A Study on Functional Analysis

Sobolev Space Theory

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Submitted by: Mehedi Hasan Nowshad

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Faculty of Science
Department of Mathematics
University of Dhaka
Dhaka-1000, Bangladesh

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Abstract

One of the most important topics in functional analysis is the Sobolev space. Sobolev space is a type of function space that consists of functions with some necessary differentiability and integrability conditions. In particular, Sobolev spaces are built to make the PDE operators bounded so that we can apply the theory of bounded operators. It is extremely useful in the analysis of partial differential equations (PDEs). This is because, Sobolev space provides tools to analyze the existence, uniqueness, regularity, and stability of solutions to various types of PDEs. In this project, we develop the notion of Sobolev spaces and study their application in the theory of PDEs. In addition, we will introduce some important notions about functional analysis that will be necessary for the study of Sobolev Spaces.

Introduction

In this paper, we are going discuss some preliminaries of functional analysis, develop the notion of Sobolev spaces, and then study their application in the theory of partial differential equations (PDEs).

Functional analysis is a branch of mathematics that deals with vector spaces of functions, maps between different function spaces, and the properties of these spaces and maps. Functional analysis allows us to define and study various function spaces that are essential for the analysis of PDEs. One of the key spaces is the Sobolev space.

Sobolev spaces are a type of function space that consists of functions with some necessary differentiability and integrability conditions. It is useful in the theory of PDEs because it provides tools to analyze the existence, uniqueness, regularity, and stability of solutions to various types of PDEs.

There is no general theory known concerning the solvability of all partial differential equations [3]. Such a theory is extremely unlikely to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations that are important for applications within and outside of mathematics, with the hope that insight from the origins of these PDEs can give clues as to their solutions.

The theory of PDEs is nowadays a huge area of active research [1], and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The contents of Sobolev space and theory of PDEs are very broad and deserve a book of its own. So for the sake of simplicity and brevity, we will limit our focus to linear PDEs, especially elliptic PDEs. Of course, we are not discussing here how to solve particular types of PDEs, rather will focus on the analytic aspects of it like the existence and uniqueness of solutions, smoothness (if a function is infinitely many times differentiable, we call it smooth) of solutions, etc. No way this paper is a comprehensive discussion of this topic, rather it is just an introductory tool that helps us to get accustomed to the discipline.

The material is often subtle and will seem largely unmotivated, but ultimately will prove extremely useful once we are done with it. This paper is primarily based on three books which are "Partial Differential Equations" [3] by Lawrence C Evans, "Real Analysis" [2] by Gerald B Folland, and "A Course in Modern Mathematical Physics" [5] by Peter Szekeres. One can give them a look if they want to explore them in detail as proofs of many theorems have been omitted due to their length and their lack of relevance to the focus of the paper.
Also for easy reading, the definitions are given in green boxes, examples in red boxes, and theorems in blue boxes.

Contents

Chapter 1 : Few First words

Contents

§1.1 PDEs, Well Posedness and Some Motivation

In simple terms, a partial differential equation (PDE) is a equation involving functions and their derivatives with respect to one or more independent variables .

§1.1.i Basic definitions in PDE

PDEs can be classified based on the order of their highest derivative (which we also call the order of the PDE) and the number of independent variables they involve.

A *solution* to a PDE is a function that satisfies the equation for all values of the independent variables. There are different types of solutions, depending on the boundary and initial conditions of the problem.

A *boundary value problem* (BVP) is a type of PDE problem that involves specifying the value of the solution at the boundary of the domain.

An *initial value problem* (IVP) is a type of PDE problem that involves specifying the value of the solution and its derivative(s) at some initial time or position.

Definition 1.1.1 (Elliptic PDE) — Let $Lu = -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u = f$ be a linear second-order PDE in n dimensions. We say that the operator L is elliptic if there exists some real valued function $\lambda(x) > 0$ such that the following condition holds:

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \lambda(x)|\xi|^2, \quad \lambda(x) > 0$$

for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. When $\lambda(x)$ can be taken as constant, we call that pde to be **Uniformly Elliptic PDE** .

This ellipticity and uniform ellipticity conditions will prove to be extremely helpful when we will discuss the analysis of PDEs in Chapter 6 .

Example 1.1.1

Here $\Delta u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ is the laplacian operator on the function u. This operator can also be extended into n-dimension .

Two of the most well known elliptic PDEs are Laplace's equation ($\Delta u = 0$) and Poisson's equation ($\Delta u = f$) with appropriate boundary conditions .

In some cases, a PDE may not have a unique solution that satisfies the boundary and/or initial conditions. In such cases, we may need to consider *generalized solutions*, which are functions that satisfy the PDE in a weaker sense.

§1.1.ii Well-posedness:

Definition 1.1.2 (Well-posedness of PDEs) — A partial differential equation (PDE) is said to be *well-posed* if it satisfies three conditions: existence, uniqueness, and continuous dependence on initial data.

(i) Existence

The existence condition requires that there is at least one solution to the PDE.

(ii) Uniqueness

The uniqueness condition requires that the solution to the PDE is unique. In other words, there is only one function that satisfies the PDE.

(iii) Continuous Dependence on Initial or boundary Data (Stability condition)

The continuous dependence on initial or boundary data condition requires that small changes in the initial or boundary data lead to small changes in the solution to the PDE. In other words, if we change the initial or boundary data by a small amount, the solution to the PDE should also change by a small amount.

Well-posedness is an important concept in PDEs because it guarantees the stability and reliability of the solution to the equation. If a PDE is not well-posed, it may have multiple solutions or no solutions at all. In addition, even if a PDE has a unique solution, small changes in the initial data may lead to large changes in the solution, making it difficult to predict or control the behavior of the system. As our physical experiments are not error free, we would be then in a big trouble if PDE involving some physical phenomena is not stable .

Example 1.1.2 (Examples of not Well Posed Problems [4])

Though the examples here may consist of ODEs, don't forget that ODEs are special types of PDEs too.

1. Consider the $1D$ problem
$$u'' = f \text{ in } (a, b)$$
$$u'(a) = u'(b) = 0$$
If a solution u exists to this problem, then it is not unique as $u + c$ is also a solution where c is a constant. So the ODE/PDE is not well-posed.

2. Consider the $1D$ problem
$$u'' + u = 0 \text{ in } (0, 2\pi)$$

$$u(0) = a \ , \ u(2\pi) = b$$

It is known that a solution of the equation has the form

$$u(x) = c_1 \ sin(x) + c_2 \ cos(x) \quad \text{ with } c_1, c_2 \in \mathbb{R}$$

hence is a 2π-periodic function. The choice of the constants is determined by the boundary conditions.
Since $u(0) = u(2\pi)$, if $a \neq b$ problem above has no solution .

By ensuring that a PDE is well-posed, we can have confidence in the accuracy and usefulness of its solutions. This is especially important in applications where the behavior of a system is critical, such as in engineering, physics, and finance. Our one key goal in this paper is to show why certain PDEs are well posed . And throughout the paper, we are going to keep building necessary frameworks for them.

§1.1.iii Multi Index Notation

Definition 1.1.3 (Multi Index Notation) — A *multi index* is a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$ is called its *order* or *length* .

Let Ω be a domain in $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a k-times continuously differentiable function. We write $D^\alpha f$ for the partial derivative of f with respect to x_1 a total of α_1 times, with respect to x_2 a total of α_2 times, and so on, up to $\partial^\alpha f / \partial x_n^{\alpha_n}$.

The *multi index notation* is a compact way of writing partial derivatives of a function f using multi indices. For example, we write

$$D^\alpha f(x) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f(x)$$

for the partial derivative of f of order $|\alpha|$ with respect to x.

Even to write more compactly, we often use the notation ∂_i to denote $\frac{\partial}{\partial x_i}$. Thus in this notation, the above equation can be written as :

$$D^\alpha f(x) = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f(x)$$

Example 1.1.3

Consider the following PDE:

$$u_{xx} + u_{yy} + u_{zz} + u_{xy} + u_{yz} + u_{zx} + u_x + u_y + u_z + u = 0$$

Using multi-index notation and summation notation, we can write this big formula very shortly as following :

$$\sum_{|\alpha| \leq 2} D^\alpha u = 0$$

Multi index notation helps us to write PDES in a very compact way, which saves a lots of space and energy for us.

§1.2 Quick Review of Some Preliminaries

§1.2.i Metric Space

Metric spaces basically generalizes the concept of Euclidean space by abstracting out the distance function on arbitrary sets .

Definition 1.2.1 — A metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ is a function called the metric on X, satisfying the following properties:

1. $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Intuitively, a metric space is a set equipped with a notion of distance between any two points in the set.

Example 1.2.1
1. For any set X, we can define a discrete metric by $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$, thus creating a metric space.
2. The set $\mathbb{R}^n$ with the Euclidean distance $d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$ is a metric space.

Now we are going to see some key definitions, examples and results regarding metric space .

Definition 1.2.2 —
- A sequence (x_n) in a metric space (X, d) is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for all $n, m \geq N$.
- A metric space (X, d) is called *complete* if every Cauchy sequence in X converges to a point in X.
- The *completion* of a metric space (X, d) is a complete metric space $(\tilde{X}, \tilde{d})$ such that X is a dense subset of $\tilde{X}$, and $\tilde{d}|_{X \times X} = d$.

In simple terms, we call $(\tilde{X}, \tilde{d})$ is a completion of (X, d) if every element in $(\tilde{X}, \tilde{d})$ can be approximated by a sequence in (X, d) .

Theorem 1.1
Every metric space (X, d) has a completion.

Example 1.2.2
The set of rational numbers $\mathbb{Q}$ is not complete but it has a completion which is the complete metric space of real numbers $\mathbb{R}$.

Definition 1.2.3 (Open Set) — Let (X, d) be a metric space and let $E \subseteq X$. We say that E is **open** if for every $x \in E$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq E$, where $B(x, \epsilon)$ is the open ball of radius ϵ centered at x.

Definition 1.2.4 (Closed Set) — Let (X, d) be a metric space and let $E \subseteq X$. We say that E is **closed** if its complement $X \setminus E$ is open.

Definition 1.2.5 (Interior) — Let (X, d) be a metric space and let $E \subseteq X$. The **interior** of E, denoted by $Int(E)$, is the union of all open sets contained in E.

Definition 1.2.6 (Closure) — Let (X, d) be a metric space and let $E \subseteq X$. The **closure** of E, denoted by $\overline{E}$, is the intersection of all closed sets containing E.

Definition 1.2.7 (Boundary) — By $\partial\Omega$ we denote the boundary of Ω and it is defined as $\partial\Omega = \overline{\Omega} - Int(\Omega)$.

Example 1.2.3
1. In the real numbers $\mathbb{R}$ with the standard metric, the interval $(0, 1)$ is an open set and the interval $[0, 1]$ is a closed set as it includes its boundary $\{0, 1\}$.
2. In the real numbers $\mathbb{R}$ with the standard metric, the interval $[0, 1)$ is neither open nor closed. Its interior is $(0, 1)$ and closure is $[0, 1]$.

Definition 1.2.8 (Function restricted to a set) — Let $f : A \to B$ is a function and $S \subset A$. Then by saying f is restricted to S, we mean that we now consider S as our domain and we usually denote this restriction as $f|_S$. Formally speaking, $f|_S(x) = f(x)$ whenever $x \in S \subset A$.

Throughout the chapter, we are going to use the notation $u|_{\partial\Omega}$ a lot, where Ω is a subset (usually open) of $\mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$. By $u|_{\partial\Omega}$, we mean that u restricted to a boundary .

Definition 1.2.9 (Continuity) — Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. We say that f is *continuous at* $x_0 \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$. We say that f is *continuous on* X if f is continuous at every point $x \in X$.

Example 1.2.4
1. The function $f(x) = x^2$ is continuous on the interval $[0, 1]$ in $\mathbb{R}$.
2. The function
$$h(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x \geq 0, \end{cases}$$
is discontinuous at $x = 0$ in $\mathbb{R}$. This function is known as **Heaviside function** or **Step up function** .
3. The function
$$\mathbf{1}_{\mathbb{Q}}(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{else} \end{cases}$$
is discontinuous at every point in $\mathbb{R}$. This function is known as **Dirichlet function** which is also known as the characteristic function of the rationals.

Definition 1.2.10 (Lipschitz function) — A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz continuous if there exists a constant $L \geq 0$ such that for all $x, y \in \mathbb{R}$, the following

Lipschitz condition holds:

$$|f(x) - f(y)| \leq L|x - y|$$

The constant L is called the Lipschitz constant.

One should notice that any Lipschitz function is also continuous .

Example 1.2.5

1. Consider the function $f(x) = 2x$ defined on $\mathbb{R}$. This function is Lipschitz continuous with Lipschitz constant $L = 2$, as $|f(x) - f(y)| = 2|x - y|$.
2. The function $f(x) = \sqrt{x}$ defined on $[0, \infty)$ is continuous, but it is not Lipschitz continuous. As x approaches 0, the slope of the function becomes unbounded, violating the Lipschitz condition.

Definition 1.2.11 (Compactness) — A subset K of a metric space (X, d) is said to be *compact* if every open cover of K has a finite subcover. That is, if $\{U_i\}_{i\in I}$ is an open cover of K, then there exist finitely many indices $i_1, i_2, \ldots, i_n$ such that $K \subseteq U_{i_1} \cup U_{i_2} \cup \cdots \cup U_{i_n}$.

Theorem 1.2 (Heine-Borel Theorem)
Let $E \subseteq \mathbb{R}^n$ be a subset. Then E is compact if and only if it is closed and bounded.

Example 1.2.6
1. The closed interval $[0, 1]$ is a compact set in $\mathbb{R}$.
2. The open interval $(0, 1)$ is a non-compact set in $\mathbb{R}$.
3. The set of natural numbers in $\mathbb{R}$ is not compact in $\mathbb{R}$ even though it is closed in $\mathbb{R}$.

Definition 1.2.12 — U, V, W usually denote the open sets in $\mathbb{R}^n$. We write

$$V \subset\subset U$$

if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact and say V **is compactly contained in** U .

Example 1.2.7
1. Consider the interval $(0, 1)$ in $\mathbb{R}$, which is compactly contained in the interval $(-1, 2)$.
2. The interval $(0, 1)$ in $\mathbb{R}$ is not compactly contained in the interval $(0, 2)$ since the closure of $(0, 1)$ includes 0, which is outside $(0, 2)$.

Definition 1.2.13 —
- The **support** of a function $f : \Omega \to \mathbb{R}$, denoted by $supp(f)$, is defined as the closure of the set $\{x \in \Omega : f(x) \neq 0\}$.
- A function $f : \Omega \to \mathbb{R}$ has **compact support** if $supp(f)$ is a compact set.
- The space of smooth functions (smooth means the function is infinitely many times differentiable) with compact support on Ω, denoted by "$C_c^\infty(\Omega)$", is defined as the set of functions $f : \Omega \to \mathbb{R}$ that are infinitely differentiable and have compact support.

Example 1.2.8

The following function is a smooth function with compact support.

$$\eta(x) = \begin{cases} C\ e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where C is chosen to be a normalizing constant such that $\int_{\mathbb{R}^n} \eta\ dx = 1$.

This function $\eta(x)$ is known as **Standard mollifier** which is very important in smoothing a function which we will discuss later at chapter 4 . They are extremely useful for constructing smooth approximation to some functions .

All the definitions made here can also be made with more general space called topological space. But to avoid complexity, we have restricted ourselves to metric space only. Interested readers can give it a look to reference [2] and [5] to explore about topological stuff.

§1.2.ii Linear Space

Intuitively speaking, a set V is considered as a **vector space** or **Linear space** over a field $\mathbb{F}$ if it is closed under linear combination i.e for any $u, v \in V$ and for any $a, b \in \mathbb{F}$, we have $au + bv \in V$. Though we expect that the reader has some "standard" ideas about Linear Algebra, let's recall some stuff which will be useful in our whole discussion .

Definition 1.2.14 (Normed space) — A normed space $(V, \|\cdot\|)$ is a vector space V equipped with a norm $\|\cdot\|$, which is a function that assigns to each vector u in V a non-negative real number $\|u\|$ satisfying the following axioms:

(a) $\|u\| = 0$ if and only if $u = \mathbf{0}$.

(b) $\|\alpha u\| = |\alpha|\|u\|$ for all scalars α in $\mathbb{F}$ and all vectors u in V.

(c) $\|u + v\| \leq \|u\| + \|v\|$ for all vectors u and v in V (the triangle inequality).

Definition 1.2.15 (Inner Product Space) — Let V be a vector space over the field $\mathbb{R}$. An **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that satisfies the following properties for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$:

1. $\langle u, v \rangle = \langle v, u \rangle$ (symmetry)
2. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ (linearity in the first argument)
3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ (additivity in the first argument)
4. $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0$ if and only if $u = 0$ (positive definiteness)

Given any inner product, we can define a norm on the vector space, where the norm is defined as $\|u\| = \sqrt{\langle u, u \rangle}$. Thus any inner product space is also a normed space . Also if a norm can be defined in this way from an inner product, then we say the norm is induced from an inner product .

Theorem 1.3 (Parallelogram Law)

A norm $\|\cdot\|$ is induced by an inner product if and only if it satisfies the following equation which is known as Parallelogram Law .

$$2\left(\|u\|^2 + \|v\|^2\right) = \|u+v\|^2 + \|u-v\|^2$$

Example 1.2.9

If we define a norm on $\mathbb{R}^2$ by $\|(x,y)\| = |x| + |y|$, then we can show that it is indeed a norm and isn't induced from any inner product .

Example 1.2.10

Let f, g be two integrable functions. Then inner product between them can simply be defined as $\langle f, g\rangle = \int fg$ if the integration is well defined. This inner product will prove to be very useful in our later chapters.

Definition 1.2.16 (Linear functional) — Let $\mathcal{V}$ be a Linear space over $\mathbb{R}(or\ \mathbb{C})$. A linear functional on $\mathcal{V}$ is a linear map $T : \mathcal{V} \to \mathbb{R}(or\ \mathbb{C})$.

Definition 1.2.17 (Dual Space) — Given a vector space V over a field $\mathbb{R}(or\ \mathbb{C})$, the dual space V^* is the set of all linear functionals from V to $\mathbb{R}(or\ \mathbb{C})$. In other words, V^* consists of all linear maps $f : V \to \mathbb{R}(or\ \mathbb{C})$.

Dual space of V is also denoted as V' often . Note that a normed linear space is also a metric space with the metric defined as $d(x,y) = \|x-y\|$. Hence we can talk about continuity of linear functionals too . Continuous Linear functionals are used to define distributions, which we will see in Chapter 4 of Distribution theory .

Definition 1.2.18 (Dual Basis) — Let V be a vector space with basis $\{v_1, v_2, \ldots, v_n\}$. The dual basis $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ of the dual space V^* is defined by:

$$\varphi_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Example 1.2.11

1. Consider the vector space $\mathbb{R}^3$. The dual space $(\mathbb{R}^3)^*$ consists of linear functionals $f : \mathbb{R}^3 \to \mathbb{R}$, which can be written as $f(x,y,z) = ax + by + cz$ for some scalars $a, b, c \in \mathbb{R}$.
2. Let $V = \mathbb{R}^3$ with the standard basis $\{(1,0,0), (0,1,0), (0,0,1)\}$. The dual basis $\{\varphi_1, \varphi_2, \varphi_3\}$ is given by:

$$\varphi_1(x,y,z) = x, \quad \varphi_2(x,y,z) = y, \quad \varphi_3(x,y,z) = z.$$

§1.2.iii Balls in higher dimension

Definition 1.2.19 (n-balls) — Let $p = (p_1, \cdots, p_n)$ be a point and $r > 0$. An n-dimensional ball $B^n(p,r)$ in Euclidean space $\mathbb{R}^n$ centered at p with radius r is defined by :

$$B^n(p,r) = \{x \in \mathbb{R}^n \mid (x_1 - p_1)^2 + (x_2 - p_2)^2 + \cdots + (x_n - p_n)^2 \leq r^2\}$$

In particular, we denote $B^n(\mathbf{0},1)$ by B^n , which is known as standard unit ball. Also by $B^n(r)$ we usually denote $B^n(\mathbf{0},r)$. Infact if the dimension is clearly known, we often ignore the superscript and only use the notation $B(p,r)$. And we denote the surface of $B^n(p,r)$ by $\partial B^n(p,r)$ and it consists of those points x such that $\|x-p\| = r$. The surface area is denoted by $S^{n-1}(r)$.

Theorem 1.4 (Volume and Surface area of n-Dimensional Ball)
The volume of an n-dimensional ball B^n with radius r is given by:

$$V_n(r) = \text{Volume}(B^n(r)) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot r^n$$

where Γ denotes the gamma function.

The surface area of the boundary of an n-dimensional ball $B^n(r)$ with radius r is given by:

$$S_{n-1}(r) = \text{Surface Area}(B^n(r)) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \cdot r^{n-1}$$

By $\alpha(n)$ we denote the volume of n-dimensional standard unit ball. Then we have,

$$\text{Volume}(B^n(r)) = \alpha(n)r^n$$
$$\text{Surface Area}(B^n(r)) = n\alpha(n)r^{n-1}$$

Also note that if we were about to integrate a function g over some ball $B(r)$, we can do it in following way :

$$\int_{B(r)} g\, dx = \int_0^r \left(\int_{\partial B(\rho)} g\, dS_x \right) d\rho$$

where dx is a infinitesimal volume element and dS_x is a infinitesimal surface element .

These stuff will prove to be very useful in our chapter 6, when we are going to do regularity analysis of PDEs .

Obviously, in particular, we have :

$$\text{Volume}(B^n(r)) = \alpha(n)r^n = \int_{B(r)} dx$$

$$\text{Surface Area}(B^n(r)) = n\alpha(n)r^{n-1} = \int_{\partial B(r)} dS_x$$

Chapter 2 : Function Spaces

Contents

We will now see some specific types of Linear Spaces and their properties, which will be pretty useful in our discussion throughout the paper.

§2.1 Banach and Hilbert Space

§2.1.i Some Useful Definitions and Examples

Definition 2.1.1 (Banach Space) — A Banach space is a complete normed vector space, i.e., a vector space X equipped with a norm $\|\cdot\|_X$ such that every Cauchy sequence in X converges to a limit in X.

Example 2.1.1

The space $C(\Omega)$ of continuous functions on a compact set $\Omega \subseteq \mathbb{R}^n$ is a Banach space with the supremum norm $\|f\|_\infty = \sup_{x\in\Omega} |f(x)|$ where $|f(x)|$ is its Euclidean norm .

The supremum norm in above example is also sometimes called **uniform norm or $C(\Omega)$ norm** and is used extensively in many areas of analysis . And from now on, whenever we talk about any norm on $C(\Omega)$, we will mean this norm unless stated otherwise .

Definition 2.1.2 (Embedding) — Let B_1 and B_2 be two Banach spaces. We say that B_1 is embedded into B_2 and write $B_1 \to B_2$, if for any $u \in B_1$ we have $u \in B_2$ and $\|u\|_{B_2} \leq c\|u\|_{B_1}$, where the constant c does not depend on $u \in B_1$.

In chapter 5, we will discuss about Sobolev embeddings and see how some function spaces are embedded in much smoother function spaces . These kinds of results are very useful in discussing regularity analysis of PDEs .

Definition 2.1.3 (Hilbert Space) — A Hilbert space is a complete inner product space, i.e., a vector space H equipped with an inner product $\langle\cdot,\cdot\rangle_H$ such that every Cauchy sequence in H converges to a limit in H.

Of course, we see that every Hilbert space is also a Banach Space .

Example 2.1.2

The space $L^2([0,1])$ of square integrable functions on $[0,1]$ is a Hilbert space with the inner product $\langle f,g\rangle_{L^2} = \int_0^1 f(x)g(x)dx$.
Details about these kinds of spaces will be discussed in next chapter.

Definition 2.1.4 (Bounded linear functional) — Let $\mathcal{V}$ be a normed space. A linear functional $T:\mathcal{V}\to\mathbb{R}$ is said to be bounded if there exists a constant $C>0$ such that for all $x\in\mathcal{V}$,

$$|T(x)| \leq C\|x\|_{\mathcal{V}}.$$

The norm of T is then defined as

$$||T||_{\mathcal{V}*} = sup_{x\in\mathcal{V}} \frac{|T(x)|}{\|x\|_{\mathcal{V}}}$$

The set of all bounded linear functionals on $\mathcal{V}$ is denoted by $\mathcal{B}(\mathcal{V})$.

Example 2.1.3

1. Consider the Euclidean space $\mathbb{R}^n$ with the Euclidean norm. The linear functional $f(v)=v_i$, where $v=(v_1,v_2,\ldots,v_n)$, is bounded since

$$\|f\|_{\text{op}} = \sup_{\|v\|\leq 1} |v_i| = 1 < \infty.$$

2. Consider the vector space $L^2[0,1]$ of square-integrable functions on $[0,1]$. The linear functional $f:L^2[0,1]\to\mathbb{R}$ defined by $f(g)=\int_0^1 g(x)\,dx$ is unbounded, as $|f(g)|$ can be arbitrarily large for functions with small norm.
3. The derivative operator $\frac{d}{dx}: C^1(\mathbb{R})\to\mathbb{R}$ is an unbounded linear function with respect to the supremum norm.

Definition 2.1.5 (Bilinear Forms) — Let V and W be vector spaces over a field $\mathbb{F}$. A function $B: V\times W\to\mathbb{F}$ is called a **bilinear form** if it satisfies the following properties for all $u,v\in V$ and $w,z\in W$ and all $a,b\in\mathbb{F}$:

1. $B(u+v,w) = B(u,w)+B(v,w)$
2. $B(u,w+z) = B(u,w)+B(u,z)$
3. $B(au,w) = aB(u,w)$
4. $B(u,bw) = bB(u,w)$

Example 2.1.4

Consider a Linear 2nd order PDE of the following form :

$$Lu = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u = f$$

Then we define an **associated bilinear form** related to this PDE as following :

$$B(u,v) = \int\left(\sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j}a_{ij}(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i}v + c(x)uv\right)$$

It can be proven that $B(u, v)$ defined above is indeed a Bilinear form.

In chapter 4, we will see detailed discussion on why associated bilinear form related to some Linear 2nd order PDE is defined like this .

Definition 2.1.6 (Bounded Bilinear Forms) — A bilinear form $B : V \times W \to \mathbb{F}$ on two normed linear spaces V and W is called **bounded** if there exists a constant $C > 0$ such that

$$|B(u, w)| \leq C|u|_V|w|_W$$

for all $u \in V$ and $w \in W$. Here $|\cdot|_V$ and $|\cdot|_W$ denote the norms on V and W, respectively.

Definition 2.1.7 (Coercive Bilinear Forms) — A bilinear form $B : V \times V \to \mathbb{F}$ is called **coercive** if there exists a constant $C > 0$ such that

$$B(u, u) \geq C|u|_V^2$$

for all $u \in V$.

Example 2.1.5

In chapter 6, we will see the proof that associated bilinear form related to Laplace and Poisson equation is both bounded and coercive .

In the next subsection, we will discuss a very important theorem in our discussion called Lax Milgram theorem, which uses Boundedness and Coercivity of bilinear forms as a necessary condition . As Lax Milgram Theorem would be very useful in our later discussion, we will sketch the proof ideas to it too. Proofs or Proof sketches of rest of the theorems in the following subsection will be omitted .

§2.1.ii Some Useful Theorems

Theorem 2.1 (Cauchy-Schwarz Inequality)

Let H be a Hilbert space and $u, v \in H$. Then,

$$|\langle u, v\rangle| \leq ||u||\ ||v||$$

Theorem 2.2 (Young's Inequality)

Let a and b be non-negative reals on $\mathbb{R}^n$ and let $1 \leq p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, Young's inequality states:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Theorem 2.3 (Riesz Representation Theorem)

Let $\mathcal{H}$ be a Hilbert space and L is a bounded linear functional on $\mathcal{H}$. Then there exists a unique $y_L \in \mathcal{H}$ such that

$$L(x) = \langle x, y_L\rangle \quad \text{for all } x \in \mathcal{H}.$$

Theorem 2.4 (Bounded Inverse Theorem)
Let X, Y be Banach spaces. Let $T : X \to Y$ be an invertible bounded operator. Then T^{-1} is also bounded.

Theorem 2.5 (Lax-Milgram Theorem)
Let H be a Hilbert space, $B : H \times H \to \mathbb{R}$ be a bilinear form that is bounded and coercive and let $F : H \to \mathbb{R}$ be a bounded linear functional i.e $F \in H^*$. Then there exists a unique $u \in H$ such that

$$B(u, v) = F(v) \quad \text{for all } v \in H. \tag{2.1}$$

Proof Sketch :

1. Let $T_u(v) = B(u, v)$. Then it can be shown that T_u is a bounded linear functional .
2. By the Riesz Representation Theorem, there exists a unique $w \in H$ so that for a fixed u and for all $v \in H$, we have :

   $$\langle w, v \rangle = T_u(v) = B(u, v)$$

   Then we can construct a map $A : H \to H$ such that for all v, we have

   $$B(u, v) = \langle Au, v \rangle$$

3. From the linearity of the bilinear form B , it can be shown that A is linear .
4. From the boundedness of B, it can be shown that A is bounded .
5. From the coercivity of B, it can be shown that A is injective .
6. From the boundedness of A, it can be shown that range of A is closed .
7. Using coercivity of B, it can be shown that range of A is H .
8. Combining all the results so far and along with Bounded Inverse Theorem, we can conclude that A has a bounded linear inverse, A^{-1} .
9. By Riesz Representation Theorem, there exists a unique $w \in H$ so that for all $v \in H$, we have :

   $$F(v) = \langle w, v \rangle$$

   Then one can show that $u = A^{-1}w$ is our one desired solution .
10. Then using the coercivity of B again, one can show the uniqueness of u .
    Thus completing our proof .

■

Remark. *Note that Lax Milgram theorem talks about some existence and uniqueness related stuff. This should give one an intuitive idea on why they might be useful in the proofs of well posedness of particuar PDEs by showing the boundedness and coercivity of associated bilinear form related to some PDE. In chapter 6 of Applications of Sobolev space, we will see exactly how this idea is useful . But before that, we still have to develop a lots of framework to discuss about that .*

§2.2 Some Useful Function Spaces

Below are some Important function spaces which would be enormously used in our later discussions. So one should bear them in mind .

1. $$C(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \;\; continuous\}$$

2. $$C\left(\overline{\Omega}\right) = \{u : \Omega \to \mathbb{R} \mid u \;\; uniformly \;\; continuous\}$$

3. $$C^k(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \;\; is \;\; k \;\; times \;\; continuously \;\; differentiable\}$$

4. $$C^k\left(\overline{\Omega}\right) = \{u : \Omega \to \mathbb{R}; \mid D^\alpha u \;\; is \;\; uniformly \;\; continuous \;\; for \;\; for \;\; all \;\; |\alpha| \leq k\}$$

5. $$C^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \;\; is \;\; infinitely \;\; differentiable\}$$

6. $$C^\infty\left(\overline{\Omega}\right) = \{u : \Omega \to \mathbb{R} \mid u \;\; is \;\; infinitely \;\; and \;\; uniformly \;\; differentiable\}$$

7. By $C_c(\Omega)$, $C_c^k(\Omega)$, $C_c^\infty(\Omega)$, etc , we denote exactly those functions which have compact support in $C(\Omega)$, $C^k(\Omega)$, $C^\infty(\Omega)$ etc respectively.

8. In Chapter 3, we are gonna see what $L^p(\Omega)$ spaces are. In intuitive sense, they are collection of all functions $f : \Omega \to \mathbb{R}$ such that $|f|^p$ is integrable in Ω .

9. In Chapter 5, we are gonna see what Sobolev spaces $W^{k,p}(\Omega)$ and $H^k(\Omega)$ are . In intuitive sense, $W^{k,p}(\Omega)$ are collections of all functions $f : \Omega \to \mathbb{R}$ such that f has weak derivatives (the concept of weak differentiability is discussed in chapter 3) of order α for all $|\alpha| \leq k$ and all the weak derivatives are in $L^p(\Omega)$.

§2.3 Holder Spaces

In this section, we will see some "Weakening" definition of Lipschitz continuity and spaces related to them .

Definition 2.3.1 (Holder Continuity) — A function $f : \Omega \to \mathbb{R}$ is said to be Hölder continuous with exponent $\alpha \in (0, 1]$ on a domain $\Omega \subset \mathbb{R}^n$ if there exists a constant $C > 0$ such that for all $x, y \in \Omega$,

$$|f(x) - f(y)| \leq C\|x - y\|^\alpha,$$

Note that any holder continuous function is also continuous .

Example 2.3.1

The function $f(x) = x^\beta$ (with $\beta \leq 1$) defined on $[0, 1]$ is a prototypical example of a function that is $C^{0,\alpha}$ Hölder continuous for $0 < \alpha \leq \beta$, but not for $\alpha > \beta$.

Definition 2.3.2 (Holder Norm) — The Hölder semi-norm of a function f with exponent α is defined as

$$[f]_{C^{0,\alpha}(\overline{\Omega})} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

The Hölder norm of a function f with exponent α in a domain Ω is defined as

$$\|f\|_{C^{0,\alpha}(\overline{\Omega})} = \|f\|_{C(\overline{\Omega})} + [f]_{C^{0,\alpha}(\overline{\Omega})},$$

Definition 2.3.3 (Holder Spaces) — The Hölder space $C^{k,\alpha}(\Omega)$ consists of all functions $u \in C^k\left(\overline{\Omega}\right)$ for which the following norm is finite :

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\alpha}(\overline{\Omega})}$$

It can be shown that the holder spaces are also Banach spaces .

In chapter 5 of Sobolev Spaces, we will see Morrey's Inequality which will basically talk about how some particular Sobolev spaces are embedded in some Holder Space .

Chapter 3 : L^p Space

Contents

§3.1 Measurable space and Measures

Definition 3.1.1 (Measurable Space) — Let Ω be a set and $\mathcal{F}$ be a collection of subsets of Ω. The pair $(\Omega, \mathcal{F})$ is called a measurable space if $\mathcal{F}$ satisfies the following properties:

1. $\Omega \in \mathcal{F}$ (the whole set is measurable).
2. If $A \in \mathcal{F}$, then the complement $A^c = \Omega \setminus A$ is also in $\mathcal{F}$.
3. If $A_1, A_2, \ldots \in \mathcal{F}$, then their union $\bigcup_{n=1}^{\infty} A_n$ is also in $\mathcal{F}$.

Here $\mathcal{F}$ is also called a sigma (σ) algebra on Ω .

Definition 3.1.2 (Measure) — Let $(\Omega, \mathcal{F})$ be a measurable space. A measure μ on $\mathcal{F}$ is a non-negative function $\mu : \mathcal{F} \to [0, \infty]$ that satisfies:

1. $\mu(\emptyset) = 0$.
2. If $A_1, A_2, \ldots$ are pairwise disjoint sets in $\mathcal{F}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Measures allow us to assign a size or weight to subsets of a set. For example, the Lebesgue measure is a measure on $\mathbb{R}^n$ that assigns a size to subsets of $\mathbb{R}^n$.

Example 3.1.1 (Borel Measurable Space)

The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ on $\mathbb{R}$ is the smallest σ-algebra containing all open intervals. A set $E \subseteq \mathbb{R}$ is Borel measurable if $E \in \mathcal{B}(\mathbb{R})$.

Measures provide a way to assign a "size" to subsets of a set X. For example, if X is a subset of $\mathbb{R}^n$, we can define measures such as the Lebesgue measure, which assigns a measure to subsets of $\mathbb{R}^n$ based on their "size" or volume.

One of the key reasons we need to define measure and integration with respect to a measure so abstractly, is because we can then integrate much more functions in this way than we can do in Riemann way. In fact, the space of Lebesgue integrable functions are complete while the space of Riemann integrable functions are not. Thus we have the tools of Banach spaces to deal with the Lebesgue integrable functions.

§3.2 Lebesgue Measure and Integration

We will now define lebesgue measure and lebesgue measurable space on real numbers $\mathbb{R}$.

Definition 3.2.1 (Lebesgue Outer Measure) — The Lebesgue outer measure of the interval (a, b) is $\mu^*((a, b)) = b - a$.

The Lebesgue outer measure of the open set $G = \bigcup(a_k, b_k)$ is $\mu^*(G) = \sum(b_k - a_k)$.

where open sets (a_k, b_k)'s are disjoint.

The Lebesgue outer measure of an arbitrary set A is :

$$\mu^*(A) = inf\{\mu^*(G)| \text{ G is open } \}$$

Example 3.2.1

Any countable set of real numbers has outer measure of zero. Because :

Consider a countable set $C = \{x_1, x_2, x_3, \ldots\}$ in $\mathbb{R}$. For each x_i, we can find an open interval I_i of length $\epsilon/2^i$ containing x_i, where $\epsilon > 0$ is a chosen small positive constant. Then,

$$C \subseteq \bigcup_{i=1}^{\infty} I_i$$

and we have

$$\sum_{i=1}^{\infty} |I_i| \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$$

Since ϵ can be arbitrarily small, we have $m^*(C) = 0$, which means the outer measure of the countable set C is zero.

Definition 3.2.2 (Lebesgue Measurable Set and Lebesgue Measure) — A set E is said to be lebesgue measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

for any set A .

The Lebesgue measure of E is then simply set to be

$$\mu(E) = \mu^*(E)$$

Theorem 3.1

Any set of outer measure zero is measurable. In particular, any countable set is measurable.

Theorem 3.2

The collection of all lebesgue measurable sets forms a $\sigma-$algebra .

Definition 3.2.3 (Lebesgue Measurable Space) — The collection of all lebesgue measurable set is lebesgue measurable space

Definition 3.2.4 (Lebesgue Measurable function) — An extended real valued (extended reals mean $\mathbb{R} \cup \{-\infty, \infty\}$) function f is said to be lebesgue measurable if its domain is lebesgue measurable and pre-image of any open set under this function is lebesgue measurable.

Definition 3.2.5 (Characteristic Function) — The characteristic function χ_E of a set E is defined as:

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases}$$

Theorem 3.3

Characteristic functions of lebesgue measurable spaces are lebesgue measurable functions.

Definition 3.2.6 (Almost Everywhere (a.e.)) — Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A property holds almost everywhere (a.e.) if it holds for all points in Ω except possibly on a set of measure zero. Formally, a property holds almost everywhere if there exists a set $N \in \mathcal{F}$ with $\mu(N) = 0$ such that the property holds for all points in $\Omega \setminus N$.

Example 3.2.2

Characteristic function of a countable set is 0 almost everywhere .

Definition 3.2.7 (Simple Function) — A simple function $s : \Omega \to \mathbb{R}$ is a function that takes on a finite number of distinct values. It can be written as a linear combination of characteristic functions:

$$s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

where a_i are constants and E_i are disjoint measurable sets.

Definition 3.2.8 (Lebesgue Integration) — Let $s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ be a simple function with E_i being disjoint measurable sets and $a_i \neq 0 \implies \mu(E_i) < \infty$. Then the Lebesgue integral of s with respect to lebesgue measure μ is given by:

$$\int_\Omega s \, d\mu = \sum_{i=1}^{n} a_i \mu(E_i)$$

For arbitrary Lebesgue measurable function $f : \Omega \to \mathbb{R}$ with $f > 0$, If every step function s, with $s \geq 0$ and $s \leq f$, is integrable and $\int s \, d\mu$ is finite, then we say that f is Lebesgue integrable. The value of the Lebesgue integral of f is

$$\int_\Omega f \, d\mu = sup\{\int_\Omega s \, d\mu\}$$

Then, for arbitrary Lebesgue measurable function $f : \Omega \to \mathbb{R}$, $\int f = \int f^+ - \int f^-$ with $f^+ = \frac{f+|f|}{2}$ and $f^- = \frac{|f|-f}{2}$.

Theorem 3.4

Any Riemann integrable function is also Lebesgue integrable.

Example 3.2.3

Consider the interval $[0, 1]$ and let A be the set of rational numbers in $[0, 1]$. The characteristic function $\chi_A(x)$ of the rationals on $[0, 1]$ is given by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Since the rationals are countable and the irrationals are uncountable, the set of rational points has Lebesgue measure zero. Therefore, the Lebesgue integral becomes:

$$\int_0^1 \chi_A(x)\, dx = 1 \times \mu(\mathbb{Q} \cap [0, 1]) + 0 \times \mu([0, 1] - \mathbb{Q}) = 1 \times 0 + 0 \times 1 = 0$$

Remark. *Note that the above function is not Riemann integrable but it is Lebesgue integrable. So Lebesgue integral kind of allows us to integrate "many more" functions than Riemann integral .*

The concept of lebesgue measure and integration can be extended to arbitrary Euclidean Space $\mathbb{R}^n$ simply by defining the outer measure of a window in following way

$$\mu^*([a_1, b_1] \times \cdots \times [a_n, b_n]) = (b_1 - a_1) \times \cdots \times (b_n - a_n)$$

And all the other results remain same!

From now on, all our integrals would be in lebesgue sense!

§3.3 L^p Space

Definition 3.3.1 (L^p Space) — The L^p space $L^p(\Omega)$ is defined as the set of all measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f|^p d\mu < \infty$.

Remember, here the integration is in lebesgue sense .

Definition 3.3.2 ($L^p Norm$) — The L^p norm of a function f is defined as:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, d\mu(x) \right)^{1/p}, \quad 1 \le p < \infty,$$

whereas for $p = \infty$,

$$\|f\|_{L^\infty(X)} = \sup_{x \in X} |f(x)| = \inf\{M \ge 0 : |f(x)| \le M \text{ a.e. on } \Omega\}.$$

Example 3.3.1
Let $\Omega = (0,1)$ and $f(x) = \frac{1}{\sqrt{x}}$. Then clearly $f \in L^1(\Omega)$ but $f \notin L^2(\Omega)$. Its L^1 norm is 2 .

Theorem 3.5
(i) L^p space is a banach space with L^p norm.
(ii) L^2 space is a hilbert space with L^2 norm and inner product defined as the following:

$$< f, g >= \int fg$$

Theorem 3.6 (Holder's Inequality)
Hölder's inequality is a generalization of the Cauchy-Schwarz inequality. Let $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if $f \in L^p$ and $g \in L^q$, we have

$$\langle |f|, |g| \rangle = \int |fg| \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

In this case, we also call p and q to be holder conjugate of each other .

In particular, if $q = 1$, then its holder conjugate is $p = \infty$, from holder's inequality, we can conclude :

$$\int |fg| \leq \|f\|_{L^\infty} \cdot \|g\|_{L^1} = \sup_{x \in \Omega} |f(x)| \cdot \int |g|$$
$$\implies \left| \int fg \right| \leq \int |fg| \leq \|f\|_{L^\infty} \cdot \int |g|$$

This result will be useful in Chapter 6 when we will study energy estimates of PDE operators .

Theorem 3.7 (Minkowski's Inequality)
Let $1 \leq p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then if $f, g \in L^p$, we have

$$\|f + g\|_{L^p} \leq \|f\|_{L^p} \cdot \|g\|_{L^p}$$

Theorem 3.8
If Ω is of finite lebesgue measure and $1 \leq p < q$, then there exists some constant C such that :

$$\|u\|_{L^p(\Omega)} < C \|u\|_{L^q(\Omega)}$$

In other words for $\mu(\Omega) < \infty$ and $1 \leq p < q$, we have :

$$L^q(\Omega) \subset L^p(\Omega)$$

In chapter 5, we will see how the above theorem helps us to prove Poincare inequality .

A locally integrable function (sometimes also called locally summable function) is a function which is integrable (so its integral is finite) on every compact subset of its domain of definition.

Definition 3.3.3 (Locally integrable function and its space) — Let Ω be an open subset of

$\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a measurable function. We say that f is *locally integrable* if for every compact subset $K \subset \Omega$, the integral

$$\int_K |f(x)| \, dx$$

exists and is finite.

The space of *locally integrable functions* on Ω, denoted by $L^1_{loc}(\Omega)$, is the set of all locally integrable functions $f : \Omega \to \mathbb{R}$. That is,

$$L^1_{loc}(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ is locally integrable}\}.$$

Similarly one can also define $L^p_{loc}(\Omega)$.

Example 3.3.2

Let $\Omega = (0, 1)$ and $f(x) = \frac{1}{x}$. Clearly f is not integrable in the whole domain but it is locally integrable. In notational terms, $f \notin L^1(\Omega)$ but $f \in L^1_{loc}(\Omega)$.

Theorem 3.9

Any L^p function is locally integrable i.e

$$L^p(\Omega) \subset L^1_{loc}(\Omega)$$

This theorem is very crucial as it basically means if some property is true for all locally integrable functions, then it's also true for any L^p functions. In the next chapter of Distribution Theory, we will see how this theorem helps us to conclude all L^p functions can be associated with some distribution .

Chapter 4 : Distribution Theory

Contents

In classical analysis, the concept of derivatives is central to studying functions and solving differential equations. However, there are certain functions for which traditional derivatives are not well-defined or do not possess the desired properties. This motivates the need for distribution theory, which provides a framework for working with generalized functions and introducing the concept of weak derivatives.

§4.1 Limitations of Classical Derivatives

Classical derivatives have limitations when applied to certain classes of functions. Consider the **Heaviside step function** defined as:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases} \tag{4.1}$$

The derivative of the Heaviside step function is not well-defined at $x = 0$ since the function is discontinuous at that point. If it had some derivative, it would be something like :

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \end{cases}$$

This is known as **dirac delta function** but clearly this is not a function in usual sense as it takes infinity as a value .

This highlights a limitation of usual derivatives in handling discontinuous functions and we will see how distribution theory helps us tackle this.

§4.2 Basic Definitions

Definition 4.2.1 (Test function) — A test function on an open set $\Omega \subseteq \mathbb{R}^n$ is a smooth function $\phi : \Omega \to \mathbb{C}$ such that $\phi \in C_c^\infty(\Omega)$.

Test functions are often called bump functions too. And we often use the notation $D(\Omega)$ to denote $C_c^\infty(\Omega)$. It can be shown that $D(\Omega)$ forms a Banach space when equipped with the supremum norm .

Example 4.2.1

The function $\phi(x) = \begin{cases} e^{\frac{2a}{x^2-a^2}} & \text{if } |x| \leq a \\ 0 & \text{otherwise} \end{cases}$

is a test function with compact support in the interval $[-a, a]$.

Definition 4.2.2 (Distribution) — A distribution is a continuous linear functional on the space of test functions $D(\Omega) = C_c^\infty(\Omega)$ equipped with supremum norm .

The set of all distributions is often denoted as $D'(\Omega)$.

For a distribution F, we use the notation $\langle F, \phi \rangle$, where $\phi \in D(\Omega)$, so that it is understood that F acts on test functions from the space $D(\Omega)$.

The utility of distributions arises from the fact that they are generalized functions, which allows for operations, such as differentiation and convolution, on objects that fail to be functions. Distributions also have the 'nice' property that they act on a space of test functions whose elements are smooth and zero outside of some closed and bounded set.

Example 4.2.2

Let $f \in L^1_{loc}(\Omega)$. Define $T_f : C_c^\infty(\Omega) \to \mathbb{R}$ (or $\mathbb{C}$) as following :

$$T_f(\phi) = \int_\Omega f \ \phi = \langle f, \phi \rangle \qquad \text{, for all test functions } \phi$$

Then it can be shown that T_f is linear and continuous and hence a distribution .
Thus any locally integrable function can be associated with some distribution. And in the last chapter , we saw $L^p \subset L^1_{loc}$. Thus any L^p function is also associated with some distribution .

With some abuse of language, some also say, "Every locally integrable function is a distribution " .

Definition 4.2.3 (Regular Distributions) — A distribution T is called regular if there exists some locally integrable function f such that $T(\phi) = \int_\Omega f \ \phi$. We may call T is induced by f .

We can also define distributional derivative on the set of distributions. The following definition of distributional derivatives are primarily motivated from integration by parts formula.

Definition 4.2.4 (Distributional derivative) — Let T be a distribution on Ω and α be a multi index. The α-th distributional derivative of T is denoted by $D^\alpha T$ and is defined by the relation

$$D^\alpha T(\phi) = (-1)^{|\alpha|} T(D^\alpha \phi)$$

for all test functions ϕ .

Distributional derivatives allow us to extend the concept of derivatives to a broader class of functions that may not be classically differentiable, such as functions with singularities.

Example 4.2.3

The Dirac delta function is an important distribution that is not a traditional function but can be defined as a distributional derivative of heaviside function that we defined earlier. It is defined as :

$$\delta(\phi) = \phi(0) \tag{4.2}$$

for any test function $\phi \in D(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ and it can be shown that δ is indeed a distribution and it is not a regular distribution.

Even though dirac delta "function" $\delta(x)$ is neither a regular distribution nor a function, informally, we also write it as the following form :

$$\int_\Omega \delta(x)\phi(x)dx = \phi(0)$$

In particular, for Heaviside step function $H(x)$, we have :

$$\int_{-\infty}^{x} \delta(y)dy = H(x)$$

In general, if F is a distribution and ϕ is a test function, we often informally use the notation $\left(\int F\phi\right)$ to denote $\langle F, \phi\rangle$ even if F is not regular . In this case, we will say the integration is done in distributional sense . Also this idea of integration in distributional sense is somewhat valid due to the following theorem .

Theorem 4.1 (Approximation by regular distributions [6])
If F is a distribution i.e $F \in D'(\Omega)$, then there exists sequence of test functions (ψ_k) in $D(\Omega)$ such that for any test function ϕ we have :

$$\langle F, \phi\rangle = \lim_{k\to\infty} \int_\Omega \psi_k \phi$$

Later in this chapter, we will see that the dirac delta distribution can be approximated by sequence of test functions called standard mollifiers .

The concept of distributional derivatives arises from the desire to extend the notion of derivatives to functions that are not traditionally differentiable. The motivation comes from integration by parts, where the derivative of a function is related to the integral of its distributional derivative. They have enormous application in the analysis of PDEs .

§4.3 Weak Formulation of PDEs

We will now see some "Weakening" definition of differentiability which will allow us to differentiate "many more" functions, at least in weak sense, which wouldn't be differentiable in classical sense.

Definition 4.3.1 (Weak Derivatives) — Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{\text{loc}}(\Omega)$. We say that $v \in L^1_{\text{loc}}(\Omega)$ is a **weak derivative** of u if for all test functions $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u(x)D\phi(x)dx = -\int_\Omega v(x)\phi(x)dx = - < v, \phi >$$

If α is a multi index, we say that $v \in L^1_{\text{loc}}(\Omega)$ is a α**-th weak derivative** of u if for all test functions $\phi \in C_c^\infty(\Omega)$ we have

$$\int_\Omega u(x)D^\alpha\phi(x)dx = (-1)^{|\alpha|}\int_\Omega v(x)\phi(x)dx = (-1)^{|\alpha|} < v, \phi >$$

The above definition is inspired from repeated use of integration parts and using the fact that ϕ has compact support i.e ϕ is 0 outside a compact set.(Think how).

If such weak derivatives exists, it is unique up to sets of measure zero. A function u is said to be k times weakly differentiable if all of its weak derivatives $D^\alpha u$ exist for $|\alpha| \leq k$.

Also notice that in the definition of weak derivatives, the function v in the right hand side doesn't require to be integrable on the whole space as ϕ has compact support i.e ϕ is 0 outside a compact set. Thus we can loosen the condition by requiring v only to be locally integrable and the RHS still remains well defined. By removing the necessity of v to be integrable over whole space, we create the opportunity for many more functions to have weak derivatives.

Example 4.3.1

1. Consider the functions
$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$$
and,
$$v(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 1 < x < 2 \end{cases}$$
It can be shown that v is the weak derivative of u i.e $Du = v$.
2. It can be shown that the following function has no weak derivatives
$$u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$$

Theorem 4.2

Let f has some weak derivative $D^\alpha f$. Then,

$$D^\alpha T_f = T_{D^\alpha f}$$

where T_f is the distribution induced by f .

The **Sobolev Space** $W^{k,p}(\Omega)$ consists of those functions in $L^p(\Omega)$ such that weak derivatives of all order upto k exist for that function and all such weak derivatives are also in $L^p(\Omega)$. More treatment of Sobolev spaces will be discussed in next chapter.

Now consider a 2nd order linear pde of the form $Lu = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^n b^i(x)\frac{\partial u}{\partial x_i} + c(x)u = f$. Multiplying both sides with a function v in Sobolev space and applying integration by parts enough times, we should have the following :

$$\int v\ Lu = \int \left(-\sum_{i,j=1}^{n} v \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + v\sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} + v\ c(x)u \right) = \int v\ f =< f, v >$$

$$\implies \int \left(\sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j} a_{ij}(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} v + c(x)uv \right) =< f, v > \ \text{[By integration by parts]}$$

$$\implies B(u, v) =< f, v >$$

where $B(u, v) = \int \left(\sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j} a_{ij}(x)\frac{\partial u}{\partial x_i} + \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x_i} v + c(x)uv \right)$ and it's easy to check that $B(u, v)$ is a bilinear form. Here, we call $B(u, v)$ as the **associated bilinear form** to the pde $Lu = f$. Also we see that if the pde has a solution u, then it satisfies the equation $B(u, v) =< f, v >$. In other words, if there is no solution to the equation $B(u, v) =< f, v >$, then we can conclude that there is no solution to the pde $Lu = f$.

The above discussion motivates us define the following definition which will be useful in the next chapter to prove some existence and uniqueness theorem.

Definition 4.3.2 (Weak Solution of PDEs) — Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}$ be a given function in $L^2(\Omega)$. We say that a pde $Lu = f$ has a **weak solution** u in some Sobolev space,if for all v in that sobolev space, the following is true:

$$B(u, v) =< f, v >$$

where $B(u, v)$ is the associated bilinear form to the pde $Lu = f$.

Remark. *Remember that having weak solution doesn't guarantee that the original PDE has any classical solution. Because, even if we can get any weak solution, it doesn't necessarily mean the solution would be differentiable in classical sense. Infact those weak solutions might not even be continuous. In the next two chapters, we will see some embedding and regularity (in intuitive sense, regularity is related to continuity, differentiability and smoothness of the functions) theorems, which will fix these issues given that the pde follows some "nice enough" conditions .*

Remark. *Recall that in Chapter 3 of L^p Space, we saw that by defining Lebesgue Integrability, we can then integrate "many more" functions, which wouldn't be integrable in Riemann sense. Similarly, now by defining weak derivatives, we can now differentiate "many more" functions, at least in weak sense. And then the existence stuff usually get much easier to prove as one has "many more" functions to work on and find solution from. This is exactly why we need to define weak derivatives. Also luckily, we will see later that some embedding theorem infact implies that weak solutions to some particular PDEs, are indeed a classical solution infact !*

In the next section, we will discuss some tools like mollifiers, convolution which will help us later in the proofs of regularity of PDEs .

§4.4 Smoothing a function by Mollifiers

Definition 4.4.1 (Convolution) — Let f and g be integrable functions on $\mathbb{R}^n$. The convolution of f and g is denoted as $(f * g)$ and is defined by:

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy.$$

Example 4.4.1

Let $\psi(x)$ be a test function and $\delta(x)$ is dirac delta function/distribution . Then it can be shown that :

$$(\delta * \psi)(x) = \int_{\mathbb{R}^n} \delta(x-y)\psi(y)\,dy = \psi(x)$$

where the integration is done in distributional sense .

Definition 4.4.2 (Standard mollifier) — A standard mollifier η is a smooth function with compact support defined as following:

$$\eta(x) = \begin{cases} C\, e^{\frac{1}{|x|^2-1}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where C is chosen to be a normalizing constant such that $\int_{\mathbb{R}^n} \eta\,dx = 1$.
$\eta(x)$ looks like a bell shaped curve which peaks at $x = 0$ and vanishes outside the interval $[-1, 1]$.

Then we define $\eta_\epsilon(x)$ as following :

$$\eta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right), & \text{if } |x| < \epsilon, \\ 0, & \text{if } |x| \geq \epsilon, \end{cases}$$

Note that we also have η_ϵ to be a smooth function with compact support with $\int_{\mathbb{R}^n} \eta_\epsilon(x)\,dx = 1$.

It can be shown that $\lim_{\epsilon\to 0} \eta_\epsilon(x) = \delta(x)$ where $\delta(x)$ is the dirac delta distribution .

Now let $f : \Omega \to \mathbb{R}$ be a locally integrable function and let $\Omega_\epsilon = \{x \in \Omega \parallel distance(x, \delta\Omega) > \epsilon\}$.

To have a smooth approximation to a function f on Ω_ϵ using a standard mollifier, we perform the convolution $(f * \eta_\epsilon)(x)$. Usually by f^ϵ we denote $f * \eta_\epsilon$ and we call this process "mollification of f on Ω_ϵ " . The following results shows us how mollification helps us to have some smooth approximation to f

Theorem 4.3 (Properties of mollifiers)

1. $f^\epsilon \in C^\infty(\Omega_\epsilon)$.
2. $f^\epsilon \to f$ almost everywhere as $\epsilon \to 0$.
3. if $f \in C(\Omega)$, then $f^\epsilon \to f$ uniformly on compact subsets of Ω.
4. if $f \in L^p_{loc}(\Omega)$, then $f^\epsilon \to f$ in $L^p_{loc}(\Omega)$ as $\epsilon \to 0$.

A **cutoff function** is usually a smooth function which takes values between 0 and 1 and the value of the function is 0 outside a compact set and they are used to create some smooth approximation of a function through convolution. Hence, standard mollifiers are also cut-off functions. Standard mollifiers along with any other possible cutoff functions are extremely useful in dealing with the proof of regularity of PDEs .

§4.5 Green's Function

In this section, we will discuss very shortly and informally one key concept in distribution theory which is very helpful in constructing solutions of PDEs .

Consider a PDE $Lu = g$ defined on $\Omega \subset \mathbb{R}^n$ with some fixed boundary condition . A function $G : \Omega^2 \to \mathbb{R}$ is called a **green function** corresponding to the PDE, if for any integrable real valued function f, we can construct a solution to the PDE as follows :

$$u(x) = \int_\Omega G(x, y)f(y)dy$$

When we can get such green function, it is very helpful as even if we fail to find some analytic solution to the PDE, there are many numerical tools which helps us to find at least numerical solution and in most of the cases, that is enough for our practical purpose .

Now let F be a distribution such that $LF = \delta(x)$ where L is some linear PDE operator. Note that as dirac delta function $\delta(x)$ is a pure distribution and not a real function, so all the derivatives in LF must be done in distributional sense . Then it can be shown that :

$$L(F * g) = L\left(\int_\Omega F(x - y)g(y)\ dy\right) = g$$

Again the integration should be viewed in the sense of distribution .

Thus we see that if we take $u(x) = (F * g)(x)$, it is indeed a solution to the PDE $Lu = g$, at least in distributional sense . And if the distributional solution is regular, it infact becomes a weak solution .

In this way, distributions help us to construct solutions to PDE and prove the existence of solutions .

Chapter 5 : Sobolev Space

Contents

We are now in our Key chapter of Sobolev Space. We will see some necessary definitions and related properties of it, which will help us to do analysis of PDEs in next chapter .

§5.1 Basics of Sobolev Spaces

Definition 5.1.1 (Sobolev Space) — Let Ω be an open subset of $\mathbb{R}^n$. The Sobolev space $W^{k,p}(\Omega)$ is defined as the set of all functions $f \in L^p(\Omega)$ such that all weak derivatives up to order k are in $L^p(\Omega)$, i.e.,

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) : D^\alpha f \in L^p(\Omega), \text{ for all } |\alpha| \leq k\},$$

where $D^\alpha f$ denotes the weak derivative of f of order α and α is a multi-index.

Of course derivatives here are in weak or distributional sense!

Definition 5.1.2 (Sobolev Norm) —

$$\|f\|_{W^{k,p}} = \left(\sum_{|\alpha|\leq k} \int_\Omega |D^\alpha f(x)|^p dx\right)^{\frac{1}{p}}$$

Theorem 5.1
The Sobolev space $W^{k,p}(\Omega)$ is a Banach space.

Definition 5.1.3 — The space $W_0^{k,p}(\Omega)$ is the completion/closure of the space $C_c^\infty(\Omega)$ with respect to the Sobolev norm.
It basically means, $u \in W_0^{k,p}(\Omega)$ iff $u \in W^{k,p}(\Omega)$ and there exists a sequence of functions $u_n \in C_c^\infty(\Omega)$ such that the sequence of functions (u_n) converges to u in $W_0^{k,p}(\Omega)$ with respect to the sobolev norm.

Intuitively, the space $W_0^{k,p}(\Omega)$ consists of those functions in $W^{k,p}(\Omega)$ that "vanish" on the boundary $\partial\Omega$ with all their derivatives up to order $k-1$.

Definition 5.1.4 — We use the notation H_0^k to denote the space $W_0^{k,2}$. Also by H^k we denote $W^{k,2}$.

In fact for the rest of the chapter, we are particularly interested in the space $H_0^1(\Omega)$.

Theorem 5.2
The Sobolev space H^k and $H_0^k(\Omega)$ are Hilbert spaces.

§5.2 Approximations

In order to study the deeper properties of Sobolev spaces, we therefore need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions.

Theorem 5.3 (Local Approximation)
Let $u \in W^{k,p}(\Omega)$ and $u^\epsilon = u * \eta_\epsilon$ in Ω_ϵ .

Then,

1. $u \in C^\infty(\Omega_\epsilon)$
2. $u^\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$.

Proof. Part 1 is a direct consequence of property of mollifiers. So we will only focus on proof of part 2 . Now,

$$\begin{aligned} D^\alpha u^\epsilon(x) &= D^\alpha \int_\Omega \eta_\epsilon(x-y)u(y) \\ &= \int_\Omega D_x^\alpha \eta_\epsilon(x-y)u(y) \\ &= (-1)^{|\alpha|} \int_\Omega D_y^\alpha \eta_\epsilon(x-y)u(y) \end{aligned}$$

If we consider x to be fixed now, then,

$$\int_\Omega D_y^\alpha \eta_\epsilon(x-y)u(y) = (-1)^{|\alpha|} \int_\Omega \eta_\epsilon(x-y)D^\alpha u(y)$$

Thus we can say.

$$\begin{aligned} D^\alpha u^\epsilon(x) &= (-1)^{|\alpha|+|\alpha|} \int_\Omega \eta_\epsilon(x-y)D^\alpha u(y) \\ \implies D^\alpha u &= D^\alpha u * \eta_\epsilon \end{aligned}$$

Then by propertied of mollifiers, if $V \subset\subset \Omega$ we have $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$. Hence we conclude :

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\le k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \to 0 \quad \text{as } \epsilon \to 0$$

Thus completing our proof! ■

Theorem 5.4 (Global Approximation)
Let Ω is bounded and $u \in W^{k,p}(\Omega)$, where $1 \le p < \infty$. Then there exists a sequence of smooth functions $u_n \in C^\infty(\Omega) \cap W^{k,p}(\Omega)$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

Theorem 5.5 (Global Approximation up to the Boundary)
Let Ω is bounded and $\partial\Omega$ is C^1. Also suppose $u \in W^{k,p}(\Omega)$, where $1 \le p < \infty$. Then there exists a sequence of smooth functions $u_n \in C^\infty\left(\overline{\Omega}\right)$ such that $u_n \to u$ in $W^{k,p}(\Omega)$.

§5.3 Extensions and Traces

The question of extending a Sobolev function in some proper open subset $\Omega \subset \mathbb{R}^n$ to the entire space may be delicate, since the extended function must have integrable weak derivatives across the boundary $\partial\Omega$. Observe for instance that if we extend $u \in W^{1,p}(\Omega)$ by defining u to be zero in $(\mathbb{R}^n - \Omega)$, it will not work in general, as we may thereby create such a bad discontinuity along $\partial\Omega$ that the extended function no longer has a weak partial derivative of first order . We must instead invent a way to extend u which "preserves the weak derivatives across $\partial\Omega$ " . Fortunately, smooth approximation up to the boundary allows one to construct such extensions, at least when $\partial\Omega$ is C^1 .

Theorem 5.6 (Extension Theorem)
Let Ω is bounded and $\partial\Omega$ is C^1. Select a bounded open set V such that $\Omega \subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that for each $u \in W^{1,p}(\Omega)$:
(i) $Eu = u$ a.e in Ω .
(ii) Eu has support within V .
(iii) $||Eu||_{W^{1,p}(\mathbb{R}^n} \le C||u||_{W^{1,p}(\Omega)}$

Definition 5.3.1 — We call Eu as an extension of u to $\mathbb{R}^n$.

A lot of the PDE problems we are interested in are boundary value problems, namely we want to solve a PDE subject to the function taking some prescribed values on the boundary . But since Sobolev spaces are defined as spaces of integrable functions with integrable weak derivatives, in general it doesn't make sense to talk about the value of a Sobolev function at a point, or on a set of zero measure. In particular, values of a Sobolev function in Ω are not well-defined on the boundary $\partial\Omega$, as $\partial\Omega$ is typically a set of measure zero . However, approximation by smooth functions up to the boundary provides a way of "restricting" Sobolev functions to the boundary. We would hope that if we require u to have more regularity, then perhaps it now makes sense to define the value at the boundary.
This is resolved by our following trace theorem .

Theorem 5.7 (Trace Theorem)
Let Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that
(i) $Tu = u|_{\partial\Omega}$ if $u = W^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right)$.
(ii) $||Tu||_{L^p(\partial\Omega)} \leq C||u||_{W^{1,p}(\Omega)}$

Definition 5.3.2 — We call Tu the trace of u on $\partial\Omega$.

§5.4 Sobolev Inequalities and Embeddings

Recall the definition of embedding which we discussed in Chapter 2 . Our goal in this section is to discover embeddings of various Sobolev spaces into other function spaces . The crucial analytic tools here will be certain so-called "Sobolev- type inequalities". They are the key results which will help us to deal with the regularity of PDEs as they discuss how some sobolev space is embedded in much smoother space .

Definition 5.4.1 (Sobolev Conjugate) — Let $p \geq 1$ be a real number. The Sobolev conjugate exponent p^* is defined as

$$p^* = \frac{np}{n-p},$$

where n is the dimension of the underlying space.

Note that we have,

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \quad \textit{and} \quad p^* > p$$

Theorem 5.8 (Gagliardo-Nirenberg-Sobolev Inequality)
If $1 \leq p \leq n$, then there exists some constant C depending on p and n only such that we have the inequality :

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for any function $u \in C^1_c(\mathbb{R}^n)$.

The theorem also remains valid if we replace $\mathbb{R}^n$ by any bounded and open $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ being C^1 . And the constant C depending only on p, n, and Ω .

Motivation for Gagliardo-Nirenberg-Sobolev inequality :
We first demonstrate that if any inequality of the form

$$\|u\|_{L^q(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)}$$

for certain constants $C > 0$, $1 \leq q \leq \infty$ and functions $u \in C^\infty_c(\mathbb{R}^n)$ holds, then the number q cannot be arbitrary. Let $u \in C^\infty_c(\mathbb{R}^n), u \neq 0$ and define for $\lambda > 0$,

$$u_\lambda(x) := u(\lambda x)$$

Then if our desired inequality holds, we would have :

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}$$

Then we have :

$$\int_{\mathbb{R}^n} \|u_\lambda(x)\|^q = \int_{\mathbb{R}^n} \|u(\lambda x)\|^q = \frac{1}{\lambda^n} \int_{\mathbb{R}^n} \|u(x)\|^q$$

And

$$\int_{\mathbb{R}^n} \|Du_\lambda(x)\|^p = \lambda^p \int_{\mathbb{R}^n} \|Du(\lambda x)\|^p = \frac{\lambda^p}{\lambda^n} \int_{\mathbb{R}^n} \|Du(x)\|^p$$

Then if our desired Gagliardo-Nirenberg-Sobolev inequality holds, we would have :

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du_\lambda\|_{L^p(\mathbb{R}^n)}$$

$$\implies \left(\int_{\mathbb{R}^n} \|u_\lambda(x)\|^q\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} \|Du_\lambda(x)\|^p\right)^{\frac{1}{p}}$$

$$\implies \left(\frac{1}{\lambda^n}\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} \|u(x)\|^q\right)^{\frac{1}{q}} \leq C\left(\frac{\lambda^p}{\lambda^n}\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} \|Du_\lambda(x)\|^p\right)^{\frac{1}{p}}$$

$$\implies \left(\int_{\mathbb{R}^n} \|u(x)\|^q\right)^{\frac{1}{q}} \leq C\, \lambda^{1-\frac{n}{p}+\frac{n}{q}} \left(\int_{\mathbb{R}^n} \|Du_\lambda(x)\|^p\right)^{\frac{1}{p}}$$

$$\implies \|u\|_{L^q(\mathbb{R}^n)} \leq C\, \lambda^{1-\frac{n}{p}+\frac{n}{q}} \|Du\|_{L^p(\mathbb{R}^n)}$$

If $1 - \frac{n}{p} + \frac{n}{q} \neq 0$, we can obtain a contradiction by sending λ to 0 or ∞, depending on whether $1-\frac{n}{p}+\frac{n}{q} > 0$ or $1-\frac{n}{p}+\frac{n}{q} < 0$. Thus, if in fact the desired inequality holds, we must necessarily have $1-\frac{n}{p}+\frac{n}{q} = 0$, which eventually implies $\frac{1}{q} = \frac{1}{p} - \frac{1}{n}$ or $q = \frac{np}{n-p}$. Thus we get the motivation for defining sobolev conjugate like this .
Here we also see that q is positive iff $p < n$. Thus we also see why the condition $1 \leq p \leq n$ is necessary .

Here is another similar embedding theorem which discusses the case for $n < p \leq \infty$.

Theorem 5.9 (Morrey's Inequality)
If $n < p \leq \infty$, then there exists some constant C depending on p and n only such that we have the inequality :

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for any function $u \in C^1(\mathbb{R}^n)$. where $\gamma = 1 - \frac{n}{p}$

Here also Morrey's Inequality still remains valid if we replace $\mathbb{R}^n$ by any bounded and open $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ being C^1 . And the constant C depending only on p, n, and Ω .

This basically means that if $u \in W^{1,p}(\Omega)$ and $n < p \leq \infty$, then u is in fact Holder continuous in the interior of Ω, after possibly being redefined on a set of measure zero. In other words, $W^{1,p}(\Omega)$ is embedded in $C^{0,\gamma}(\Omega)$ whenever $n < p \leq \infty$.

Next we are going to discuss Poincare inequality. There many different types of poincare inequality, but here we will discuss the simplest one, which will be useful later in our proof of existence and uniqueness of solutions of some specific elliptic PDEs .

Theorem 5.10 (Poincare Inequality : Simplest version)
Assume that Ω is a bounded subset of $\mathbb{R}^n$ and hence it is of finite lebesgue measure. Let $u \in W_0^{1,p}(\Omega)$ for some $1 \leq p < n$. Then we have the estimate:

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)},$$

for all $q \in [1, p^*]$, and the constant C depending only on n, p, q and Ω .

Proof. Here we are going to see the use of smooth approximation of Sobolev functions and Gagliardo-Nirenberg-Sobolev Inequality which we discussed earlier in this chapter.

We know there exists sequence of functions $u_m \in C_c^\infty(\Omega)$ (for $m = 1, 2, 3, \cdots$) which converges to u in $W^{1,p}(\Omega)$. We extend each function u_m to be 0 on $\mathbb{R}^n - \Omega$ and apply Gagliardo-Nirenberg-Sobolev Inequality to get

$$\|u\|_{L^{p^*}(\Omega)} \leq C_1 \|Du\|_{L^p(\Omega)},$$

for some constant C_1 .

Then recall the chapter of L^p space. as $1 \leq q < p^*$, we have :

$$\|u\|_{L^q(\Omega)} \leq C_2 \|u\|_{L^{p^*}(\Omega)},$$

for some constant C_2 .

Then combining all the results, we get our desired inequality :

$$\|u\|_{L^q(\Omega)} \leq C \|Du\|_{L^p(\Omega)},$$

■

Corollary 5.1

There is some $C > 0$ such that for all $u \in H_0^1(\Omega)$, we have :

$$\|u\|_{H_0^1(\Omega)} \leq C \|Du\|_{L^2(\Omega)},$$

Proof. We see that here $p = 2$ and the sobolev conjugate $p^* = \frac{n \times 2}{n-2} > 2$. And from Poincare Inequality , there exists some constant $C_1 > 0$ such that :

$$\|u\|_{L^2(\Omega)} \leq C_1 \|Du\|_{L^2(\Omega)},$$

$$\implies \|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)} \leq C_1 \|Du\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}$$
$$\implies \|u\|_{H_0^1(\Omega)} \leq (C_1 + 1) \|Du\|_{L^2(\Omega)},$$

Taking $C = C_1 + 1$, we get our desired inequality . ■

This corolarry will prove to be extremely useful in our next chapter. Also in view of this estimate, on $W_0^{1,p}(\Omega)$ the norm $\|Du\|_{L^p(\Omega)}$ is equivalent to $\|u\|_{W_0^{1,p}(\Omega)}$ if Ω is bounded .

Finally we finish this section discussing a more general and important embedding theorem (with proof) in the study of regularity analysis of PDEs .

Theorem 5.11 (General Sobolev Inequalities)

Let U be a bounded open subset of $\mathbb{R}^n$, with a C^1 boundary. Assume $u \in W^{k,p}(U)$

(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. We have in addition the estimate :

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{k,p}(U)}$$

the constant C depending only on k, p, n and U

(ii) If $k > \frac{n}{p}$ then $u \in C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}\left(\overline{U}\right)$, where $\lfloor x \rfloor$ = largest integer lower than x and

$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p} & ; \quad \text{if } \frac{n}{p} \text{ is not integer} \\ \text{any positive number} < 1\,; & \text{if } \frac{n}{p} \text{ is an integer} \end{cases}$$

We have in addition to estimate :

$$\|u\|_{C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}\left(\overline{U}\right)} \leq C\,\|u\|_{W^{k,p}(U)}$$

the constant C depending only on k, p, n, γ and U.

Proof. (i) Assume $k < \frac{n}{p}$. Then since $D^{\alpha}u \in L^p(U)$ for all $|a| = k$, the Sobolev-Niremberg-Gagliardo inequalities implies

$$\left\|D^{\beta}u\right\|_{L^{p^*}(U)} \leq C\,\|u\|_{W^{k,p}(U)} \qquad \text{if} \quad |\beta| = k-1,$$

and so $u \in W^{k-1,p^*}(U)$. Similarly $u \in W^{k-2,p^{**}}(U)$, where $\frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$. Continuing we eventually discover after k steps that $u \in W^{0,q}(U) = L^q(U)$, for $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. The stated estimate $\|u\|_{L^q(U)} \leq C\,\|u\|_{W^{k,p}(U)}$ follows from multiplying the relevant estimates at each stage of the above argument.
(ii) Assume now $k > \frac{n}{p}$ holds and $\frac{n}{p}$ is not an integer. Then as above we see

$$u \in W^{k-l,r}(U) \tag{5.1}$$

for

$$\frac{1}{r} = \frac{1}{p} - \frac{l}{n} \tag{5.2}$$

for $lp < n$. We choose the integer l so that

$$l < \frac{n}{p} < l+1 \tag{5.3}$$

that is , we set $l = \left\lfloor \frac{n}{p} \right\rfloor$. Consequently these imply $r = \frac{pn}{n-pl} > n$.
Then equation 5.1 in above and Morrey's inequality imply that $D^{\alpha}u \in C^{0,1-\frac{n}{r}}\left(\overline{U}\right)$ for all $|a| \leq k-l-1$. Observe also that $1-\frac{n}{r} = 1-\frac{n}{p}+l = \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}$. Thus $u \in C^{k-\lfloor \frac{n}{p} \rfloor -1, \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}}\left(\overline{U}\right)$ and the stated estimate follow easily.
Assume now

$$\|u\|_{C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}\left(\overline{U}\right)} \leq C\,\|u\|_{W^{k,p}(U)}$$

holds and $\frac{n}{p}$ is an integer. Set $l = \left\lfloor \frac{n}{p} \right\rfloor - 1 = \frac{n}{p} - 1$. Consequently, we have as $u \in W^{k-l,r}(U)$ for $r = \frac{pn}{n-pl} = n$. Hence the Sobolev-Nirenberg-Gagliardo inequality shows $D^{\alpha}u \in L^q(U)$ for all $n \leq q < \infty$ and all $|a| \leq k-l-1 = k - \frac{n}{p}$. Therefore Morrey's inequality further implies $D^{\alpha}u \in C^{0,1-\frac{n}{q}}\left(\overline{U}\right)$ for all $n < q < \infty$ and all $|\alpha| \leq k - \left\lfloor \frac{n}{p} \right\rfloor - 1$
Consequently $u \in C^{k-\lfloor \frac{n}{p} \rfloor -1,\gamma}\left(\overline{U}\right)$ for each $0 < \gamma < 1$. As before, the stated estimate follows as well. ∎

§5.5 Difference Quotients

Definition 5.5.1 (Difference Quotients) — Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a function defined on an open set Ω. Also assume $u : \Omega \to \mathbb{R}$ be a locally summable function and $V \subset\subset \Omega$. Then $i-$th difference quotient of size h is defined as

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}.$$

for $x \in V$ and $h \in \mathbb{R}$ with $0 < |h| < dist(V, \partial\Omega)$ and here $dist(,)$ is the distance function.

Also we write

$$D^h u = (D_1^h u, D_2^h u, \cdots, D_n^h u)$$

Theorem 5.12 (Difference quotients and weak derivatives)
(i) Let $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then for each $V \subset\subset \Omega$,

$$||D^h u||_{L^p(V)} \leq C||Du||_{L^p(\Omega)}$$

for some constant C and all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$

(ii) Let $1 \leq p < \infty$ and $u \in L^p(V)$ and there exists a constant C such that :

$$||D^h u||_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$. Then :

$$u \in W^{1,p}(V) \quad \text{with } ||Du||_{L^p(V)} \leq C$$

We are not going to discuss difference quotients in details, but results in above theorem proves to be very useful in regularity analysis of PDEs along with convolution through Standard Mollifiers or other Cutoff functions .

§5.6 Fourier Transform and Sobolev Space

Theorem 5.13 (Characterization of H^k by Fourier transform)
Let k be a nonnegative integer. Then a function $u \in L^2(\mathbb{R}^n)$ is in $H^k(\mathbb{R}^n)$ iff

$$(1 + |y|^k)\widehat{u} \in L^2(\mathbb{R}^n)$$

where $\widehat{u}$ is the fourier transform of u .

This theorem allows us to define sobolev space on non-integer real values.

Definition 5.6.1 — Assume $0 < s < \infty$ and $u \in L^2(\mathbb{R}^n)$. Then u is said to be in $H^s(\mathbb{R}^n)$ if

$$(1 + |y|^s)\widehat{u} \in L^2(\mathbb{R}^n)$$

and we set the sobolev norm to be :

$$||u||_{H^s(\mathbb{R}^n)} = ||(1 + |y|^s)\widehat{u}||_{L^2(\mathbb{R}^n)}$$

Also one should notice that H^0 is simply L^2 . Furthermore, we can define sobolev space for non-negative reals as following :

Definition 5.6.2 — Let $0 < s < \infty$. Then we define $H^{-s}(\Omega)$ as the dual space of $H_0^s(\Omega)$. Furthermore, we define sobolev norm of $f \in H^{-s}(\Omega)$ as following :

$$||f||_{H^{-s}(\Omega)} = sup\{< f, u > |u \in H_0^s(\Omega) \ \ and \ \ ||u||_{H_0^s(\Omega)} \leq 1\}$$

Chapter 6 : Applications in Theory of PDEs

Contents

Here in this chapter, for the sake of brevity, we will restrict our discussion only to Analysis of Elliptic PDEs . We are going to investigate the solvability of uniformly elliptic, second- order partial differential equations, subject to prescribed boundary conditions. We will exploit two essentially distinct techniques, energy methods within Sobolev spaces and maximum principle methods.

§6.1 Energy Estimates

Probably one of the key reasons the methods discussed here are called Energy Estemates, is that, it tries to estimate or bound the L^2 norm of the related PDEs with the help of integration by parts and other necessary inequalities. And one might intuitively make sense that L^2 norms are somewhat can be attributed to be "energy" .

Now first, we are going to start off with analyzing the existence and uniquness of a particular type of Uniformly Elliptic PDE which will illustrate us the idea of Energy methods .

Recall the definition of Uniform ellipticity from Chapter 1 and Lax Milgram Theorem from Chapter 2 . We are now going to see their use in next theorem .

Theorem 6.1
Let $-\sum a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u = f$ be a uniformly elliptic PDE where $c(x) \geq 0$. Then weak solution to this PDE exists in the space $H_0^1(\Omega)$ and the solution is unique .

Consider the associated bilinear form $B(u, v)$ to the given PDE. Since $H_0^1(\Omega)$ is a Hilbert space, due to Lax Milgram Theorem , it's enough to show the followings :

(i) Boundedness condition i.e $|B(u, v)| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$ for all $u, v \in H_0^1(U)$ for some $\alpha \geq 0$

(ii) Coercivity condition i.e $\beta \|u\|^2_{H_0^1(U)} \leq B(u, u)$. for some $\beta \geq 0$

Proof. (i)

$$
\begin{aligned}
|B(u,v)| = & \left| \int \left(\sum_{i,j=1}^{n} \frac{\partial v}{\partial x_j} a_{ij}(x) \frac{\partial u}{\partial x_i} + c(x) uv \right) \right| \\
\leq & \sum_{i,j} \left\| a^{ij} \right\|_{L^\infty(U)} \int_U |\mathrm{D}u||\mathrm{D}v| \mathrm{d}x + \|c\|_{L^\infty(U)} \int_U |u||v| \mathrm{d}x \\
& \qquad \text{[By Holder's inequality]} \\
\leq & c_1 \|\mathrm{D}u\|_{L^2(U)} \|\mathrm{D}v\|_{L^2(u)} + c_3 \|u\|_{L^2(U)} \|v\|_{L^2(u)} \\
& \qquad \text{[By Cauchy-Schwarz inequality]} \\
\leq & \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}
\end{aligned}
$$

for some $\alpha \geq 0$.

(ii) We start from uniform ellipticity. This implies

$$
\begin{aligned}
\theta \int_U |\mathrm{D}u|^2 \, \mathrm{d}x \leq & \int_U \sum_{i,j=1}^{n} a^{ij}(x) u_{x_i} u_{x_j} \, \mathrm{d}x \\
= & B(u,u) - \int_U cu^2 \, \mathrm{d}x \\
\leq & B(u,u) \\
& \qquad [As\ c \geq 0\ and\ u^2 \geq 0]
\end{aligned}
$$

Then along with the corollary of poincare inequality we saw in last chapter, we get our desired result

$$
\beta \|u\|^2_{H_0^1(U)} \leq B(u,u)
$$

for some $\beta \geq 0$ ∎

Corollary 6.1

Unique weak solutions exist to Laplace and Poisson equation .

The results in above theorem can be more generalized as following :

Theorem 6.2 (Energy Estimates)

There exist constants $\alpha, \beta > 0$ and $\gamma \geq 0$ such that

1. $|B(u,v)| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$
2. $\beta \|u\|^2_{H_0^1(U)} \leq B(u,u) + \gamma \|u\|^2_{L^2(U)}$.

for all $u, v \in H_0^1(U)$.

Theorem 6.3

There exists a $\lambda \geq 0$ fuch that for each $\mu \geq \lambda$ and $f \in L^2(U)$, there exists a unique weak solution $u \in H_0^1(U)$ to the BVP

$$
Lu + \mu u = f \quad \text{ in } U
$$

$$u|_{\partial U} = 0$$

§6.2 Maximum Principles

We say a pde follows **maximum (minimum) principle** if the maximum (minimum) value of any solution function to the pde occurs at the boundary. This principle is incredibly useful in proving the uniqueness or stability of PDEs.

Theorem 6.4

Consider the following pde $Lu = f$ with $u|_{\partial\Omega} = g$. Then if the pde follows both the maximum and minimum principle, then if a solution exists to the pde, it's unique and stable.

Proof.

Uniqueness part :
Let u_1 and u_2 be two solutions to the PDE. Let $u = u_1 - u_2$. Then we have $Lu = 0$ with $u|_{\partial\Omega} = 0$, Then we have $max(u) = max(u|_{\partial\Omega}) = 0 = min(u|_{\partial\Omega}) = min(u)$ by the maximum and minimum principle and hence $u = 0$ everywhere in Ω . So $u_1 = u_2$ and we're done.

Stability part :
Let $Lu_1 = f$ with $u_1|_{\partial\Omega} = g_1$ and $Lu_2 = f$ with $u_2|_{\partial\Omega} = g_2$. Also assume $||g_1 - g_2|| < \epsilon$. Let $u = u_1 - u_2$ and $g = g_1 - g_2$. Then it's enough to show $||u|| < \epsilon$.
Now we have ,

$$\begin{aligned} & Lu = f \quad \text{with } \; u|_{\partial\Omega} = g \;\text{ and }\; ||g|| < \epsilon \\ \implies & -\epsilon < \min(g) \le u \le \max(g) < \epsilon \quad \text{[By maximum-minimum-principle]} \\ \implies & ||u|| < \epsilon \end{aligned}$$

And we're done! ■

There are many weak and strong maximum principles related to many PDEs. Here we are only going to state some such theorems (without proof) related to elliptic PDEs .

§6.2.i Weak maximum principle for Elliptic PDEs

Theorem 6.5 (for $c = 0$)

Assume $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and $c = 0$.
Suppose also Ω is connected, open and bounded. Then,

1. If $Lu \le 0$ in Ω then $\quad \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
2. Similarly if $Lu \ge 0$ in Ω then $\quad \min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

Theorem 6.6 (for $c \ge 0$)

Assume $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and $c \ge 0$.
Suppose also Ω is connected . Then,

1. If $Lu \le 0$ in Ω then $\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u^+$.
2. Similarly if $Lu \ge 0$ in Ω then $\min_{\overline{\Omega}} u \ge$ -$\max_{\partial\Omega} u^-$.

where $u^+ = max(0, u)$ and $u^- = max(0, -u)$.

§6.2.ii Strong maximum principle for Elliptic PDEs

Theorem 6.7 (for $c = 0$)
Assume $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and $c = 0$.
Suppose also Ω is connected, open and bounded. Then,

1. If $Lu \le 0$ in Ω and u attains its maximum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.
2. Similarly if $Lu \ge 0$ in Ω and u attains its minimum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.

Theorem 6.8 (for $c \ge 0$)
Assume $u \in C^2(\Omega) \cap C\left(\overline{\Omega}\right)$ and $c \ge 0$.
Suppose also Ω is connected . Then,

1. If $Lu \le 0$ in Ω and u attains its *nonnegative* maximum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.
2. Similarly if $Lu \ge 0$ in Ω and u attains its *nonpositive* minimum over $\overline{\Omega}$ at an interior point, then u is constant within Ω.

§6.3 Regularity Analysis

Here we start off with the regularity analysis of Laplace equation as they are easy to deal with and yet illustrates us the idea that how regularity analysis is usually done. We will first state the Mean value formula for Laplace equation without proof, and then use this theorem along with standard mollifier to prove the regularity of Laplace Equation . Also one should recall our discussion on Balls in higher dimension before moving on to following theorem .

Theorem 6.9 (Mean value formula for Laplace equation)
Let u be a harmonic function (i.e., a solution of the Laplace equation $\Delta u = 0$) on a ball $B_r(a) \subset \mathbb{R}^n$. Then, for any point x in the ball, the value of $u(x)$ is the average of u over the surface of the ball:

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)\, dS(y),$$

where $|\partial B_r(a)|$ is the surface area of the boundary of the ball and $dS(y)$ is the differential surface area element. Furthermore, we also have :

$$u(x) = \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$$

Theorem 6.10
If $u \in C(\Omega)$ satisfies the mean-value property for each ball $B(x,r) \subset \Omega$, then

$$u \in C^\infty(\Omega)$$

Note carefully that u may not be smooth, or even continuous, up to $\partial\Omega$.

Proof. Let $u^\epsilon = u * \eta_\epsilon$ in Ω_ϵ. It's then enough to show $u = u^\epsilon$ in Ω_ϵ as we get our desired result if we let $\epsilon \to 0$. Indeed we have,

$$\begin{aligned}
u_\epsilon(x) &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)u(y)dy \\
&= \frac{1}{\epsilon^n}\int_{B(x,\epsilon)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dy \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \left(\int_{\partial B(x,r)} \eta\left(\frac{|x-y|}{\epsilon}\right)u(y)dSy\right)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \left(\int_{\partial B(x,r)} \eta\left(\frac{|r|}{\epsilon}\right)u(y)dSy\right)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{|r|}{\epsilon}\right)\left(\int_{\partial B(x,r)} u(y)dSy\right)dr \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{|r|}{\epsilon}\right)n\alpha(n)r^{n-1}u(x)dr \qquad \text{[By Mean value formula]} \\
&= \frac{1}{\epsilon^n}\int_0^\epsilon \eta\left(\frac{|r|}{\epsilon}\right)u(x)\left(\int_{\partial B(x,r)} dSy\right)dr \\
&= u(x)\frac{1}{\epsilon^n}\int_{B(0,\epsilon)} \eta\left(\frac{|y|}{\epsilon}\right)dy \\
&= u(x)\int_{B(0,\epsilon)} \eta_\epsilon(|y|)dy \\
&= u(x)
\end{aligned}$$

■

Now we are going to state some more regularity theorems, but without proof for the sake of brevity .

Theorem 6.11 (Interior H^2 Regularity)
Assume $a^{ij} \in C^1(U)$, $b^i, c \in L^\infty(U)$, $(i,j = 1,2,\cdots,n)$ and $f \in L^2(U)$. Also let, $u \in H^1(U)$ be a weak solution to the PDE $Lu = f$ in U.

Then $u \in H^2_{loc}(U)$ and for each open $V \subset\subset U$, we have the estimate

$$||u||_{H^2(V)}|| \leq C\left(||f||_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

the constant C depending only on U, and the coefficients of L.

Note carefully that we do not require $u \in H^1_0(\Omega)$; that is, we are not necessarily assuming the boundary condition $u = 0$ on $\partial\Omega$ in the trace sense.

Theorem 6.12 (Infinite Differentiability for Elliptic PDE in interior)
Let $a^{ij}, b^i, c, f \in C^\infty(U)$. If $u \in H^1(U)$ is a weak solution to the PDE $Lu = f$ in U , then we also have $u \in C^\infty(U)$

Now there are similar-looking theorems, which extend the ideas of regularity of solutions to the boundary too.

Theorem 6.13 (Boundary H^2 Regularity)
Assume $a^{ij} \in C^1\left(\overline{U}\right)$, $b^i, c \in L^\infty(U)$, $(i, j = 1, 2, \cdots, n)$ and $f \in L^2(U)$. Also let, $u \in H_0^1(U)$ be a weak solution to the elliptic boundary value PDE $Lu = f$ in U with $u = 0$ in ∂U.

Then $u \in H^2(U)$ and we have the estimate

$$||u||_{H^2(U)}|| \leq C \left(||f||_{L^2(U)} + \|u\|_{L^2(U)}\right)$$

the constant C depending only on U, and the coefficients of L.

See carefully that now we are assuming the boundary condition $u = 0$ on $\partial\Omega$ in the trace sense.

Theorem 6.14 (Infinite Differentiability for Elliptic PDE in Boundary)
Let $a^{ij}, b^i, c, f \in C^\infty\left(\overline{U}\right)$. If $u \in H_0^1(U)$ is a weak solution to the elliptic boundary value PDE $Lu = f$ in U with $u = 0$ in ∂U. Also, assume that ∂U is C^∞. Then we also have $u \in C^\infty\left(\overline{U}\right)$

§6.4 Some Conclusory Remarks

In summary, Sobolev spaces provide a powerful framework for studying the analytic aspects of PDEs and their applications in mathematics and physics. The study of Sobolev spaces is still an active area of research with many open questions and directions for future research. Also considering the enormous varieties of PDEs in both Mathematical and Physical realms, it can be expected that it is also going to remain an active area of research for many decades or even centuries .

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