Pattern Recognition Homework 1

Ali Mehmandoost

November 4, 2017

Contents

1 1: **2**
- 1.1 : 2
- 1.2 : 2
- 1.3 : 2
- 1.4 : 2
- 1.5 : 3
- 1.6 : 3
- 1.7 : 3

2 2: **3**

3 3: **3**
- 3.1 p(y): 3
- 3.2 p(x): 4
- 3.3 p(x|y): 5
- 3.4 p(y|x): 5
- 3.5 Drawing: 5

4 4: **6**
- 4.1 : 6
- 4.2 : 6
- 4.3 : 7
- 4.4 : 7

5 5: **8**
- 5.1 : 8
- 5.2 : 9
- 5.3 : 10

5.3.1 P(Y) = N(μ_{X_1} + μ_{X_2}, $\sigma^2_{X_1}$+$\sigma^2_{X_2}$) 10
5.3.2 Proof: 11
5.3.3 Drawing: 12

1 1:

1.1 :

$$\int_{-\infty}^{+\infty}(x+y)f(x,y)dxdy = \int_{-\infty}^{+\infty}xf(x,y)dxdy + \int_{-\infty}^{+\infty}yf(x,y)dxdy$$
$$= \int_{-\infty}^{+\infty}xf_X(x,y)dx + \int_{-\infty}^{+\infty}yf_Y(x,y)dy = E[X] + E[Y]$$

1.2 :

$$\sigma^2 = E[(X-\mu_x)^2] = E[X^2 - 2X\mu_x + \mu_x^2] =$$
$$E[X^2] - 2E[X]\mu + E[\mu^2] = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2$$

1.3 :

Let X,Y be two independent random variable, that implies E[XY] = E[X]E[Y].
now Cov(X,Y) = E[XY]-E[X]E[Y] = E[X]E[Y] - E[X]E[Y] = 0 $\implies X, Y are Uncorrelated.$

1.4 :

Let X be a random variable with an estimated value greater than zero.
Let Y be a random variable independent of X with a zero estimated value.
Let Z be a random variable as Z = XY, therefore Z is dependent on both X and Y.
E[Z] = E[XY] = E[X]E[Y] = E[X] 0 = 0
Cov (Z, X) = E[ZX] - E[Z]E[X] = 0 - 0.E[X] = 0 $\implies X, Z are Uncorrelated.$

1.5 :

$$X, Y are uncorolated \implies E[XY] = E[X]E[Y]$$

$$\sigma^2 = E[(X+Y)^2] - E[X+Y]^2 = E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2 =$$

$$E[X^2] + 2E[XY] + E[Y^2 - E[X]^2 - E[Y]^2 - 2E[X]E[Y]$$

$$= E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 + 2E[XY] - 2E[XY] = E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2 = \sigma_x + \sigma_y$$

1.6 :

X_1 and X_2 are continuous random variables that implies P(X) for a single givin point is equal to zero.

$$P[X_1 \leq X2] = P[X_1 < X_2] + P[x_1 = X_2] = P[X_1 < X_2] + 0 = P[X_1 < X_2] \tag{4}$$

X_1 and X_2 are independent and identically distributed So:
P[X_1 < X_2] = P[X_1 > X{2}]

$$\int_{\infty}^{+\infty} p(X) = 1 \implies 2Px_1 \leq X_2] = 1 \implies P[x_1 \leq X_2] = \frac{1}{2}$$

1.7 :

For Discrete random variable we can not assume P(X) of any single point is equal to zero So we need P[X_2] and P[X_1].

2 2:

[]

3 3:

3.1 p(y):

$$p(y) = \int_{-\infty}^{+\infty} p(x,y)dx = \int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{(y-\mu)^2)}{2a^2} + \frac{(x-y)^2)}{2b^2})} dx =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{(y-\mu)^2}{2a^2}} e^{\frac{(x-y)^2}{2b^2})} dx =$$

$$\frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(y-\mu)^2}{2a^2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi b^2}} e^{-\frac{(x-y)^2}{2b^2}} dx =$$

$$\frac{1}{\sqrt{2\pi a^2}} e^{-\frac{(y-\mu)^2}{2a^2}} = \mathcal{N}(\mu,\, a^2)$$

3.2 p(x):

$$p(y) = \int_{-\infty}^{+\infty} p(x,y) dy = \int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{(y-\mu)^2)}{2a^2} + \frac{(x-y)^2)}{2b^2})} dy =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{b^2(y-\mu)^2 + a^2(x-y)^2}{2a^2b^2})} dy =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{(a^2+b^2)y^2 - 2(a^2x+b^2\mu)y + (b^2\mu^2+a^2x^2)}{2a^2b^2})} =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{y^2 - 2\frac{a^2x+b^2\mu}{a^2+b^2}y + \frac{b^2\mu^2+a^2x^2}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} dy =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{y^2 - 2\frac{a^2x+b^2\mu}{a^2+b^2}y + \frac{b^2\mu^2+a^2x^2}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} dy =$$

$$\int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{y-\frac{a^2x+b^2\mu}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} e^{-(\frac{\frac{b^2\mu^2+a^2x^2}{a^2+b^2} - \frac{a^2x+b^2\mu}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})dy=}$$

$$\frac{1}{1\pi ab}\sqrt{2\pi\frac{a^2b^2}{a^2+b^2}} e^{-(\frac{\frac{b^2\mu^2+a^2x^2}{a^2+b^2} - \frac{a^2x+b^2\mu}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} \int_{-\infty}^{+\infty} \frac{1}{2\pi ab} e^{-(\frac{y-\frac{a^2x+b^2\mu}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} dy =$$

$$\frac{1}{\sqrt{2\pi(a^2+b^2)}} e^{-(\frac{\frac{b^2\mu^2+a^2x^2}{a^2+b^2} - \frac{a^2x+b^2\mu}{a^2+b^2}}{2\frac{a^2b^2}{a^2+b^2}})} =$$

$$\frac{1}{\sqrt{2\pi(a^2+b^2)}} e^{-(\frac{(x-\mu)^2}{2(a^2+b^2)})} =$$

$$\mathcal{N}(\mu,\, a^2+b^2)$$

3.3 p(x|y):

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{\frac{1}{2\pi ab}e^{-(\frac{(y-\mu)^2)}{2a^2}+\frac{(x-y)^2)}{2b^2})}}{\frac{1}{\sqrt{2\pi a^2}}e^{-\frac{(y-\mu)^2}{2a^2}}} = \frac{1}{\sqrt{2\pi b^2}}e^{-\frac{(x-y)^2}{ab^2}} = \mathcal{N}(y, b^2)$$

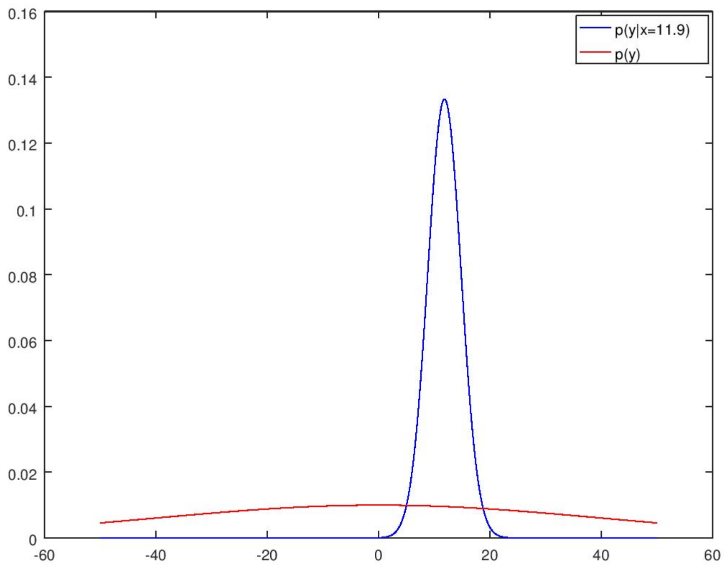
3.4 p(y|x):

$$p(y|x) = \frac{p(x,y)}{p(y)} = \frac{\frac{1}{2\pi ab}e^{-(\frac{(y-\mu)^2)}{2a^2}+\frac{(x-y)^2)}{2b^2})}}{\frac{1}{\sqrt{2\pi(a^2+b^2)}}e^{-(\frac{(x-\mu)^2}{2(a^2+b^2)})}} = \frac{1}{\sqrt{2\pi\frac{a^2b^2}{a^2+b^2}}}e^{-(\frac{(y-\frac{a^2x+b^2\mu}{a^2+b^2})^2}{2\frac{a^2b^2}{a^2+b^2}})} = \mathcal{N}(\frac{a^2x+b^2\mu}{a^2+b^2}, \frac{a^2b^2}{a^2+b^2})$$

3.5 Drawing:

Following octave code will produce graphics for this question.

```
mu = 0;
a = 40;
b = 3;
x = 11.9

Y=[-50:0.1:50];

mean = ((a^2)*x+(b^2)*mu)/(a^2 + b^2);
var = ((a*b)^2)/(a^2+b^2);
Py_x = (1.0/sqrt(2*pi*var))*exp(-((Y-mean).^2)/(2*var));

var2 = a^2;
Py = (1.0 / sqrt(2*pi*var2))*exp(-((Y-mu).^2)/(2*var2));

plot(Y, Py_x, "b;p(y|x=11.9);");
hold on;
plot(Y,Py,'r;p(y);');
```



4 4:

4.1 :

A valid covariance matrix is symmetric and positive semi-definite.
$\Sigma = \Sigma^{\mathrm{T}}$ Its symmetric.
Σ is positive semi-definite if all its eigenvalues are non-negative.[Next Part]

4.2 :

$$\Sigma e = \lambda e$$

$$(\Sigma - I)e = 0$$

$$det(\Sigma - \lambda I) = 0 \implies (64 - \lambda)^2 - (-25)^2 = 0$$

$$64^2 - 2 \times 64\lambda + \lambda^2 + 25^2 = 0$$

$$\lambda^2 - 128\lambda + 3471 = 0$$

$$\lambda = 39, 89$$

$$(\Sigma - 39\lambda)e = \begin{bmatrix} 64-39 & -25 \\ -25 & 64-39 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a = b \implies \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(\Sigma - 89\lambda)e = \begin{bmatrix} 64-89 & -25 \\ -25 & 64-89 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -25 & -25 \\ -25 & -25 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$a = -b \implies \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4.3 :

Octave code returned normalized eigenvector.

```
S = [64, -25; -25, 64];
[a, b] = eig(S)
```

a =
-0.70711 -0.70711
-0.70711 0.70711

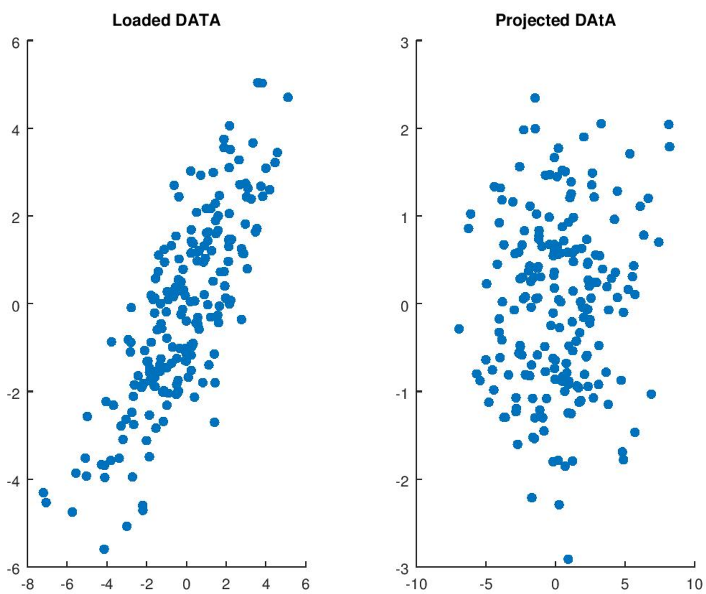
b =
Diagonal Matrix
39 0
0 89

4.4 :

I couldn't find any data attached I built my own data in data.mat.

```
S = [64, -25; -25, 64];
load ('data');

[eigvec, eigval] = eig(S);

Projected_data = (eigvec* X')';
```

```
subplot(121)
scatter(X(:,1), X(:,2), 'filled')
title ('Loaded DATA');

subplot(122)
scatter(Projected_data(:,1), Projected_data(:,2), 'filled')

title ('Projected DAtA');
```



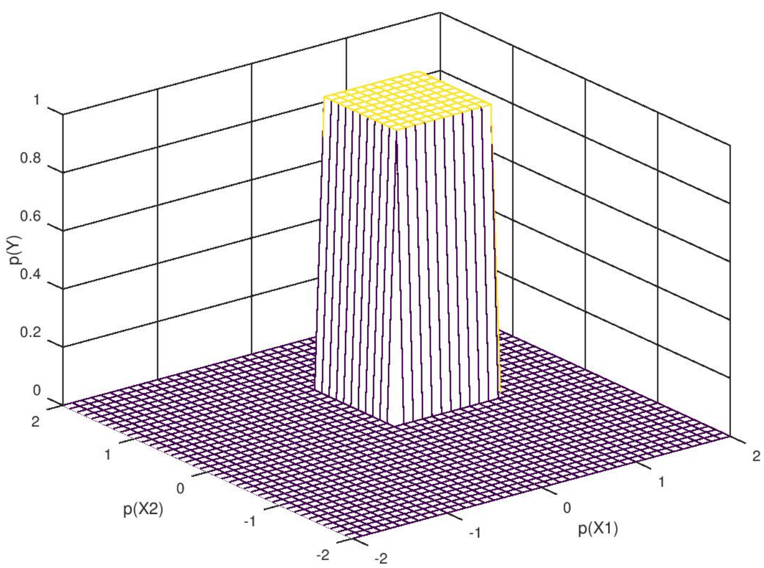
5 5:

5.1 :

$\mathrm{P(Y)} = \mathrm{P(X_1)}\ \mathrm{P(X_2)}$

```
function a = p_x(n)
a = zeros(size(n));
a( n <= 1) =1;
a( n < 0) = 0;
endfunction
```

```
X = [-2:0.1:2];
[X1, X2] = meshgrid (X, X);
Y = p_x(X1).*p_x(X2);
mesh (X1, X2, Y);
xlabel ("p(X1)");
ylabel ("p(X2)");
zlabel ("p(Y)");
```

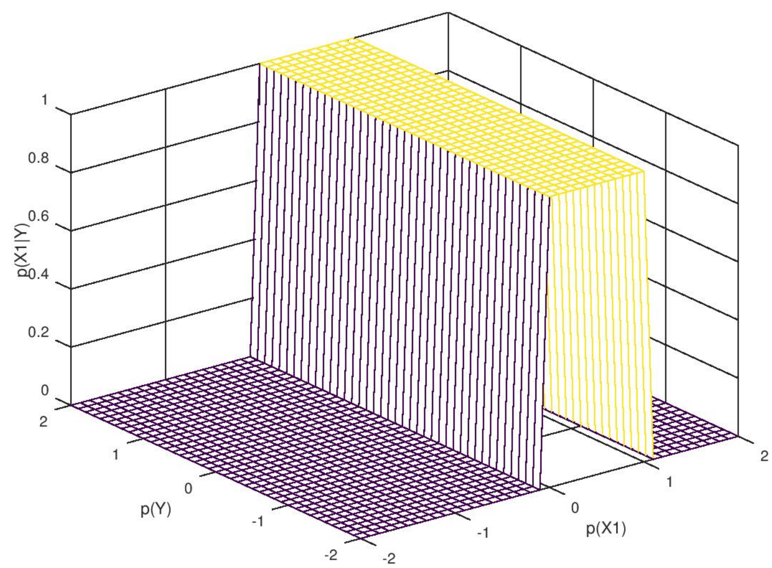


5.2 :

$$P(X_1|y = \frac{P(X_1)P(X_2)}{P(X_1)} = P(X_2) \tag{12}$$

```
function a = p_x(n)
a = zeros(size(n));
a( n <= 1) =1;
a( n < 0) = 0;
endfunction

X = [-2:0.1:2];
[X1, Y] = meshgrid (X, X);
P_X_Y = p_x(X1);
```

```
mesh (X1, Y, P_X_Y);
xlabel ("p(X1)");
ylabel ("p(Y)");
zlabel ("p(X1|Y)");
```



5.3 :

5.3.1 $\mathbf{P(Y) = N(\mu_{X_1} + \mu_{X_2}, \sigma^2_{X_1} + \sigma^2_{X_2})}$

$$X_1\ \mathcal{N}(\mu_{X_1}, \sigma^2_{X_1})$$

$$X_2\ \mathcal{N}(\mu_{X_2}, \sigma^2_{X_2})$$

$$Z\ X + Y$$

$$Z\ \mathcal{N}(\mu_{X_1} + \mu_{X_2}, \sigma^2_{X_1} + \sigma^2_{X_2})$$

5.3.2 Proof:

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\left[-\frac{(z-x-\mu_Y)^2}{2\sigma_Y^2}\right] \frac{1}{\sqrt{2\pi}\sigma_X} \exp\left[-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{\sigma_X^2(z-x-\mu_Y)^2 + \sigma_Y^2(x-\mu_X)^2}{2\sigma_X^2\sigma_Y^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{\sigma_X^2(z^2+x^2+\mu_Y^2-2xz-2z\mu_Y+2x\mu_Y) + \sigma_Y^2(x^2+\mu_X^2-2x\mu_X)}{2\sigma_Y^2\sigma_X^2}\right] dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{2\pi}\sigma_X\sigma_Y} \exp\left[-\frac{x^2(\sigma_X^2+\sigma_Y^2)-2x(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X)+\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{2\sigma_Y^2\sigma_X^2}\right] dx$$

$$Defining\ \sigma_Z = \sqrt{\sigma_X^2+\sigma_Y^2}$$

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{x^2 - 2x\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2} + \frac{\sigma_X^2(z^2+\mu_Y^2-2z\mu_Y)+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2 - \left(\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2 + \frac{\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2}{\sigma_Z^2}}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left[-\frac{\sigma_Z^2\left(\sigma_X^2(z-\mu_Y)^2+\sigma_Y^2\mu_X^2\right)-\left(\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X\right)^2}{2\sigma_Z^2\left(\sigma_X\sigma_Y\right)^2}\right] \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x-\frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}\right] \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\frac{\sigma_X\sigma_Y}{\sigma_Z}} \exp\left[-\frac{\left(x - \frac{\sigma_X^2(z-\mu_Y)+\sigma_Y^2\mu_X}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_X\sigma_Y}{\sigma_Z}\right)^2}\right] dx$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left[-\frac{(z-(\mu_X+\mu_Y))^2}{2\sigma_Z^2}\right]$$

5.3.3 Drawing:

```
X = [-10:0.1:10];
plot (X, normpdf(X,0,2))
```