TABLE 8.12 Parameter table for Metro Water District

			Cost (Tens of Millions of Dollars) per Unit Distributed					
			Destination					
			Berdoo (min.) 1	Berdoo (extra) 2	Los Devils 3	San Go 4	Hollyglass 5	Supply
Source	Colombo River	1	16	16	13	22	17	50
	Sacron River	2	14	14	13	19	15	60
	Calorie River	3	19	19	20	23	M	50
	Dummy	4(D)	M	0	M	0	0	50
Demand			30	20	70	30	60	

This problem will be solved in the next section to illustrate the solution procedure presented there.

8.2 A STREAMLINED SIMPLEX METHOD FOR THE TRANSPORTATION PROBLEM

Because the transportation problem is just a special type of linear programming problem, it can be solved by applying the simplex method as described in Chap. 4. However, you will see in this section that some tremendous computational shortcuts can be taken in this method by exploiting the special structure shown in Table 8.6. We shall refer to this streamlined procedure as the **transportation simplex method.**

As you read on, note particularly how the special structure is exploited to achieve great computational savings. This will illustrate an important OR technique—streamlining an algorithm to exploit the special structure in the problem at hand.

Setting Up the Transportation Simplex Method

To highlight the streamlining achieved by the transportation simplex method, let us first review how the general (unstreamlined) simplex method would set up a transportation problem in tabular form. After constructing the table of constraint coefficients (see Table 8.6), converting the objective function to maximization form, and using the Big M method to introduce artificial variables $z_1, z_2, \ldots, z_{m+n}$ into the $m + n$ respective equality constraints (see Sec. 4.6), typical columns of the simplex tableau would have the form shown in Table 8.13, where all entries *not shown* in these columns are *zeros.* [The one remaining adjustment to be made before the first iteration of the simplex method is to algebraically eliminate the nonzero coefficients of the initial (artificial) basic variables in row 0.]

After any subsequent iteration, row 0 then would have the form shown in Table 8.14. Because of the pattern of 0s and 1s for the coefficients in Table 8.13, by the *fundamental insight* presented in Sec. 5.3, u_i and v_j would have the following interpretation:

u_i = multiple of *original* row i that has been subtracted (directly or indirectly) from *original* row 0 by the simplex method during all iterations leading to the current simplex tableau.

8.2-21.) Therefore, any *BF solution* appears on a transportation simplex tableau with exactly $m + n - 1$ circled *nonnegative* allocations, where the sum of the allocations for each row or column equals its supply or demand.[1]

The procedure for constructing an initial BF solution selects the $m + n - 1$ basic variables one at a time. After each selection, a value that will satisfy one additional constraint (thereby eliminating that constraint's row or column from further consideration for providing allocations) is assigned to that variable. Thus, after $m + n - 1$ selections, an entire basic solution has been constructed in such a way as to satisfy all the constraints. A number of different criteria have been proposed for selecting the basic variables. We present and illustrate three of these criteria here, after outlining the general procedure.

General Procedure[2] for Constructing an Initial BF Solution. To begin, all source rows and destination columns of the transportation simplex tableau are initially under consideration for providing a basic variable (allocation).

1. From the rows and columns still under consideration, select the next basic variable (allocation) according to some criterion.
2. Make that allocation large enough to exactly use up the remaining supply in its row or the remaining demand in its column (whichever is smaller).
3. Eliminate that row or column (whichever had the smaller remaining supply or demand) from further consideration. (If the row and column have the same remaining supply and demand, then arbitrarily select the *row* as the one to be eliminated. The column will be used later to provide a *degenerate* basic variable, i.e., a circled allocation of zero.)
4. If only one row or only one column remains under consideration, then the procedure is completed by selecting every *remaining* variable (i.e., those variables that were neither previously selected to be basic nor eliminated from consideration by eliminating their row or column) associated with that row or column to be basic with the only feasible allocation. Otherwise, return to step 1.

Alternative Criteria for Step 1

1. *Northwest corner rule:* Begin by selecting x_{11} (that is, start in the northwest corner of the transportation simplex tableau). Thereafter, if x_{ij} was the last basic variable selected, then next select $x_{i,j+1}$ (that is, move one column to the *right*) if source i has any supply remaining. Otherwise, next select $x_{i+1,j}$ (that is, move one row *down*).

Example. To make this description more concrete, we now illustrate the general procedure on the Metro Water District problem (see Table 8.12) with the northwest corner rule being used in step 1. Because $m = 4$ and $n = 5$ in this case, the procedure would find an initial BF solution having $m + n - 1 = 8$ basic variables.

[1]However, note that any feasible solution with $m + n - 1$ nonzero variables is *not necessarily* a basic solution because it might be the weighted average of two or more degenerate BF solutions (i.e., BF solutions having some basic variables equal to zero). We need not be concerned about mislabeling such solutions as being basic, however, because the transportation simplex method constructs only legitimate BF solutions.

[2]In Sec. 4.1 we pointed out that the simplex method is an example of the algorithms (systematic solution procedures) so prevalent in OR work. Note that this procedure also is an algorithm, where each successive execution of the (four) steps constitutes an iteration.

TABLE 8.16 **Initial BF solution from the Northwest Corner Rule**

		Destination						
		1	**2**	**3**	**4**	**5**	**Supply**	u_i
	1	16 30	16 20	13	22	17	50	
	2	14	14 0	13 60	19	15	60	
Source	3	19	19	20 10	23 30	M 10	50	
	4(D)	M	0	M	0	0 50	50	
Demand		30	20	70	30	60	$Z = 2{,}470 + 10M$	
	v_j							

As shown in Table 8.16, the first allocation is $x_{11} = 30$, which exactly uses up the demand in column 1 (and eliminates this column from further consideration). This first iteration leaves a supply of 20 remaining in row 1, so next select $x_{1,1+1} = x_{12}$ to be a basic variable. Because this supply is no larger than the demand of 20 in column 2, all of it is allocated, $x_{12} = 20$, and this row is eliminated from further consideration. (Row 1 is chosen for elimination rather than column 2 because of the parenthetical instruction in step 3.) Therefore, select $x_{1+1,2} = x_{22}$ next. Because the remaining demand of 0 in column 2 is less than the supply of 60 in row 2, allocate $x_{22} = 0$ and eliminate column 2.

Continuing in this manner, we eventually obtain the entire *initial BF solution* shown in Table 8.16, where the circled numbers are the values of the basic variables ($x_{11} = 30$, . . . , $x_{45} = 50$) and all the other variables (x_{13}, etc.) are nonbasic variables equal to zero. Arrows have been added to show the order in which the basic variables (allocations) were selected. The value of Z for this solution is

$$Z = 16(30) + 16(20) + \cdots + 0(50) = 2{,}470 + 10M.$$

2. *Vogel's approximation method:* For each row and column remaining under consideration, calculate its **difference,** which is defined as *the arithmetic difference between the smallest and next-to-the-smallest unit cost c_{ij} still remaining in that row or column.* (If two unit costs tie for being the smallest remaining in a row or column, then the *difference* is 0.) In that row or column having the *largest difference,* select the variable having the *smallest remaining unit cost.* (Ties for the largest difference, or for the smallest remaining unit cost, may be broken arbitrarily.)

Example. Now let us apply the general procedure to the Metro Water District problem by using the criterion for Vogel's approximation method to select the next basic variable in step 1. With this criterion, it is more convenient to work with parameter tables (rather