Stochastic Processes HW02 [Draft]

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Contents

1 Coin flipping

$$F(x,y) = \begin{cases} \frac{1}{4}, & (0,0) \\ \frac{1}{4}, & (0,1) \\ \frac{1}{4}, & (1,0) \\ \frac{1}{4}, & (1,1) \end{cases}$$

2 Divergent integral

The boundaries in Question doesn't make any sense.

3 Estimation of the minimum

$$Y = min(X_1, X_2, ..., X_n)$$

$$F(y) = P(Y < y) = 1 - P(Y > y) = 1 - P(min(X_1, X_2, ..., X_n) > y)$$

$$min(X_1, X_2, ..., X_n) > y \text{ exactly when } X_i > y \implies$$

$$F(y) = 1 - P(X_1 > y)P(X_2 > y)...P(X_n > Y) = 1 - P(X > y)^n = 1 - (\frac{b-y}{b-a})^n$$

$$F(y) = \begin{cases} 1 - (\frac{b-y}{b-a})^n, y \in (a,b) \\ 0, y < a \\ 1, y > b \end{cases}$$

$$f(y) = \frac{dF(y)}{dy} = F(y) = \begin{cases} (\frac{n}{b-a})(\frac{b-y}{b-a})^(n-1), y \in (a,b) \\ 0, otherwise \end{cases}$$

$$E[y] = \int_{-\infty}^{\infty} yf(y)dy = \frac{b+na}{n+1}$$

4 Conditional estimation

$$P(X=2)=\sum_{y=0}^{2}\frac{e^{-2}}{y!(2-y)!}=\frac{e^{-2}}{0!2!}+\frac{e^{-2}}{1!1!}+\frac{e^{-2}}{2!0!}=\frac{e^{-2}}{2}+\frac{e^{-2}}{1}+\frac{e^{-2}}{2}=\frac{4e^{-2}}{2}=2e^{-2}$$

$$P(Y|X=2)=\frac{P(Y,X=2)}{P(X=2)}=\frac{\frac{e^{-2}}{y!(2-y!)}}{2e^{-2}}=\frac{1}{2y!(2-y)!},\ for\ y=0,1,2$$

$$E[Y|X=2]=\sum_{y=0}^{1}y\frac{2}{2y!(2-y)!}=0\frac{1}{(2\ 0!)(2-0)!}+1\frac{1}{(2\ 1!)(2-1)!}+2\frac{1}{(2\ 2!)(2-2)!}=$$

$$0+\frac{1}{2}+\frac{1}{2}=1$$

5 5 Accidents happen

Success = an accident-free day

$$\lambda=2$$

$$P(success)=P(X=0)=\frac{e^(-2)2^0}{0!}=e^{-2}=0.1353$$

$$P(4\text{ Success in }3)=\binom{7}{4}(e^{-2})^4(1-e^{-2})^3=0.0075$$

6 Fire Station

We want to minimize E[|PlaceOfFire - FireStationLocation|]
Let PlaceOfFire be x and FireStationLocation be s

$$E[|x-S|]=\int_0^s(s-x)\lambda e^{-\lambda x}dx+\int_s^\infty(x-s)\lambda e^{-\lambda x}dx=s+\frac{1}{\lambda}2e^{-s\lambda}+\frac{1}{\lambda}$$

$$\frac{d[E[|x-S|]}{ds}=1-2e^{-s\lambda}=0\implies s=\frac{ln}{\lambda}$$

7 Passenger frequency

Average Passanger per bus $\frac{1}{19}\sum_{i=1}^{19} i = 10$.
Number of bus per day = 30.
Average number of passengers entering station per day = 10 * 30 = 300

8 independent distribution

8.1 A

$$\int_0^\infty \int_0^y e^{-y} dx dy = \int_0^\infty ye^{-y} = (-y-1)e^{-y}| = 0 - (-1) = 1$$

8.2 B

$$f_x(x) = \int_x^\infty e^{-y} dy = e^{-x}$$

$$f_y(y) = \int_0^y e^{-y} dx = ye^{-y}$$

8.3 C

No

$$e^{-y} \neq ye^{-(x+y)}$$

$$f(x,y) \neq f_x(x)F_y(y)$$

9 Papoulis#4

9.1 Papoulis#4-6

9.1.1 A

$$F(R) = \frac{1}{105-95}(R-95) \implies p = 0.1(104-95) - 0.1(96-95) = 0.8$$

9.1.2 B

$$p = G(2.5) - G(-2.5) = 0.9876$$

9.2 Papoulis#4-26

$$p = e^{0/T} - e^{-\frac{\frac{T}{4}}{T}} = 1 - e^{-\frac{1}{4}} = 0.22,\ np = 220,\ npq = 171.6k_2 = 100$$

$$\frac{k_2 - np}{\sqrt{npq}} = -9.16$$

$$P(0 \leq k \leq 100) = G(9.16) = 0$$

9.3 Papoulis#4-36

9.3.1 A

$$P_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}$$

10 Papoulis#5

10.1 Papoulis#5-21

if y $>$ 0 then

$$F_y(y|x \geq 0) = F_x(\sqrt{y}|x \geq 0) + F_x(-\sqrt{y}|x \geq 0) = F_x(\sqrt{y}|x \geq 0)$$

$$F_x(\sqrt{y}|x \geq 0) = \frac{P(0 < x < \sqrt{y})}{P(x \geq 0)} = \frac{F_x(\sqrt{y}) - F_X(0)}{1 - F_{X(0)}}$$

$$f_y(y|x \geq 0) = \frac{d}{dy} F_y(\sqrt{y}|x \geq 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1 - F_x(0)]}$$

10.2 Papoulis#5-50

$$P(X=k) = P(\text{"}TT..TH\text{"} \cup \text{"}HH...HT\text{"}) = P(TT...TH) + P(HH...HT) = q^k p + q^k p$$

$$k = 1, 2, ...$$

$$E(X) = \sum_{k=1}^{\infty} Kp(X=K) = \sum_{k=1}^{\infty} Kq^k p + \sum_{k=1}^{\infty} kp^k q =$$

$$pq(\sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1}) =$$

$$pq(\frac{d}{dq}\sum_{k=1}^{\infty} q^k + \frac{d}{dp}\sum_{k=1}^{\infty} p^k) = \frac{p}{q} + \frac{q}{p}$$

10.3 Papoulis#5-51

10.3.1 A

$$p = \frac{M}{N}$$

$$q = 1 - p = \frac{N-M}{M} < 1$$

$$P(X=K) = \binom{n}{k} p^k q^{n-k}, k = 0, 1, 2, ..., n$$

10.3.2 B

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{nk}}{\binom{N}{n}}$$

10.3.3 C

$$P(X=k)=\frac{M!}{k!(M-k)!}\frac{(N-M)!}{(n-k)!(N-M-n+k)!}\frac{n!(N-n)!)}{N!}=$$
$$\binom{n}{k}(\frac{M}{N})^k(\frac{N-M}{N})^{n-k}=\binom{n}{k}p^k(1-p)^{n-k}$$

10.4 Papoulis#5-52

10.4.1 A

$$P(X=k)=\binom{k-1}{r-1}p^{r-1}q^{k-r}p=\binom{k-1}{r-1}p^rq^{k-r},\ k=r,r+1,...$$

10.4.2 B

1. I
$$\binom{k-1}{r-1}$$

II
$$1$$

III
$$\binom{m+n-k}{n-r}$$

P
$$P(X=k)\binom{k-1}{r-1}\frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}},\ k=r,r+1,...$$

10.4.3 C

$$P(x=k)=\binom{k-1}{r-1}\frac{(m+n-k)!}{(n-r)!(m-k+r)!}\frac{n!m!}{(m+n)!}$$
$$\approx\binom{k-1}{r-1}(\frac{n}{m+n})^r(\frac{m}{m+n-r})(\frac{m-1}{m+n-r-1})...(\frac{m-k+r+1}{m+n-k+1})$$
$$\approx\binom{k-1}{r-1}(\frac{n}{m+n})^r(\frac{m}{m+n})^{k-r}=\binom{k-1}{r-1}p^rq^{k-r}$$

11 Papoulis#6

11.1 Papoulis#6-5

11.1.1 A

$$F_z(Z) = \int_{y=-z}^{z} \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{xy} dx dy = \frac{z}{\sigma^2} e^{-z/2\sigma^2} U(z)$$

11.1.2 B

$$F_z(Z) = \int_{y=-z}^{z} \frac{1}{\sqrt{z-y^2}} (2. \frac{1}{2\pi\sigma^2} e^{(z-y^2+y^2)/2\sigma^2}) dy$$

$$= \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}} dy = \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\pi/2} \frac{\sqrt{z} cos}{\sqrt{z} cos} d\theta = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} U(z)$$

11.1.3 C

$$U = X - Y = N(0, 2\sigma^2) \iff Var(U) = Var(X) + Var(Y) = 2\sigma^2$$

11.2 Papoulis#6-16

11.2.1 A

$$y = y_1 < g(x) \implies$$
$$F(x, y) = Px \leq x, y \leq y_1 = Py \leq y_1 = F_y(y_1)$$
$$y = y_2 > g(x) \implies$$
$$F(x, y) = Px \leq x, y \leq y_2 = Px \leq x = F_x(x)$$

11.2.2 B

$$y = y_1 < g(x) \implies$$
$$F(x, y) = Px \leq x, y \leq y_1 = 0$$
$$y = y_2 > g(x) \implies$$
$$F(x, y) = Px \leq x, y \leq y_2 = Px \leq x - Py > y_2 = F_x(x) - [1 - F_y(y_2)]$$

11.3 Papoulis#6-25

$$f_{XY}(x,y) = f_X(x)F_Y(y) = \frac{1}{\lambda^2}e^{\frac{-(x+y)}{\lambda}}U(x)U(y)$$

$$Z = X + Y$$

$$\phi_Z(\omega) = \phi_X(\omega)\frac{1}{(1-j\omega\lambda)^2}$$

$$f_Z(z) = \frac{z}{\lambda^2}e^{-\frac{x}{y}}U(z)$$

$$P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{2}{\lambda^2}e^{-\frac{z}{\lambda}}dz = 0.406$$

$$W = Y - X$$

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w)dw = \int_0^{\infty} \frac{1}{\lambda^2}e^{-\frac{w+2y}{\lambda}}dy = \frac{1}{2\lambda}e^{-\frac{2}{\lambda}}$$

$$P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda}e^{-\frac{w}{\lambda}}dw = \frac{1}{2e}$$