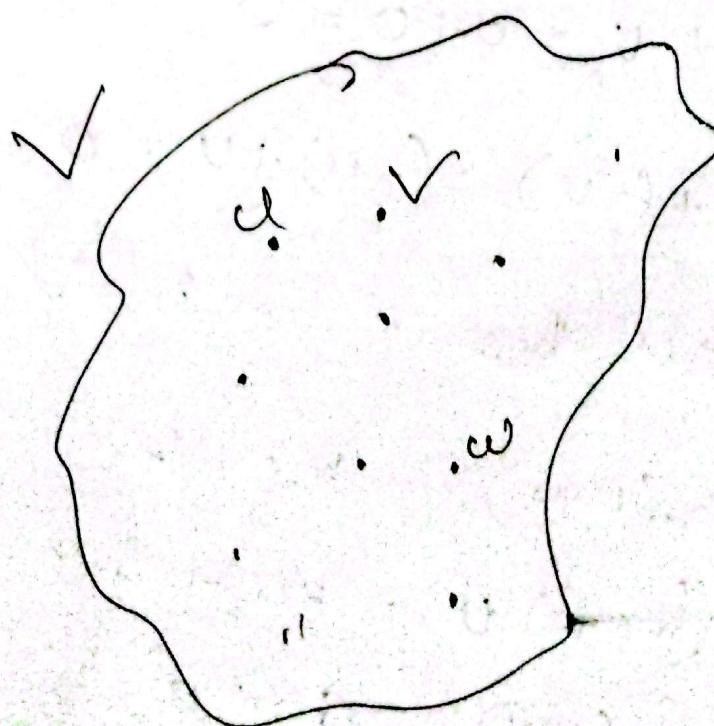


## Vector Spaces

Let  $V$  be an arbitrary non empty set of objects.

We can call  $V$  a vector space if two operations are defined for  $V$  one is addition and other one is multiplication.

Let's,  $u, v, w$  are objects in  $V$



If we want to call  $V$  a vector space  
then following 10 axioms has to be  
satisfied by  $v, u, w$ .

1. If  $u$  and  $v$  are in  $V$ , then

$u+v$  has to be in  $V$

2.  $u+v = v+u$

3.  $u+(v+w) = (u+v)+w$

4.  $0+u = u+0 = u$

5.  $u+(-u) = (-u)+u = 0$

6.  $ku$  has to be in  $V$

7.  $k(u+v) = ku+kv$

8.  $(k+m)u = ku+mu$

9.  $k(mu) = (km)u$

10.  $1.u = u$

Q If axioms 1 and 6 hold. Remaining axioms automatically holds.

Ex: zero vectors space.

Sol:

Axioms 1:  $0+0 = 0$

Axioms 6:  $k \cdot 0 = 0$

Ex:  $\mathbb{R}^n$  is a vector space

Sol:

$$V = \mathbb{R}^n$$

$$\begin{aligned} u+v &= (u_1, u_2, \dots, u_n) + (v_1, v_2, \dots, v_n) \\ &= (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n) \end{aligned}$$

$$ku = (ku_1, ku_2, \dots, ku_n)$$

Ex:  $\mathbb{R}^N$  is a vector space or not?

$$V = \mathbb{R}^N$$

$$u = (u_1, u_2), v = (v_1, v_2)$$

let's define addition, multiplication.

$$u+v = (u_1+v_1, u_2+v_2)$$

$$ku = (ku_1, 0) \rightarrow [\text{define}]$$

From axioms we know,

$$1 \cdot u = u$$

$$\therefore 1 \cdot u$$

$$= 1(u_1, u_2)$$

$$= u_1, 0$$

$$\neq u$$

So,  $\mathbb{R}^N$  is not a vector space

• If  $u$  and  $v$  are objects in  $V$ , then  
 $u+v$  is in  $V$

↳ closure under addition

• If  $k$  is any scalar and  $u$  is any  
object in  $V$ , then  $ku$  is in  $V$ .

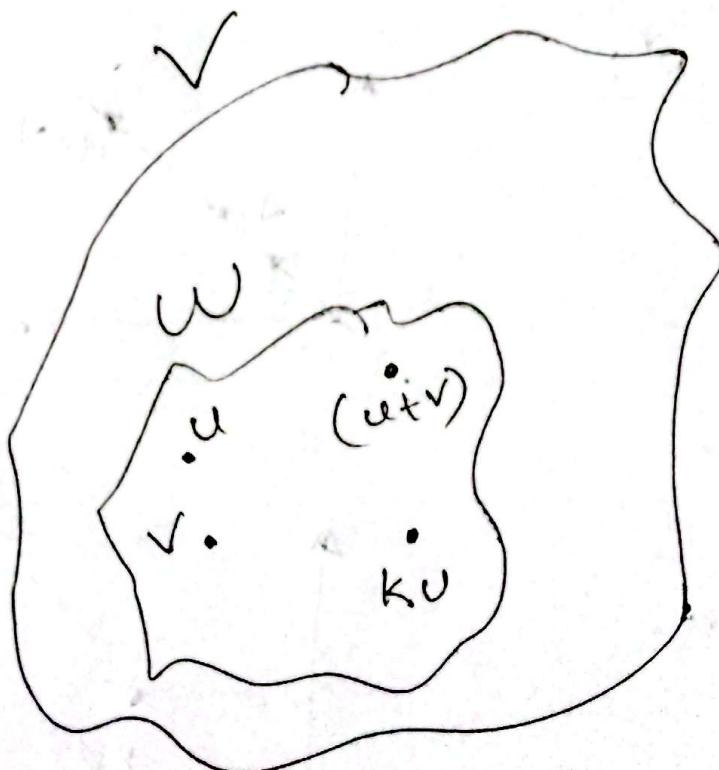
↳ closure under scalars  
Multiplication

## Subspace

if  $w$  is a subspace of  $V$ , if and only if:

a) If  $u$  and  $v$  are in  $w$  and  $u+v$  also in  $w$

b)  $ku$  also in  $w$ .



Ex: The zero subspace

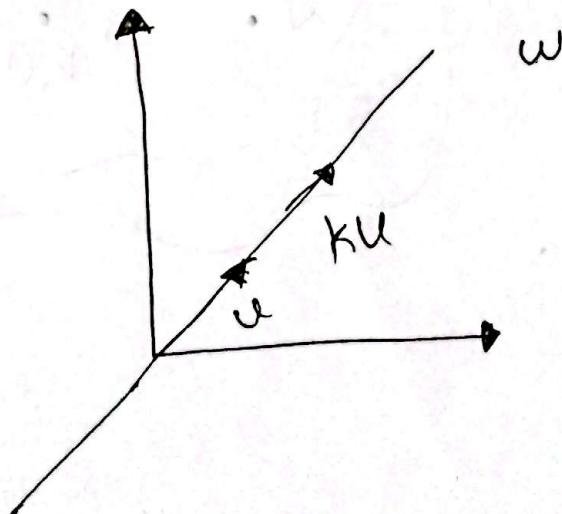
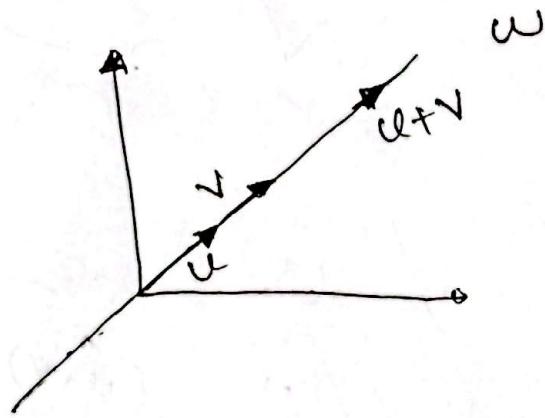
$$\omega = \{0\}$$

$$0+0 = 0$$

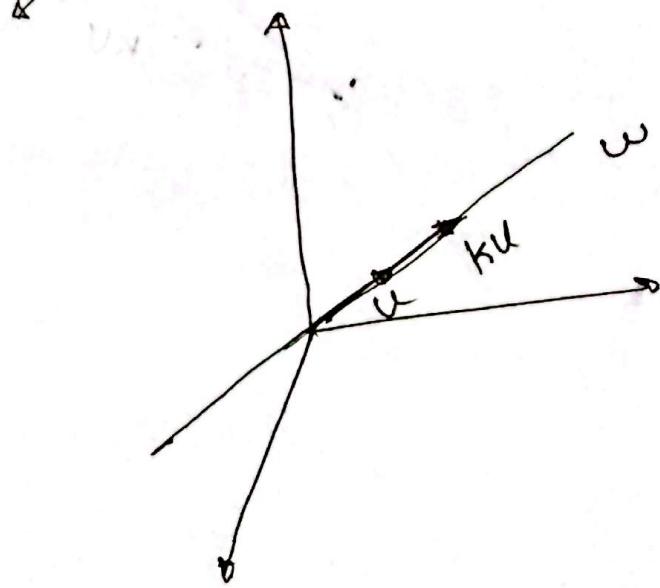
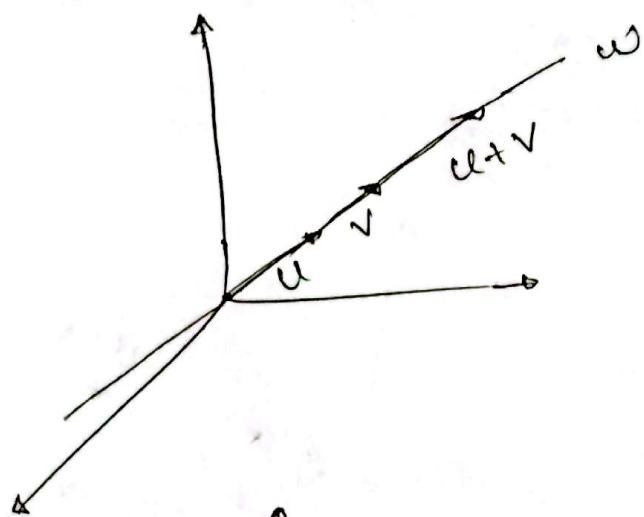
$$k0 = 0$$

Ex: Lines through origin.

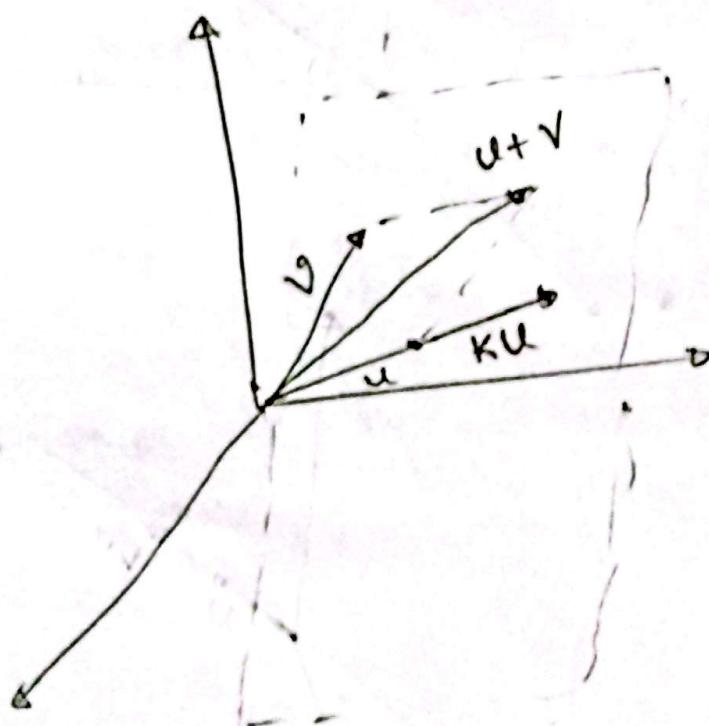
for  $\mathbb{R}^n$ :



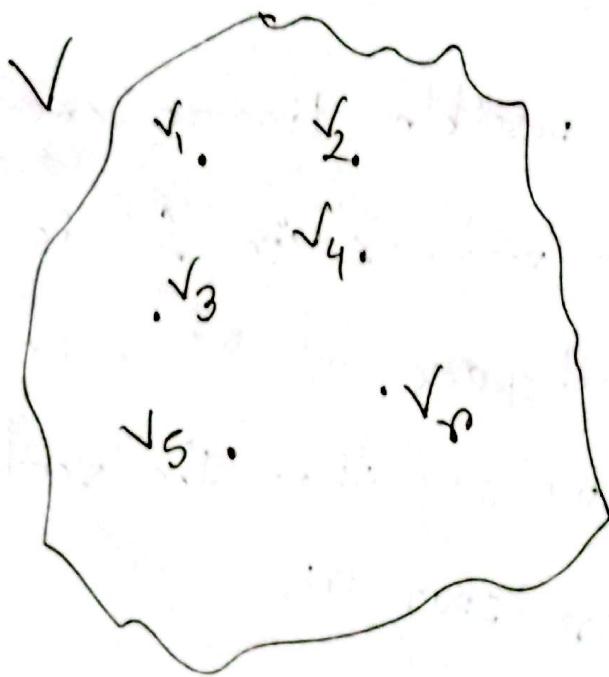
For  $\mathbb{R}^3$ :



Ex: Planes through the origin in  $\mathbb{R}^3$



If  $w$  is a vector in a vector space  $V$ , then  $w$  is said to be a linear combination of the vectors  $v_1, v_2, \dots, v_r$  in  $V$  if  $w$  can be expressed,



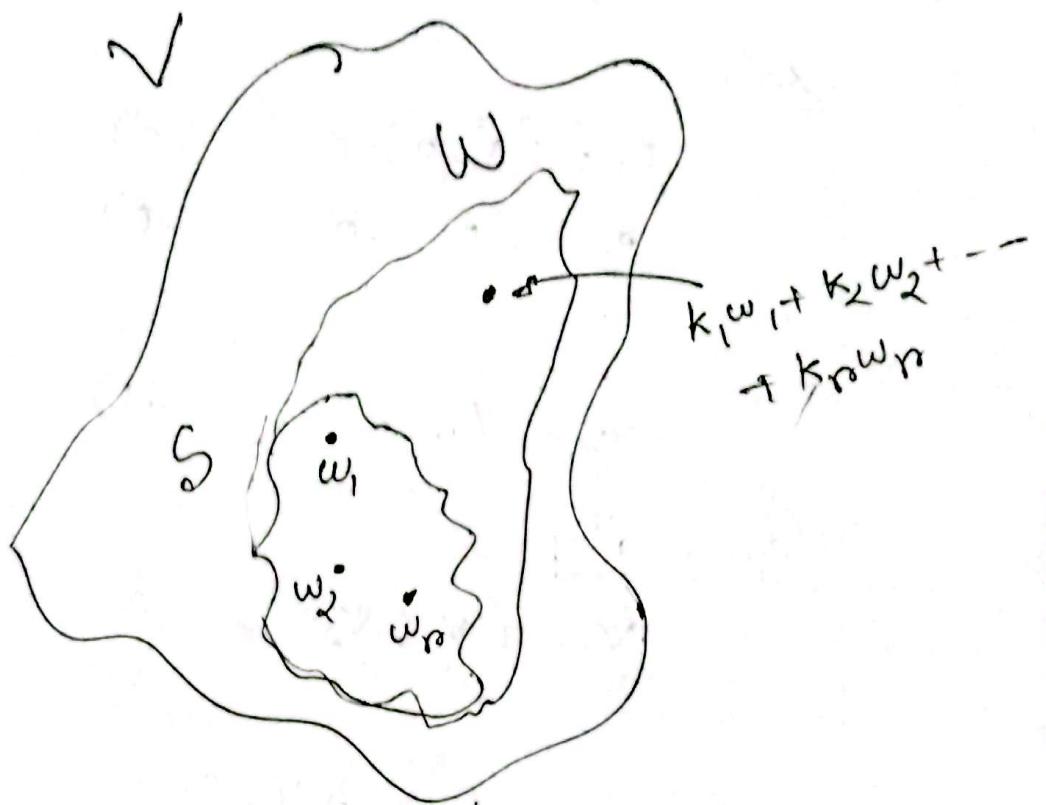
$$w = k_1 v_1 + k_2 v_2 + \dots + k_r v_r$$

If  $S = \{w_1, w_2, \dots, w_r\}$  is a nonempty set of vectors in a vector space  $V$ , then:

the subspace  $W$  of  $V$  that consists of all possible linear combinations of the vectors in  $S$  is called the subspace of  $V$  generated by  $S$ , and we say that the vectors  $w_1, w_2, \dots, w_r$  span  $W$ .

$$W = \text{span}\{w_1, w_2, \dots, w_r\}$$

$$W = \text{span}(S)$$



$\boxed{\text{The standard unit vectors span } \mathbb{R}^n}$

Sol:

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

:

$$e_n = (0, 0, \dots, -1)$$

Span  $\mathbb{R}^n$ :

$$v = (v_1, v_2, \dots, v_n) \text{ in } \mathbb{R}^n$$

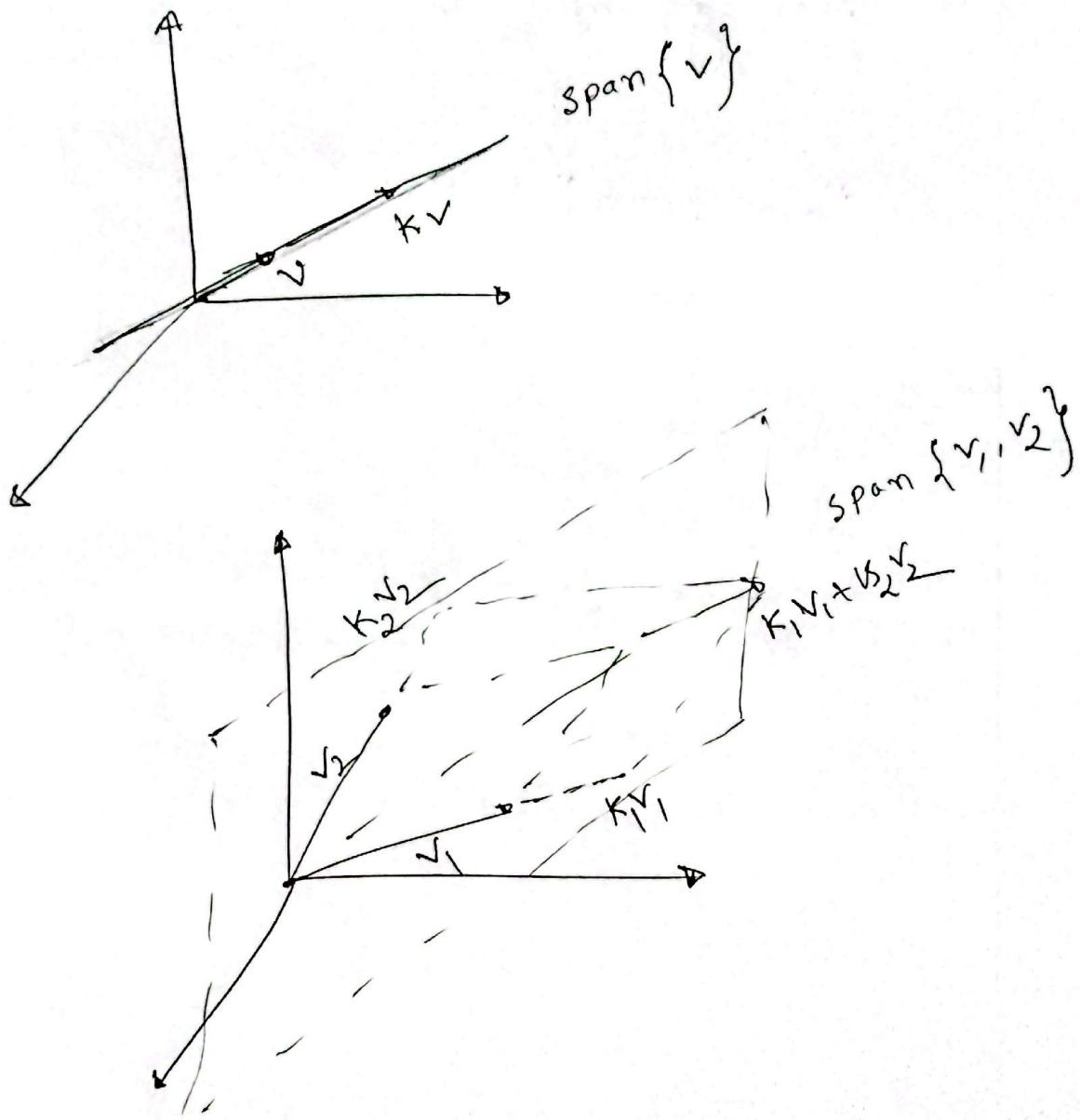
$$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n$$

Span  $\mathbb{R}^3$ :

$$v = (a, b, c) \text{ in } \mathbb{R}^3$$

$$v = a(1, 0, 0) + b(0, 1, 0) \\ + c(0, 0, 1)$$

$$v = \hat{a}\hat{i} + \hat{b}\hat{j} + \hat{c}\hat{k}$$



④ spanning of  $P_n$

$$P_n = \text{span} [1, x, x^2, \dots, x^n]$$

Example: If  $u = (1, 2, -1)$ ,  $v = (6, 4, 2)$  in  $\mathbb{R}^3$   
 show that  $w = (9, 2, 7)$  is a linear combination and  $w' = (4, -1, 8)$  not

Sol:

$$w = k_1 u + k_2 v$$

$$\Rightarrow (9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2)$$

$$\Rightarrow (9, 2, 7) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{array}{l} \therefore k_1 + 6k_2 = 9 \\ \quad 2k_1 + 4k_2 = 2 \\ \quad -k_1 + 2k_2 = 7 \end{array} \quad \left\{ \begin{array}{l} k_1 = -3 \\ k_2 = 2 \end{array} \right.$$

$w$  is a linear combination

and,  $(4, -1, 8) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$

$$\begin{array}{l} k_1 + 6k_2 = 9 \\ 2k_1 + 4k_2 = -1 \\ -k_1 + 2k_2 = 8 \end{array} \quad \left\{ \begin{array}{l} \text{It's inconsistent.} \\ \text{So, } w' \text{ is not linear combinations.} \end{array} \right.$$

Q) Determine whether  $v_1 = (1, 1, 2)$ ,  $v_2 = (1, 0, 1)$

$v_3 = (2, 1, 3)$  span  $\mathbb{R}^3$ .

Soln: If  $b = (b_1, b_2, b_3)$  in  $\mathbb{R}^3$  then  
it's has to be expressed ~~as~~ as  
linear combination,

$$b = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$\therefore (b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

$$\Rightarrow (b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

$$k_1 + k_2 + 2k_3 = b_1$$

$$k_1 + k_3 = b_2$$

$$2k_1 + k_2 + 3k_3 = b_3$$

$$\therefore A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$\det(A) = 0$ , so linear combinations not possible

that's why  $v_1, v_2, v_3$  do not span  $\mathbb{R}^3$

◻ Solution set of a homogenous linear system  $Ax = 0$  of  $m$  equations in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .

Proof: Let  $W$  be the solution set of the system. The set  $W$  is not empty because it contains at least the trivial solution  $x = 0$ .

$Ax = 0$   
 $x_1, x_2$  is a solution of the system,

$$\therefore Ax_1 = 0$$

$$Ax_2 = 0$$

$$\begin{aligned} A(x_1 + x_2) &= Ax_1 + Ax_2 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} Akx_1 \\ = k \cdot Ax_1 \\ = k \cdot 0 \\ = 0 \end{aligned}$$

∴ Solution set of a homogeneous system in  $n$  unknowns is a subspace of  $\mathbb{R}^n$ .

## Linear Independence

If  $S = \{v_1, v_2, \dots, v_n\}$  is a set of two or more vectors in a vector space  $V$ , then  $S$  is said to be a linearly independent set if no vector in  $S$  can be expressed as a linear combination of the others. A set that is not linearly independent is said to be linearly dependent.

◻ A nonempty set  $S = \{v_1, v_2, \dots, v_n\}$  in a vector space  $V$  is linearly independent if and only if the vector equation  $k_1 v_1 + k_2 v_2 + \dots + k_n v_n = 0$  has only trivial solution.

$$k_1 = k_2 = \dots = 0$$

Ex: Linear Independence of the standard Unit Vectors in  $\mathbb{R}^n$

$$\begin{array}{ll} e_1 = (1, 0, 0, \dots, 0) & i = (1, 0, 0) \\ e_2 = (0, 1, 0, \dots, 0) & j = (0, 1, 0) \\ \vdots & k = (0, 0, 1) \\ e_n = (0, 0, 0, \dots, 1) & \end{array}$$

$$\begin{aligned} k_1 i + k_2 j + k_3 k &= 0 \\ k_1 (1, 0, 0) + k_2 (0, 1, 0) + k_3 (0, 0, 1) &= 0 \\ \Rightarrow (k_1, k_2, k_3) &= (0, 0, 0) \\ k_1 = 0, k_2 = 0, k_3 &= 0 \end{aligned}$$

■ Determine whether the vectors:  
 $v_1 = (1, -2, 3)$ ,  $v_2 = (5, 6, -1)$ ,  $v_3 = (3, 2, 1)$   
 are linearly independent or linearly  
 dependent in  $\mathbb{R}^3$ .

Sol:

For, linear independence,

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0.$$

$$k_1(1, -2, 3) + k_2(5, 6, -1) + k_3(3, 2, 1) = (0, 0, 0)$$

$$k_1 + 5k_2 + 3k_3 = 0$$

$$-2k_1 + 6k_2 + 2k_3 = 0$$

$$3k_1 - k_2 + k_3 = 0$$

$$A = \begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 3 & -1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ -2 & 6 & 2 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix} \quad \text{iii} \leftarrow \text{iii} - 1 \times \text{iii}$$

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & -16 & -8 & 0 \end{bmatrix} \quad \text{ii} \leftarrow \text{ii} + 1 \times \text{iii}$$

$$= \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & 16 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{iii} \leftarrow \text{ii} + \text{iii}$$

Let's,  $k_3 = t$

$$16k_2 + 8k_3 = 0$$

$$\therefore k_2 = -\frac{1}{2}t$$

$$k_1 + 5k_2 + 3k_3 = 0$$

$$\Rightarrow k_1 - \frac{5}{2}t + 3t = 0$$

$$k_1 = -\frac{1}{2}t$$

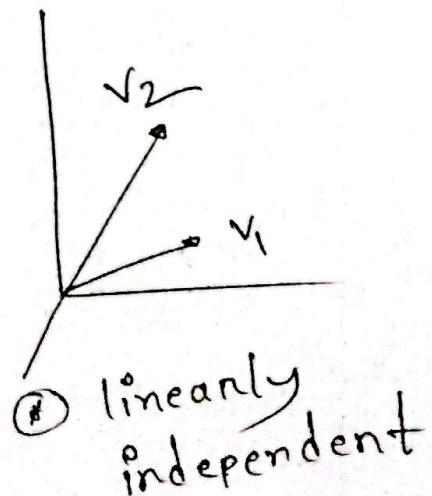
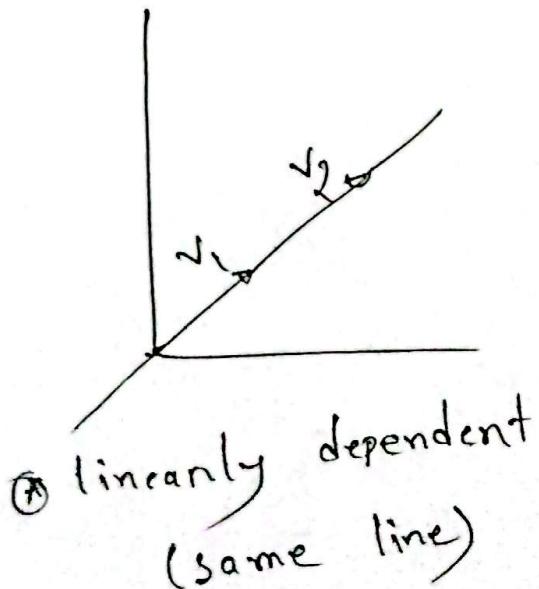
so, Infinit solutions, linearly dependent

◻ A finite set that contains  $0$  is linearly dependent.

→ A set with ~~of~~ only one vector is linearly independent if that vector is not  $0$ .

→ A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

◻



$\boxed{\text{#}}$  Let  $S = \{v_1, v_2, \dots, v_r\}$  be a set of vectors in  $\mathbb{R}^n$ . If  $r > n$ , then  $S$  is linearly dependent.

Determine whether the polynomials,

$$P_1 = 1-x, P_2 = 5+3x-2x^2, P_3 = 1+3x-x^2$$

are linearly dependent or linearly independent in  $P_2$ .

Sol:

$$\begin{aligned} k_1 P_1 + k_2 P_2 + k_3 P_3 &= 0 \\ \Rightarrow k_1(1-x) + k_2(5+3x-2x^2) + k_3(1+3x-x^2) &= 0 \\ \Rightarrow (k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + \\ &\quad (-2k_2 - k_3)x^2 = 0 \end{aligned}$$

$$k_1 + 5k_2 + k_3 = 0 \quad \det(A) = 0$$

$$-k_1 + 3k_2 + 3k_3 = 0$$

$$-2k_2 - k_3 = 0$$

$\therefore$  so, linearly  
dependent

$$A = \begin{bmatrix} 1 & 5 & 1 \\ -1 & 3 & 3 \\ 0 & -2 & -1 \end{bmatrix}$$

## Coordinates and Basis

if  $S = \{v_1, v_2, \dots, v_n\}$  is a set of vectors in a finite-dimensional vector space  $V$ , then  $S$  is called a basis for  $V$  if:

- a]  $S$  spans  $V$
- b]  $S$  is linearly independent

∴ Standard unit vectors are basis of  $\mathbb{R}^n$ :

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

$$\vdots e_n = (0, 0, 0, \dots, 1)$$

Let's  ~~$\mathbb{R}^n$~~   $v$  in  $\mathbb{R}^n$

$$v = (v_1, v_2, \dots, v_n)$$

$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n$  can  
possible so,  $e_1, e_2, \dots, e_n$  are  
basis of  $\mathbb{R}^n$ .

next,  $k_1 e_1 + k_2 e_2 + \dots + k_n e_n = 0$

$$\Rightarrow k_1 + \dots + k_n = 0$$

$\therefore k_1 = k_2 = \dots = k_n = 0$   
so, linearly independent

∴ Standard unit vectors are basis of  $\mathbb{R}^n$ .

>Show that the vectors  $v_1 = (1, 2, 1)$ ,  $v_2 = (2, 1, 0)$  and  $v_3 = (3, 3, 4)$  form a basis for  $\mathbb{R}^3$ .

Soln: we have to show the vectors are linearly independant and span  $\mathbb{R}^3$ .

For linear independence, we have to show that,  $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$  has only trivial solution.

For, span  $\mathbb{R}^3$ , we must show that every vector  $b = (b_1, b_2, b_3)$  in  $\mathbb{R}^3$  can be expressed as,

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = b$$

$$c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4) = 0$$

$$\therefore c_1 + 2c_2 + 3c_3 = 0$$

$$2c_1 + 9c_2 + 3c_3 = 0$$

$$c_1 + 4c_3 = 0.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$$

$$\text{from, } c_1(1, 2, 1) + c_2(2, 9, 0) + c_3(3, 3, 4) = (b_1, b_2, b_3)$$

$$c_1 + 2c_2 + 3c_3 = b_1$$

$$2c_1 + 9c_2 + 3c_3 = b_2$$

$$c_1 + 4c_3 = b_3$$

$$\det(A) = -1$$

Let  $(A) \neq 0$ , From equivalent statements,

$Ax=0$  has only trivial soln,

$Ax=b$  is consistent,

$\therefore v_1, v_2, v_3$  form a basis for  $R^3$ .

◻ If  $S = \{v_1, v_2, \dots, v_n\}$  is a basis for a vector space  $V$ , then every vector  $v$  in  $V$  can be expressed in the form  $v = c_1v_1 + c_2v_2 + \dots + c_nv_n$  in exactly one way.

proof:

Let's two different ways,

$$v = c_1v_1 + c_2v_2 + \dots + c_nv_n \quad \text{--- (i)}$$

$$v = k_1v_1 + k_2v_2 + \dots + k_nv_n \quad \text{--- (ii)}$$

$$\text{(i) - (ii)} \quad 0 = (c_1 - k_1)v_1 + (c_2 - k_2)v_2 + \dots + (c_n - k_n)v_n$$

$$\therefore c_1 - k_1 = 0 \quad | \quad c_2 - k_2 = 0 \quad | \quad c_n - k_n = 0$$

$$\Rightarrow c_1 = k_1 \quad | \quad \Rightarrow c_2 = k_2 \quad | \quad \Rightarrow c_n = k_n$$

田 If  $S = \{v_1, v_2, \dots, v_n\}$  is a basis for a vector space  $V$ , and

$$v = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

then,

$c_1, c_2, \dots, c_n$  are called the coordinates of  $v$  relative to the basis  $S$ . The vector  $(c_1, c_2, \dots, c_n)$  in  $\mathbb{R}^n$  constructed from these coordinates is called the coordinate vector of  $v$  relative to  $S$ : it is denoted

by

$$(v)S = (c_1, c_2, \dots, c_n)$$

$\square \quad v_1 = (1, 2, 1), v_2 = (2, 0, 0), v_3 = (3, 3, 4)$

form a basis for  $\mathbb{R}^3$ . Find the coordinate  
vector of  $v = (5, -1, 9)$  relative to the  
basis  $S = \{v_1, v_2, v_3\}$

Soln:

$$(5, -1, 9) = c_1(1, 2, 1) + c_2(2, 0, 0) + c_3(3, 3, 4)$$

$$c_1 + 2c_2 + 3c_3 = 5$$

$$2c_1 + 0c_2 + 3c_3 = -1$$

$$c_1 + 4c_3 = 9$$

$$c_1 = 1, c_2 = -1, c_3 = 2$$

$$(v)_S = (1, -1, 2)$$

## Dimension

All bases for a finite-dimensional vector space have the same number of vectors.

④ dimension of a finite-dimensional vector space  $V \rightarrow \dim(V)$

④ For zero vector space  $\rightarrow \dim(V) = 0$

④  $\dim(\mathbb{R}^n) = n$  [The standard basis has  $n$  vectors]

④  $\dim(P_n) = n+1$  [The standard basis has  $n+1$  vectors]

④  $\dim(M_{mn}) = mn$  [The standard basis has  $mn$  vectors]

◻ If  $S = \{v_1, v_2, \dots, v_n\}$  then every vector in  $\text{span}(S)$  is expressible as a linear combination of vectors in  $S$ . Thus, if the vectors in  $S$  are linearly independent, they automatically form a basis for  $\text{span}(S)$

$$\dim [\text{span} \{v_1, v_2, \dots, v_n\}] = n$$

◻ The dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

Find a basis for and the dimension of the solution space of the homogeneous system

$$x_1 + 3x_2 - 2x_3 + 2x_5 = 0$$

$$2x_1 + 6x_2 - 5x_3 - 2x_4 + 9x_5 - 3x_6 = 0$$

$$5x_3 + 10x_4 + 15x_6 = 0$$

$$2x_1 + 6x_2 + 8x_4 + 4x_5 + 18x_6 = 0$$

Sol:  $x_1 = -3s - 4t - 2r, x_2 = r, x_3 = -2s, x_4 = s$

$$x_5 = t, x_6 = 0$$

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3s - 4t - 2r, r, -2s, s, t, 0)$$

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

$$v_1 = (-3, 1, 0, 0, 0, 0), v_2 = (-4, 0, -2, 1, 0, 0)$$

$$v_3 = (-2, 0, 0, 0, 1, 0)$$

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0$$

$$\Rightarrow k_1(-3, 1, 0, 0, 0, 0) + k_2(-4, 0, -2, 1, 0, 0) + k_3(-2, 0, 0, 1, 0, 0) = 0$$

$$-3k_1 - 4k_2 - 2k_3 = 0$$

$$k_1 = 0$$

$$-2k_2 = 0$$

$$\Rightarrow k_2 = 0, k_1 = 0$$

so, linearly independent

∴ The solution space has dimension 3

### Plus/Minus Theorem

◻ If  $S$  is a linearly independent set, and if  $v$  is a vector in  $V$  that is outside of  $\text{span}(S)$ , then the set  $S \cup \{v\}$  that results by inserting  $v$  into  $S$  is still linearly independent.

◻ If  $v$  is a vector in  $S$  that is expressible as a linear combination of other vectors in  $S$ , and  ~~$S - \{v\}$~~

so, then,

$$\text{span}(S) = \text{span}(S - \{v\})$$

Let  $V$  be an  $n$ -dimensional vector space, and let  $S$  be a set in  $V$  with exactly  $n$  vectors. Then  $S$  is a basis for  $V$  if and only if  $S$  spans  $V$  or  $S$  is linearly independent.

If  $W$  is a subspace of a finite-dimensional vector space  $V$ , then:

a)  $W$  is finite-dimensional.

b)  $\dim(W) \leq \dim(V)$

c)  $W = V$  if and only if  $\dim(W) = \dim(V)$

•  $\square$  If  $A$  is an  $m \times n$  matrix, then the subspace of  $\mathbb{R}^n$  spanned by the row vectors of  $A$  is called the row space of  $A$ .

The subspace of  $\mathbb{R}^m$  spanned by the column vectors of  $A$  is called the column space of  $A$ .

The solution space of the homogeneous system of equations  $Ax = 0$ , which is a subspace of  $\mathbb{R}^n$ , is called the null space of  $A$

$\square$   $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$\begin{aligned} r_1 &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix} \\ r_2 &= \begin{bmatrix} a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \\ &\vdots \\ r_m &= \begin{bmatrix} a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{aligned}$$

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, c_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

◻ A system of linear equations  $Ax = b$  is consistent if and only if  $b$  is in the column space of  $A$ .

⇒ We can express,

$$Ax = x_1 c_1 + x_2 c_2 + \cdots + x_n c_n$$

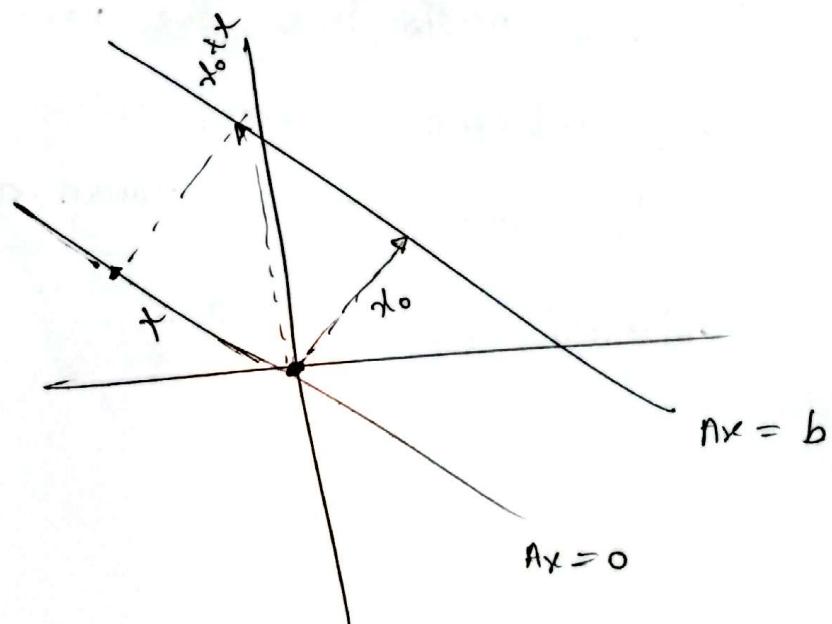
$$\therefore x_1 c_1 + x_2 c_2 + \cdots + x_n c_n = b$$

$\square$  If  $x_0$  is a solution of  $Ax=b$  and  $x'$  is a solution of  $Ax=0$  then the remaining solution of  $Ax=b$  is

$$x = x_0 + x'$$

If  $S = \{v_1, v_2, \dots, v_k\}$  is a basis for the null space of  $A$ . Then,

$$x = x_0 + c_1 v_1 + c_2 v_2 + \dots + c_k v_k$$



- ⊕ Elementary row operations do not change the null space of a matrix.
- ⊕ Elementary row operations do not change the row space of a matrix.
- ⊕ If a matrix  $R$  is in row echelon form, then the row vectors with the leading 1's form a basis for the row space of  $R$ . The column vectors with the leading 1's of the row vectors form a basis for the column space of  $R$ .

If A and B are now equivalent matrices,  
then:

- a) A given set of column vectors of A is linearly independent if and only if the corresponding column vectors of B are linearly independent.
- b) A given set of column vectors of A forms a basis for the column vectors of B form a basis for the column space of B.

◻ The row space and the column space of a matrix A have the same dimension.

◻ The common dimension of the row space and column space of a matrix A is called the rank of A.

◻ The dimension of the null space of A is called the nullity of A.

◻  $A \rightarrow mxn$  matrix

$$\text{rank}(A) \leq \min(m, n)$$

$n \rightarrow$  columns,

$$\text{rank}(A) + \text{nullity}(A) = n$$

④ If  $A$  is an  $m \times n$  matrix, then,  
 $\text{rank}(A)$  = the number of leading variables  
in the general solution of  $Ax=0$   
 $\text{nullity}(A)$  = the number of parameters in  
the general solution of  $Ax=0$

④ If  $Ax=b$  is a consistent linear system  
of  $m$  equations in  $n$  unknowns, and if  
 $A$  has rank  $r$ , then the general solution  
of the system contains  $n-r$  parameters

④ The Fundamental Space of a Matrix.

row space of  $A$

column space of  $A$

null space of  $A$

null space of  $A^T$

$\square$  If  $A$  is any matrix, then  $\text{rank}(A) = \text{rank}(A^T)$

$\square$   $A_{m \times n}$

$$\text{rank}(A^T) + \text{nullity}(A^T) = m$$

$$\text{rank}(A) + \text{nullity}(A^T) = m$$

If  $\text{rank}(A) = r$ ,

$$\dim[\text{row}(A)] = r, \quad \dim[\text{col}(A)] = r$$

$$\dim[\text{null}(A)] = n - r, \quad \dim[\text{null}(A^T)] = m - r$$

$\square$  If  $w$  is a subspace of  $\mathbb{R}^n$ , then the set of all vectors in  $\mathbb{R}^n$  that are orthogonal to every vector in  $W$  is called the orthogonal component of  $w$ , denoted by  $w^\perp$ .

- The only common vector in  $W$  and  $W^\perp$  is 0.
- The null space of  $A$  and the row space of  $A$  are orthogonal complements in  $\mathbb{R}^n$ .
- The null space of  $A^\top$  and the column space of  $A$  are orthogonal complements in  $\mathbb{R}^m$ .

## Equivalent statements:

If  $A$  is  $n \times n$  matrix,

a]  $A$  is invertible.

b]  $Ax = 0$  has only the trivial solution.

c] The reduced row echelon form of  $A$  is  $I_n$ .

d]  $A$  is expressible as a product of elementary

matrices.

e]  $Ax = b$  is consistent for every  $n \times 1$  matrix

$b$ .

f]  $Ax = b$  has exactly one solution for every

$n \times 1$  matrix  $b$ .

g]  $\det(A) \neq 0$ .

h] The column vectors of  $A$  are linearly independent.

i] The row vectors of  $A$  are linearly independent.

j] The column vectors of  $A$  span  $\mathbb{R}^n$ .

- k] The row vectors of A span  $\mathbb{R}^n$ .
- l] The column vectors of A form a basis for  $\mathbb{R}^n$
- m] The row vectors of A form a basis for  $\mathbb{R}^m$ .
- n] A has rank n.
- o] A has nullity 0.
- p] The orthogonal complement of the null space of A is  $\mathbb{R}^n$ .
- q] The orthogonal complement of the row space of A is  $\{0\}$ .

finding a Basis for the Null Space of a Matrix,

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 9 & 18 \end{bmatrix}$$

Soln:

$$Ax = 0$$

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 9 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = -3s - 4t, \quad x_2 = s, \quad x_3 = -2s, \quad x_4 = 5$$

$$x_5 = t, \quad x_6 = 0$$

$$v_1 = (-3, 1, 0, 0, 0, 0)$$

$$v_2 = (-4, 0, -2, 1, 0, 0)$$

$$v_3 = (-2, 0, 0, 0, 1, 0)$$

Find a basis for the row space  
and for the column space of the matrix.

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Sol<sup>n</sup>:

Row echelon form,

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The non zero row vectors of R form a basis for the row space of R and hence form a basis for the row space of A.

$\therefore$  These basis vectors are,

$$v_1 = [1 \ -3 \ 4 \ -2 \ 5 \ 4]$$

$$v_2 = [0 \ 0 \ 1 \ 3 \ -2 \ -6]$$

$$v_3 = [0 \ 0 \ 0 \ 0 \ 1 \ 5]$$

④ A and R have different column space so, we can not find a basis for the column space of A directly from the column vectors of R.

If we find ~~the~~ a set of vectors of R that forms a basis for the column space of R, then the corresponding column vectors of A will form a basis for the column space of A.

Since first, third, fifth columns of R contain the leading 1's of the row vectors,

the vectors,

$$c_1' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_3' = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_5' = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the column space of R.

Thus, corresponding column vectors of A,

which are,

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix}, \quad c_5 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Q1 Find a subset of the vectors;

$$\underline{v_1} = (-1, 0).$$

$$v_1 = (1, -2, 0, 3), v_2 = (2, -5, -3, 6)$$

$$v_3 = (0, 1, 3, 0), v_4 = (2, -1, 4, -7)$$

$$v_5 = (5, -8, 1, 2)$$

that forms a basis for the subspace of  $\mathbb{R}^4$  spanned by these vectors.

Express each vector not in the basis as a linear combination of the basis vectors.

Sol:

$$A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 9 & 1 \\ 3 & 6 & 0 & -7 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$
$$v_1, v_2, v_3, v_4, v_5$$



$$R = \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$  basis for the  
 $\{\omega_1, \omega_2, \omega_4\} \rightarrow$  column space of R  
 $\therefore \{v_1, v_2, v_4\} \rightarrow$  is a basis for the  
 column space of A.

we can express,

$$\omega_3 = 2\omega_1 - \omega_2$$

$$\omega_5 = \omega_1 + \omega_2 + \omega_4$$

$$\therefore v_3 = 2v_1 - v_2$$

$$v_5 = v_1 + v_2 + v_4$$

Find the rank and nullity of the matrix,

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Sol:

now echelon form of A is

$$\begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row and column spaces are two dimensional

$$\text{rank}(A) = 2$$

solution,

$$x_1 = 4r + 28s + 37t - 13u$$

$$x_2 = 2r + 12s + 16t - 5u$$

$$x_3 = r \quad : \quad x_5 = t$$

$$x_4 = s \quad x_6 = u$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{nullity}(A) = 4$$

A matrix has 6 columns.

$$\text{rank}(A) + \text{nullity}(A) = 2 + 4 = 6$$