An Exploration in Elliptic Genus of a Manifold

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Abstract

In this thesis work, we have explored the interplay between manifold geometry and number theory. More specifically, we have worked on some particular types of elliptic genera that assign some invariants to a manifold. Most importantly, those invariants take values in modular forms, and as modular forms have many nice number-theoretic properties, they can provide us with many number-theoretic aspects of manifolds. To illustrate the importance of the elliptic genus, we have shown its application in proving Ochanine's theorem, which says that the signature of certain manifolds is always divisible by 16. Thus, by virtue of the elliptic genus, we can find interesting results about a manifold's signature, which itself is a significant manifold invariant, as it helps to show many interesting results about manifolds, like constructing spaces that are homeomorphic but not diffeomorphic [Shi]. The key result we have studied in this thesis is Ochanine's theorem, which first appeared in the 1987 paper ([Och87]) by Serge Ochanine. Even though the proof of this theorem was known much before without the use of the elliptic genus, the introduction of the elliptic genus really offered a more concise and elegant proof of the theorem. The review work of this thesis is primarily based on the excellent book by Hirzebruch Et al. named "Manifolds and Modular Forms" [HBJ94].

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Chapter 1 : Introduction

Manifolds are a generalization of Euclidean spaces that allows us to do smooth geometry in a more abstract manner. Modular forms, on the other hand, are functions with certain holomorphicity conditions on the complex upper half-plane that satisfy some particular transformation rules under the action of $SL_2(\mathbb{Z})$ or some other specific subgroups of it [Mil90]. It turns out that they have many interesting number-theoretic aspects, with deep connections to elliptic curves and their $L-$functions.

The elliptic genus (Plural genera) is a tool to connect these two fields, which we have introduced in this thesis paper. We have illustrated its significance by showing its use in the proof of Ochanine's theorem, which says that the signature of an $8k+4$ dimensional smooth compact spin manifold is always a multiple of 16 [HBJ94]. We have also introduced the notions to understand the specific terms mentioned in this theorem.

Here we have assumed that the reader is already familiar with the basics of linear algebra, rings, and modules. Especially, ideas about bilinear forms, dual vectors/covectors, tensor product, exterior product, graded ring/algebra, representation of groups and algebras are required. As this thesis is by no means intended to give a full-rigorous introduction to differential geometry, we have just barely introduced the necessary notions to understand the basic terms, and we have barely included any proofs that are solely related to differential geometry. Therefore, it would also be better if the reader already has some familiarity with these differential geometry notions. However, even though we omitted most of the proofs, we have tried to justify it either with examples/heuristic arguments/historical background as much as possible. To know about differential geometry in more rigorous detail, one can follow the references [Tu], [Sch15], and [Sze04]. In particular, chapters 2-4 and Appendix A are almost without any proof, while we have included as much proof as possible in chapters 5-7 based on pedagogical relevance.

Chapter 2 is an extremely oversimplified review of manifold geometry; hence, it is better if the reader is already familiar with at least some basics of it. In this chapter, we have discussed how we can extend many useful geometric aspects, like vector fields, directional derivatives, to manifold geometry. We have also defined important tools like homology and cohomology (following [May99], [MS74], [Tu]), which roughly count the number of holes in a space. We ended that chapter with some symmetric function theory, which is not necessarily geometric in itself but will be useful in many future results, especially in characteristics classes of manifolds.

In Chapter 3, we introduced the notion of characteristic classes, which are used to define topological invariants of a manifold. Manifold invariants are important to consider as they have use in many important results like the classification of spaces. We have mostly followed the structure of Milnor's book ([MS74]) here, as well as Chapter 1 of [HBJ94]. Even though we ignored many of the proofs related to characteristics classes, in the final section of this chapter, we have proved all the theorems related to some generating functions, as they will be heavily used throughout our thesis paper, and they are not usually introduced in any standard text of characteristics classes.

In Chapter 4, notions of cobordism ring and multiplicative genera are introduced (following from [MS74] and Chapter 1 of [HBJ94], which will be essential in defining elliptic genera. Two manifolds are said to be cobordant if their disjoint union forms a boundary of a manifold, and cobordant classes possess a ring structure with addition being disjoint union and multiplication being Cartesian product. A genus is then a ring homomorphism from the cobordism ring to any other ring of convenience. In the final section of this chapter, we introduced the signature and $\widehat{A}-$genus of a manifold, which are one of the core ingredients of our thesis. We have also computed the signature and $\widehat{A}-$genus for a particular bundle, which will be used in our definition of signature elliptic genus and $\widehat{A}-$elliptic genus (these are not standard names; we have just given these names for convenience), respectively.

In Chapter 5, we have discussed the structure and some bare minimum results about modular forms, Jacobi forms, and theta functions, which will appear in our discussion of the elliptic genus. The transformation properties of theta functions will help us to conclude that some particular elliptic genera indeed assign a modular form to a manifold. Appendix I of [HBJ94] is enough for details; however, to know more about modular forms, one can see [Mil90, Con16]. Here, most of the theorems are introduced with either the entire proof or at least the sketch of the proof, which will be relevant in our study of the elliptic genus. However, if any proof is just straightforward definition chasing, simple but tedious to do the calculations, we skipped the proof.

In Chapter 6, we focus on two elliptic genera, namely the signature elliptic genus and $\widehat{A}-$elliptic genus, and discuss their properties (mostly with proofs/sketches of proofs), which will be relevant in proving Ochanine's theorem. By the name, we can guess that these two genera will be related to the signature and $\widehat{A}-$genus of a manifold, respectively.

In chapter 7, we will finish it off by proving Ochanine's theorem, which is our key goal in the thesis. Contents of both chapters 6 and 7 will be solely followed from [HBJ94].

In Appendix A, we will give a very oversimplified overview (so better if the reader already has some familiarity) of spin geometry and the Dirac operator. The $\widehat{A}-$genus of a spin manifold is related to the index of the Dirac operator on a spin manifold. Thus $\widehat{A}-$genus of a spin manifold will indeed be an integer. In fact, in our desired case, we have a better result that $\widehat{A}-$genus appears to be an even integer, which is necessary to have in Ochanine's theorem. We will follow it mostly from [Bar11, FO10, FO06].

Chapter 2 : Some Preliminaries

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§2.1 Manifolds

Definition 2.1.1 (Locally Euclidean space and charts) — A topological space M is said to be locally Euclidean of dimension n if for every point $p \in M$, there exists a neighborhood U around p such that there exists a homeomorphism ϕ from U to an open set of $\mathbb{R}^n$. The pair (U, ϕ) is called a chart or coordinate system.

Definition 2.1.2 (Topological manifold) — A topological manifold is a Hausdorff, second-countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

Definition 2.1.3 (Compatible charts) — Two charts (U, ϕ) and (V, ψ) are said to be C^r-compatible if $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^r maps in the sense of multivariable calculus. If $U \cap V$ is empty, then the two charts are automatically C^r compatible.

Definition 2.1.4 (Atlas) — A C^r atlas, or simply an atlas, is a cover of a locally Euclidean space by a collection of charts which are pairwise C^r compatible.

A maximal atlas is an atlas that is not contained in a larger atlas. It can be shown that any atlas can be extended to a maximum atlas.

Definition 2.1.5 (Smooth manifold) — A smooth manifold is a topological manifold with a maximal C^∞ atlas.

Example 2.1.1

Euclidean spaces $\mathbb{R}^n$, spheres $\mathbb{S}^n$, $n-$torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ are some examples of smooth manifolds.

An n-dimensional manifold can be roughly viewed as a space which locally looks like $\mathbb{R}^n$ and has a smooth structure.

Definition 2.1.6 (Smooth morphism) — A map $f : M \to N$ between two manifolds is said to be smooth if for any chart (U, ϕ) of M and any chart (V, ψ) of N, the map $\psi \circ f \circ \phi^{-1}$ is smooth whenever the map is defined on a suitable domain.

If f is invertible and the inverse is also smooth, we call it a diffeomorphism between M and N.

If $N = \mathbb{R}$, the map f is said to be a smooth function. We can then define the following infinite-dimensional vector space:

$$C^\infty(M) = \{f : M \to \mathbb{R} \mid \text{f is smooth}\}$$

If U is a open neighborhood of M, $C^\infty(U)$ can be analogously defined.

A smooth map $\gamma : \mathbb{R} \to M$ or $\gamma : I \to M$ (with I being an open interval of M) is said to be a smooth curve on M.

Definition 2.1.7 (directional derivative/tangent vector) — Let γ be a smooth curve through $p \in M$ and without loss of generality, $\gamma(0) = p$. The directional derivative at p along γ is defined to be the following linear map from $C^\infty(M)$ to $\mathbb{R}$.

$$X_{\gamma,p}(f) = (f \circ \gamma)'(0)$$

where the derivative is simply the usual derivative of a smooth function from $\mathbb{R}$ to $\mathbb{R}$.

Note that $X_{\gamma,p} = X_{\gamma',p}$ if and only if γ and γ' coincides in a small neighborhood around 0. So it only depends on the local picture of the curve around that point. Also, the map $X_{\gamma,p}$ can be intuitively understood as the velocity of the curve at the point p. Hence, $X_{\gamma,p}$ is also said to be tangent/velocity vector at p.

Definition 2.1.8 (Tangent space) — The tangent space of a manifold M at a point p is defined to be the following vector space:

$$T_pM = \{X_{\gamma,p} \,|\gamma \text{ is a smooth curve through } p\}$$

One can see [Sch15] to see how the bases of the tangent space look locally or to see any other local constructions of the concepts that we are going to mention in this chapter. However, for our purpose, this co-ordinate free definition is more than enough.

We can then define the cotangent space at $p \in M$ to be T_pM^*.

Definition 2.1.9 (Push forward) — Let $f : M \to N$ be a smooth map. Then push-forward is a linear map $f_* : T_pM \to T_{f(p)}N$, where the map f_* sends $X_{\gamma,p} \in T_pM$ to a vector in $T_{f(p)}N$ in following way:

$$(f_*X_{\gamma,p})(g) := X_{\gamma,p}(f \circ g) \quad , \quad g \in C^\infty(N)$$

Definition 2.1.10 (Pull back) — Let $f : M \to N$ be a smooth map. Then pull-back is a linear map $f^* : T_{f(p)}N^* \to T_pM^*$, where the map f^* sends $\xi \in T_{f(p)}N^*$ to a covector in T_pM^* in following way:

$$(f^*\xi)(v) = \xi(f_*v) \quad , \quad v \in T_pM$$

§2.2 Vector Bundles

Definition 2.2.1 (Bundle) — A bundle is a triple (E, π, M) where E and M are topological manifolds called the **total space** and the **base space**, respectively, and π is a continuous, surjective map $\pi : E \to M$ called the **projection map**.

Given a bundle, the **fibre** at $p \in M$ is defined to be $F_p := \text{preimage}_\pi(\{p\})$.

Definition 2.2.2 (Fibre bundle) — A fibre bundle is a quadruple (E, π, M, F) where each fibre F_p is homeomorphic to F and for each $p \in M$, there is a neighborhood U of p such that $\pi^{-1}(U) \cong U \times F$. Such neighborhoods are called **trivialisations** of the fibre bundle.

Definition 2.2.3 (Vector bundle) — A fibre bundle is called a vector bundle if the fibre is a vector space.

The dimension of the fibre is called the rank of the vector bundle.

Given some vector bundles on a manifold M, we can take fibrewise dual/product/direct sum/tensor product/exterior product, and we can create new vector bundles in this way. An elegant proof of this can be found in Chapter 3 of [MS74].

Example 2.2.1 (Tautological bundle in real projective space)
Consider the $n-$dimensional **real projective space** $\mathbb{R}P^n$.

$$\mathbb{R}P^n = (\mathbb{R}^{n+1} - \{0\})/\sim$$

where $x \sim y$ with $x, y \in (\mathbb{R}^{n+1} - \{0\})$, if and only if for some $\mu \in (\mathbb{R} - \{0\})$, we have

$$x = \mu y$$

This indeed forms an equivalence relation and the equivalence classes form the $n-$dimensional manifold $\mathbb{R}P^n$ where the topology and smooth structures are inherited from that of $(\mathbb{R}^{n+1} - \{0\})$ through taking topological quotient.

Now $\mathbb{R}P^n$ has a natural line bundle (it means fibres are isomorphic to $\mathbb{R}$) on it, called the **tautological bundle**, where the total space is defined as follows:

$$E = \{(w, v) \in \mathbb{R}P^n \times \mathbb{R}^{n+1} \ : \ v = \lambda x \ \text{ for some } \lambda \in \mathbb{R}\}$$

where $x \in \mathbb{R}^{n+1}$ is a representative of the equivalence class $w \in \mathbb{R}P^n$. The projection is the natural projection on $\mathbb{R}P^n$. One can easily see that it indeed forms a line bundle, and we denote it by γ.

This is a very important example of a vector bundle, and we will need this in later chapters.

Definition 2.2.4 (section) — A section of a fibre bundle is a smooth map $s : M \to E$ such that $\pi \circ s = id$.

Sections of a bundle $\pi : E \to M$ is denoted as $\Gamma(E)$. If we consider sections only on $U \subset M$, we will denote it as $\Gamma(E|_U)$.

A section of a vector bundle can be interpreted as a vector field.

Definition 2.2.5 (Tangent bundle) — The tangent bundle on a manifold M is a vector bundle with total space $TM = \bigsqcup_p T_pM$ and the projection map sends $v \in TM$ to $p \in M$ if $v \in T_pM$.
It can be shown that it is indeed a fibre bundle i.e, trivialisations exist.

Definition 2.2.6 (Cotangent bundle) — Cotangent bundle on a manifold M is a vector bundle with total space $T^*M = \bigsqcup_p T_pM^*$ and the projection map sends $\xi \in T^*M$ to $p \in M$ if $\xi \in T_pM^*$.

Definition 2.2.7 (Riemannian manifold) — A Riemannian metric g on M assigns to each $p \in M$ a smooth inner product

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

Here smooth means g is actually a section of the bundle $T^*M \otimes T^*M$.
A smooth manifold M equipped with a Riemannian metric g is a *Riemannian manifold*, denoted by (M, g).

A Riemannian vector bundle can also be defined analogously, where the tangent bundle is replaced by the vector bundle. We may call it a Riemannian structure on the bundle.

Note that when we have a Riemannian metric, we have a natural way to identify the tangent space with the cotangent one by sending the vector v to the covector $g(-, v)$, which is in fact an isomorphism for the two spaces. This **identification process** will prove useful when we have to identify the two spaces, like in the Clifford multiplication in Appendix A.

Definition 2.2.8 (Complex structure) — A *complex vector bundle* E is a fiber bundle with fiber $\mathbb{C}^n$ and structure group $GL(n, \mathbb{C})$. Alternatively, E is a real vector bundle with a bundle automorphism $J : E \to E$ satisfying $J^2 = -id$ where J is a fibrewise linear map. (any vector space with a linear map J satisfying $J^2 = -id$ has an $\mathbb{R}$-basis identifying it with $\mathbb{C}^n$ and J with the multiplication by i). Then J is called a *almost complex structure* on E.

A complex vector bundle automatically inherits a Riemannian structure. A complex manifold is a manifold where the transition function between the chart is holomorphic.

Example 2.2.2 (Tautological bundle in Complex projective space)

Consider the complex $n-$dimensional (real $2n-$dimensional) **complex projective space** $\mathbb{C}P^n$.

$$\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\sim$$

where $x \sim y$ with $x, y \in (\mathbb{C}^{n+1} - \{0\})$, if and only if for some $\mu \in (\mathbb{C} - \{0\})$, we have

$$x = \mu y$$

This indeed forms an equivalence relation and the equivalence classes form the complex $n-$dimensional manifold $\mathbb{C}P^n$.

Now $\mathbb{C}P^n$ has a natural complex line bundle ((it means fibres are isomorphic to $\mathbb{C}$)) on it, called the **tautological bundle**, where the total space is defined as follows:

$$E = \{(w, v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \ : \ v = \lambda x \ \text{ for some } \lambda \in \mathbb{C}\}$$

where $x \in \mathbb{C}^{n+1}$ is a representative of the equivalence class $w \in \mathbb{C}P^n$. The projection is the natural projection on $\mathbb{C}P^n$. One can easily see that it indeed forms a complex line bundle, and we denote it by γ.

This is also a very important example of a vector bundle, and we will need this in later chapters.

Definition 2.2.9 (Bundle Morphism) — Let $\pi_E : E \to B$ and $\pi_{E'} : E' \to B'$ be vector bundles. A **bundle morphism** consists of a pair of smooth maps $(f, \tilde{f})$ such that:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_E"'] & E' \arrow[d, "\pi_{E'}"] \\ B \arrow[r, "f"'] & B' \end{tikzcd} \quad \text{with} \quad \pi_{E'} \circ \tilde{f} = f \circ \pi_E.$$

Furthermore, for each $x \in B$, the map $\tilde{f}$ restricts to a linear map between the fibers:

$$\tilde{f}_x : E_x \to E'_{f(x)}.$$

Definition 2.2.10 (Pullback Bundle) — Let $\pi : E \to B$ be a vector bundle and let $f : X \to B$ be a smooth map. The **pullback bundle** f^*E is the vector bundle over X defined by:

$$f^*E = \{(x, v) \in X \times E \mid f(x) = \pi(v)\},$$

with projection $\pi' : f^*E \to X$ given by $\pi'(x, v) = x$.

Then we have the follwing pullback diagram:

$$\begin{tikzcd}
f^*E \arrow[r, "\tilde{f}"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\
X \arrow[r, "f"'] & B
\end{tikzcd}$$

The map $\tilde{f} : f^*E \to E$ sends $(x, v) \mapsto v$, and $\pi' : f^*E \to X$ is the projection onto the first component.

Definition 2.2.11 (Structure group) — Let $E \to M$ be a vector bundle of rank n over a smooth manifold M. A *structure group* of the vector bundle E is a group $G \subset \mathrm{GL}(n, \mathbb{R})$ (or $\mathrm{GL}(n, \mathbb{C})$ in the complex case) such that there exists an open cover $\{U_\alpha\}$ of M and local trivializations

$$\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{R}^n$$

with the property that on overlaps $U_\alpha \cap U_\beta$, we have the **transition functions**

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

which satisfy the following property:

$$\phi_\alpha \circ \phi_\beta^{-1}(x, v) = \big(x,\ g_{\alpha\beta}(x)\, v\big).$$

They further satisfy the following compatibility conditions:

$$g_{\alpha\beta} \cdot g_{\beta\gamma} = g_{\alpha\gamma}$$

$$g_{\beta\alpha} = g_{\alpha\beta}^{-1}$$

In other words, the vector bundle E admits a trivialization whose transition functions all take values in G. The minimal such G is often called *the structure group* of the bundle.

Example 2.2.3

Every smooth real vector bundle of rank n has $\mathrm{GL}(n, \mathbb{R})$ as a structure group, because any change of trivializations can be expressed by invertible $n \times n$ matrices.

In fact, we have something in the opposite direction, too. That means, given a cover (U_α) of M and transition functions $g_{\alpha\beta}$ (which are in some subgroup of GL_n) that satisfy the above compatibility conditions, then we can form a rank n vector bundle with them too. Here is how we form the total space:

$$E = \left(\bigsqcup_\alpha (U_\alpha \times \mathbb{R}^n) \right) / \sim$$

where the equivalence relation $\sim$ identifies $(x, v) \in (U_\alpha \times \mathbb{R}^n)$ with $(x, w) \in (U_\beta \times \mathbb{R}^n)$ for some $x \in (U\alpha \ \cap \ U_\beta)$, if we have $v = g_{\alpha,\beta}\, w$.
Then the projection is defined as the natural projection. One can see that it indeed forms

a vector bundle.

This is an amazing result as given a vector bundle, we have its transition functions $(g_{\alpha\beta})$ which are group elements for some matrix group G, and then if we further have a representation ρ of the group G on another space $\mathbb{R}^m$, then we can make a new rank m vector bundle through the transition functions $(\rho g_{\alpha\beta})$. We give some examples below.

Example 2.2.4

Let U be any element of the structure group G. Then we have the following bundles for the given representations ρ.

1. $E^* \cong E$ for $\rho(U) = \overline{U}$
2. $\Lambda^k E$ for $\rho(U)(v_1 \wedge \cdots \wedge v_k) = Uv_1 \wedge \cdots \wedge Uv_k$ where $v_1 \wedge \cdots \wedge v_k \in \Lambda^k \mathbb{C}^n$ (Λ^k denotes the antisymmetric tensors)
3. $S^k E$ for $\rho(U)(v_1 \otimes \cdots \otimes v_k) = Uv_1 \otimes \cdots \otimes Uv_k$ where $v_1 \otimes \cdots \otimes v_k \in S^k \mathbb{C}^n$ (S^k denotes the symmetric tensors)

§2.3 Algebra of differential forms

Definition 2.3.1 (differential $n-$form) — A differential $n-$form is a section of the bundle $\Lambda^n(T^*M)$ where Λ^n denotes the $n-$th exterior product i.e, fibres at $p \in M$ of the bundle $\Lambda^n(T^*M)$ would be simply $n-$th exterior product of the cotangent space T_pM^*.The space of all differential $n-$forms is denoted as Ω^n_M. The space Ω^0_M is simply the space of all smooth functions on M which is just $C^\infty(M)$.

As $\omega \in \Omega^n_M$ is a section i.e a smooth map from M to $\Lambda^n(T^*M)$, we denote its value at $p \in M$ simply as ω_p.

Given a bundle $\pi : E \to M$, an $E-$valued differential $n-$form is a section of the bundle $\Lambda^n(T^*M) \otimes E$ and the space of all $E-$valued differential $n-$forms is denoted as $\Omega^n_M(E)$.

Pullback of differential $n-$forms can be defined analogously. Given a smooth map $f : M \to N$ and a bundle $E \to N$, we can define the pullback bundle as the bundle where the fibre at $p \in M$ will simply be the fibre of $f(p) \in N$. Then for $\omega \in \Omega^n_N(E)$, we can define its pullback to be a form in $\Omega^n_M(E)$ as follows:

$$(f^*\omega)_p(X_1, \cdots, X_n) := \omega(f_*X_1, \cdots, f_*X_n) \quad , \ X_i \in T_pM$$

Definition 2.3.2 (wedge/exterior product of forms) — Given $\omega \in \Omega^n_M$ and $\sigma \in \Omega^m_M$, we define their wedge or exterior product $\omega \wedge \sigma$ as the usual exterior product fibrewise. In other words, we have:

$$\omega \wedge \sigma(X_1, \cdots, X_n) = \frac{1}{n!m!} \sum_{\pi \in S_{n+m}} \operatorname{sgn}(\pi)(\omega \otimes \sigma)(X_{\pi(1)}, \cdots, X_{\pi(n+m)})$$

Theorem 2.1

For smooth $f : M \to N$, $\omega \in \Omega_N^n$ and $\sigma \in \Omega_N^n$, we have,

$$f^*(\omega \wedge \sigma) = f^*\omega \wedge f^*\sigma$$

Definition 2.3.3 (Grassmann algebra) — For a smooth manifold M, its grassman algebra is defined as the following $C^\infty(M)$ module:

$$\Omega_M = \bigoplus_n \Omega_M^n$$

where the addition is simply the addition of forms fibrewise, and multiplication is given by the exterior product.

Recall that a directional derivative or tangent vector in T_pM was defined as a linear functional on $C^\infty(M)$, and a tangential vector field is an element in $\Gamma(TM)$, which is the space of all sections in TM. Then a vector field can be understood as an operator $C^\infty(M)$ to $C^\infty(M)$.

Definition 2.3.4 (Commutator/Lie bracket) — Given $X, Y \in \Gamma(TM)$, its Lie bracket is a linear map $[X, Y] : C^\infty(M) \to C^\infty(M)$ defined as follows:

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \quad , \; f \in C^\infty(M)$$

Definition 2.3.5 (exterior derivative) — Exterior derivative is a linear operator $d : \Omega_M^n \to \Omega_M^{n+1}$ which satisfies the Leibniz rule. The operator $d : \Omega_M^0 \to \Omega_M^1$ is defined as

$$df(X) = X(f) \quad , \; X \in TM, \; f \in \Omega_M^0 = C^\infty(M)$$

Then $d : \Omega_M^n \to \Omega_M^{n+1}$ is defined inductively as follows:

$$d\omega(X_1, \cdots, X_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k+1} X_k(\omega(X_1, \cdots, \widehat{X_k}, \cdots, X_{n+1}))$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \cdots, \widehat{X_i}, \cdots, \widehat{X_j}, \cdots, X_{n+1})$$

where the symbol $\widehat{X_k}$ simply means that the term X_k is missing in the list.

Example 2.3.1

For $M = \mathbb{R}^3$, the exterior derivative operators on Ω_M^0, Ω_M^1 and Ω_M^2 are gradient, curl, and divergence, respectively.

Theorem 2.2

We have the following properties of the exterior derivative operator:

1. $d^2 = 0$.
2. For smooth $f : M \to N$, we have $f^* \circ d = d \circ f^*$.

§2.4 Homology and Cohomology

Definition 2.4.1 (Chain Complex and Homology) — Let the following be a sequence of morphisms of modules/groups

$$0 \leftarrow C_0 \xleftarrow{\partial_0} C_1 \xleftarrow{\partial_1} C_2 \xleftarrow{\partial_2} C_3 \xleftarrow{\partial_3} \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$.
Then the sequence is called a chain complex and denoted as (C_*, ∂).
Then the $n-th$ homology group of this chain complex is defined to be:

$$H_n(C_*) = \mathrm{Ker}(\partial_{n-1})/\mathrm{im}(\partial_n)$$

Definition 2.4.2 (Cochain Complex and Cohomology) — Let the following be a sequence of morphisms of modules/groups

$$0 \to C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} C_3 \xrightarrow{d_3} \cdots$$

such that $d_{n+1} \circ d_n = 0$.
Then the sequence is called a cochain complex and denoted as (C^*, d).
Then the $n-th$ cohomology group of this chain complex is defined to be:

$$H^n(C_*) = \mathrm{Ker}(d_n)/\mathrm{im}(d_{n-1})$$

If the context is clear, we often ignore the subscripts from ∂_n and d_n and we simply write them as ∂ and d.

If the homology/cohomology groups are all trivial, we say that the sequence is exact.

Definition 2.4.3 (de Rham Cohomology) — In the last section, we saw that the exterior derivative has the property $d^2 = 0$, and so we have the following cochain complex:

$$0 \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \Omega^2_M \xrightarrow{d} \Omega^3_M \xrightarrow{d} \cdots$$

This is known as de Rham complex, and the cohomology of this complex is called the de Rham cohomology. The $k-$th de Rham cohomology group is expressed by $H^k_{dR}(M)$.

Example 2.4.1
The manifold $\mathbb{R}^3$ has the following de Rham Complex, which is in fact an exact sequence.

$$C^\infty(\mathbb{R}^3) \xrightarrow{\text{gradient}} \Omega^1(\mathbb{R}^3) \xrightarrow{\text{curl}} \Omega^2(\mathbb{R}^3) \xrightarrow{\text{div}} \Omega^3(\mathbb{R}^3)$$

One can easily see the exactness by basic results in vector analysis. Hence, the de Rham cohomology groups are zero.

We shall now discuss some important homology and cohomology defined on a topological space which is not necessarily a smooth manifold but has some relation with de Rham cohomology when the topological space is a smooth manifold.

Definition 2.4.4 (Standard $n-$simplex) — The standard $n-$simplex is the convex set $\Delta^n \subset \mathbb{R}^n$ defined as:

$$\Delta^n = \{(x_0, x_1, \cdots, x_n) \in \mathbb{R}^n \ : \ x_0 + x_1 + \cdots + x_n = 1\}$$

Definition 2.4.5 (Singular $n-$simplex) — A continuous map $\sigma : \Delta^n \to X$ is called a singular $n-$simplex on a topological space X.

The $k-$th face of σ is defined to be the singular $(n-1)-$simplex $(\sigma \circ \phi_k) : \Delta^{n-1} \to X$ where $\phi_k : \Delta^{n-1} \to \Delta^n$ is defined as follows:

$$\phi_k(t_0, t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_n) = (t_0, t_1, \cdots, t_{k-1}, 0, t_{k+1}, \cdots, t_n)$$

Definition 2.4.6 (Singular homology) — The $n-$th singular chain group $C_n(X, A)$ on X with coefficients in a commutative ring A is defined to be the free $A-$module with one generator $[\sigma]$ for every singular $n-$simplex σ on X.
Then we can make it a chain complex with $\partial : C_n(X, A) \to C_{n-1}(X, A)$ defined as:

$$\partial[\sigma] = \sum_{k=0}^{n} (-1)^k [\sigma \circ \phi_k]$$

i.e ∂ sends a simplex to the alternating sum of its $k-$th faces.

The homology of this complex is known as the singular homology of X, and the $k-$th homology group is expressed by $H_k(X, A)$.

Definition 2.4.7 (Singular cohomology) — The $n-$th cochain group $C^n(X, A)$ is defined to be $Hom_A(C_n(X, A), A)$ and the evaluation of the cochain c on a chain γ is denoted as $\langle c, \gamma \rangle \in A$. Then we can make a cochain complex with $d : C^n(X, A) \to$

$C^{n+1}(X, A)$, which is determined by the following equation:

$$\langle dc, \gamma\rangle + (-1)^n \langle c, \partial\gamma\rangle = 0$$

The cohomology of this complex is known as the singular cohomology of X, and the $k-$th cohomology group is denoted as $H^k(X, A)$.

Theorem 2.3 (de Rham theorem)
For any smooth manifold M and nonnegative integer n, we have:

$$H^n_{dR}(M) \cong H^n(M, \mathbb{R})$$

Now we will see a ring structure on the singular cohomology groups by defining a multiplication. Let two cochains be given as $c \in C^m X$ and $c' \in C^n X$. Now we will construct a product $cc' = c \smile c' \in C^{m+n}X$. Let $\sigma : \Delta^{m+n} \to X$ be a singular simplex. The front m-face of σ is defined to be the composition $\sigma \circ \alpha_m : \Delta^m \to X$, where

$$\alpha_m(t_0, \dots, t_m) = (t_0, \dots, t_m, 0, \dots, 0).$$

Similarly the *back n-face* of σ is defined as $\sigma \circ \beta_n$ where

$$\beta_n(t_m, t_{m+1}, \dots, t_{m+n}) = (0, \dots, 0, t_m, t_{m+1}, \dots, t_{m+n}).$$

Now the product $cc' = c \smile c'$ is determined by the following identity

$$\langle cc', [\sigma]\rangle = (-1)^{mn}\langle c, [\sigma \circ \alpha_m]\rangle \cdot \langle c', [\sigma \circ \beta_n]\rangle \in A.$$

This is called **cup product** of cochains, and it is associative, bilinear, but not necessarily commutative.

So the space $H^*(M, A) = \bigoplus_{n=0}^{\infty} H^n(M, A)$ forms a ring and we call it a cohomology ring. This is analogous to Grassmann Algebra where the multiplication was given by the exterior product.

We will now mention the ring structure of the cohomology ring of two spaces in the following examples. We will omit the proof of them as they involve certain algebraic topological tools which are beyond the scope of this thesis. However, we still need to mention them as we will see some of their use later. Interested readers can see the proof in [Hat02].

Example 2.4.2

$$H^*(\mathbb{R}P^n, \mathbb{Z}_2) = \mathbb{Z}_2[\alpha]/\alpha^{n+1} \quad \text{for some } \alpha \in H^1(\mathbb{R}P^n, \mathbb{Z}_2)$$

Example 2.4.3

$$H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1} \quad \text{for some } \alpha \in H^2(\mathbb{C}P^n, \mathbb{Z})$$

§2.5 Category Theory

Definition 2.5.1 (Category) — A **category** $\mathcal{C}$ consists of:

- a class of objects, $\mathrm{Ob}(\mathcal{C})$,
- for every pair of objects $A, B \in \mathrm{Ob}(\mathcal{C})$, a set of arrows (most of the times, arrows are just morphisms, but in generally they are not) $\mathrm{Hom}_{\mathcal{C}}(A, B)$,
- for every triple of objects A, B, C, a composition law $\circ : \mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$,
- for every object A, an identity morphism $\mathrm{id}_A \in \mathrm{Hom}(A, A)$,

such that:

1. **Associativity:** For $f \in \mathrm{Hom}(A, B)$, $g \in \mathrm{Hom}(B, C)$, $h \in \mathrm{Hom}(C, D)$,
$$h \circ (g \circ f) = (h \circ g) \circ f.$$
2. **Identity:** For $f \in \mathrm{Hom}(A, B)$,
$$f \circ \mathrm{id}_A = f, \quad \mathrm{id}_B \circ f = f.$$

Example 2.5.1
1. **Set** category with objects being all the sets and arrows being all the functions between sets.
2. **Grp** category with objects being all the groups and arrows being all group homomorphisms.
3. **Ring/Alg** category with objects being all the rings/Algebra and arrows being all the ring/algebra homomorphisms.
4. **Top** category with objects being all topological spaces and arrows being all the homeomorphisms between topological spaces.

Definition 2.5.2 (Covariant Functor) — A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns:

- to each object A in $\mathcal{C}$, an object $F(A)$ in $\mathcal{D}$,
- to each morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$,

such that:

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all objects A,
2. $F(g \circ f) = F(g) \circ F(f)$ for all composable morphisms f, g in $\mathcal{C}$.

Definition 2.5.3 (Contravariant Functor) — A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ assigns:

- to each object A in $\mathcal{C}$, an object $F(A)$ in $\mathcal{D}$,

- to each morphism $f : A \to B$ in $\mathcal{C}$, a morphism $F(f) : F(B) \to F(A)$ in $\mathcal{D}$,

such that:

1. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for all objects A,
2. $F(g \circ f) = F(f) \circ F(g)$ for all composable morphisms f, g in $\mathcal{C}$.

Look, the arrows get reversed in the case of contravariant functors. The term covariant or contravariant is often ignored if the context is clear.

Example 2.5.2

The **forgetful functor** U : **Grp** $\to$ **Set** assigns:

- to each group G, the underlying set $U(G)$,
- to each group homomorphism $f : G \to H$, the same function $U(f) : U(G) \to U(H)$ between the underlying sets.

Example 2.5.3

From the category **Top** of topological spaces to the category $A-$**Mod** of $A-$modules, homology assigns a covariant functor and cohomology assigns a contravariant functor. We call them **homology functor** and **cohomology functor** respectively.

Example 2.5.4

For a manifold M, we can assign a set V_M which is the collection of all vector bundles over M. Recall that for any smooth map $f : M_1 \to M_2$, we can pullback any bundle from M_2 to M_1. Thus, assigning V_M to every manifold in fact defines a contravariant functor from the category of manifolds to the category of sets. We call this functor the **Vector bundle functor.**

Definition 2.5.4 (Natural Transformations) — Given two covariant functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \Rightarrow G$ assigns to each object A in $\mathcal{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$ such that for every morphism $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"] \\ G(A) \arrow[r, "G(f)"'] & G(B) \end{tikzcd}$$

Natural transformation between contravariant functors can be defined analogously (just the directions of the arrows $F(f)$ and $G(f)$ getting reversed).

Example 2.5.5

Consider the functors F, G : **Set** $\to$ **Set** where

- $F(A) = A \times \{0\}$, $G(A) = A \times \{1\}$,

- $\eta_A : F(A) \to G(A)$ defined by $\eta_A(a, 0) = (a, 1)$,

then η is a natural transformation from F to G since the square commutes for any function between sets.

§2.6 Orientability

Definition 2.6.1 (Orientation on a vector space) — In a finite-dimensional vector space V, two ordered bases are said to have the same orientation if their change of basis matrix has a positive determinant. This forms an equivalent relation, and thus there can be only two orientations on a vector space.

There are many equivalent ways to define orientation on a manifold. We take the following definition.

Definition 2.6.2 (Orientation on a manifold) — An orientation in a $n-$ dimension manifold is a choice of a form $\omega \in \Omega^n_M$ which is nowhere vanishing (i.e, non-zero at every point).

Note that such a nowhere vanishing form defines orientation on the tangent space at every point.

We will now introduce an alternate characterization of orientability, which we will use throughout the paper.

Let M be a closed (here closed means compact) connected oriented $n-$dimensional manifold. Then it is a basic result in algebraic topology that the top homology group is *infinite cyclic*:

$$H_n(M; \mathbb{Z}) \cong \mathbb{Z},$$

An orientation is then defined to be a choice of a generator, and the generator is called the **fundamental homology class** or shortly the fundamental class, denoted by $[M]$. There are exactly two possible choices, hence exactly two possible orientations.

In manifold theory, there is a notion of integration too, which is intuitively the summation over small pieces as usual. We will not define this in detail as we will not have to use it that much, but one can see [Tu] for details. However, it would be an injustice if the following facts related to integration on manifolds are ignored. Also, it should be mentioned that we can only integrate top forms in the case of a manifold, and the manifold must be orientable to define it.

The fundamental homology class can also define *integration over manifold M*, through evaluation by the top-degree cochains. In fact, we have:

$$\omega[M] = \langle \omega, [M] \rangle = \int_M \omega.$$

where ω in singular and de Rham cohomology cases is identified by the de Rham theorem.

We won't prove the following example, but one can see [Hat02], for details. We are mentioning it as we will need it later.

Example 2.6.1
For $M = \mathbb{C}P^n$, we have:
$$\alpha^n[M] = 1$$
Where $\alpha \in H^2(\mathbb{C}P^n, \mathbb{Z})$ is a generator of the cohomology ring of $\mathbb{C}P^n$.

To introduce manifold with boundary, let consider the *half-space* $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$.

Definition 2.6.3 — A *topological manifold with boundary* M is a second countable Hausdorff topological space which is locally homeomorphic to H^n. Its boundary ∂M is the $(n-1)$ manifold consisting of all points mapped to $x_n = 0$ by a chart, and its *interior* $\mathrm{Int}(M)$ is the set of points mapped to $x_n > 0$ by some chart. Infact we have, $M = \partial M \cup \mathrm{Int}\,(M)$.

Example 2.6.2
The ball $\{x \in \mathbb{R}^n \ : \ ||x|| \leq 1\}$ is a manifold with boundary where the boundary is the standard $(n-1)-$sphere embedded in $\mathbb{R}^n$. But the standard $(n-1)-$sphere has no boundary of itself.

Theorem 2.4 (Stokes' theorem)
Let M be an orientable smooth manifold with boundary ∂M and ω be a top form on the boundary. Then,
$$\int_{\partial M} \omega = \int_M d\omega.$$

§2.7 Connection and Curvature

The present section is primarily based on Appendix C of [MS74].

Let E be an $n-$dimensional complex bundle with base M, and let
$$TM^*_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(TM, \mathbb{C})$$
be the complexified cotangent bundle of M. Then $TM^*_{\mathbb{C}} \otimes E$ is also a complex vector bundle over M. The space of sections of this complex bundle will be denoted by $\Gamma(TM^*_{\mathbb{C}} \otimes E)$.

There is no canonical way to define directional derivative on manifolds; hence, we need to define it in an artificial way. Here we discuss the procedure.

Definition 2.7.1 — A connection on E is a linear map:

$$\nabla : \Gamma(E) \to \Gamma(TM^*_{\mathbb{C}} \otimes E)$$

satisfying the Leibniz formula

$$\nabla(fs) = df \otimes s + f\nabla(s).$$

Then the directional or covariant derivative $\nabla_X Y$ for $X \in \Gamma(TM)$ and $Y \in \Gamma(E)$ is defined as:

$$\nabla_X Y = (\nabla(Y))(X)$$

where X is taken by the element in $TM^*_{\mathbb{C}}$ through the ∇ map.

Example 2.7.1

The gradient defines a connection $C^\infty(\mathbb{R}^3)$ where the covariant derivative is simply $\nabla_v f = \nabla f \cdot v$ for $f \in C^\infty(\mathbb{R}^3)$ and $v \in \mathbb{R}^3 = T_p\mathbb{R}^3$ for any $p \in \mathbb{R}^3$.

Theorem 2.5

A connection ∇ on the trivialisation $E|_U$ is uniquely determined by $\nabla(s_1), \dots, \nabla(s_n)$ (here s_i's are the vector space basis of the trivial bundle), which can be arbitrary sections of $TM^*_{\mathbb{C}} \otimes E$. Each section $\nabla(s_i)$ can be uniquely expressed as a sum

$$\sum_j \omega_{ij} \otimes s_j$$

where $[\omega_{ij}]$ is an $n \times n$ matrix of smooth complex 1-forms on U.

Now, given a smooth map $g : M' \to M$ we can make a pullback bundle $E' = g^*E$, where the fibre in E' at $p \in M'$ is simply the fibre in E at $g(p) \in M$. Note that we have a natural $C^\infty(M)$-linear mapping

$$g^* : \Gamma(E) \to \Gamma(E').$$

Also, any 1-form on M pulls back to a 1-form on M', so there is a canonical map

$$g^* : \Gamma(TM^*_{\mathbb{C}}(M) \otimes E) \to \Gamma(TM^*_{\mathbb{C}}(M') \otimes E').$$

Theorem 2.6

For each connection ∇ on E there exists a unique connection $\nabla' = g^*\nabla$ on the induced bundle E' so that the following diagram is commutative:

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{\nabla} & \Gamma(TM^*_{\mathbb{C}}(M) \otimes E) \\ \downarrow g^* & & \downarrow g^* \\ \Gamma(E') & \xrightarrow{\nabla'} & \Gamma(TM^*_{\mathbb{C}}(M') \otimes E') \end{array}$$

Proof. For example, given sections $s_1, \dots, s_n$ over an open subset U of M with

$$\nabla(s_i) = \sum_j \omega_{ij} \otimes s_j,$$

we can form the lifted 1-forms ω'_{ij} and the lifted sections s'_j over $g^{-1}(U)$. If such a connection ∇' exists, then evidently

$$\nabla'(s'_i) = \sum_j \omega'_{ij} \otimes s'_j.$$

■

Given a connection ∇ on E, let us try to pull back the connection on the bundle $TM^*_{\mathbb{C}} \otimes E$. We will make use of ∇ along with the exterior differentiation operator $d : \Gamma(TM^*_{\mathbb{C}}) \to \Gamma(\Lambda^2 TM^*_{\mathbb{C}})$.

Theorem 2.7
Given ∇ there is one and only one $\mathbb{C}$-linear mapping

$$\widehat{\nabla} : \Gamma(TM^*_{\mathbb{C}} \otimes E) \to \Gamma(\Lambda^2 TM^*_{\mathbb{C}} \otimes E)$$

satisfying the Leibniz formula

$$\widehat{\nabla}(\theta \otimes s) = d\theta \otimes s - \theta \wedge \nabla(s)$$

for every 1-form θ and every section $s \in \Gamma(E)$. Furthermore $\widehat{\nabla}$ satisfies the identity

$$\widehat{\nabla}(f(\theta \otimes s)) = df \wedge (\theta \otimes s) + f\widehat{\nabla}(\theta \otimes s).$$

Now let us consider the composition $K = \widehat{\nabla} \circ \nabla$ of the two $\mathbb{C}$-linear mappings

$$\Gamma(E) \xrightarrow{\nabla} \Gamma(TM^*_{\mathbb{C}} \otimes E) \xrightarrow{\widehat{\nabla}} \Gamma(\Lambda^2 TM^*_{\mathbb{C}} \otimes E)$$

Theorem 2.8
Let $s \in \Gamma(E)$. Then $K(s) = \widehat{\nabla}(\nabla(s))$ at a particular $x \in M$ depends only on $s(x)$, not on the nearby values of s at other points. Hence, the following correspondence defines a smooth section on $\mathrm{Hom}(E, \Lambda^2 TM^*_{\mathbb{C}} \otimes E)$.

$$s(x) \mapsto K(s)(x)$$

Definition 2.7.2 — This section $K = K_\nabla$ of the bundle

$$\mathrm{Hom}(E, \Lambda^2 TM^*_{\mathbb{C}} \otimes E) \cong \Lambda^2 TM^*_{\mathbb{C}} \otimes \mathrm{Hom}(E, E)$$

is called the *curvature tensor* for the connection ∇.

Note that the curvature of a curve is roughly the 2nd derivative of that curve, and here the operator $K = \widehat{\nabla} \circ \nabla$ can also be interpreted as a 2nd derivative on the vector fields. Hence, we can find some justification to name it a curvature tensor.

In terms of a basis of sections $s_1, \ldots, s_n$ on $E|_U$, with $\nabla(s_i) = \sum \omega_{ij} \otimes s_j$, we have the following:

$$K(s_i) = \widehat{\nabla}\left(\sum \omega_{ij} \otimes s_j\right) = \sum \Omega_{ij} \otimes s_j$$

where we have set

$$\Omega_{ij} = d\omega_{ij} - \sum_{\alpha} \omega_{i\alpha} \wedge \omega_{\alpha j}.$$

Thus K can be locally expressed as a 2-form valued matrix $\Omega = [\Omega_{ij}]$ in much the same way that ∇ is locally expressed by the 1-form valued matrix $\omega = [\omega_{ij}]$. In matrix notation, we have

$$\Omega = d\omega - \omega \wedge \omega.$$

We call Ω the **curvature matrix** for the connection ∇ on the trivialisation $E|_U$.

One can see the examples of computations of connections and curvature in detail in [Sze04]. However, we won't explore this more, as for us, this is enough to know that curvature takes the form of a matrix on a trivialisation.

Now let P(X) be a polynomial where X takes values in $n \times n$ matrices, and the output of $P(X)$ is a scalar quantity. We call it an invariant polynomial if for any invertible matrix T, we have:

$$P(T^{-1}XT) = P(X)$$

For an invariant polynomial, we can define a valid de Rham cohomology class $P(K)$ with K being the curvature, where we compute $P(K)$ for some representative matrix Ω of the curvature on some trivialisation. The choice of the trivialisation does not matter as if Ω_1 and Ω_2 are two matrix representations of the curvature in two different trivialisations, then they are related by $\Omega_1 = T^{-1}\Omega_2 T$ for some invertible matrix T. Thus, we have $P(\Omega_1) = P(\Omega_2)$. We have introduced this notion as we will need this in the next chapter when we describe the relation between characteristics classes and curvature. For the details of this, see Appendix C of [MS74]. Interesting thing about $P(K)$ is that it is independent of the connection we choose to have the curvature, and it is, in fact, a topological invariant.

We will finish this section with an important theorem, which we will need to define the spinor connection in Appendix A.

Theorem 2.9 (Levi Civita Connection)
For a given Riemannian manifold (M, g), there is unique connection ∇ on the tangent bundle which satisfies the following conditions:

$$\nabla g = 0$$

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad for \;\; X, Y \in \Gamma(TM)$$

We call this connection the Levi Civita Connection for (M, g).

§2.8 A bit of symmetric function theory

The present section is not geometric in itself; however, we are introducing it here as it will be needed in our discussion of Chern roots in the next chapter.

Definition 2.8.1 (Symmetric function) — A function $f(x_1, \cdots, x_n)$ in n variables is symmetric if for any permutation $\sigma \in S_n$, we have:

$$f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$$

Definition 2.8.2 (Elementary symmetric function) — The $k-th$ elementary symmetric function in n variables is the $k-$th homogenous part of the following symmetric function:

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n)$$

Theorem 2.10
Any symmetric function can be expressed as a function of the elementary symmetric functions.

Example 2.8.1
Consider $f(x_1, x_2) = x_1^2 + x_2^2$, a symmetric function in 2 variables. Note that,

$$(1 + x_1)(1 + x_2) = 1 + (x_1 + x_2) + x_1 x_2$$

Thus elementary symmetric functions in 2 variables are $1, (x_1 + x_2)$ and $x_1 x_2$. Then we see that,

$$f(x_1, x_2) = x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2$$

which is indeed expressed as a function of elementary symmetric functions.

Chapter 3 : Characteristic Classes

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§3.1 Characteristics classes

Characteristics classes assign important topological/manifold invariants to some space which takes values in cohomology classes. Topological invariants, or in fact more generally invariants in different areas of mathematics, are crucial to consider, as they are useful tools in classifying spaces. Classification of mathematical objects is a key area of research in mathematics as it gives a thorough picture/understanding of these objects. Tackling classification problems in mathematics is usually very hard; hence, mathematicians introduce invariants because if two objects have different values of the invariants, they have to be in different classes. Moreover, studying mathematical invariants can also be physically interesting, as they can potentially relate to energy or other conserved quantities in physical sciences.

Definition 3.1.1 (Characteristic Class) — A characteristic class is a natural transformation from a vector bundle functor to a cohomology functor.

Now we will discuss some theorems related to the existence of some particular characteristics classes with examples. Even though we will not prove them and will not even see what those classes look like, we will illustrate throughout the thesis how effectively we can play with them just knowing their properties. There are two approaches to characteristics classes, in the sense of algebraic topology and in the sense of differential geometry. We will mainly focus on the algebraic topology approach, though we will discuss some aspects of the geometric part. For the algebraic topology approach, one can find a detailed exposition at [Hat03]. For the geometric approach, one can see Chapter 6 of [Tu17].

Theorem 3.1 (Existence of Stiefel-Whitney classes)
For a rank n **real** vector bundle $E \to M$, there exists unique cohomology classes $w_k(E) \in H^{2k}(M;\mathbb{Z}_2)$ (Called $k-$th Stiefel-Whitney class) such that the following properties hold:

- $w_0(E) = 1$ and $w_i(E) = 0$ for $i > n$
- $w(E \oplus F) = w(E) \smile w(F)$ where $w(E) = \sum_k w_k(E)$ and it is called the total Stiefel-Whitney class of E.
- $w_k(f^*E) = f^*w_k(E)$
- $w(\gamma) = 1 + \alpha$ where γ is the tautological bundle over $\mathbb{R}P^1$ and $\alpha \in H^2(\mathbb{R}P^1;\mathbb{Z}_2)$ is the algebra generator of $H^*(\mathbb{R}P^1;\mathbb{Z}_2)$

By $w(M)$, we mean the total Stiefel-Whitney class for the tangent bundle of the manifold M.

Note that the third condition means Stiefel-Whitney classes are indeed examples of characteristic classes (i.e, natural transformation) while the last condition can be considered as a “normalization condition” to ensure that the class is unique.

Example 3.1.1
Note that any rank n trivial bundle E over a manifold M can be formed as a pull back of the trivial rank n bundle over a point manifold. However, all the higher cohomology groups of a point manifold is 0, hence all of its positive degree Stiefel-Whitney classes will be also zero. Thus if we pull back the positive degree Stiefel-Whitney classes to the bundle E, they will be also zero i.e $w(E) = 1$.

Hence we have shown that all positive degree Stiefel-Whitney classes of a trivial bundle must vanish. In other words, they can measure the obstruction for a bundle to be trivial. Infact all the other characteristics classes we will discuss here, they will have trivial characteristics classes for any trivial bundle in the same way. Stiefel-Whitney classes give us even a better obstruction which we state in the following theorem.

Theorem 3.2
A manifold is orientable if and only if its first Stiefel-Whitney class w_1 is 0.

See [MS74] for details.

Theorem 3.3 (Existence of Chern Classes)
For a **complex** rank $n-$ vector bundle $E \to M$, there exists unique cohomology classes $c_k(E) \in H^{2k}(M;\mathbb{Z})$ (Called $k-$th Chern class) such that the following properties hold:

- $c_0(E) = 1$
- $c(E \oplus F) = c(E) \smile c(F)$, where $c(E) = \sum_k c_k(E)$ and it is called the total Chern class of E.
- $c_k(f^*E) = f^*c_k(E)$
- $c(\gamma) = 1 - \alpha$, where γ is the tautological bundle over $\mathbb{C}P^n$ and $\alpha \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is the algebra generator of $H^*(\mathbb{C}P^n;\mathbb{Z})$

By $c(M)$, we mean the total Chern class for the tangent bundle of the manifold M.

Example 3.1.2
For $M = \mathbb{C}P^n$, we have:

$$c(M) = c(\mathbb{C}P^n) = (1+\alpha)^{n+1}$$

where α is the generator of the cohomology ring.
We will not prove this in full generality; rather, we will sketch out the proof. For details, refer to Chapter 14 of [MS74].
Let ϵ be the bundle $\mathrm{Hom}(\gamma,\gamma)$ and $\overline{\gamma}$ be the complex conjugate bundle of γ (i.e taking fibrewise complex conjugate).
It turns out that ϵ is a trivial bundle, and:

$$TM \oplus \epsilon = \overline{\gamma} \oplus \overline{\gamma} \oplus \cdots \oplus \overline{\gamma} \quad (n+1 \text{ times summation})$$

By the argument we used in the previous example of the Stiefel-Whitney class, we can say that the higher Chern classes vanish in the case of a trivial bundle. So we must have the total Chern class $c(\epsilon) = 1$.
Also there is a theorem that if $\overline{\omega}$ is the complex conjugate bundle for some vector bundle ω, then we have:

$$c(\overline{\omega}) = 1 - c_1(\omega) + c_2(\omega) - c_3(\omega) + c_4(\omega) - \cdots$$

Thus in our case,

$$c(\overline{\gamma}) = 1 - c_1(\gamma) = 1 + \alpha$$

Now putting these all together, we have:

$$\begin{aligned} & c(TM \oplus \epsilon) = c(\overline{\gamma} \oplus \overline{\gamma} \oplus \cdots \oplus \overline{\gamma}) \quad (n+1 \text{ times summation}) \\ \Longrightarrow\ & c(TM) \smile c(\epsilon) = c(\overline{\gamma})^{n+1} \\ \Longrightarrow\ & c(TM) = (1+\alpha)^{n+1} \end{aligned}$$

■

Definition 3.1.2 (Pontrjagin classes) — For a **real** vector bundle $E \to M$, its $k-$th Pontrjagin class $p_k \in H^{4k}(M;\mathbb{Z})$ is defined as

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C})$$

Let $p(E) = \sum_k p_k(E)$ and call it total Pontrjagin class of E.

Theorem 3.4

Pontrjagin classes have the following properties:

- $p_0(E) = 1$
- $p(E \oplus F) = p(E) \smile p(F)$ modulo 2 torsion (i.e $2p(E \oplus F) = 2p(E) \smile p(F)$)
- $p_k(f^*E) = f^*p_k(E)$
- $c(\gamma_{\mathbb{R}}) = 1 + \alpha^2$, where $\gamma_{\mathbb{R}}$ is the underlying real vector bundle of the tautological bundle over $\mathbb{C}P^n$ i.e, we just forget the complex structure of the tautological bundle.

By $p(M)$, we mean the total Pontrjagin class for the tangent bundle of the manifold M.

Example 3.1.3

$$p(\mathbb{C}P^n) = (1+\alpha^2)^{n+1}$$

In this case, we will also sketch out the proof only. For details, see Chapter 15 of [MS74].

If ω is a complex rank n bundle $\omega_{\mathbb{R}}$ is its underlying real bundle, then it turns out that $\omega_{\mathbb{R}} \otimes \mathbb{C} = \omega \oplus \overline{\omega}$. Hence, by some simple algebraic manipulations, we have the following identity for the complex bundle ω (where we omitted the ω in parentheses inside $p_k(\omega)$ for notational simplicity):

$$1 - p_1 + p_2 - p_3 + \cdots \pm p_n = (1 - c_1 + c_2 - c_3 + \cdots \pm c_n) \smile (1 + c_1 + c_2 + c_3 + \cdots)$$

Then, taking $\omega = T\mathbb{C}P^n$, and using the results from our previous example, we have:

$$\begin{aligned} 1 - p_1 + p_2 - p_3 + \cdots \pm p_n &= c(\overline{T\mathbb{C}P^n}) \smile c(T\mathbb{C}P^n) \\ &= (1-\alpha)^{n+1} \smile (1+\alpha)^{n+1} \\ &= (1-\alpha^2)^{n+1} \end{aligned}$$

$$\Longrightarrow\ p(\mathbb{C}P^n) = 1 + p_1 + p_2 + \cdots + p_n = (1+\alpha^2)^{n+1}$$

■

§3.2 Chern root and Chern Character

Note that if $E_i \to M$ are complex line bundle for $i = 1, 2, \cdots, n$, then $E \to M$ is a rank n complex bundle where $E = E_1 \oplus E_2 \oplus \cdots \oplus E_n$, and for the total Chern class, we have:

$$c(E) = c(E_1) \smile c(E_2) \smile \cdots \smile c(E_n) = (1 + x_1) \smile (1 + x_2) \smile \cdots \smile (1 + x_n)$$

where $1 + x_i$ is the total Chern class of the line bundle $E_i \to M$. Thus, the $k-$th Chern class of $E \to M$ is the $k-$th elementary symmetric polynomial in the x_i's (discussed in the final section of the previous chapter).

Even if a rank n complex bundle $E \to M$ doesn't split into n line bundles with $E = E_1 \oplus \cdots \oplus E_n$, we can still formally factorize the total Chern class as $c(E) = (1+x_1)(1+x_2)\cdots(1+x_n)$ (here we have ignored the cup product for notational convenience), and we would consider the $k-$th Chern class as $k-$th elementary symmetric polynomial in the x_i's. Notice that here x_i's are just formal variables and do not necessarily have an implicit meaning, but their $k-$th elementary symmetric polynomials denote the Chern classes.

We call these x_i's as the **Chern roots** of the bundle $E \to M$.

As any symmetric function in the x_i's can be expressed as a function in the elementary symmetric polynomials of $x_i's$, thus any symmetric function in the Chern roots can be expressed as functions in the Chern class. Having said that, we have the following definition:

Definition 3.2.1 (Chern Character) — The Chern character of a rank n complex bundle $E \to M$ is given by

$$ch(E) = \sum_{i=1}^{n} e^{x_i}$$

Note that $ch(E)$ is a symmetric function in the $x_i's$ and so can be expressed in the Chern classes, thus $ch(E)$ is a valid cohomology class.

The following theorem will be very useful in many of our future computations.

Theorem 3.5
Let E_1 and E_2 be two complex vector bundles over M. Then,

$$ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$$

$$ch(E_1 \otimes E_2) = ch(E_1)ch(E_2)$$

§3.3 Relation with Curvature

Here we will briefly discuss how characteristics classes and Chern character can also be understood through the curvature matrix of a Riemannian manifold, which will provide us with some heuristic insights for the next chapter (details at [Tu17]). Of course, in this case, the classes are defined on de Rham cohomology; however, it is not an issue due to de Rham theorem [2.3].

Let K be the curvature of a manifold M with respect to any connection on it, with a representative matrix Ω for the curvature on some trivialisation. Recall from our discussion of connection and curvature in Chapter 2 that we can get valid de Rham cohomology classes by evaluating the curvature matrix on some invariant polynomial. We will now construct Chern Classes and Chern character using some specific invariant polynomials.

The Chern classes $c_i \in H^{2i}(M)$ (for $i = 0, 1, 2, \ldots, k$), are defined via the expansion:

$$\det\left(1 + \frac{\sqrt{-1}}{2\pi}\Omega\right) = c_0 + c_1 + c_2 + \cdots .$$

where c_j is the $j - th$ degree differential form part in the expansion.

The Chern characters $\mathrm{ch}_i(\Omega) \in H^{2i}(M)$ (for $i = 0, 1, 2, \ldots$), are introduced as

$$\mathrm{ch}_i(\Omega) = \frac{1}{i!} \operatorname{tr}\left(\left(\frac{\sqrt{-1}}{2\pi}\Omega\right)^i\right).$$

Occasionally, one encounters other types of characteristic classes, such as Todd classes, Hirzebruch L-polynomials, and the $\hat{A}$-polynomials. These are based on various invariant polynomials and offer alternative perspectives on curvature.

To express them, one often diagonalizes the curvature form Ω, writing its eigenvalues as $x_1, \ldots, x_k$, such that:

$$\frac{\sqrt{-1}}{2\pi}\Omega \sim \mathrm{diag}(x_1, \ldots, x_k).$$

As an example, the total Chern class can then be succinctly written as:

$$c(\Omega) = \det\left(1 + \frac{\sqrt{-1}}{2\pi}\Omega\right) = \prod_{i=1}^{k}(1 + x_i).$$

Thus Chern roots can be viewed formally as the eigenvalues of the matrix $\frac{\sqrt{-1}}{2\pi}\Omega$.

§3.4 How Chern roots change through representations

In this section, we will briefly discuss how Chern roots change when we change a bundle through some representation of the structure group. We will not go into the rigorous details of these as it will take a very long explanation which will not be particularly illuminating for this thesis. Rather, we will give some heuristic arguments about how the Chern roots change in some specific cases of changing a vector bundle through a representation. Then we will use it to deduce some generating functions which will appear very useful later.

Let E be a complex vector bundle of rank n over a differentiable manifold X. Assume that the structure group of E is reduced to $U(n)$. Let ρ be a unitary representation of $U(n)$ of dimension m, so that

$$\rho : U(n) \to U(m)$$

is a homomorphism. The group $U(n)$ then acts on $\mathbb{C}^m$ by means of ρ. We consider the vector bundle ρE, associated to E, with fibre $\mathbb{C}^m$. Recall from Chapter 2 that we have the following examples.

Example 3.4.1

Let U be any element of the structure group G. Then we have the following bundles for the given representations ρ.

1. $E^* \cong E$ for $\rho(U) = \overline{U}$
2. $\Lambda^k E$ for $\rho(U)(v_1 \wedge \cdots \wedge v_k) = Uv_1 \wedge \cdots \wedge Uv_k$ where $v_1 \wedge \cdots \wedge v_k \in \Lambda^k \mathbb{C}^n$ (Λ^k denotes the antisymmetric tensors)
3. $S^k E$ for $\rho(U)(v_1 \otimes \cdots \otimes v_k) = Uv_1 \otimes \cdots \otimes Uv_k$ where $v_1 \otimes \cdots \otimes v_k \in S^k \mathbb{C}^n$ (S^k denotes the symmetric tensors)

Let the bundle E have Chern classes $c_j(E) \in H^{2j}(X; \mathbb{Z})$. We want to express the Chern classes or the Chern roots of the bundle ρE in terms of the $c_j(E)$ or the Chern roots in the bundle E. In fact, finding how the Chern root changes is enough as they can define Chern classes. Roughly speaking, how the Chern root changes is exactly the same as how the eigenvalues of the curvature matrix change under the representation (This is true due to our discussion in the previous section). We will illustrate what we mean by this in the following examples, which are the only examples we need for the thesis. Also, recall that we have:

$$c(E) = (1 + x_1)(1 + x_2) \cdots (1 + x_n)$$

We represent the Chern characters of ρE as $y_{(j)}$ for a suitable index (j) according to the basis vectors.

Example 3.4.2

For our special representations at the start of this section, we obtain:

1. For $\rho E = E^*$, the eigenvalue simply gets negative and we have:

$$y_i = -x_i \quad \text{and} \quad c(E^*) = \prod_{i=1}^{n} (1 - x_i) = \sum_{i=1}^{n} (-1)^i c_i(E).$$

2. For $\rho E = \Lambda^k E$, we have eigenvalue $x_{i_1} + \cdots + x_{i_k}$ for each basis vector $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ of $\Lambda^k E$. Thus, we have:

$$y_{i_1,\ldots,i_k} = x_{i_1} + \cdots + x_{i_k} \quad \text{and} \quad c(\Lambda^k E) = \prod_{1 \le i_1 < \cdots < i_k \le n} (1 + (x_{i_1} + \cdots + x_{i_k}))\,.$$

3. For $\rho E = S^k E$, we have eigenvalue $x_{i_1} + \cdots + x_{i_k}$ for each basis vector $dx_{i_1} \otimes \cdots \otimes dx_{i_k}$ of $S^k E$. Thus, we have:

$$y_{i_1,\ldots,i_k} = x_{i_1} + \cdots + x_{i_k} \quad \text{and} \quad c(S^k E) = \prod_{1 \le i_1 \le \cdots \le i_k \le n} (1 + (x_{i_1} + \cdots + x_{i_k}))\,.$$

   Note that we have used " $<$ " symbol under the Π sign in the previous example, but used $\le$ symbol in this example. this is because in $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, indices cannot be repeated, while in $dx_{i_1} \otimes \cdots \otimes dx_{i_k}$, indices can be repeated.

4. For $\rho E = \Lambda^k E^*$, we have the following:

$$y_{i_1,\ldots,i_k} = -(x_{i_1} + \cdots + x_{i_k}) \quad \text{and} \quad c(\Lambda^k E^*) = \prod_{1 \le i_1 < \cdots < i_k \le n} (1 - (x_{i_1} + \cdots + x_{i_k}))\,.$$

Now we will use these heuristically understood results in proving some useful theorems in the following section, which have a very nice combinatorial feature.

§3.5 Some useful generating functions

Definition 3.5.1 —

$$\mathrm{ch}(\Lambda_t E) := \sum_{k=0}^{\infty} \mathrm{ch}(\Lambda^k E) t^k$$

$$\mathrm{ch}(S_t E) := \sum_{k=0}^{\infty} \mathrm{ch}(S^k E) t^k.$$

Theorem 3.6

$$ch(\Lambda_t(E)) = \prod (1 + te^{x_i})$$

$$ch(S_t(E)) = \prod \frac{1}{1 - te^{x_i}}$$

Proof. For the first equation, we have:

$$\begin{aligned}
\mathrm{ch}(\Lambda_t E) &= \sum_{k=0}^{\infty} \mathrm{ch}(\Lambda^k E)t^k \\
&= 1 + t\left(\sum_{1\le i\le n} e^{x_i}\right) + t^2\left(\sum_{1\le i<j\le n} e^{x_i+x_j}\right) + \cdots + t^n e^{x_1+\cdots+x_n} \\
&= \prod_{i=1}^{n}(1+te^{x_i})
\end{aligned}$$

For the 2nd equation, we have:

$$\begin{aligned}
\mathrm{ch}(S_t E) &= \sum_{k=0}^{\infty} \mathrm{ch}(S^k E)t^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{1\le i_1\le\cdots\le i_k} e^{x_{i_1}+\cdots+x_{i_k}}\right)t^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{\substack{k_1+\cdots+k_n=k \\ k_i\ge 0}} \sum_{i_{k_1}<\cdots<i_{k_n}} e^{k_1x_{k_1}+\cdots+k_nx_{k_n}}\right)t^k \\
&= \prod_{i=1}^{n}(1+te^{x_i}+t^2e^{2x_i}+t^3e^{3x_i}+\cdots) \\
&= \prod_{i=1}^{n}\sum_{k=0}^{\infty}(te^{x_i})^k \\
&= \prod_{i=1}^{n}\frac{1}{1-te^{x_i}}
\end{aligned}$$

■

When $E = TM_{\mathbb{C}} = TM \otimes C$ for a $4k$-dimensional manifold M, we know that its Chern roots have the following form:

$$x_1, \ldots, x_{2k}, -x_1, \ldots, -x_{2k}.$$

Thus from the previous theorem, we get the following corollary.

Corollary 3.1

$$\mathrm{ch}(\Lambda_t TM_{\mathbb{C}}) = \prod_{i=1}^{2k}(1+te^{x_i})(1+te^{-x_i}) = \prod_{i=1}^{2k}(1+te^{\pm x_i})$$
$$\mathrm{ch}(S_t TM_{\mathbb{C}}) = \prod_{i=1}^{2k}\frac{1}{(1-te^{x_i})(1-te^{-x_i})} = \prod_{i=1}^{2k}\frac{1}{1-te^{\pm x_i}}$$

where we have used the following shortcut notations:

$$1+te^{\pm x_i} := (1+te^{x_i})(1+te^{-x_i})$$
$$1-te^{\pm x_i} := (1-te^{x_i})(1-te^{-x_i})$$

Theorem 3.7

For a $4k$ dimensional manifold

$$ch\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty}\Lambda_{q^n}TM_{\mathbb{C}}\right) = \prod_{n=1}^{\infty}\prod_{k=1}^{2k}\frac{1+q^n e^{\pm x_i}}{1-q^n e^{\pm x_i}}$$

Proof.

$$\begin{aligned}\mathrm{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty}\Lambda_{q^n}TM_{\mathbb{C}}\right) &= \prod_{n=1}^{\infty}\mathrm{ch}(S_{q^n}TM_{\mathbb{C}})\ \mathrm{ch}(\Lambda_{q^n}TM_{\mathbb{C}}) \\ &= \prod_{n=1}^{\infty}\prod_{i=1}^{2k}\frac{1+q^n e^{\pm x_i}}{1-q^n e^{\pm x_i}}\end{aligned}$$

■

Theorem 3.8

For a $4k$ manifold

$$ch\left(\bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty}\Lambda_{-q^n}TM_{\mathbb{C}}\right) = \prod_{n=1}^{\infty}\prod_{k=1}^{2k}\left(\frac{1}{1-q^n e^{\pm x_i}}\right)^{(-1)^n}$$

Proof.

$$
\begin{aligned}
ch\left(\bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty} \Lambda_{-q^n}TM_{\mathbb{C}}\right) &= \prod_{\substack{n=1\\ n\ even}}^{\infty} \mathrm{ch}(S_{q^n}TM_{\mathbb{C}}) \cdot \prod_{\substack{n=1\\ n\ odd}}^{\infty} \mathrm{ch}(\Lambda_{-q^n}TM_{\mathbb{C}}) \\
&= \prod_{\substack{n=1\\ n\ even}}^{\infty} \prod_{i=1}^{2k} \frac{1}{1-q^n e^{\pm x_i}} \cdot \prod_{\substack{n=1\\ n\ odd}}^{\infty} \prod_{i=1}^{2k} (1-q^n e^{\pm x_i}) \\
&= \prod_{n=1}^{\infty} \prod_{i=1}^{2k} \left(\frac{1}{1-q^n e^{\pm x_i}}\right)^{(-1)^n}
\end{aligned}
$$

■

Chapter 4 : Multiplicative Genera and Index Theorems

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§4.1 Cobordism and genera

Definition 4.1.1 (Unoriented Cobordism) — Two $n-$dimensional manifolds M and M' are said to be unorientedly cobordant if there exists an $n+1-$dimensional manifold W such that $\partial W = M \sqcup M'$.

Definition 4.1.2 (Oriented Cobordism) — Two $n-$dimensional oriented manifolds M and M' are said to be orientedly cobordant if there exists a $n+1-$dimensional manifold W such that $\partial W = M \sqcup (-M')$ where the manifold $-M'$ is the same manifold as M but with opposite orientation.

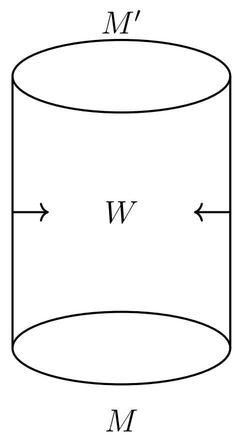

Figure 4.1: A cylinder is a cobordism between two circles.

Theorem 4.1 (Cobordism Ring)
Let Ω^n be the cobordism classes of n-dimensional, compact, oriented, differentiable manifolds. Then it is an abelian group with the addition being the disjoint union of manifolds. As a convention, we say that every manifold with boundary is cobordant to the empty set (which is also trivially a manifold) so that each cobordism group Ω^n has the same additive identity.
Let $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$. Then it is a ring with unity where the addition is the disjoint union, multiplication is the cartesian product and multiplicative identity is the class of singleton sets, which are $0-$dimensional manifolds.
Ω is known as the oriented cobordism ring, which has an obvious $\mathbb{Z}-$grading.

Theorem 4.2

$$\Omega \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \cdots]$$

Here tensoring with $\mathbb{Q}$ is done to kill the torsion subgroup.

See [GWY16] for details.

To know about cobordism theory in full generality, one can see [Sto15].

Definition 4.1.3 (Genus) — A genus is an algebra homomorphism $\varphi : \Omega \otimes \mathbb{Q} \to R$ where R is a $\mathbb{Q}-$algebra.

The following theorem is a trivial consequence of Stokes' theorem.

Theorem 4.3

The Pontrjagin numbers are cobordism invariants and hence can be used to define genus.

§4.2 Genera from multiplicative sequence

Let $Q(x)$ be an even power series with coefficients in R, starting with 1.
For indeterminates x_i of weight 2 with $1 \leq i \leq n$, $Q(x_1)Q(x_2)\cdots Q(x_n)$ is symmetric in x_i^2, and hence every weight $4r$ component of it can be expressed as a homogenous polynomial $K_r(p_1, p_2, \cdots p_r)$ of weight $4r$ in the elementary symmetric functions p_j of the x_i^2. Thus,

$$Q(x_1)Q(x_2)\cdots Q(x_n) = 1 + K_1(p_1) + K_2(p_1, p_2) + K_3(p_1, p_2, p_3) + \cdots$$

One should notice that the polynomials K_r do not depend on n, and they are called the multiplicative sequence of polynomials associated to the power series $Q(x)$.

Definition 4.2.1 (Genus corresponding to a power series) — The genus φ_Q corresponding to an even power series (starting with 1) Q is defined, for every compact, oriented, differentiable manifold M of dimension $4n$, by

$$\varphi_Q(M) = K_n(p_1, p_2, \cdots, p_n)[M] \in R$$

where p_i's are Pontrjagin classes of the manifold.
If the dimension of the manifold is not divisible by 4, the genus is set to be 0.

Theorem 4.4

Such a genus φ_Q is a well-defined genus. In fact, every genus can be obtained from such an even power series starting with 1.

The proof is not particularly illuminating for this thesis, and interested readers can see Chapter 3 of [Goe23] for this.

However, we will still describe the even power series starting with 1 that corresponds to a given particular genus (we have already seen the other direction). We will do so as it will be very useful in computing some genus. Let's have the following definition first.

Definition 4.2.2 (Logaritm of a genus) — Let $Q(x)$ denote the *even power series* corresponding to a genus

$$\varphi\colon \Omega^{SO} \otimes \mathbb{Q} \longrightarrow R,$$

The *logarithm* of φ is the formal power series $\log_\varphi(x)$ which is defined as the formal inverse (under composition) of the following function

$$f(x) = \frac{x}{Q(x)}.$$

Theorem 4.5

Let φ be a genus with corresponding even power series $Q(x)$. Its logarithm satisfies:

$$\log_\varphi(x) = x + \frac{\varphi([\mathbb{CP}^2])}{3}x^3 + \frac{\varphi([\mathbb{CP}^4])}{5}x^5 + \frac{\varphi([\mathbb{CP}^6])}{7}x^7 + \cdots = \sum_{n=0}^{\infty} \frac{\varphi([\mathbb{CP}^{2n}])}{2n+1}x^{2n+1}.$$

Now, if we think in the other way, given a genus φ (whose corresponding even power series is unknown to us a priori), we can have its value at the even complex dimensional projective spaces to get its logarithm, and then we can take the inverse of that logarithm which is just $\frac{x}{Q(x)}$. Thus, we can find the corresponding even power series.

Now we will see the application of logarithms to compute the genera defined in the following examples for complex projective spaces.

Example 4.2.1

The $\hat{A}$-genus corresponds to the power series given by

$$\widehat{A}(x) = \frac{x/2}{\sinh(x/2)} = 1 - \frac{1}{24}x^2 + \frac{7}{5760}x^4 - \frac{31}{967680}x^6 + \cdots$$

This genus is closely linked to the Dirac operator, assuming the manifold admits a spin structure.

Then we find its logarithm to be,

$$\log_{\widehat{A}}(x) = \left(\frac{x}{\frac{x/2}{\sinh(x/2)}}\right)^{-1} = (2\sinh(x/2))^{-1} = 2\sinh^{-1}(x/2) = \sum_{n=0}^{\infty} \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} \cdot \frac{1}{2n+1} y^{2n+1}$$

Thus we clearly have:

$$\hat{A}([\mathbb{CP}^{2n}]) = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}.$$

∎

Example 4.2.2

The L-genus corresponds to the power series given by

$$L(x) = \frac{x}{\tanh(x)} = 1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \frac{2}{945}x^6 + \cdots$$

This genus is linked to the *signature* of a manifold, which is a crucial topological invariant in the study of $4n$-dimensional manifolds.

Then we find its logarithm to be,

$$\log_L(x) = \left(\frac{x}{\frac{x}{\tanh(x)}}\right)^{-1} = \tanh^{-1}(x/2) = \sum_{n=0}^{\infty} \frac{1}{2n+1} y^{2n+1}$$

Hence, we have:

$$L[\mathbb{CP}^{2n}] = 1, \quad \forall n \in \mathbb{N}.$$

We will discuss more about these two genera in the next section, which will be important in our study of the elliptic genus.

§4.3 Index Theorems

Let $D : C^\infty(E) \to C^\infty(F)$ be an elliptic differential operator between smooth sections of vector bundles E and F over a compact manifold M. We will not bother about what ellipticity condition means for now as it is not much important for the pedagogical purpose now, however, interested reader can see Chapter 1 of [Bar11]. The **index** of D is defined as:

$$\operatorname{index}(D) = \dim \ker D - \dim \operatorname{coker} D.$$

Coker D can be understood as the kernel of the adjoint of the operator D.

Definition 4.3.1 (Signature of a manifold) — Let M be a closed, oriented, smooth $4k$-dimensional manifold. The cup product induces a symmetric bilinear form:

$$Q : H^{2k}(M, \mathbb{R}) \times H^{2k}(M, \mathbb{R}) \to \mathbb{R},$$

given by

$$Q(\alpha, \beta) = (\alpha \smile \beta)[M]$$

The **signature** of M, denoted by $sign(M)$, is the signature of this form (i.e., the number of positive eigenvalues minus the number of negative ones).
In fact, we do not lose any generality if we replace $H^{2k}(M, \mathbb{R})$ above by $H^{2k}(M, \mathbb{Z})$.

Example 4.3.1
Recall from Chapter 2 that:

$$H^*(\mathbb{C}P^n, \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^{n+1} \quad , \ \alpha \in H^2(\mathbb{C}P^n, \mathbb{Z})$$

$$\alpha^n[\mathbb{C}P^n] = 1$$

Let $n = 2k$ and hence $\mathbb{C}P^n$ is a complex $2k-$dimensional manifold, but its real dimension is $4k$.
As $H^{2k}(\mathbb{C}P^n, \mathbb{Z})$ is generated by single element α^k, thus its signature is completely

determined by whether $(\alpha^k \smile \alpha^k)[\mathbb{C}P^n]$ is positive or negative. But we have:

$$(\alpha^k \smile \alpha^k)[\mathbb{C}P^n] = \alpha^{2k}[\mathbb{C}P^n] = \alpha^n[\mathbb{C}P^n] = 1$$

So we have,

$$sign(\mathbb{C}P^n) = 1$$

Theorem 4.6

The **Hirzebruch Signature Theorem** states that:

$$sign(M) = L(M)$$

where $L(M)$ denotes the L-genus of the tangent bundle TM.

Proof. It is known that the signature of a manifold is cobordism invariant, multiplicative under the Cartesian product, and additive under the disjoint union (see [May99] for example). Thus, the signature is indeed a genus. As the oriented cobordism ring is given to be:

$$\Omega \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{C}P^2, \mathbb{C}P^4, \mathbb{C}P^6, \cdots]$$

Thus, to prove our theorem, it is enough to show that the signature and the $L-$genus coincide on complex even-dimensional projective spaces. But in our earlier examples in this chapter, we have already seen that:

$$\text{sign}(\mathbb{C}P^n) = 1 = L(\mathbb{C}P^n)$$

Hence, we are done! ■

Definition 4.3.2 (Twisted signature) — Suppose E is a complex vector bundle over a manifold M. We define the *twisted signature of M with coefficients in E* as

$$\text{sign}(M, E) := \left(\text{ch}(E) \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i/2)} \right) [M].$$

Twisted signature is related to the index of a twisted signature operator; hence, it will give us integer values. However, we will not discuss it here. To know about the signature operator, one can see Chapter 5 of [HBJ94].

Example 4.3.2

$$
\begin{aligned}
&\operatorname{sign}\left(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM_{\mathbb{C}}\right) \\
&= \left(\operatorname{ch}\left(\bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM_{\mathbb{C}}\right) \cdot \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i/2)}\right)[M] \\
&= \left(\prod_{n=1}^{\infty} \prod_{i=1}^{2k} \frac{1+q^n e^{\pm x_i}}{1-q^n e^{\pm x_i}} \cdot \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i/2)}\right)[M]
\end{aligned}
$$

This example will appear to be very crucial when we discuss the signature elliptic genus.

Now let us discuss some index theory in spin geometry. For now, we can assume that spin manifolds are some special type of manifolds with special bundle (called the spinor bundle) on it and a special operator (called the Dirac operator) defined on that bundle. This is enough for the moment, for some brief sketch of these notions, one can see Appendix A of this thesis, or even better if one goes through the references [Bar11, FO10].

Let M be a closed spin manifold and D^+ the Dirac operator. The celebrated **Atiyah–Singer Index Theorem** for the Dirac operator states:

$$
\operatorname{index}(D^+) = \widehat{A}(M)
$$

where $\widehat{A}(M)$ denotes the $\widehat{A}-$*genus* of M.

Definition 4.3.3 (Twisted $\widehat{A}-$genus) — Consider a complex vector bundle E defined over the manifold M. The twisted $\hat{A}$-genus of M is accordingly expressed as

$$
\hat{A}(M, E) := \left(\operatorname{ch}(E) \prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)}\right)[M].
$$

Twisted $\widehat{A}-$genus is related to an index of an operator called twisted Dirac operator on a spin manifold. Hence, it will be an integer too in case of a spin manifold.

Example 4.3.3

$$
\begin{aligned}
&\widehat{A}\left(M, \bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}}\right) \\
&= \left(\operatorname{ch}\left(\bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}}\right) \cdot \prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)}\right)[M] \\
&= \left(\prod_{n=1}^{\infty} \prod_{i=1}^{2k} \left(\frac{1}{1-q^n e^{\pm x_i}}\right)^{(-1)^n} \cdot \prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)}\right)[M]
\end{aligned}
$$

This example will appear to be very crucial when we discuss the $\widehat{A}-$elliptic genus.

Theorem 4.7

Let E be the complex extension of a real vector bundle over a compact, oriented, differentiable spin manifold M with dimension $8k+4$. Then,

$$\widehat{A}(M,E) \in 2\mathbb{Z}$$

See Appendix A for some sketch.

Chapter 5 : Modular Forms

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§5.1 Modular Forms

The space $\mathcal{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ is called the upper half-plane. There is an action of the group $SL_2(\mathbb{R})$ on the upper half-plane where the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on τ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}$$

We are especially interested in the action of the group $\Gamma = SL_2(\mathbb{Z})$. If we denote the point at infinity by $i\infty$, then the action on it is defined as:

$$\frac{a(i\infty) + b}{c(i\infty) + d} := \frac{a}{c}$$

$$\frac{a\left(\frac{-d}{c}\right) + b}{c\left(\frac{-d}{c}\right) + d} := i\infty$$

Γ acts on $\mathbb{Q} \cup \{i\infty\}$ in a similar way.

Definition 5.1.1 (Principal Congruence Subgroup) — Let $N \in \mathbb{N}$. Then the principal congruence subgroup of level N is defined by:

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$$

As $\Gamma(N)$ is the kernel of the natural projection $\pi : SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_N)$, this is a normal subgroup of Γ.

Definition 5.1.2 (Congruence subgroup) — Γ' is called a congruence subgroup of level N if $\Gamma(N) \subseteq \Gamma' \subseteq SL_2(\mathbb{Z})$.

Definition 5.1.3 (Cusps) — The cusps of a congruence subgroup Γ' is defined as the orbits of the action of Γ' on $\mathbb{Q} \cup \{i\infty\}$.

One should note that the action of Γ is transitive on $\mathbb{Q} \cup \{i\infty\}$, hence it has just one cusp. Without loss of generality, we call it a cusp at infinity (as $i\infty$ is representative element of the orbit). To denote a cusp, we will always choose a convenient representative like we have chosen cusp at infinity incase of the congruence group Γ.

Definition 5.1.4 (Modular Transformation) — Given a function f on upper half plane, its modular transformation of weight k on congruence subgroup Γ' is defined as follows for all $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$:

$$f[A]_k(\tau) := f(A\tau)(c\tau + d)^{-k}$$

Now suppose f is a function which is invariant under modular transformation of weight k on congruence subgroup Γ', i.e

$$f[A]_k(\tau) = f(\tau)$$

Note that $T_N = \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(N)$, so if f is invariant under modular transformation of weight k, we have:

$$f(T_N\tau)(0\tau + 1)^k = f(\tau + N) = f(\tau)$$

Therefore it is periodic with period N and has the following fourier expansion:

$$f(\tau) = \sum_{n=-\infty}^{\infty} a_n q^{n/N}$$

where $q = e^{2\pi i \tau}$.
Let $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$, and then we can expand $f = \sum a_n q_{n/N}$ to $\overline{\mathcal{H}}$ too. Note that the correspondence $\tau \to q = e^{2\pi i \tau}$ maps $\overline{\mathcal{H}}$ to the closed unit disk D in $\mathbb{C}$ with $\mathcal{H}$ maps to $D \backslash (S^1 \cup \{0\})$, $\mathbb{Q}$ maps to unit circle and $i\infty$ maps to 0.

Definition 5.1.5 — The function f is said to be holomorphic at $i\infty$ if all the negative fourier coefficients are 0.

Now let s be a cusp for some congruence subgroup Γ'. As action of Γ is transitive on $\mathbb{Q} \cup \{i\infty\}$, there is some $B \in \Gamma$, such that $B(i\infty) = s$.

Definition 5.1.6 (Holomorphicity at cusp) — A function is said to be holomorphic at cusp s of a congruence subgroup Γ' if $f[B]_k$ is holomorphic at $i\infty$ where $B(i\infty) = s$.

It can be shown with easy but tedious calculation that the holomorphicity does not depend on the choice of B. One can see Chapter 5 of [Des14] for details.

Definition 5.1.7 — A function is said to be a modular form of weight k on Γ' if it is invariant under modular transformation of weight k on congruence subgroup Γ' and is holomorphic at all of the cusps of Γ'.

The space of modular forms of weight k on Γ' is denoted as $M_k(\Gamma')$. It is a complex vector space under pointwise addition and scalar multiplication. The following space is a $\mathbb{Z}-$graded ring where the multiplication is pointwise multiplication:

$$M(\Gamma') = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma')$$

For the rest of the chapter, we will consider the following two congruence subgroups.

$$\Gamma_0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod 2 \right\}$$

$$\Gamma^0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \pmod 2 \right\}$$

Theorem 5.1

The congruence subgroup $\Gamma_0(2)$ has exactly two cusp, at $i\infty$ and 0. Also, $S(i\infty) = 0$ where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Proof. Note that $A(i\infty) = 0$ implies that

$$\frac{a(i\infty) + b}{c(i\infty) + d} = \frac{a}{c} = 0 \implies a = 0$$

But there is no matrix in $\Gamma_0(2)$ such that $a = 0$. Hence 0 and $i\infty$ belong to two different cusps.

Now for any rational $q = \frac{m}{n}$ with $\gcd(m, n) = 1$, Bézout's identity assures the existence of integers x, y such that

$$mx + my = 1.$$

Then if n is even (hence m and x must be odd). Consider the following matrix, which is in $\Gamma_0(2)$ then.

$$A = \begin{pmatrix} -x & -y \\ n & -m \end{pmatrix} \in \Gamma_0(2),$$

which satisfies

$$Aq = \frac{-x\left(\frac{m}{n}\right) - y}{n\left(\frac{m}{n}\right) - m} = \frac{\frac{mx+ny}{n}}{m - m} = i\infty.$$

On the other hand, if n is odd, then consider

$$A = \begin{pmatrix} n & -m \\ x & y \end{pmatrix},$$

Then,

$$Aq = \frac{n\left(\frac{m}{n}\right) - m}{x\left(\frac{m}{n}\right) + y = \frac{m-m}{\frac{mx+ny}{n}} = 0}$$

For the matrix A to be in $\Gamma_0(2)$, x need to be even. But we can take x to be even as the equation in the Bezout identity has all the solutions in the following form:

$$m(x + kn) + n(y - km) = 1, \quad \forall k \in \mathbb{Z},$$

Hence, we are done! ■

§5.2 Jacobi forms

Definition 5.2.1 — A meromorphic function f is elliptic if it is doubly periodic with respect to a lattice $L = \mathbb{Z}\tau + \mathbb{Z} \subset \mathbb{C}$ i.e if $(z - w) \in L$, then $f(z) = f(w)$.

Definition 5.2.2 (Jacobi Forms) — Let $\Psi(\tau, x)$ be an elliptic function, $k \in \mathbb{Z}$, $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup. Then Ψ is a jacobi form of weight k on Γ if

$$\Psi\left(\gamma(\tau), \frac{x}{c\tau + d}\right)(c\tau + d)^{-k} = \Psi(\tau, x) \qquad , \forall\, \gamma \in \Gamma$$

Example 5.2.1

Weierstrass $\wp$-function defined as following is a jacobi form of weight 2 on $\Gamma = SL_2(\mathbb{Z})$.

$$\wp(\tau, x) = \frac{1}{x^2} + \sum_{\substack{\omega \in 2\pi i(\mathbb{Z}\tau + \mathbb{Z}) \\ \omega \neq 0}} \left(\frac{1}{(x - \omega)^2} - \frac{1}{\omega^2}\right)$$

One can easily see that:

$$\wp(\tau, x) = \frac{1}{x^2} + O(x)$$

and it has poles only at lattice points and poles are of order 2.

Theorem 5.2

Let Ψ be a Jacobi form of weight k on Γ, $n \in \mathbb{Z}$, $\alpha, \beta \in \mathbb{R}$ and let $g_n(\tau)$ be the $n-$th coeffiecient of laurent expantion of Ψ at $2\pi i(\alpha\tau + \beta)$. Then we have,

$$g_n[\gamma]_{k+n} = g_n \qquad , \forall \gamma \in \Gamma \quad with \quad (\alpha, \beta)\gamma \equiv (\alpha, \beta) \quad mod \quad \mathbb{Z}^2$$

Proof. By Cauchy integral formula, we have,

$$g_n(\tau) = \frac{n!}{2\pi i} \oint \frac{\Psi(\tau, x + 2\pi i(\alpha\tau + \beta))}{x^{n+1}} dx$$

Then,

$$\begin{aligned}
g_n[\gamma]_{k+n} &= g_n(\gamma(\tau))(c\tau + d)^{-(k+n)} \\
&= \frac{n!}{2\pi i}(c\tau + d)^{-(k+n)} \oint \frac{\Psi(\gamma(\tau), x + 2\pi i(\alpha\gamma(\tau) + \beta))}{x^{n+1}} dx \\
&= \frac{n!}{2\pi i} \oint \frac{\Psi\left(\gamma(\tau), x + 2\pi i(\alpha\frac{a\tau + b}{c\tau + d} + \beta\right)(c\tau + d)^{-k}}{(x(c\tau + d))^{n+1}} d(x(c\tau + d))
\end{aligned}$$

Taking $y = x(c\tau + d)$ and $(\alpha', \beta') = (\alpha, \beta)\gamma$, and using the definition of Jacobi form, we have the following,

$$g_n[\gamma]_{k+n} = \frac{n!}{2\pi i} \oint \frac{\Psi(\tau, y + 2\pi i(\alpha'\tau + \beta'))}{y^{n+1}} dy$$

∎

Corollary 5.1

Let $e_1(\tau) = \wp(\tau, \pi i)$,$e_2(\tau) = \wp(\tau, \pi i\tau)$,$e_3(\tau) = \wp(\tau, \pi i(\tau+1)$. Then,

$$e_1 \in M_2(\Gamma_0(2)), \qquad e_2 \in M_2(\Gamma^0(2)), \qquad e_3 \in M_2\left(\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \Gamma_0(2) \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\right)$$

Moreover, we have,

$$e_1 \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_2 (\tau) = e_2$$

$$e_2 \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_2 (\tau) = e_1$$

$$e_3 \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_2 (\tau) = e_3$$

Theorem 5.3

Let $q = e^{2\pi i\tau}$. Then we have,

$$\wp(\tau, x) = \sum_{n\in\mathbb{Z}} \frac{1}{(q^{n/2}e^{x/2} - q^{-n/2}e^{-x/2})^2} - \left(-\frac{1}{12} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(q^{n/2} - q^{-n/2})^2}\right)$$

In the proof, we will use the idea that if two elliptic functions have identical zeros/poles and coefficients of the first term in Laurent series are equal, then they differ by a constant function. This is nothing but a consequence of Liouville's theorem. The idea is very recurrent in the theory of elliptic functions.

Proof. Note that the first summand on the RHS is periodic in x with respect to $2\pi i(\mathbb{Z}\tau + \mathbb{Z}$. Also, it has a pole at x iff for some $n \in \mathbb{Z}$, we have,

$$\begin{aligned} & q^{n/2}e^{x/2} - q^{-n/2}e^{-x/2} = 0 \\ \iff & q^{n/2}e^{x/2} = q^{-n/2}e^{-x/2} \\ \iff & e^{-x} = q^n = e^{2\pi i\tau n} \\ \iff & e^{2\pi i\tau n + x} = 1 = e^{2\pi i m} \\ \iff & x = 2\pi i(\tau n - m) \end{aligned}$$

Thus, the first summand has pole only at lattice points. By similar calculations, it can be easily seen that the derivative of $q^{n/2}e^{x/2} - q^{-n/2}e^{-x/2}$ is not zero at the lattice points, and so the pole is of order 2. Now let see its local behaviour at $x = 0$, Note that,

$$e^{x/2} - e^{-x/2} = (1 + \frac{x}{2} + \frac{x^2}{8} + \frac{x^3}{48} + \cdots) - (1 - \frac{x}{2} + \frac{x^2}{8} - \frac{x^3}{48} \cdots) = x + \frac{x^3}{24} + O(x^4)$$

Thus the local behaviour of the first summand is $\frac{1}{x^2}$. But we also have $\wp(\tau, x) = \frac{1}{x^2} + O(x)$.

Now note that,

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{(q^{n/2}e^{x/2}-q^{-n/2}e^{-x/2})^2} &= \frac{1}{e^{x/2}-e^{-x/2}} + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x) \\
&= \frac{1}{(x+x^3/24+O(x^4))^2} + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x) \\
&= \frac{1}{x^2}\cdot\frac{1}{(1+x^2/24+O(x^2))^2} + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x) \\
&= \frac{1}{x^2}\cdot\frac{1}{1+x^2/12+O(x^4)} + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x) \\
&= \frac{1}{x^2}(1-\frac{x^2}{12}+O(x^4)) + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x) \\
&= \frac{1}{x^2} - \frac{1}{12} + \sum_{n\neq 0} \frac{1}{(q^{n/2}-q^{-n/2})^2} + O(x)
\end{aligned}$$

Thus we must have,

$$\wp(\tau,x) = \sum_{n\in\mathbb{Z}} \frac{1}{(q^{n/2}e^{x/2}-q^{-n/2}e^{-x/2})^2} - \left(-\frac{1}{12} + \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1}{(q^{n/2}-q^{-n/2})^2}\right)$$

■

Corollary 5.2

For $|q| < min(|e^x|, |e^{-x}|)$, we have:

$$\wp(\tau,x) = \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n=1}^{\infty}\left(\sum_{d|n} d(e^{dx}+e^{-dx})\right)q^n + \frac{1}{12}\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)$$

Proof. We have the following result for $|a| < 1$,

$$\frac{1}{(a^{1/2}-a^{-1/2})^2} = \frac{a}{(1-a)^2} = \sum_{k=1}^{\infty} ka^k$$

Applying it to previous theorem, we have,

$$\begin{aligned}
\wp(\tau,x) &= \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n\neq 0} \frac{1}{(q^{n/2}e^{x/2}-q^{-n/2}e^{-x/2})^2} + \frac{1}{12} - \sum_{n\neq 0}\frac{1}{(q^{n/2}-q^{-n/2})^2} \\
&= \frac{1}{(e^{x/2}-e^{-x/2})^2} + \sum_{n>0}\sum_{d=1}^{\infty} d(e^{dx}+e^{-dx})q^{dn} + \frac{1}{12} - 2\sum_{d=1}^{\infty} dq^{dn} \\
&= \frac{1}{(e^{x/2}-e^{-x/2})^2} - e^{-x/2})^2 + \sum_{n=1}^{\infty}\left(\sum_{d|n} d(e^{dx}+e^{-dx})\right)q^n + \frac{1}{12}\left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)
\end{aligned}$$

■

Then evaluating $\wp$–function at $\pi i, \pi i\tau$ and $\pi i(\tau+1)$ with some algebraic manipulation, we have the following corollary.

Corollary 5.3

$$e_1(\tau) = \wp(\tau, \pi i) = -\frac{1}{6}\left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{odd}(n)q^n\right)$$
$$e_2(\tau) = \wp(\tau, \pi i\tau) = \frac{1}{12}\left(1 + 24\sum_{n=1}^{\infty}\sigma_1^{odd}(n)q^{n/2}\right)$$
$$e_3(\tau) = \wp(\tau, \pi i(\tau+1)) = \frac{1}{12}\left(1 + 24\sum_{n=1}^{\infty}(-1)^n\sigma_1^{odd}(n)q^{n/2}\right)$$

Definition 5.2.3 — We define the $\tilde{.}$ operator of weight k for a function $f(\tau, x)$ as follows:

$$\tilde{f}(\tau, x) = f\left(\frac{-1}{2\tau}, \frac{x}{2\tau}\right)(2\tau)^{-k}$$

If f is a function in τ only, then,

$$\tilde{f}(\tau) = f\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_k (2\tau) = f\left(\frac{-1}{2\tau}\right)(2\tau)^{-k}$$

Theorem 5.4

$\tilde{.}$ operator on f simply computes the exapansion at 0 cusp first and then composes it with $\tau \to 2\tau$. Also we have

$$f\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_k (\tau) \in M_k(\Gamma^0(2)) \quad \text{and} \quad \tilde{f}(\tau) \in M_k(\Gamma_0(2))$$

Corollary 5.4

We have the following modular forms:

$$\delta = -\frac{3}{2}e_1 = \frac{1}{4} + 6q + \cdots \in M_2(\Gamma_0(2))$$

$$\epsilon = (e_1 - e_2)(e_1 - e_3) = \frac{1}{16} - q + \cdots \in M_4(\Gamma_0(2))$$

$$\tilde{\delta} = -\frac{1}{8} - 3\sum_{n=1}^{\infty}\sigma_1^{odd}(n)q^n \in M_2(\Gamma_0(2))$$
$$\tilde{\epsilon} = q + \cdots \in M_4(\Gamma_0(2))$$

Theorem 5.5

$$M_*(\Gamma_0(2)) = \mathbb{C}[\delta, \epsilon] = \mathbb{C}[\tilde{\delta}, \tilde{\epsilon}]$$

The proof of the above theorem uses valence formula and it's a bit long which is not particularly illuminating for this thesis. Interested readers can see the proof at appendix I of [HBJ94].

§5.3 Theta Function

Definition 5.3.1 (Theta Function) — A meromorphic function f is said to be a Theta function with respect to lattice L if $\frac{d^2}{dx^2} log\ f(x)$ is elliptic with respect to L.

Example 5.3.1 (Trivial theta functions)
e^{ax^2+bx+c}

Example 5.3.2 (Weierstrass $\sigma-$ function)

$$\sigma(\tau, x) = x \prod_{\substack{\omega \in 2\pi i(\mathbb{Z}\tau+\mathbb{Z}) \\ \omega \neq 0}} \left(1 - \frac{x}{\omega}\right) exp\left(\frac{x}{\omega} + \frac{1}{2}\left(\frac{x}{\omega}\right)^2\right)$$

This is a theta function as $\frac{d^2}{dx^2} log\ \sigma = -\wp(\tau, x)$.

By simple algebraic manipulation, we can observe the following theorem.

Theorem 5.6
The following is true for all $\gamma \in SL_2(\mathbb{Z})$.

$$\sigma\left(\gamma(\tau), \frac{x}{c\tau + d}\right)(c\tau + d) = \sigma(\tau, x)$$

Thus σ is also a jacobi for of weight -1.

Theorem 5.7

$$\sigma(\tau, x) = exp\left(\frac{1}{2}G_2(\tau)x^2\right)\left(e^{x/2} - e^{-x/2}\right)\prod_{n=1}^{\infty}\frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2}$$

where,

$$G_2(\tau) = \sum_{(c,d)\neq(0,0)} \frac{1}{(2\pi i(c\tau + d))^2}$$

The proof of the above theorem can be done easily by applying the operator $\frac{d^2}{dx^2}$log operator to the right hand side along with the expansion $G_2(\tau) = -\frac{1}{12} + 2\sum_{n=1}^{\infty}\sigma_1(n)q^n$, and then comparing it with the expansion of $\wp(\tau, x)$ in corollary of the theorem 5.3.

Definition 5.3.2 (Φ−function) —

$$\begin{aligned}\Phi(\tau,x) &= \sigma(\tau,x)exp\left(-\frac{1}{2}G_2(\tau)x^2\right)\\ &= \left(e^{x/2}-e^{-x/2}\right)\prod_{n=1}^{\infty}\frac{(1-q^ne^x)(1-q^ne^{-x})}{(1-q^n)^2}\\ &= 2\sinh\left(\frac{x}{2}\right)\cdot\prod_{n=1}^{\infty}\frac{(1-q^ne^x)(1-q^ne^{-x})}{(1-q^n)^2}\end{aligned}$$

Note that Φ−function is also a theta function and it will very useful in our discussion of elliptic genera.

Theorem 5.8 (i) For $\gamma \in SL_2(\mathbb{Z})$, we have:

$$\Phi\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d) = exp\left(\frac{cx^2}{4\pi i(c\tau+d)}\right)\Phi(\tau,x)$$

(ii) For $\lambda,\mu \in \mathbb{Z}$, we have,

$$\Phi(\tau,x+2\pi i(\lambda\tau+\mu)) = q^{-\lambda^2/2}e^{-\lambda x}(-1)^{\lambda+\mu}\,\Phi(\tau,x)$$

Proof. (i) The first one is true as $\sigma(\tau,x)$ is a Jacobi form and the extra factor comes from $\exp(-\frac{1}{2}G_2(\tau)\cdot x^2)$ and the following pseudo modular transformation formula:

$$G_2[\gamma]_2(\tau) = G_2(\tau) - \frac{c}{2\pi i(c\tau+d)}.$$

Thus we have,

$$\begin{aligned}\Phi\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d) &= \sigma\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d)exp\left(-\frac{1}{2}G_2(\gamma\tau)\left(\frac{x}{c\tau+d}\right)^2\right)\\ &= \sigma(\tau,x)exp\left(-\frac{1}{2}G_2[\gamma]_2(\tau)(c\tau+d)^2\left(\frac{x}{c\tau+d}\right)^2\right)\\ &= \sigma(\tau,x)exp\left(-\frac{1}{2}\left(G_2(\tau)-\frac{c}{2\pi i(c\tau+d)}\right)(c\tau+d)^2\left(\frac{x}{c\tau+d}\right)^2\right)\\ &= \sigma(\tau,x)exp\left(-\frac{1}{2}G_2(\tau)x^2\right)exp\left(\frac{cx^2}{4\pi i(c\tau+d)}\right)\\ &= exp\left(\frac{cx^2}{4\pi i(c\tau+d)}\right)\Phi(\tau,x)\end{aligned}$$

(ii) For the 2nd part, proving for $\lambda > 0$ is sufficient as since changing left hand side to right hand side we get the formula for $(-\lambda,-\mu)$. First note that,

$$e^{x+2\pi i\cdot(\lambda\tau+\mu)} = e^{2\pi i\tau\lambda}e^{2\pi i\mu}e^x = q^\lambda(-1)^\mu e^x$$

Using this identity along with the definition of $\Phi-$function, we have:

$$
\begin{aligned}
\Phi(\tau, x+2\pi i\cdot(\lambda\tau+\mu)) &= (q^{\lambda/2}(-1)^{\mu}e^{x/2} - q^{-\lambda/2}(-1)^{\mu}e^{-x/2})\prod_{n=1}^{\infty}\frac{(1-q^{n+\lambda}e^{x})(1-q^{n-\lambda}e^{-x})}{(1-q^n)^2} \\
&= (-1)^{\mu}\cdot(-1)q^{-\lambda/2}e^{-x/2}(1-q^{\lambda}e^{x})\cdot\prod_{n=\lambda+1}^{\infty}(1-q^n e^x) \\
&\quad\cdot\prod_{n=1-\lambda}^{0}(-q^n e^{-x})(1-q^{-n}e^{x})\cdot\prod_{n=1}^{\infty}(1-q^n e^x)\cdot\prod_{n=1}^{\infty}(1-q^n)^{-2} \\
&= (-1)^{\mu}\cdot(-1)^{\lambda+1}q^{\lambda/2-\lambda(\lambda-1)/2}e^{-\lambda x}e^{-x/2}(1-e^x) \\
&\quad\cdot\prod_{n=1}^{\infty}\frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2} \\
&= q^{-\lambda^2/2}e^{-\lambda x}(-1)^{\lambda+\mu}(e^{x/2}-e^{-x/2})\cdot\prod_{n=1}^{\infty}\frac{(1-q^n e^x)(1-q^n e^{-x})}{(1-q^n)^2} \\
&= q^{-\lambda^2/2}e^{-\lambda x}(-1)^{\lambda+\mu}\cdot\Phi(\tau,x)
\end{aligned}
$$

■

Chapter 6 : Elliptic Genus

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§6.1 Elliptic Genus

Elliptic genera first originated with the motivation of computing the partition function of some infinite dimensional loop spaces. However, we won't take that historical approach, and even though our definitions are going to be a little bit artificial and unintuitive, yet it will provide us with some relative ease to introduce the notions with minimal prerequisites and develop the results necessary for this thesis work. We will focus on two important elliptic genera, namely signature elliptic and $\widehat{A}-$elliptic genus, that will play a crucial role in our proof of Ochanine's theorem as they assign modular form-based invariants to a manifold that encodes some number theoretic aspects. We have introduced the definition of these two elliptic genera slightly differently from most mathematical literature, as I find it a more intuitive way to introduce them; nevertheless, it will not have any effect on the necessary results we need.

There are many equivalent ways to define the elliptic genus. We take the following definition.

Definition 6.1.1 — A genus φ_Q is called an elliptic genus if $f(x) = \frac{x}{Q(x)}$ satisfies the following equation:

$$(f')^2 = 1 - 2\delta f^2 + \varepsilon f^4 \quad ; \ \delta, \varepsilon \in R$$

§6.2 Signature elliptic genus

Definition 6.2.1 — Signature elliptic genus of a closed oriented $4k-$manifold M is defined as following :

$$\varphi_1(M) := c * \operatorname{sign}\left(\bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM_{\mathbb{C}}\right)$$

where the constant c is chosen so that the power series defining the genus is normalized.

We will now see what the value of c should be. Recall from the last section of Chapter 4 that:

$$\operatorname{sign}\left(M, \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n} TM_{\mathbb{C}}\right) = \left(\prod_{n=1}^{\infty}\prod_{i=1}^{2k} \frac{1+q^n e^{x_i}}{1-q^n e^{x_i}} \cdot \prod_{i=1}^{2k} \frac{x_i}{\tanh(x_i/2)}\right)[M]$$

$$= 2^{2k}\left(\prod_{n=1}^{\infty}\prod_{i=1}^{2k} \frac{1+q^n e^{\pm x_i}}{1-q^n e^{\pm x_i}} \cdot \prod_{i=1}^{2k} \frac{x_i/2}{\tanh(x_i/2)}\right)[M]$$

Also recall that $\frac{x}{tanh\ (x/2)}$ already defines the $L-$genus and the power series starts with 1, therefore the power series of $\frac{x/2}{tanh\ (x/2)}$ also starts with 1. Hence to make $\varphi_1(M)$ a well defined genus, i.e to make its corresponding power series starting with 1, we must have the value of c to be the reciprocal of $2^{2k}\prod_{n=1}^{\infty}\prod_{i=1}^{2k}\frac{1+q^ne^{\pm x_i}}{1-q^ne^{\pm x_i}}$ evaluated at $x_i = 0$ for all i. Thus,

$$c = 2^{-2k}\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^{4k}$$

Then we have:

$$\varphi_1(M) = 2^{-2k}\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^{4k}\cdot \operatorname{sign}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}}\otimes\bigotimes_{n=1}^{\infty}\Lambda_{q^n}TM_{\mathbb{C}}\right)$$
$$\implies \varphi_1(M) = \left(\prod_{i=1}^{2k}\frac{x_i}{2}\cdot\frac{e^{x_i/2}+e^{-x_i/2}}{e^{x_i/2}-e^{-x_i/2}}\prod_{n=1}^{\infty}\frac{1+q^ne^{\pm x_i}}{1-q^ne^{\pm x_i}}\cdot\frac{(1-q^n)^2}{(1+q^n)^2}\right)[M]$$

Let $\varphi_1(\tau,x) = \frac{1}{2}\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}\prod_{n=1}^{\infty}\frac{(1+q^ne^x)\left(1+q^ne^{-x}\right)/(1+q^n)^2}{(1-q^ne^x)(1-q^ne^{-x})/(1-q^n)^2}$. Then the above genus can simply be written as:

$$\varphi_1(M) = \left(\prod_{i=1}^{2k}x_i\varphi_1(\tau,x_i)\right)[M]$$

Theorem 6.1 (i)

$$\varphi_1(\tau,x) = \frac{\Phi(\tau,x-\pi i)}{\Phi(\tau,x)\Phi(\tau,-\pi i)}$$

(ii)

$$\varphi_1\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d)^{-1} = \varphi_1(\tau,x)\quad\text{for all }\ \gamma\in\Gamma_0(2)$$

(iii)

$$\varphi_1(\tau,x)^2 = \wp(\tau,x)-e_1(\tau)$$

Proof. (i) This is simply comparing the expansion of φ_1 with the expansion of Φ in the definition of $\Phi-$function.

(ii) Recall from theorem 5.8 that:

$$\Phi\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d) = exp\left(\frac{cx^2}{4\pi i(c\tau+d)}\right)\Phi(\tau,x)$$

Now applying it to part (i) of this theorem, we have:

$$\varphi_1\left(\gamma\tau, \frac{x}{c\tau+d}\right)(c\tau+d)^{-1} = \frac{\Phi(\gamma\tau, \frac{x}{c\tau+d} - \pi i)}{\Phi(\gamma\tau, \frac{x}{c\tau+d})\Phi(\gamma\tau, -\pi i)} \cdot (c\tau+d)^{-1}$$

$$= \frac{exp\left(\frac{c(x-\pi i c\tau - \pi i d)^2}{4\pi i(c\tau+d)}\right)\Phi(\tau, x - \pi i c\tau - \pi i d)}{exp\left(\frac{cx^2}{4\pi i(c\tau+d)}\right)\Phi(\tau,x) exp\left(\frac{c(-\pi i c\tau - \pi i d)^2}{4\pi i(c\tau+d)}\right)\Phi(\tau, -\pi i c\tau - \pi i d)}$$

$$\cdot (c\tau+d)^{-1}\frac{(c\tau+d)^{-1}}{(c\tau+d)^{-1}\cdot(c\tau+d)^{-1}}$$

$$= exp\left(\frac{cx}{2}\right)\frac{\Phi(\tau, x-\pi i c\tau - \pi i d)}{\Phi(\tau,x)\Phi(\tau, -\pi i c\tau - \pi i d)}$$

As $\gamma \in \Gamma_0(2)$, there are integers λ and μ such that $c = 2\lambda$ and $d = 2\mu + 1$. Again recall from theorem 5.8 that:

$$\Phi(\tau, x + 2\pi i(\lambda\tau + \mu)) = q^{-\lambda^2/2}e^{-\lambda x}(-1)^{\lambda+\mu}\,\Phi(\tau, x)$$

Applying to the expression of φ_1, we have:

$$\begin{aligned}
\varphi_1\left(\gamma\tau, \frac{x}{c\tau+d}\right)(c\tau+d)^{-1} &= exp\left(\frac{cx}{2}\right)\frac{\Phi(\tau, x-\pi i c\tau - \pi i d)}{\Phi(\tau,x)\Phi(\tau, -\pi i c\tau - \pi i d)} \\
&= exp\left(\frac{2\lambda x}{2}\right)\frac{\Phi(\tau, x - \pi i + 2\pi i(-\lambda\tau - \mu))}{\Phi(\tau,x)\Phi(\tau, -\pi i + 2\pi i(-\lambda\tau-\mu))} \\
&= exp\left(\lambda x\right)\frac{q^{-\lambda^2/2}e^{\lambda(x-\pi i)}(-1)^{\lambda+\mu}\Phi(\tau, x-\pi i)}{\Phi(\tau,x)\Phi(\tau,-\pi i)q^{-\lambda^2/2}e^{\lambda(-\pi i)}(-1)^{\lambda+\mu}} \\
&= \frac{\Phi(\tau, x-\pi i)}{\Phi(\tau,x)\Phi(\tau,-\pi i)} \\
&= \varphi_1(\tau, x)
\end{aligned}$$

(iii) We will not prove this part in detail. For the sketch of the proof, recall that $e_1(\tau) = \wp(\tau, \pi i)$. With some algebraic manipulation, one can show that both $\varphi_1(\tau,x)^2$ and $\wp(\tau,x) - e_1(\tau)$ have zeros at $\pi i, \pi i(1+2\tau)$ and poles at $\pi i(2\tau), 0$. Also, both behave like $\frac{1}{x} + O(1)$ as $x \to 0$. Thus, they must be the same.

■

Theorem 6.2

For a $4k-$dimensional manifold M,

(i) $\varphi_1(M)$ indeed defines an elliptic genus.

(ii) $\varphi_1(M)$ is a modular form of weight $2k$ on $\Gamma_0(2)$

Proof. (i) Note that here $Q(x) = x\varphi_1(\tau, x)$. So it is enough to show that $f = \frac{x}{Q(x)} = \frac{1}{\varphi_1(x)} =$ satisfies the differential equation for elliptic genus i.e $(f')^2 = 1 - 2\delta f^2 + ef^4$.

From the last theorem, we have that, $\varphi_1(\tau,x)^2 = \wp(\tau,x) - \varepsilon_1(\tau)$. So, $f = \frac{1}{\sqrt{\wp(\tau,x)-e_1(\tau)}}$. Now define,

$$(u,v) = (\wp(x), \wp'(x)) \quad \text{and} \quad (w,z) = (f(x), f'(x)).$$

Then we have,

$$w = \frac{1}{\sqrt{u-e_1}} \qquad \Longleftrightarrow \qquad u = w^{-2} + e_1,$$

$$z = \frac{d}{dx}\left(\frac{\wp(x)-e_1}{2}\right)^{-\frac{1}{2}} = -\frac{\wp'(x)}{2\left(\wp(x)-e_1\right)^{\frac{3}{2}}} = -\frac{vw^3}{2} \Longleftrightarrow v = -2zw^{-3}.$$

Now with some algebraic manipulation, it can be shown that $\wp-$function satisfies the following differential equation:

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$$

Thus we have,

$$\begin{aligned} v^2 &= 4(u-e_1)(u-e_2)(u-e_3), \\ 4z^2w^{-6} &= 4(w^{-2})(w^{-2}+e_1-e_2)(w^{-2}+e_1-e_3). \end{aligned}$$

Multiplying both sides by $w^6/4$, and using $e_1+e_2+e_3=0$, we have,

$$\begin{aligned} z^2 &= (1+(e_1-e_2)w^2)(1+(e_1-e_3)w^2) \\ &= 1+(e_1-e_2+e_1-e_3)w^2+(e_1-e_2)(e_1-e_3)w^4 \\ &= 1+3e_1w^2+(1+e_1)(e_1-e_2)ew^4 \\ &= 1-2\delta w^2+\varepsilon w^4 \end{aligned}$$

Thus we get the differential equation for f by choosing $\delta = -\frac{3}{2}e_1$ and $\varepsilon = (e_1 - e_2)(e_1-e_3)$. Hence, φ_Q is indeed an elliptic genus.

(ii) Here we will omit the holomorphicity proofs and focus on the transformation rule only. Let

$$x\varphi_1(\tau,x) = Q_r(x) = \sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n}, \quad \text{where } a_0 = 1.$$

Then,

$$\varphi_1(M) = \left(\prod_{i=1}^{2k} Q_r(x_i)\right)[M] = \left(\prod_{i=1}^{2k}\left(\sum_{n=0}^{\infty} a_{2n}(\tau)x_i^{2n}\right).\right)[M]$$

It is then enough to prove that for all $n \in \mathbb{N}$, a_{2n} transforms as a modular form of weight $2n$ for $\Gamma_0(2)$, i.e.,

$$a_{2n}(\gamma\tau) = (c\tau+d)^{2n}a_{2n}(\tau), \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(2).$$

By using part (ii) of the last theorem, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} a_{2n}(\gamma\tau)(c\tau+d)^{-2n}x^{2n} &= \sum_{n=0}^{\infty} a_{2n}(\gamma\tau)\left(\frac{x}{c\tau+d}\right)^{2n} \\ &= \frac{x}{c\tau+d}\varphi(\gamma\tau, \frac{x}{c\tau+d}) \\ &= x\varphi(\tau, x) \\ &= \sum_{n=0}^{\infty} a_{2n}(\tau)x^{2n}\end{aligned}$$

Comparing coefficients on both sides, we conclude that,

$$a_{2n}(\gamma\tau) = (c\tau+d)^{2n}a_{2n}(\tau).$$

Thus, the signature elliptic genus indeed assigns a modular form to a manifold.

■

§6.3 $\widehat{A}-$elliptic genus

Definition 6.3.1 — $\widehat{A}-$elliptic genus of a closed oriented $4k-$manifold M is defined as following :

$$\varphi_2(M) := c \cdot \widehat{A}\left(\bigotimes_{\substack{n=1 \\ n \text{ even}}}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Lambda_{-q^n}TM_{\mathbb{C}}\right)$$

Here c is again a normalizing constant. Recall from the last section of Chapter 4 that:

$$\widehat{A}\left(M, \bigotimes_{\substack{n=1 \\ n \text{ even}}}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \Lambda_{-q^n}TM_{\mathbb{C}}\right) = \left(\prod_{n=1}^{\infty}\prod_{i=1}^{2k}\left(\frac{1}{1-q^n e^{\pm x_i}}\right)^{(-1)^n} \cdot \prod_{i=1}^{2k}\frac{x_i/2}{\sinh(x_i/2)}\right)[M]$$

Like in the approach of previous section, we can now show that:

$$c = \prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k}$$

Thus we have

$$\begin{aligned}\varphi_2(M) &= \prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k}\cdot \widehat{A}\left(\bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty}\Lambda_{-q^n}TM_{\mathbb{C}}\right)\\ &= \left(\prod_{i=1}^{2k}\frac{x_i}{e^{x_i/2}-e^{-x_i/2}}\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^2}{1-q^ne^{\pm x_i}}\right)^{(-1)^n}\right)[M]\end{aligned}$$

Let $\varphi_2(\tau,x) = \left(\prod_{i=1}^{2k}\frac{1}{e^{x_i/2}-e^{-x_i/2}}\prod_{n=1}^{\infty}\left(\frac{(1-q^n)^2}{1-q^ne^{\pm x_i}}\right)^{(-1)^n}\right)$. Then the above genus can simply be written as:

$$\varphi_2(M) = \left(\prod_{i=1}^{2k} x_i\varphi_2(\tau,x_i)\right)[M]$$

The following two results are just like the signature elliptic genus, hence we omit the proof.

Theorem 6.3 (i)

$$\varphi_2(\tau,x) = \frac{\Phi(\tau,x-\pi i\tau)}{\Phi(\tau,x)\Phi(\tau,-\pi i\tau)}$$

(ii)

$$\varphi_2\left(\gamma(\tau),\frac{x}{c\tau+d}\right)(c\tau+d)^{-1} = \varphi_2(\tau,x) \quad \text{for all} \quad \gamma\in\Gamma_0(2)$$

(iii)

$$\varphi_2(\tau,x)^2 = \wp(\tau,x)-e_2(\tau)$$

Theorem 6.4

For a $4k-$dimensional manifold M,

(i) $\varphi_2(M)$ indeed defines an elliptic genus.

(ii) $\varphi_2(M)$ is a modular form of weight $2k$ on $\Gamma_0(2)$

Recall the following definition and a theorem related to it from the Modular Forms chapter:

Definition 6.3.2 — We define the $\tilde{.}$ operator of weight k for a function $f(\tau,x)$ as follows:

$$\tilde{f}(\tau,x) = f\left(\frac{-1}{2\tau},\frac{x}{2\tau}\right)(2\tau)^{-k}$$

If f is a function in τ only, then,

$$\tilde{f}(\tau) = f\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_k (2\tau) = f\left(\frac{-1}{2\tau}\right)(2\tau)^{-k}$$

Theorem 6.5

$\tilde{.}$ operator on f simply computes the exapansion at 0 cusp first and then composes it with $\tau \to 2\tau$. Also we have

$$f\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_k (\tau) \in M_k(\Gamma^0(2)) \quad \text{and} \quad \tilde{f}(\tau) \in M_k(\Gamma_0(2))$$

We will now see that the signature elliptic genus and $\widehat{A}-$elliptic genus are related to the different cusp expansion of the same modular form.

Theorem 6.6

$$\tilde{\varphi_1}(\tau, x) = \varphi_1\left(\frac{-1}{2\tau}, \frac{x}{2\tau}\right)(2\tau)^{-1} = \varphi_2(\tau, x)$$
$$\varphi_2(M) = \tilde{\varphi_1}(M)$$

Proof. Recall that $\varphi_1(\tau, x)^2 = \wp(\tau, x) - e_1(\tau)$ and $\varphi_2(\tau, x)^2 = \wp(\tau, x) - e_2(\tau)$. Now,

$$\tilde{\wp}(\tau, x) - \tilde{e_2}(\tau) = \wp(\frac{-1}{\tau}, \frac{x}{\tau})\tau^{-2} - e_1\left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]_2 (\tau) = \wp(\tau, x) - e_2(\tau)$$

where the last equality is due to the corollary of theorem 5.2.

Thus we have,

$$\tilde{\varphi_1}(\tau, x)^2 = \varphi_2(\tau, x)^2$$

As both power series start with constant term 1, we must have:

$$\tilde{\varphi_1}(\tau, x) = \varphi_2(\tau, x)$$

And consequently,

$$\varphi_2(M) = \tilde{\varphi_1}(M)$$

■

Chapter 7 : Ochanine's Theorem

Theorem 7.1 (Ochanine)
The signature of a smooth closed spin manifold of dimension $8k+4$ is divisible by 16.

Proof. Recall that we have the following expression for $\widehat{A}-$elliptic genus:

$$\begin{aligned}\varphi_2(M) &= \prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k} \cdot \widehat{A}\left(\bigotimes_{\substack{n=1\\ n\ even}}^{\infty} S_{q^n} TM_{\mathbb{C}} \otimes \bigotimes_{\substack{n=1\\ n\ odd}}^{\infty} \Lambda_{-q^n} TM_{\mathbb{C}}\right)\\ &= \prod_{n=1}^{\infty}(1-q^n)^{(-1)^n 4k} \cdot \left(\widehat{A}(M) - q\ \widehat{A}(M, TM_{\mathbb{C}}) + \cdots\right)\end{aligned}$$

As M is a spin manifold of dimension $8k+4$, all the twisted $\widehat{A}-$genera are even integers, hence the coefficients in the $q-$expansion of the $\widehat{A}-$elliptic genus are even integers.

As $\varphi_2(M)$ is a modular form of weight $2k$ on $\Gamma_0(2)$, it can be expressed as a homogenous polynomial of weight $2k$ in $8\tilde{\delta}$ and $\tilde{\epsilon}$.

$$\begin{aligned}\varphi_2(M) &= P(8\tilde{\delta}, \tilde{\epsilon})\\ &= \sum_{\substack{a,b\in\mathbb{N}\\ 2a+4b=2k}} c_{a,b}(8\tilde{\delta})^a(\tilde{\epsilon})^b\\ &= \sum_{\substack{a,b\in\mathbb{N}\\ 2a+4b=2k}} c_{a,b}\left(-1-24\sum_{n=1}^{\infty}\sigma_1^{odd}(n)q^n\right)^a (q+\cdots)^b\\ &= \sum_{\substack{a,b\in\mathbb{N}\\ 2a+4b=2k}} c_{a,b}\left((-1)^a q^b + \text{ higher order terms}\right)\end{aligned}$$

As the coefficients in the $q-$expansion of the $\widehat{A}-$elliptic genus are even integers, it can be easily seen by induction that $c'_{a,b}$s are even integers too.

Now as the signature elliptic genus $\varphi_1(M)$ is also a modular form of weight $2k$ on $\Gamma_0(2)$, we can write it as:

$$\varphi_1(M) = \sum_{\substack{a,b\in\mathbb{N}\\ 2a+4b=2k}} k_{a,b}(8\delta)^a(\epsilon)^b$$

Now we have,

$$
\begin{aligned}
\varphi_2(M) &= \tilde{\varphi}_1(M) \\
&= (2\tau)^{-2k}\varphi_1(M)\left(\frac{-1}{2\tau}\right) \\
&= (2\tau)^{-2k} \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}\left(8\delta\left(\frac{-1}{2\tau}\right)\right)^a \left(\epsilon\left(\frac{-1}{2\tau}\right)\right)^b \\
&= \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}\left(8(2\tau)^{-2}\delta\left(\frac{-1}{2\tau}\right)\right)^a \left((2\tau)^{-4}\epsilon\left(\frac{-1}{2\tau}\right)\right)^b \\
&= \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}\left(8\tilde{\delta}\right)^a (\tilde{\epsilon})^b
\end{aligned}
$$

Thus we have $k_{a,b} = c_{a.b}$ and $k'_{a,b}$s are even integers too.

Now,

$$
\begin{aligned}
\varphi_1(M) &= 2^{-2k}\prod_{n=1}^{\infty}\left(\frac{1-q^n}{1+q^n}\right)^{4k} \cdot \operatorname{sign}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}}\right) \\
\implies \operatorname{sign}\left(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}}\right) &= 2^{2k}\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^{4k} \cdot \varphi_1(M) \\
\implies \operatorname{sign}(M) + 2q\cdot\operatorname{sign}(M, TM_{\mathbb{C}}) + \cdots &= 2^{2k}\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^{4k} \cdot \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}(8\delta)^a(\epsilon)^b \\
&= \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^{4k} \cdot \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}\ 2^{2a+4b}\ (8\delta)^a(\epsilon)^b \\
&= \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^{4k} \cdot \sum_{\substack{a,b\in\mathbb{N}\\2a+4b=2k}} k_{a,b}(32\delta)^a(16\epsilon)^b \\
&= \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^{4k} \cdot P(32\delta, 16\epsilon)
\end{aligned}
$$

As $P(\delta,\epsilon)$ is a weight $4k+2$ modular form,, we have $\delta | P(\delta,\epsilon)$ and as we also have $k'_{a,b}$s are even, thus we get,

$$64\delta \mid P(32\delta, 16\epsilon)$$

But,

$$64\delta = 64\left(\frac{1}{4} + 6q + \cdots\right) = 16\left(1 + 24q + \cdots\right)$$

Thus coefficients of sign $(\bigotimes_{n=1}^{\infty} S_{q^n}TM_{\mathbb{C}} \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}TM_{\mathbb{C}})$ is divisible by 16
and in particular $16 \mid$ sign (M) .

■

Chapter A : Spin Geometry Overview

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We require the concepts in Chapter 2 of this thesis as the minimal set of prerequisites for this chapter. Also, for the final section in this chapter where we discussed the index of a Dirac operator, we need the ideas of Characteristics classes and genera from Chapters 3 and 4, respectively.

§A.1 Clifford Algebra

Definition A.1.1 (Clifford Algebra) — Let V be a vector space with β being a symmetric bilinear form on it. Then Clifford Algebra $Cl(V,\beta)$ is defined as $T(V)/I$ where $T(V)$ is the tensor algebra on V and I is the ideal in $T(V)$ generated by the following elements:

$$v \otimes w + w \otimes v + 2\beta(v,w) \cdot 1 \quad , \; v, w \in V$$

The equation $v \otimes w + w \otimes v = -2\beta(v,w)$ is said to be a clifford relation.

Note that V is naturally embedded in $Cl(V,\beta)$. If $V = \mathbb{R}^n$ with bilinear form being the standard Euclidean inner product, then we denote $Cl(V,\beta)$ as Cl_n.

Theorem A.1

$Cl(V,\beta)$ is $\mathbb{Z}_2-$graded algebra with the grading $Cl^0(V,\beta) = T^{even}(V)/I$ and $Cl^1(V,\beta) = T^{odd}(V)/I$.

Theorem A.2

$$dim(Cl^0(V,\beta)) = dim(Cl^1(V,\beta)) = \frac{dim(Cl(V,\beta))}{2} = 2^{dim(V)-1}$$

Here, the basis elements of $Cl(V,\beta)$ take the form $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n = \dim(V)$ where $\{v_i\}_{i=1}^n$ forms a bsis of V. Thus, there are exactly 2^n basis elements of $Cl(V,\beta)$.

§A.2 Spin group

Let $v \in \mathbb{R}^n \backslash \{0\} \subset Cl_n \backslash \{0\}$. Then from clifford relation, we have that,

$$v^2 = -|v|^2 \cdot 1$$

$$\implies -\frac{v}{|v|^2} \cdot v = v \cdot \left(-\frac{v}{|v|^2}\right) = 1$$

Hence $\mathbb{R}^n \backslash \{0\}$ is contained in the subgroup of multiplicatively invertible elements in Cl_n.

Definition A.2.1 (Pin group) —

$$Pin(n) = \{v_1 \cdot v_2 \cdot \; \cdots \; \cdot v_m \in Cl_n \mid v_k \in \mathbb{S}^{n-1} \subset \mathbb{R}^n\}$$

Definition A.2.2 (Spin group) —

$$Spin(n) = Pin(n) \cap C\ell_n^0$$

For a fixed $v \in S^{n-1} \subset \mathbb{R}^n$ and any $x \in \mathbb{R}^n$, we have:

$$v \cdot x \cdot v^{-1} = -v \cdot x \cdot v = -(-x \cdot v - 2\langle x, v\rangle) \cdot v = -(x - 2\langle x, v\rangle v).$$

The map

$$x \mapsto x - 2\langle x, v\rangle v$$

is the reflection about the hyperplane $v^\perp$ perpendicular to v. In particular,

$$(x \mapsto v \cdot x \cdot v^{-1}) \in O(n).$$

For any

$$a = v_1 \cdots v_m \in \mathrm{Spin}(n),$$

the map

$$x \mapsto a \cdot x \cdot a^{-1} = v_1 \cdots v_m \cdot x \cdot v_m^{-1} \cdots v_1^{-1}$$

consists of an even number of hyperplane reflections and is thus contained in $\mathrm{SO}(n)$. We have thus defined a group homomorphism

$$\varrho : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

by

$$\varrho(a) := x \mapsto a \cdot x \cdot a^{-1}. \tag{2.3}$$

The kernel of this map is $\{1, -1\}$.

Theorem A.3

$Spin(n)$ is the double cover of $SO(n)$ i.e the following sequence is exact:

$$1 \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to 1$$

§A.3 Some motivation

Suppose we have the following Laplace operator:

$$\Delta f = -\left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2}\right) = -\sum \frac{\partial^2 f}{\partial x_k^2}$$

This is a 2nd order differential operator. Now suppose we want to find a 1st order differential operator $Df = c_1 \frac{\partial f}{\partial x_1} + c_2 \frac{\partial f}{\partial x_2} + \cdots + c_n \frac{\partial f}{\partial x_n} = \sum c_k \frac{\partial f}{\partial x_k}$ (c_k's are some constants) such that if we compose the operator with itself, we get the Laplace operator i.e $D^2 = \Delta$. Then we must have,

$$-\sum_k \frac{\partial^2 f}{\partial x_k^2} = \sum_m c_m \frac{\partial}{\partial x_m}\left(\sum_n c_n \frac{\partial f}{\partial x_n}\right)$$
$$= \sum_{m,n} c_m c_n \frac{\partial^2 f}{\partial x_m \partial x_n}$$

Comparing term by terms, we have that:

$$c_m c_n = \begin{cases} -1 & \text{if } \ m = n \\ 0 & \text{otherwise} \end{cases}$$

Note that the above equation is just Clifford relation for the standard inner product on $\mathbb{R}^n$. Thus $c_k's$ must take values in Cl_n.

The operator D is called the Dirac operator for $\mathbb{R}^n$. The reason we care about this operator is that sometimes a system satisfies Laplace equation, however, the solution which we are interested in might depend on a single initial condition only. Hence, Laplace operator is not a good choice then as the solution would require two initial conditions in that case. Thus it is reasonable to seek for a 1st order differential operator which act as a square root of the Laplace operator.

Now in practice, it is a good idea to represent the Clifford algebra elements with the elements of a matrix algebra as we know and can handle matrices very well. And for that purpose, we would love to find such matrices for which the dimension is the smallest. And when we ask for a smallest dimensional representation of an algebra or a group, we are automatically bound to look for irreducible representations.

Now if an algebra is of the form $End(V)$ or $End(V) \oplus End(V)$ where V is a finite dimensional vector space, it turns out that it has only one non-trivial irreducible representation up to isomorphism and that representation is V itself.

It turns out that Clifford algebra $\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ is also of this form, and hence it has only one non-trivial irreducible representation which we denote as Σ_n and it is called the spinor space of dimension n. Infact, this spinor space is completely embedded in the Clifford Algebra $\mathbb{C}l_n$ and there is an action of the spin group on this spinor space. We will discuss what it looks like in the next section.

There is another very good reason why we care about representation of Clifford algebra or spin group. There is a notion called principal $G-$bundle (which we will discuss later) where G is a **Lie group** (i.e a group which is also a manifold and the maps defined by multiplication and inverse are smooth), and the fibre of this bundle are isomorphic to G. Given a principal $G-$bundle over a manifold M and representation of the group G, we can construct an associate vector bundle to it. In particular, we would be interested in a principal $spin(n)$ bundle and with the representation of the spin group on the spinor space, we can construct a vector bundle called spinor bundle. This spinor bundle will allow us to extend the Dirac operator to manifolds which will be the square root of Laplace operator on manifolds. Infact, Dirac operator on manifolds also locally look like the Dirac operator on Euclidean space.

§A.4 Spinors

We first handle the even dimensional case.

Let $n = 2m$. Consider Cl_n, the Clifford algebra of $\mathbb{R}^n$ equipped with the standard Euclidean inner product, and let $\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification.

Let $e_1, \dots, e_{2m}$ denote the standard basis for $\mathbb{R}^n$. Define, for $j = 1, \dots, m$:

$$z_j = \frac{1}{2}(e_{2j-1} - ie_{2j}) \in \mathbb{C}l_n, \qquad \bar{z}_j = \frac{1}{2}(e_{2j-1} + ie_{2j}) \in \mathbb{C}l_n.$$

The monomials of the form

$$z_{j_1} \cdots z_{j_k} \bar{z}_{i_1} \cdots \bar{z}_{i_l}, \quad \text{with } k, l \geq 0, \quad 1 \leq j_1 < \cdots < j_k \leq m, \quad 1 \leq i_1 < \cdots < i_l \leq m$$

form a basis of $\mathrm{Cl}_{n,\mathbb{C}}$ as a complex vector space.

Let

$$z_{(j_1,\dots,j_k)} := z_{j_1} \cdots z_{j_k} \bar{z}_1 \cdots \bar{z}_m$$

and define

$$\Sigma_n := \operatorname{span}\left\{ z_{(j_1,\dots,j_k)} \;\middle|\; k = 0, \dots, m,\ 1 \leq j_1 < \cdots < j_k \leq m \right\} \subseteq \mathbb{C}l_n.$$

Definition A.4.1 (Spinor Space) — The subspace $\Sigma_n \subset \mathbb{C}l_n$ is a complex vector subspace of dimension 2^m, and is called the **spinor space** for dimension n. Its elements are called **spinors**.

Clifford multiplication by vectors in $\mathbb{R}^n$ preserves Σ_n, the space Σ_n forms a left module over Cl_n. In particular, Σ_n is stable under the action of the group $\mathrm{Spin}(n)$.

We define the even and odd parts:

$$\Sigma_n^+ := \operatorname{span}\{z_{(j_1,\dots,j_k)} \mid k \text{ even}\}, \quad \Sigma_n^- := \operatorname{span}\{z_{(j_1,\dots,j_k)} \mid k \text{ odd}\}.$$

Hence, the spinor space decomposes as

$$\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-.$$

Elements in $\Sigma_n^\pm$ are called spinors of **positive** and **negative chirality**, respectively.

The relations

$$\mathbb{R}^n \cdot \Sigma_n^+ \subseteq \Sigma_n^-, \quad \mathbb{R}^n \cdot \Sigma_n^- \subseteq \Sigma_n^+$$

show that Clifford multiplication by vectors of $\mathbb{R}^n$ switches chirality.

Clifford multiplication by elements of the even subalgebra Cl_n^0 preserves chirality:

$$\mathrm{Cl}_n^0 \cdot \Sigma_n^+ \subseteq \Sigma_n^+, \quad \mathrm{Cl}_n^0 \cdot \Sigma_n^- \subseteq \Sigma_n^-.$$

Hence, the restriction of the Cl_n^0-action to $\mathrm{Spin}(n) \subset \mathrm{Cl}_n^0$ gives rise to representations on $\Sigma_n^\pm$, and thus on Σ_n.

Definition A.4.2 — The map $\sigma_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n)$ (which is just multiplication by the spin group element) is called the **spinor representation**.

The representations $\sigma_n^\pm : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n^\pm)$ are called the **positive and negative spinor representations**, respectively.

Theorem A.4

Let $n = 2m$ be even. Then the map

$$\Phi : \mathrm{Cl}_n \longrightarrow \mathrm{End}(\Sigma_n), \quad \Phi(X)(z) := X \cdot z$$

is an isomorphism of complex algebras.

We now check the odd dimensional case.

Let $n = 2m - 1$. To define the spinor space Σ_n in odd dimensions, we make the following observation:

Theorem A.5

Let $n \in \mathbb{N}$. Define the linear map

$$j : \mathbb{R}^n \longrightarrow \mathrm{Cl}_{n+1}^0, \quad X \mapsto j(X) := X \cdot e_{n+1}.$$

This map induces an algebra isomorphism:

$$\alpha : \mathrm{Cl}_n \to \mathrm{Cl}_{n+1}^0.$$

For odd n, we define the spinor space Σ_n by:

$$\Sigma_n := \Sigma_{n+1}^+.$$

In both even and odd dimensions, the dimension of the spinor space is $\dim \Sigma_n = 2^{\lfloor n/2 \rfloor}$. The Clifford algebra Cl_n acts on Σ_n via:

$$X \cdot \phi := \alpha(X) \cdot \phi, \quad \text{for } X \in \mathbb{R}^n, \ \phi \in \Sigma_n,$$

where $\alpha : \mathbb{R}^n \to \mathrm{Cl}_{n+1}^0 \subset \mathrm{Cl}_{n+1}$ is the embedding.

The restriction of this action to $\mathrm{Spin}(n) \subset \mathrm{Cl}_n$ defines the **spinor representation**:

$$\sigma_n : \mathrm{Spin}(n) \to \mathrm{GL}(\Sigma_n).$$

§A.5 Principal Bundle

Definition A.5.1 — A *Lie group* is a group $(G, \cdot)$ which is also a smooth manifold and the maps

$$\mu : G \times G \to G, \quad (g_1, g_2) \mapsto g_1 \cdot g_2$$

and

$$i : G \to G, \quad g \mapsto g^{-1}$$

are both smooth.

Definition A.5.2 — Let $(G, \cdot)$ be a Lie group and let $g \in G$. The map

$$\ell_g : G \to G, \quad h \mapsto \ell_g(h) := g \cdot h \equiv gh$$

is called the *left translation* by g.

Definition A.5.3 — Let G be a Lie group. A vector field $X \in \Gamma(TG)$ is said to be *left-invariant* if

$$\forall g \in G : \quad (\ell_g)_*(X) = X.$$

The set of all left-invariant vector fields on G is denoted by $\mathcal{L}(G)$. Definitely we have,

$$\mathcal{L}(G) \subseteq \Gamma(TG).$$

Theorem A.6 (13.4)
Let G be a Lie group with identity element $e \in G$. Then

$$\mathcal{L}(G) \cong_{\text{vec}} T_eG.$$

Definition A.5.4 — A *Lie algebra* A is an algebra whose product $[\cdot, \cdot]$ is called the *Lie bracket*, which satisfies:

i) antisymmetry:

$$\forall v, w \in A : \quad [v, w] = -[w, v];$$

ii) the Jacobi identity:

$$\forall v, w, z \in A : \quad [v, [w, z]] + [w, [z, v]] + [z, [v, w]] = 0.$$

Example A.5.1
Both $\Gamma(TG)$ and $\mathcal{L}(G)$ has a Lie algebra structure with the Lie bracket being the commutator bracket $[X, Y](f) = X(Y(f)) - Y(X(f))$ where $f \in C^\infty(G)$.

Definition A.5.5 — Let G be a Lie group. The *associated Lie algebra* of G is $\mathcal{L}(G)$. From now on, we will denote this Lie Algebra simply by $\mathfrak{g}$.

Theorem A.7

$$\mathfrak{spin}(n) \cong \mathfrak{so}(n)$$

where $\mathfrak{spin}(n)$ and $\mathfrak{so}(n)$ are the associated Lie algebras of $Spin(n)$ and $SO(n)$ respectively.

Two smooth bundles (E, π, M) and (E', π', M') are isomorphic if there exist diffeomorphisms u, f such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$

Definition A.5.6 — Let G be a Lie group. A smooth bundle (E, π, M) is called a *principal G-bundle* if E is equipped with a free right G-action and

$$\begin{tikzcd} E \arrow[r, "G"] \arrow[d, "\pi"'] & E \arrow[d, "\rho"] \\ M \arrow[r, "\simeq_{\text{diffeo}}"] & E/G \end{tikzcd}$$

where ρ is the quotient map, defined by sending each $p \in E$ to its equivalence class (i.e., orbit) in the orbit space E/G.

Example A.5.2 (Frame bundle)
Given an orientable vector bundle $E \to M$, we can consider a principal $SO(n)$ bundle $P_{SO} \to M$ where the fibres are just all positively oriented orthonormal bases of the vector space in the corresponding fibre of $E \to M$. We call it a frame bundle for E.

We can analogously construct a principal G bundle if the structure group of the vector bundle is G.

We can also correspond a vector bundle for a given principal bundle, which we discuss below.

Definition A.5.7 (Associated vector bundle) — Let (P, π, M) be a principal G-bundle and let F be a smooth manifold equipped with a left G-action $\triangleright$. We define:

i) $P_F := (P \times F)/{\sim_G} = P \times_G V$, where $\sim_G$ is the equivalence relation

$$(p, f) \sim_G (p', f') \iff \exists g \in G : \begin{cases} p' = p \cdot g \\ f' = g^{-1} \triangleright f \end{cases}$$

We denote the points of P_F as $[p, f]$.

ii) The map

$$\pi_F : P_F \to M, \quad [p, f] \mapsto \pi(p),$$

which is well-defined since, if $[p', f'] = [p, f]$, then for some $g \in G$

$$\pi_F([p', f']) = \pi_F([p \cdot g, g^{-1} \triangleright f]) = \pi(p \cdot g) = \pi(p) = \pi([p, f]).$$

The *associated bundle* (to (P, π, M), F and $\triangleright$) is the bundle (P_F, π_F, M).

Let $\pi : P \to M$ be a principal G-bundle with structure group G, and consider the associated vector bundle $E = P \times_G V$, where G acts on a vector space V via a representation $\rho : G \to \mathrm{GL}(V)$.

A **connection 1-form** on P is a Lie algebra-valued 1-form $\omega \in \Omega^1(P, \mathfrak{g})$, where $\mathfrak{g}$ denotes the Lie algebra of G. This form satisfies certain natural conditions that make it compatible with the group action and defines a horizontal distribution.

To define a covariant derivative on the associated bundle E, we use this connection 1-form ω. Let $s : U \to P$ be a local section of the principal bundle, and let $\phi : U \to V$ be a local section of the associated bundle E. Then the section ϕ can be written locally in terms of s and ϕ as:

$$\phi(x) = [s(x), v(x)],$$

where $v : U \to V$ is a vector-valued function.

The **covariant derivative** of ϕ along a vector field X on M is then given by:

$$\nabla_X \phi = [s, d_X v + \rho_*(\omega(s_*(X)))v],$$

where $d_X v$ denotes the ordinary derivative of $v(x)$ in the direction of X, and $\rho_*(\omega(s_*(X)))$ is the action of the Lie algebra (ρ_* is just the pushforward map of ρ) element $\omega(s_*(X))$ on v through the differential of the representation ρ.

This expression defines the covariant derivative on the associated bundle using the connection form ω on the principal bundle P.

Now let's find a connection $1-$form on P_{SO} given an orientable vector bundle $E \to M$. Recall the following theorem from Chapter 2.

Theorem A.8

A connection ∇ on the trivialisation $E|_U$ is uniquely determined by $\nabla(s_1), \ldots, \nabla(s_n)$ (here $s_i \in \Gamma(E|_U)$ for $i = 1, 2, \cdots, n$ forms a local frame). Each section $\nabla(s_i)$ can be uniquely expressed as a sum

$$\sum_j \omega_{ij} \otimes s_j$$

where $[\omega_{ij}]$ is an $n \times n$ matrix of smooth complex 1-forms on U.

If we let s be the vector consisting of orthonormal basis $[s_1 \; s_2 \; \cdots \; s_n]$ and let w be the $1-$form valued matrix (w_{ij}), then we can define a connection on P_{SO} by:

$$\nabla s = ws$$

Thus, this w can be considered as the Lie algebra (which is, after all, an endomorphism/matrix in this case) valued $1-$form on P_{SO}.

§A.6 Spin Structure on manifolds

Definition A.6.1 (Spin Structure) — Let M be an oriented Riemannian manifold. A *spin structure* on M is a pair $(P_{\text{Spin}}(M), \varphi)$ consisting of:

- a principal $\text{Spin}(n)$-bundle $P_{\text{Spin}}(M)$ over M,
- a twofold covering map $\varphi : P_{\text{Spin}}(M) \to P_{\text{SO}}(M)$, where $P_{\text{SO}}(M)$ is the oriented orthonormal frame bundle and φ also being a bundle morphism,

such that the following diagram commutes:

$$\begin{tikzcd}
P_{\text{Spin}}(M) \times \text{Spin}(n) \arrow[r] \arrow[d, "\varphi\times\rho"] & P_{\text{Spin}}(M) \arrow[d, "\varphi"] \\
P_{\text{SO}}(M) \times \text{SO}(n) \arrow[r] & P_{\text{SO}}(M)
\end{tikzcd}$$

Here, $\rho : \text{Spin}(n) \to \text{SO}(n)$ denotes the standard 2-to-1 covering.

Definition A.6.2 (Spin Manifold) — An oriented Riemannian manifold equipped with a spin structure is called a *spin manifold*.

A manifold is called *spinnable* if it admits a spin structure.

Theorem A.9

A manifold is spinnable iff its first two Stiefel Whitney Classes vanish.

Definition A.6.3 (Spinor Bundle) — Let $\sigma_n : \text{Spin}(n) \to \text{GL}(\Sigma_n)$ be the spinor representation. Given a spin manifold M of dimension n with spin structure $P_{\text{Spin}}(M)$, define the *spinor bundle* of M as:

$$\Sigma M := P_{\text{Spin}}(M) \times_{\sigma_n} \Sigma_n.$$

This is just an associated vector bundle, which we discussed in the principal bundle section.

Sections of ΣM are called *spinor fields* on M.

If n is even and $\sigma_n^{\pm} : \text{Spin}(n) \to \text{GL}(\Sigma_n^{\pm})$ denote the positive and negative spinor representations, then we define:

$$\Sigma^{\pm} M := P_{\text{Spin}}(M) \times_{\sigma_n^{\pm}} \Sigma_n^{\pm},$$

which are called the *positive* and *negative spinor bundles* of M, respectively.

Definition A.6.4 (Spinor Connection) — Let M be a spin Riemannian manifold with Levi Civita Connection (mentioned in Chapter 2) ∇_{SO}. Then we can find a connection $1-$form w_{SO} on principal bundle P_{SO}. From our discussion of pullback connection in Chapter 2, we can pullback the connection w_{SO} to P_{Spin} to find a connection $1-$form w_{Spin} on P_{spin}. Here w_{spin} can be considered as a $\mathfrak{spin}(n)$ valued $1-$form as we have $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$. Then from our discussion of previous section, we can again define a connection ∇_Σ on the spinor bundle ΣM. We call the connection ∇_Σ to be the spinor connection.

Definition A.6.5 (Clifford multiplication) — Let M be a Riemannian spin manifold with spinor bundle ΣM, and let $x \in M$. The map

$$\begin{aligned} \cdot : T_xM \times \Sigma_x M &\longrightarrow \Sigma_x M \\ ([X] \cdot [\psi]) &\mapsto [X \cdot \psi] \end{aligned}$$

is called *Clifford multiplication.*

§A.7 Dirac Operator and Index theorem

Definition A.7.1 (Dirac Operator) — Let Σ be a Dirac bundle. The *Dirac operator* D is defined via the composition:

$$\Gamma(\Sigma) \xrightarrow{\nabla_\Sigma} \Gamma(T^*M \otimes \Sigma) \xrightarrow{\sim} \Gamma(TM \otimes \Sigma) \longrightarrow \Gamma(\Sigma),$$

where the first map is the spinor connection on the spinor bundle, the 2nd map is the identification of T^*M with TM (We discussed it in Chapter 2), and the last map is just Clifford multiplication.

When the spin manifold is even dimensional, the Dirac operator in fact decomposes into the following:

$$\begin{aligned} D^+ &: \Gamma(\Sigma^+) \to \Gamma(\Sigma^-) \\ D^- &: \Gamma(\Sigma^-) \to \Gamma(\Sigma^+) \end{aligned}$$

They are even Hilbert adjoint of each other. The celebrated Atiyah-Singer Index theorem says that:

$$Index(D^+) = Ker(D^+) - Ker(D^-) = \widehat{A}[M]$$

Definition A.7.2 — If E is another vector bundle on the spin manifold M with connection ∇_E, then there exists a twisted connection ∇_T on $E \otimes \Sigma$ defined as follows:

$$\nabla_T = \nabla_E \otimes \mathrm{Id}_\Sigma + \mathrm{Id}_E \otimes \nabla_\Sigma$$

Then the twisted Dirac operator D_E on $E \otimes \Sigma$ is defined as the following composition:

$$\Gamma(E \otimes \Sigma) \xrightarrow{\nabla_T} \Gamma(T^*M \otimes E \otimes \Sigma) \xrightarrow{\sim} \Gamma(TM \otimes E \otimes \Sigma) \longrightarrow \Gamma(E \otimes \Sigma),$$

See [Dai15] for details.

Now, another version of the Atiyah-Singer Index theorem says that the index of the operator D_E^+ from $E \otimes \Sigma^+$ to $E \otimes \Sigma^-$ can be computed using the following genus:

Definition A.7.3 (Twisted $\widehat{A}$–genus) —

$$\hat{A}(M,E) := \left(\mathrm{ch}(E) \prod_{i=1}^{2k} \frac{x_i/2}{\sinh(x_i/2)} \right) [M].$$

We will conclude this chapter with an important theorem, which will be crucial in our proof of Ochanine's theorem.

Theorem A.10
Let E be the complex extension of a real vector bundle over a compact, oriented, differentiable spin manifold M with dimension $8k+4$. Then,

$$\widehat{A}(M,E) \in 2\mathbb{Z}$$

We will not prove it here; however, we will give a sketch of the proof.

When a space is of dimension $8k+4$, its complexified Clifford Algebra has a natural quaternionic structure on it (See Clifford Algebra classification at [FO10]). Then we get an induced quaternionic structure on the spinor space and thus in the kernel of D^+ and D^- too. Any quaternionic space has a natural even-dimensional complex structure. Hence, the index of D^+ must be even, so is $\widehat{A}(M,E)$.

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