Hopf Algebra

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Motivation

- Let R be an $F-$algebra (we may think $F = \mathbb{C}$ or $\mathbb{R}$)
- M be a right $R-$module ($M \in Mod - R$)
- N be a left $R-$module ($N \in R - Mod$)
- In general, $M \otimes_R N$ does not have an $R-$module structure.
- We want to see what conditions should be imposed on R to give the tensor product a module structure.

Modules

- Now let R be a $\mathbb{C}$ algebra and $M, N \in R - Mod$
- Then, $M \otimes_{\mathbb{C}} N$ is an $R \otimes_{\mathbb{C}} R$ module

$$(r_1 \otimes r_2) \cdot (m \otimes n) := r_1 m \otimes r_2 n$$

- If we had a ring hom $\Delta : R \to R \otimes_{\mathbb{C}} R$, then we can possibly make $M \otimes_{\mathbb{C}} N$ an $R-$mod by defining:

$$a \cdot (m \otimes n) := \Delta(a) \cdot (m \otimes n)$$

Modules

- We would also like to have the $R-$mod isomorphism:

$$(L \otimes M) \otimes N \cong L \otimes (M \otimes N)$$

- Let $\Delta(a) = \sum_i a_1^i \otimes a_2^i =: a_1 \otimes a_2$ (Sweedler notation)
- Then,

$$\Delta(a) \cdot ((l \otimes m) \otimes n) = a_1 \cdot (l \otimes m) \otimes a_2 n) = (a_{11} l \otimes a_{12} m) \otimes a_2 n$$

and,

$$\Delta(a) \cdot (l \otimes (m \otimes n)) = a_1 l \otimes a_2 \cdot (m \otimes n)) = a_1 l \otimes (a_{21} m \otimes a_{22} n)$$

Modules

- To have our desired $R-$mod isomorphism, we must have:

$$(a_{11}l \otimes a_{12}m) \otimes a_2 n = a_1 l \otimes (a_{21}m \otimes a_{22}n)$$

- So, we would like to have:

$$(a_{11} \otimes a_{12}) \otimes a_2 = a_1 \otimes (a_{21} \otimes a_{22})$$

$$\implies \Delta(a_1) \otimes a_2 = a_1 \otimes \Delta(a_2)$$

$$\implies (\Delta \otimes id) \circ \Delta(a) = (id \otimes \Delta) \circ \Delta(a)$$

Modules

- We also want $\mathbb{C} \in R - mod$ such that $C \otimes M \cong M \cong M \otimes \mathbb{C}$
- So we need a ring hom $\varepsilon : R \to \mathbb{C}$
- Then we need:

$$a \cdot (c \otimes m) = a \cdot (cm) = a \cdot (m \otimes c)$$

$$\implies a_1 \cdot c \otimes a_2 m = a \cdot (cm) = a_1 m \otimes a_2 \cdot c$$

$$\implies \varepsilon(a_1)c \otimes a_2 m = a \cdot (cm) = a_1 m \otimes \varepsilon(a_2)c$$

- So, we would like to have:

$$\varepsilon(a_1) \otimes a_2 = a = a_1 \otimes \varepsilon(a_2)$$

$$\implies (\varepsilon \otimes id) \circ \Delta(a) = a = (id \otimes \varepsilon) \circ \Delta(a)$$

Bialgebra

Definition

A $\mathbb{C}-$bialgebra is a triplet (R, Δ, ε) such that:

- *R is a $\mathbb{C}-$algebra*
- *The algebra morphism $\Delta : R \to R \otimes R$ is a comultiplication map i.e it satisfies:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta \qquad (\textit{Coassociativity})$$

- *The algebra morphism $\varepsilon : R \to \mathbb{C}$ is a counit map i.e it satisfies:*

$$(\varepsilon \otimes id) \circ \Delta = id = (id \otimes \varepsilon) \circ \Delta \qquad (\textit{counit law})$$

Bialgebra example

Example

For a group G, $\mathbb{C}[G]$ becomes a bialgebra with:

- $\Delta(g) = g \otimes g$
- $\varepsilon(g) = 1$

Example

For a Lie algebra $\mathfrak{g}$, its universal algebra $U(\mathfrak{g})$ becomes a bialgebra with:

- $\Delta(v) = v \otimes 1 + 1 \otimes v$
- $\varepsilon(v) = 0$

Duals

- Now let $M \in R-$mod with its $\mathbb{C}$-vector space dual M^*
- We want to see what conditions should be imposed on R to make M^* an $R-$module
- If we wanted to define:

$$(a \cdot f)(v) := f(av)$$

Then we would have:

$$(a \cdot (b \cdot f))(v) = (b \cdot f)(av) = f(bav) \neq f(abv) = ((ab) \cdot f)(v)$$

Thus, it does not provide us with any $R-$mod structure.

Duals

- However, if we have an antialgebra morphism $S : R \to R$ i.e S(ab)=S(b)S(a) and $S(a+b) = S(a) + S(b)$,

  then we can possibly make M^* an $R-$module by defining the following action:

  $$(a \cdot f)(v) := f(S(a)v)$$

  And we have our desired relation:

  $$(a \cdot (b \cdot f))(v) = f(S(b)S(a)v) = f(S(ab)v) = ((ab) \cdot f)(v)$$

Duals

- We further want the following maps to be $R-$algebra morphisms.

$$ev_1 : M^* \otimes M \to \mathbb{C}$$
$$ev_2 : M \otimes M^* \to \mathbb{C}$$
$$coev_1 : \mathbb{C} \to M \otimes M^*$$
$$coev_2 : \mathbb{C} \to M^* \otimes M$$

- Then for all $f \otimes v \in M^* \otimes M$, we need:

$$ev_1(a \cdot (f \otimes v)) = a \cdot ev_1(f \otimes v)$$
$$\implies ev_1(a_1 f \otimes a_2 \cdot v) = a \cdot (f(v))$$
$$\implies (a_1 \cdot f)(a_2 v) = \varepsilon(a) f(v)$$
$$\implies f(S(a_1)a_2 v) = f(\varepsilon(a)v)$$
$$\implies S(a_1)a_2 = \varepsilon(a)\mathbf{1_R}$$

Duals

- Also for the map $coev_1 : \mathbb{C} \to M \otimes M^*$, we need,

$$coev_1(a \cdot 1) = a \cdot coev_1(1)$$

$$\implies coev_1(\varepsilon(a)) = a \cdot \sum_i (e_i \otimes e^i)$$

$$\implies \varepsilon(a) \sum_i (e_i \otimes e^i) = \sum_i (a_1 e_i \otimes a_2 e^i)$$

$$\implies \varepsilon(a) \sum e_i \cdot e^i(v) = \sum a_1 e_i \cdot a_2 e^i(v)$$

$$\implies \varepsilon(a) v = \sum a_1 e_1 \cdot e^i(S(a_2)v) = a_1 S(a_2) v$$

$$\implies \varepsilon(a) \mathbf{1_R} = a_1 S(a_2)$$

- If S is invertible, then the conditions from the last two results imply ev_2 and $coev_2$ are algebra morphisms too.

Hopf Algebra

Note that we can write:

$$S(a_1)a_2 = m \circ (S \otimes id) \circ \Delta(a) \quad \textit{and} \quad a_1 S(a_2) = m \circ (id \otimes S) \circ \Delta(a)$$

Definition

A Hopf Algebra is a tuple $(H, \Delta, \varepsilon, S)$ such that:

- (H, Δ, ε) *is a bialgebra*
- *Antialgebra morphism $S : R \to R$ is an antipode map i.e it satisfies:*

$$m \circ (S \otimes id) \circ \Delta = \varepsilon(\cdot)\mathbf{1_H} = m \circ (id \otimes S) \circ \Delta$$

Example

- $\mathbb{C}[G]$ *is a Hopf algebra with* $S(g) = g^{-1}$.
- $U(\mathfrak{g})$ *is a Hopf algebra with* $S(v) = -v$

Braiding

- Now we want to see when we have $H-$algebra isomorphism between $M \otimes N$ and $N \otimes M$
- Note that if $\Delta = \tau \circ \Delta$ (cocommutativity), then the $H-$algebra isomorphism can be simply given by $m \otimes n \to n \otimes m$ or in other words $x \to \tau(x)$.

  In this case, we call H a symmetric Hopf algebra.
- However, this condition is too strong. We often look for weaker conditions. For example, consider:

$$x \to \tau(\mathcal{R}x)$$

  where $\mathcal{R} = \sum \mathcal{R}_1 \otimes \mathcal{R}_2$ is an invertible element in $H \otimes H$.

Brading

- We want the map to be $R-$linear.

$$\tau(\mathcal{R}a(m \otimes n)) = a\tau(\mathcal{R}(m \otimes n)$$

$$\implies \tau(\mathcal{R}_1 a_1 m \otimes \mathcal{R}_2 a_2 n) = a\tau(\mathcal{R}_1 m \otimes \mathcal{R}_2 n)$$

$$\implies \mathcal{R}_2 a_2 n \otimes \mathcal{R}_1 a_1 m = a_1 \mathcal{R}_2 n \otimes a_2 \mathcal{R}_1 m$$

$$\implies \mathcal{R}_1 a_1 m \otimes \mathcal{R}_2 a_2 n = a_2 \mathcal{R}_1 m \otimes a_1 \mathcal{R}_2 n$$

$$\implies \mathcal{R}\Delta(a)(m \otimes n) = \tau(\Delta(a))\mathcal{R}(m \otimes n)$$

- So, we would like to have:

$$\mathcal{R}\Delta(a) = \tau(\Delta(a))\mathcal{R}$$

$$\implies \mathcal{R}\Delta(a)\mathcal{R}^{-1} = \tau(\Delta(a))$$

Braiding

- We also want the following hexagon identities to hold.

$$\begin{tikzcd}
& U \otimes (V \otimes W) \arrow[dr] & \\
(U \otimes V) \otimes W \arrow[ur] \arrow[d] & & (V \otimes W) \otimes U \arrow[d] \\
(V \otimes U) \otimes W \arrow[dr] & & V \otimes (W \otimes U) \\
& V \otimes (U \otimes W) \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& (U \otimes V) \otimes W \arrow[dr] & \\
U \otimes (V \otimes W) \arrow[ur] \arrow[d] & & W \otimes (U \otimes V) \arrow[d] \\
U \otimes (W \otimes V) \arrow[dr] & & (W \otimes U) \otimes V \\
& (U \otimes W) \otimes V \arrow[ur] &
\end{tikzcd}$$

Quasitriangular Hopf Algebra

- It can be shown that the Hexagon identities are true if:

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$$

  where, $\mathcal{R}_{13} = \mathcal{R}_1 \otimes 1 \otimes \mathcal{R}_2, \qquad \mathcal{R}_{23} = 1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2$
  and $\mathcal{R}_{12} = \mathcal{R}_1 \otimes \mathcal{R}_2 \otimes 1$

Definition

A Hopf algebra H is called quasitriangular if there is an invertible $\mathcal{R} \in H \otimes H$, such that:

- $$\mathcal{R}\Delta(a)\mathcal{R}^{-1} = \tau(\Delta(a))$$

- $$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$$

Quantized Heisenberg Algebra

Example

We know that the Heisenberg algebra is generated by X, Y, H with:

$$[H, X] = [H, Y] = 0, \qquad [X, Y] = H$$

Now, we define the $q-$Heisenberg algebra to be the algebra generated by $X, Y, K = q^{\frac{H}{2}}, K^{-1} = q^{-\frac{H}{2}}$ subject to the conditions:

$$[K^{\pm 1}, X] = [K^{\pm 1}, Y] = 0, \qquad [X, Y] = \frac{K - K^{-1}}{q - q^{-1}}$$

If we take $q = e^{\hbar}$, then taking $\hbar \to 0$ i.e. $q \to 1$ in the $q-$Heisenberg algebra, we can recover the Heisenberg algebra.

Quantized Heisenberg Algebra

Example

The $q-$Heisenberg algebra has a Hopf algebra structure.

The comultiplication map defined by:

$$\Delta(K) = K \otimes K, \qquad \Delta(X) = X \otimes K + K^{-1} \otimes X,$$

$$\Delta(Y) = Y \otimes K + K^{-1} \otimes Y$$

The counit map defined by:

$$\varepsilon(K) = 1, \qquad \varepsilon(X) = \varepsilon(Y) = 0$$

The antipode map given by:

$$S(K) = K^{-1}, \qquad S(X) = -X, \qquad S(Y) = -Y$$

Quantized Heisenberg Algebra

Example

The extended $q-$Heisenberg algebra has additional generators $L^{\pm 1} = q^{\pm N}$ with:

$$LXL^{-1} = q^{-1}X, \qquad LYL^{-1} = qY, \qquad [L, K] = 0$$

For $q \to 1$, we get the commutation relations satisfied by the number operator N.

The Hopf algebra structure is extended by:

$$\Delta(L) = L \otimes L, \qquad \varepsilon(L) = 0, \qquad S(L) = L^{-1}$$

It is in fact quasitriangular over $\mathbb{C}[q, q^{-1}]$ with

$$\mathcal{R} = q^{-(N \otimes H + H \otimes N)} e^{(q - q^{-1})(KX \otimes K^{-1}Y)}$$

Summary

Theorem (Tannaka duality)

- *R is a $\mathbb{C}-$algebra $\implies$ $R-$mod is an Abelian category*
- *R is a bialgebra $\implies$ $R-$mod is a monoidal category*
- *R is a Hopf algebra $\implies$ $R-$mod is a rigid monoidal category*
- *R is a quasi trianguar Hopf algebra $\implies$ $R-$mod is a braided rigid monoidal category*

Further reading

- Shahn Majid, Foundations of Quantum Group Theory
- Chari and Pressley, A Guide to Quantum Groups
- P Schlösser, Quantum Groups talk on YouTube, https://youtu.be/tMpZxrzYwbM?si=DXAn27y3RRGvyh9l
- Christoph Schweigert, Hopf algebras, quantum groups and topological field theory

Thank you!