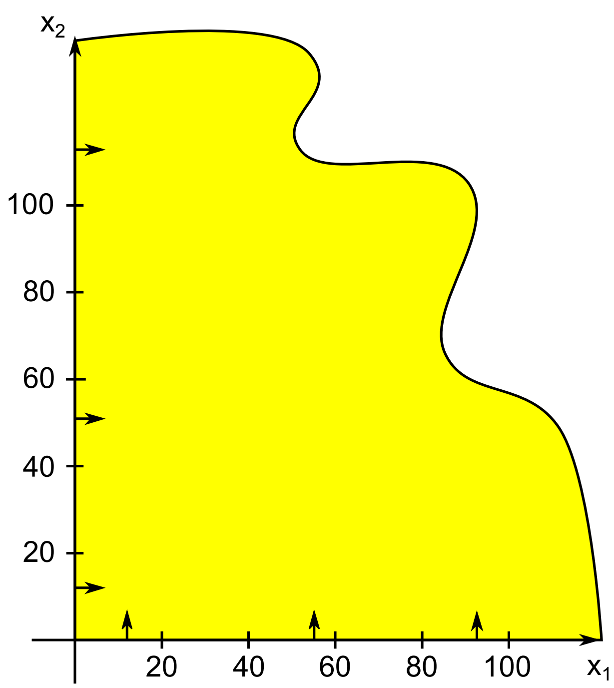
3 Solving linear programs

3.1 Graphical method

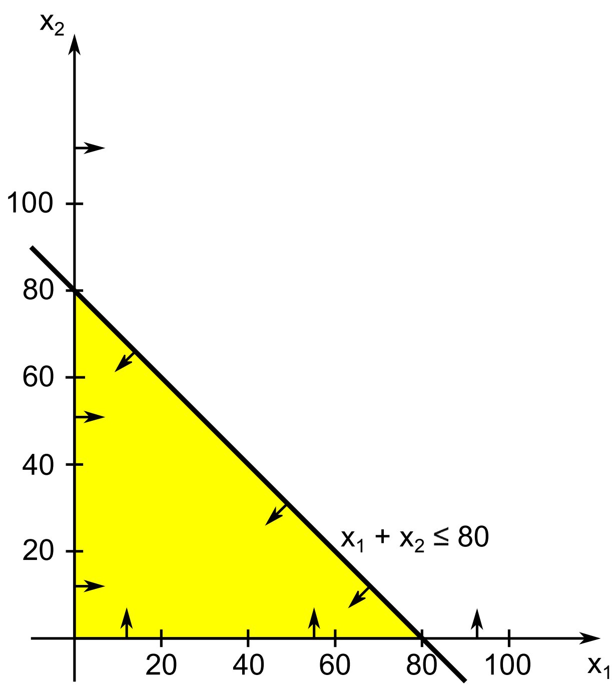
$$\begin{array}{rrrrl} \text{Max} & 3x_1 & + & 2x_2 & \\ & x_1 & + & x_2 & \leq 80 \\ & 2x_1 & + & x_2 & \leq 100 \\ & x_1 & & & \leq 40 \\ & & & x_1, x_2 & \geq 0 \end{array}$$

1. Find the feasible region.

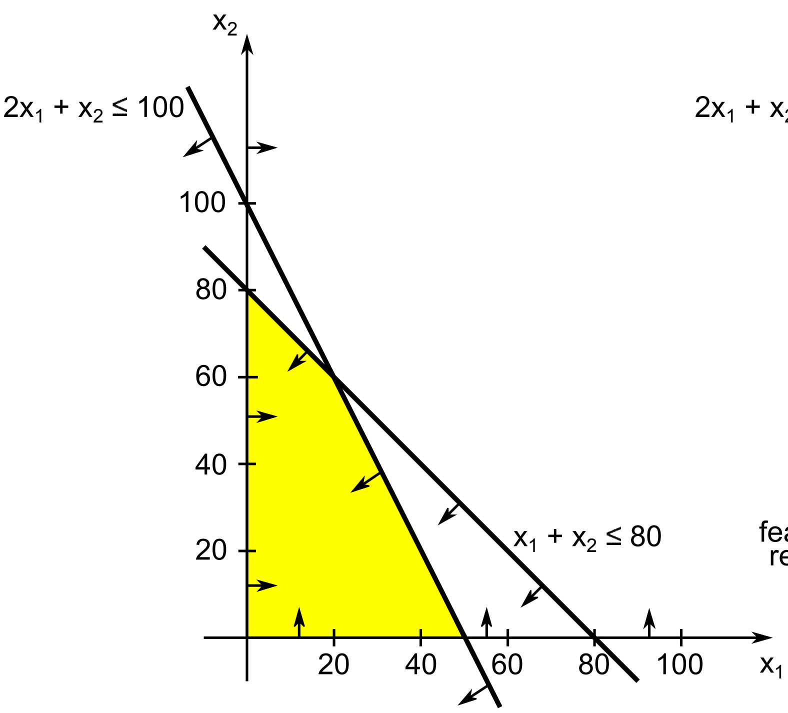
   - Plot each constraint as an equation $\equiv$ line in the plane
   - Feasible points on one side of the line – plug in (0,0) to find out which



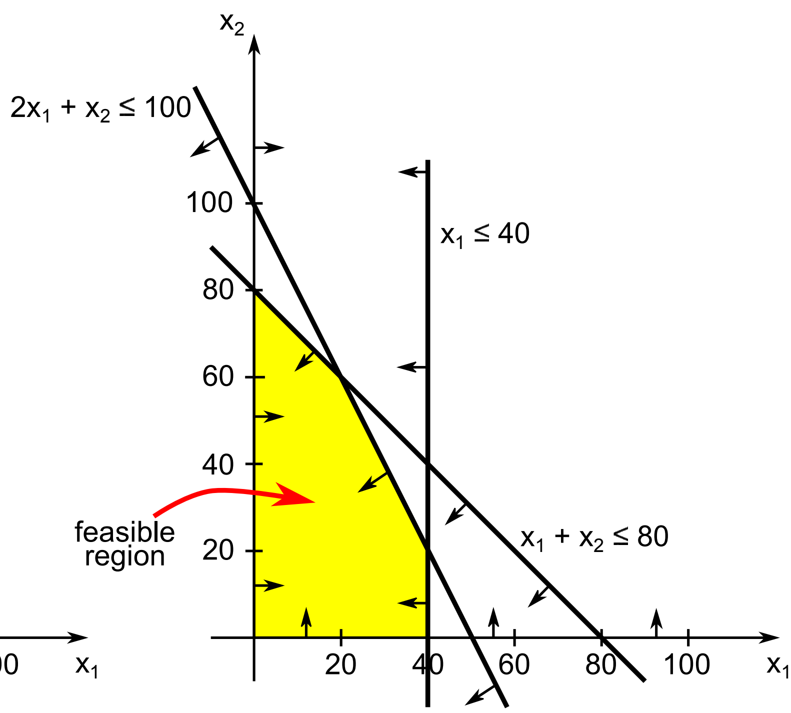
Start with $x_1 \geq 0$ and $x_2 \geq 0$



add $x_1 + x_2 \leq 80$



add $2x_1 + x_2 \leq 100$

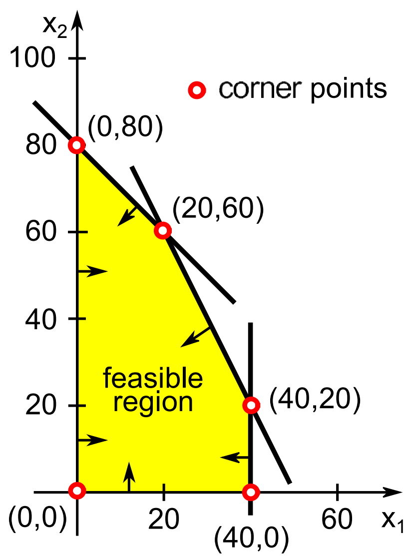


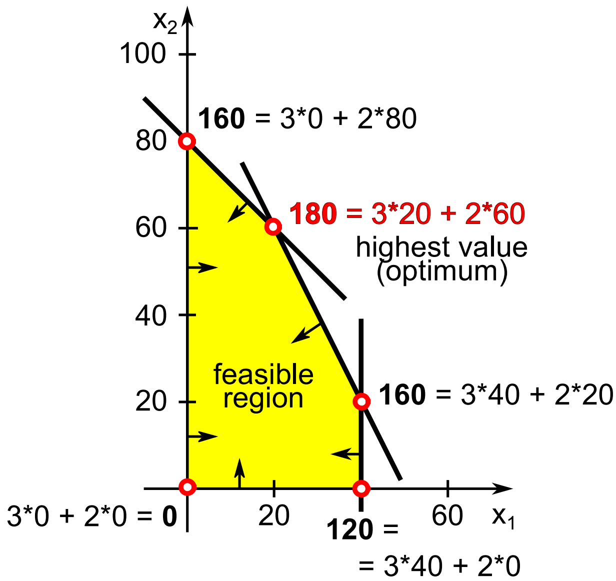
add $x_1 \leq 40$

A **corner** (extreme) point X of the region $R \equiv$ every line through X intersects R in a segment whose one endpoint is X. Solving a linear program amounts to finding a best corner point by the following theorem.

Theorem 1. *If a linear program has an* **optimal** *solution, then it also has an* **optimal solution** *that is a* **corner point** *of the feasible region.*

Exercise. Try to find all corner points. Evaluate the objective function $3x_1 + 2x_2$ at those points.





Problem: there may be too many corner points to check. There's a better way.

Iso-value line $\equiv$ in all points on this line the objective function has the same value

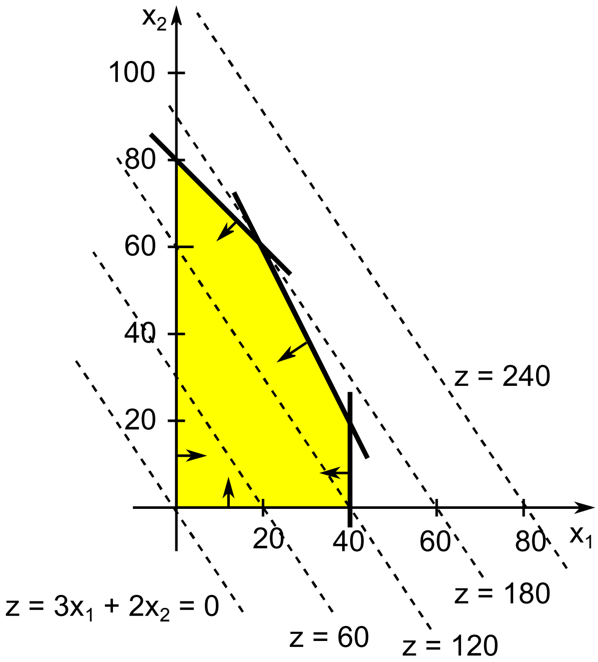
For our objective $3x_1 + 2x_2$ an iso-value line consists of points satisfying $3x_1 + 2x_2 = z$ where z is some number.

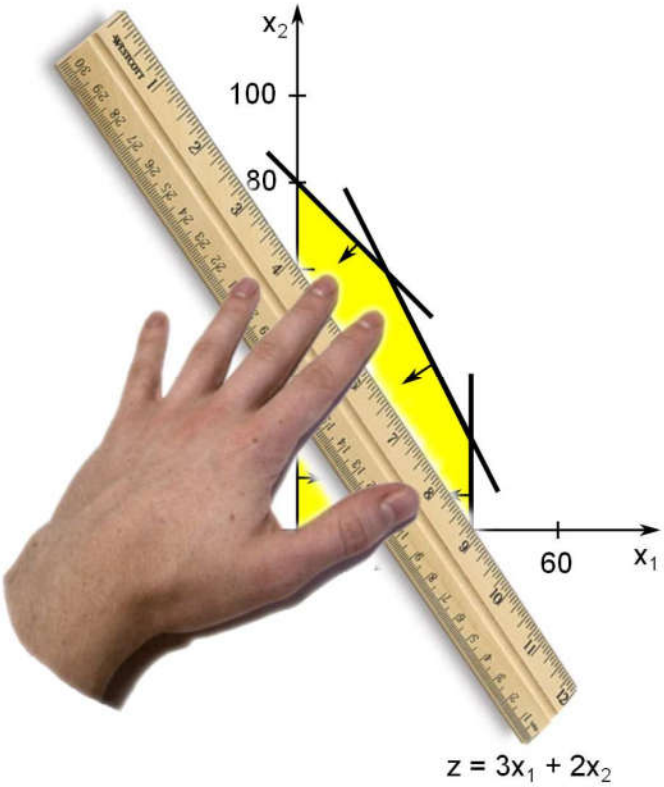
Graphical Method (main steps):

1. Find the feasible region.

2. Plot an iso-value (isoprofit, isocost) line for some value.

3. Slide the line in the direction of increasing value until it only touches the region.
4. Read-off an optimal solution.





Optimal solution is $(x_1, x_2) = (20, 60)$.

Observe that this point is the intersection of two lines forming the boundary of the feasible region. Recall that lines we use to construct the feasible region come from inequalities (the points on the line satisfy the particular inequality with equality).

Binding constraint $\equiv$ constraint satisfied with equality

For solution $(20, 60)$, the binding constraints are $x_1 + x_2 \leq 80$ and $2x_1 + x_2 \leq 100$ because $20 + 60 = 80$ and $2 \times 20 + 60 = 100$. The constraint $x_1 \leq 40$ is not binding because $x_1 = 20 < 40$.

The constraint is binding because changing it (a little) necessarily changes the optimality of the solution. Any change to the binding constraints either makes the solution not optimal or not feasible.

A constraint that is not binding can be changed (a little) without disturbing the optimality of the solution we found. Clearly we can change $x_1 \leq 40$ to $x_1 \leq 30$ and the solution $(20, 60)$ is still optimal. We shall discuss this more in-depth when we learn about Sensitivity Analysis.

Finally, note that the above process always yields one of the following cases.

Theorem 2. *Every linear program has either*

(i) a **unique** *optimal solution, or*
*(ii) multiple (***infinity***) optimal solutions, or*
(iii) is **infeasible** *(has no feasible solution), or*
(iv) is **unbounded** *(no feasible solution is maximal).*