7 Duality

7.1 Pricing interpretation

Back to our usual manufacturing LP problem. For the sake of illustration, we drop the 3rd constraint, and consider the items as *blocks of wood* and *cans of paint* (instead of shop hours).

Manufacturer	Market
$\begin{array}{rrcrl} \text{Max } 3x_1 + 2x_2 & & & & \\ x_1 + x_2 & \leq & 80 & \text{[wood]} \\ 2x_1 + x_2 & \leq & 100 & \text{[paint]} \\ x_1, x_2 & \geq & 0 & \end{array}$	Prices: $y_1 =$ price (in \$) of one block of wood $y_2 =$ price (in \$) of one can of paint

Manufacturer owns 80 blocks of wood and 100 cans of paint. He can sell his stock at market prices or buy additional stock at market prices. He can also produce and sell goods (toys) using the available stock.

What is his best strategy (assuming everything produced will be sold)?

- ⋆ Selling stock generates a profit of $80y_1 + 100y_2$.

- ⋆ If the cost (in market prices) of producing x_1 *toy soldiers* is strictly less than the sale price, i.e. if
$$y_1 + 2y_2 < 3$$
then there is **no limit** on the profit of manufacturer. He can generate arbitrarily large profit by buying additional stock to produce toy soldiers in arbitrary amounts.

  Why? The manufacturer can produce x_1 toy soldiers by purchasing x_1 blocks of wood, and $2x_1$ additional cans of paint. He pays $x_1(y_1 + 2y_2)$ and makes $3x_1$ in sales. Net profit is then $x_1(3 - y_1 - 2y_2)$. Now, if $y_1 + 2y_2 < 3$, say if $y_1 + 2y_2 \leq 2.9$, then the net profit is then $x_1(3 - y_1 - 2y_2) \geq (3 - 2.9) = 0.1x_1$. So making arbitrarily many x_1 toy soldiers generates a profit of $0.1x_1$ (arbitrarily high).

- ⋆ Similarly, **no limit** on the profit if the cost of producing x_2 *toy trains* is less than the sale price, i.e. if
$$y_1 + y_2 < 2$$
- ⋆ Market prices are **non-negative**.

Market (the competition) will not allow the manufacturer to make arbitrarily large profit. It will set its prices so that the manufacturer makes as little as possible. The market is thus solving the following:

$$\left.\begin{array}{rrcrl} \text{Min } 80y_1 + 100y_2 & & & \\ y_1 + 2y_2 & \geq & 3 & \text{[toy soldiers]} \\ y_1 + y_2 & \geq & 2 & \text{[toy trains]} \\ y_1, y_2 & \geq & 0 & \end{array}\right\} \textbf{Dual} \text{ of the manufacturing problem}$$

Estimating the optimal value

$$\begin{array}{rrcrcrl} \text{Max} & 3x_1 & + & 2x_2 & & & \\ & x_1 & + & x_2 & \leq & 80 & \text{[wood]} \\ & 2x_1 & + & x_2 & \leq & 100 & \text{[paint]} \\ & & & x_1, x_2 & \geq & 0 & \end{array}$$

Before solving the LP, the manufacturer wishes to get a quick rough estimate (upper bound) on the value of the optimal solution. For instance, the objective function is $3x_1 + 2x_2$ which is certainly less than $3x_1 + \mathbf{3}x_2$, since the variables x_1, x_2 are non-negative. We can rewrite this as $3(x_1 + x_2)$ and we notice that $x_1 + x_2 \leq 80$ by the first constraint. Together we have:

$$z = 3x_1 + 2x_2 \leq 3x_1 + 3x_2 \leq 3(x_1 + x_2) \leq 3 \times 80 = \$240$$

Conclusion is that every production plan will generate no more than \$240, i.e., the value of any feasible solution (including the optimal one) is not more than 240. Likewise we can write:

$$z = 3x_1 + 2x_2 \leq 4x_1 + 2x_2 \leq 2(2x_1 + x_2) \leq 2 \times 100 = \$200$$

since $2x_1 + x_2 \leq 100$ by the 2nd constraint. We can also combine constraints for an even better estimate:

$$z = 3x_1 + 2x_2 \leq (x_1 + x_2) + (2x_1 + x_2) \leq 80 + 100 = \$180$$

In general, we consider $y_1 \geq 0$, $y_2 \geq 0$ and take y_1 times the 1st contraint + y_2 times the 2nd constraint.

$$y_1(x_1 + x_2) + y_2(2x_1 + x_2) \leq 80y_1 + 100y_2$$

We can rewrite this expression by collecting coefficients of x_1 and x_2:

$$(y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 \leq 80y_1 + 100y_2$$

In this expression, if the **coefficient** of x_1 is **at least** 3 and the coefficient of x_2 is **at least** 2, i.e., if

$$y_1 + 2y_2 \geq 3$$
$$y_1 + y_2 \geq 2$$

then, just like before, we obtain an upper bound on the value of $z = 3x_1 + 2x_2$:

$$z = 3x_1 + 2x_2 \leq (y_1 + 2y_2)x_1 + (y_1 + y_2)x_2 = y_1(x_1 + x_2) + y_2(2x_1 + x_2) \leq 80y_1 + 100y_2$$

If we want the best possible upper bound, we want this expression be as small as possible.

$$\left.\begin{array}{rrcrcl} \text{Min} & 80y_1 & + & 100y_2 & & \\ & y_1 & + & 2y_2 & \geq & 3 \\ & y_1 & + & y_2 & \geq & 2 \\ & & & y_1, y_2 & \geq & 0 \end{array}\right\} \text{ The } \textbf{Dual} \text{ problem}$$

The original problem is then called the **Primal** problem.

$$\textbf{Primal} \quad \begin{array}{rrcrcr} \text{Max} & 3x_1 & + & 2x_2 & & \\ & x_1 & + & x_2 & \leq & 80 \\ & 2x_1 & + & x_2 & \leq & 100 \\ & & & x_1, x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcrcl} \text{Min} & 80y_1 & + & 100y_2 & & \\ & y_1 & + & 2y_2 & \geq & 3 \\ & y_1 & + & y_2 & \geq & 2 \\ & & & y_1, y_2 & \geq & 0 \end{array} \quad \textbf{Dual}$$

Matrix formulation

In general, for maximization problem with $\leq$ inequalities, the dual is obtained simply by

- transposing (flipping around the diagonal) the matrix $\mathbf{A}$,
- swapping vectors $\mathbf{b}$ and $\mathbf{c}$,
- switching the inequalities to $\geq$, and
- changing max to min.

$$\begin{array}{r} \max \mathbf{c}^T\mathbf{x} \\ \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ \mathbf{x} \geq 0 \end{array} \qquad \begin{array}{r} \min \mathbf{b}^T\mathbf{y} \\ \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ \mathbf{y} \geq 0 \end{array}$$

7.2 Duality Theorems and Feasibility

Theorem 3 (Weak Duality Theorem). *If* $\mathbf{x}$ *is any feasible solution of the primal and* $\mathbf{y}$ *is any feasible solution of the dual, then*

$$\mathbf{c}^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y}$$

In other words, the value of **any** feasible solution to the **dual** yields an upper bound on the value of any feasible solution (including the optimal) to the **primal**.

$$\mathbf{c}^T\mathbf{x} \leq (\mathbf{A}^T\mathbf{y})^T\mathbf{x} = (\mathbf{y}^T\mathbf{A})\mathbf{x} = \mathbf{y}^\mathbf{T}(\mathbf{A}\mathbf{x}) \leq \mathbf{y}^T\mathbf{b} = \mathbf{b}^T\mathbf{y}$$

Consequently, if **primal** is *unbounded*, then **dual** must be *infeasible* and likewise, if **dual** is *unbounded*, then **primal** must be *infeasible*. Note that is it possible that both **primal** and **dual** are *infeasible*. But if both are *feasible*, then neither of them is *unbounded*.

dual \ **primal**	infeasible	feasible bounded	unbounded
infeasible	✓	✗	✓
feasible bounded	✗	✓	✗
unbounded	✓	✗	✗

✓ possible

✗ impossible

Strong Duality and Complementary Slackness

Theorem 4 (Strong Duality Theorem). *If* $\mathbf{x}$ *is an* **optimal** *solution to the primal and* $\mathbf{y}$ *is an* **optimal** *solution to the dual, then*

$$\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$$

Moreover, $\mathbf{y}^T(\underbrace{\mathbf{b} - \mathbf{A}\mathbf{x}}_{\text{slack in primal}}) = 0$ *and* $\mathbf{x}^T(\underbrace{\mathbf{A}^T\mathbf{y} - \mathbf{c}}_{\text{slack in dual}}) = 0$.

In simple terms: whenever a constraint is **not tight** (has a positive slack) in the **primal**, then the **dual** variable corresponding to this constraint must be 0. Conversely, if a **primal** variable is strictly positive, then the corresponding **dual** constraint must be tight (slack is zero).

This can be seen as follows (note that $\mathbf{x} \geq 0$, $\mathbf{y} \geq 0$, $\mathbf{A}\mathbf{x} \geq \mathbf{b}$ and $\mathbf{A}^T\mathbf{y} \geq \mathbf{c}$)

$$0 \leq \mathbf{y}^T(\mathbf{b} - \mathbf{A}\mathbf{x}) = \mathbf{y}^T\mathbf{b} - \mathbf{y}^T\mathbf{A}\mathbf{x} = \mathbf{b}^T\mathbf{y} - (\mathbf{A}^T\mathbf{y})^T\mathbf{x} \leq \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = 0$$

$$0 \leq \mathbf{x}^T(\mathbf{A}^T\mathbf{y} - \mathbf{c}) = (\mathbf{y}^T\mathbf{A} - \mathbf{c}^T)\mathbf{x} = \mathbf{y}^T(\mathbf{A}\mathbf{x}) - \mathbf{c}^T\mathbf{x} \leq \mathbf{y}^T\mathbf{b} - \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y} - \mathbf{c}^T\mathbf{x} = 0$$

7.3 General LPs

If LP contains equalities or unrestricted variables, these can be also handled with ease. In particular,

equality constraint corresponds to an **unrestricted** variable, and **vice-versa**.

Why? Notice that when we produced an upper bound, we considered only non-negative $y_1 \geq 0$, since multiplying the $\leq$ constraint by a negative value changes the sign to $\geq$ and thus the upper bound becomes a lower bound instead. However, if the constraint was an equality (i.e. if we had $x_1 + x_2 = 80$ instead), we could allow negative y_1 as well and still produce an upper bound. For instance, we could write

$$3x_1 + 2x_2 \leq 5x_1 + 2x_2 = (-1) \times \underbrace{(x_1 + x_2)}_{=80} + 3 \times \underbrace{(2x_1 + x_2)}_{\leq 100} \leq -80 + 3 \times 100 = \$220$$

So we would make y_1 unrestricted.

Conversely, if some variable in our problem, say x_1, were unrestricted in sign (could be negative as well), then we could **not** conclude that $3x_1 + 2x_2 \leq 4x_1 + 2x_2$ holds for all feasible solutions, as we did in our 2nd estimate; namely if x_1 is **negative**, then this is **not true** (it is actually $>$ rather than $\leq$). However, if x_1 is unrestricted but $x_2 \geq 0$, we could still conclude that $3x_1 + 2x_2 \leq 3x_1 + 2x_2$, since the coefficient of x_1 is not changing in this expression. In our general expression, we had $(y_1 + 2y_2)x_1$ and we demanded that the coefficient $y_1 + 2y_2$ of x_1 is at least 3 for the upper bound to work. If x_1 is **unrestricted**, we can simply insist that the coefficient $y_1 + 2y_2$ **equals** 3 to make the upper bound work.

The same way we can conclude that

$\geq$ constraint corresponds to an **non-positive** variable, and **vice-versa**.

Primal (Max)	Dual (Min)
i-th constraint $\leq$	variable $y_i \geq 0$
i-th constraint $\geq$	variable $y_i \leq 0$
i-th constraint $=$	variable y_i unrestricted
$x_i \geq 0$	i-th constraint $\geq$
$x_i \leq 0$	i-th constraint $\leq$
x_i unrestricted	i-th constraint $=$

$$\begin{array}{rrrrrrl}
\text{Max} & 3x_1 & + & 2x_2 & + & x_3 & \\
& x_1 & + & x_2 & + & \frac{1}{2}x_3 & \leq 80 \\
& 2x_1 & + & x_2 & + & x_3 & = 100 \\
& x_1 & & & + & x_3 & \geq 40 \\
& & & & & x_1 & \text{unrestricted} \\
& & & & & x_2 & \leq 0 \\
& & & & & x_3 & \geq 0
\end{array}$$

Primal

$$\begin{array}{rrrrrrl}
\text{Min} & 80y_1 & + & 100x_2 & + & 40x_3 & \\
& y_1 & + & 2y_2 & + & y_3 & = 3 \\
& y_1 & + & y_2 & & & \leq 2 \\
& \frac{1}{2}y_1 & + & y_2 & + & y_3 & \geq 1 \\
& & & & & y_1 & \geq 0 \\
& & & & & y_2 & \text{unrestricted} \\
& & & & & y_3 & \leq 0
\end{array}$$

Dual

7.4 Complementary slackness

$$\begin{array}{rrrrrrrrl}
\max & 6x_1 & + & x_2 & - & x_3 & - & x_4 & \\
& x_1 & + & 2x_2 & + & x_3 & + & x_4 & \leq 5 \\
& 3x_1 & + & x_2 & - & x_3 & & & \leq 8 \\
& & & x_2 & + & x_3 & + & x_4 & = 1 \\
& & & & & & & x_2, x_3, x_4 & \geq 0 \\
& & & & & & & x_1 & \text{unrestricted}
\end{array}$$

We wish to check if one of the following assignments is an optimal solution.

a) $x_1 = 2, x_2 = 1, x_3 = 0, x_4 = 0$
b) $x_1 = 3, x_2 = 0, x_3 = 1, x_4 = 0$

To this end, we use **Complementary Slackness**. Let us discuss the theory first.

Theory

As usual, let $\mathbf{x}$ denote the vector of variables, let $\mathbf{c}$ be the vector of coefficients of variables of the objective function, let $\mathbf{A}$ be the coefficient matrix of the left-hand side of our constraints, and let $\mathbf{b}$ be the vector of the right-hand side of the constraints. Let $\mathbf{y}$ be the variables of the dual.

$$\begin{array}{rrcl}
\max & \mathbf{c}^{\mathrm{T}}\mathbf{x} & & \\
& \mathbf{A}\mathbf{x} & \leq & \mathbf{b} \quad \text{PRIMAL} \\
& \mathbf{x} & \geq & 0
\end{array}
\qquad\qquad
\begin{array}{rrcl}
\min & \mathbf{b}^{\mathrm{T}}\mathbf{y} & & \\
& \mathbf{A}^{\mathrm{T}}\mathbf{y} & \geq & \mathbf{c} \quad \text{DUAL} \\
& \mathbf{y} & \geq & 0
\end{array}$$

We say that vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_m)$ are **complementary** if

$$\mathbf{y}^T(\underbrace{\mathbf{b} - \mathbf{A}\mathbf{x}}_{\substack{\text{slack in}\\ \text{primal}}}) = 0 \text{ and } \mathbf{x}^T(\underbrace{\mathbf{A}^T\mathbf{y} - \mathbf{c}}_{\substack{\text{slack in}\\ \text{dual}}}) = 0$$

In other words,