Q1 (1)
(a) $\int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} c(x+1)\,dx + \int_{1}^{\infty} 0\,dx = 1$

$= c\left(\frac{x^2}{2} + x\right)_{-1}^{1} = 1$

$= c\left(\frac{1}{2} + 1 - \frac{1}{2} + 1\right) = 1$

$\rightarrow \boxed{c = \frac{1}{2}}$

(b) Mean = $E(x)$

$= \int_{-\infty}^{\infty} x f(x)\,dx$

$= \int_{-\infty}^{-1} 0\,dx + \int_{-1}^{1} \frac{x(x+1)}{2}\,dx + \int_{1}^{\infty} 0\,dx$

$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$

Variance = $E(x^2) - (E(x))^2$

$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} (x^3 + x^2)\,dx$

$= \frac{1}{2}\left(\frac{x^4}{4} + \frac{x^3}{3}\right)_{-1}^{1} = \frac{1}{3}$

Variance $= \frac{1}{3} - \left(\frac{1}{3}\right)^2 = \frac{2}{9}$

(2) $E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{1}^{\infty} x \cdot \frac{1}{x^2}\,dx = \int_{1}^{\infty} \frac{1}{x}\,dx = \infty$

$\therefore$ $E(x)$ doesn't exist

Q2

(i) $S_X = (0, 5/3)$

(ii) d) neither discrete nor AC

~~(iii)~~ $\cancel{\int_1^{4/3} \frac{x}{2}\,dx - \int_0^1 \frac{x}{4}\,dx}$

$\cancel{= \left(\frac{x^2}{4}\right)_1^{4/3} - \left(\frac{x^2}{8}\right)_0^1}$

$\cancel{= \left(\frac{16}{9} - 1\right)\frac{1}{4} - \left(\frac{1}{8}\right)}$

$\cancel{= \frac{7}{9 \times 4} - \frac{1}{8}}$

$\cancel{= \frac{14 - 9}{9 \times 8} - \frac{5}{72}}$

(iii)

$P(\{1 \le x \le 4/3\}) = \cancel{\int_0^{4/3} \frac{x}{2}\,dx - \int_0^1 \frac{x}{4}\,dx}$

$\left(\frac{3x}{5}\right)_{at\ x = 4/3} - \left(\frac{x}{2}\right)_{at\ x = 1}$

$= \frac{3 \times 4/3}{5} - 1/2$

$= \frac{4}{5} - 1/2$

$= \frac{3}{10}$

$P(\{1/2 \le X \le 3/4\}) = \left(\frac{x}{4}\right)_{at\ x = 3/4} - \left(\frac{x}{4}\right)_{at\ x = 1/2}$

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Q3

$1^{st}$ $F(x)$
discrete

$p(0) = 1/2$
$p(2) = 3/4 - 1/2 = 1/4$
$p(3) = 1 - 3/4 = 1/4$.

$2^{nd}$ $F(x)$
continous

$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x$
thus
$p(0 \le x < 1) = x$
and
$\frac{d}{dx}\left(\frac{x}{2}\right) = 1/2$ thus,
$p(1 < x < 2) = 1/2$

Q4

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} f_x(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t-1)}$$

$$E(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda}\lambda \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda}$$

$$= \lambda$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 f_x(x) = \sum_{x=0} \frac{x^2 e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \left( x(x-1) + x \right) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} +$$

$$\sum_{x=0}^{\infty} \frac{x\, e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \lambda^2 \frac{e^{-\lambda} \lambda^{x-2}}{(x-2)!} + \lambda$$

$$= \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$E(x^3) = \sum_{x=0}^{\infty} x^3 \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \left( x(x-1)(x-2) + 3x(x-1) + x \right) \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^3 \lambda^{x-3}}{(x-3)!} + \frac{3e^{-\lambda} \lambda^2 \lambda^{x-2}}{(x-2)!}$$

$$+ \frac{e^{-\lambda} \lambda \lambda^{x-1}}{(x-1)!} = \lambda^3 + 3\lambda^2 + \lambda$$

$$Var(X) = E(x^2) - (E(x))^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

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Q5 ① $E(X) = \sum_{x=0}^{\infty} x \cdot p(1-p)^x$

$$= p(1-p) \sum_{x=0}^{\infty} x(1-p)^{x-1}$$

$$= p(1-p) \cdot \frac{1}{p^2} = \frac{1-p}{p}$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 \cdot p(1-p)^x$$

$$= \sum_{x=0}^{\infty} (x(x-1) + x)\, p(1-p)^x$$

$$= p(1-p)^2 \sum_{x=0}^{\infty} x(x-1)(1-p)^{x-2} + p(1-p) \sum_{x=0}^{\infty} x(1-p)^{x-1}$$

$$= p(1-p)^2 \cdot \frac{2}{p^3} + \frac{(1-p)}{p}$$

$$= \frac{2(1-p)^2}{p^2} + \frac{(1-p)}{p} = \frac{2 + 2p^2 - 4p + p - p^2}{p^2}$$

$$= \frac{p^2 - 3p + 2}{p^2}$$

$$Var(x) = \frac{p^2 - 3p + 2}{p^2} - \frac{(1 + p^2 - 2p)}{p^2}$$

$$= (1-p)/p^2$$

$$M_x(t) = \sum_{x=0}^{\infty} e^{tx} p(1-p)^x = p \sum_{x=0}^{\infty} (e^t(1-p))^x$$

$$= \frac{p}{1 - e^t(1-p)}$$

② $E(X) = \int_0^\infty x \cdot \frac{1}{\lambda} e^{-x/\lambda} \, dx = \lambda$

$E(x^2) = \int_0^\infty x^2 \cdot \frac{1}{\lambda} e^{-x/\lambda} \, dx = 2\lambda^2$

$Var(x) = E(x^2) - (E(X))^2 = 2\lambda^2 - \lambda^2 = \lambda^2$

$M_x(t) = \int_0^\infty e^{tx} \frac{1}{\lambda} e^{-x/\lambda} \, dx = \frac{1}{1-\lambda t}$

Q5 Applications

For the 1st distribution :-

① Modelling the number of trials until the first sucess in a series of Bernoulli trials

② Number of coin flips until the first heads, number of trials in clinical test before succes

For the 2nd distribution -

① Time until radioactive decay, lifespan of an electron. component, time between phone calls at a call centre