SOLVED EXERCISE 4.2

Resolve into partial fractions.

(1) $\dfrac{x^2-3x+1}{(x-1)^2(x-2)}$

Solution:

Let $\dfrac{x^2-3x+1}{(x-1)^2(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}+\dfrac{C}{x-2}$

Multiplying both sides by $(x-1)^2(x-2)$, we get

$$x^2-3x+1=A(x-1)(x-2)+B(x-2)+C(x-1)^2 \qquad \text{———— (1)}$$

$$x^2-3x+1=A(x^2-3x+2)+B(x-2)+C(x^2-2x+1)$$

$$x^2-3x+1=Ax^2-3Ax+2A+Bx-2B+Cx^2-2Cx+C$$

$$x^2-3x+1=Ax^2+Cx^2-3Ax+Bx-2Cx+2A-2B+C \qquad \text{———— (2)}$$

To find C, we put $x-2=0 \Rightarrow x=2$ in eq. (1), we get

$$(2)^2-3(2)+1=A(2-1)(2-2)+B(2-2)+C(2-1)^2$$

$$4-6+1=A(1)(0)+B(0)+C(1)^2$$

$$5-6=A(0)+B(0)+C$$

$$-1=C$$

Or $\quad C=-1$

To find B, we put $(x-1)^2=0 \Rightarrow x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$(1)^2-3(1)+1=A(1-1)(1-2)+B(1-2)+C(1-1)^2$$

$$1-3+1=A(0)(-1)+B(-1)+C(0)$$

$$2-3=A(0)+B(-1)+C(0)$$

Or $\quad -1=-B$

$\Rightarrow \quad B=1$

To find A, equating coefficient of x^2 on both sides of (2), we get

$A + C \quad = 1$
$A + (-1) = 1$
$A = 1 + 1$
$A = 2$

Thus required partial fractions are $\frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{-1}{x-2}$

Hence, $\frac{x^2 - 3x + 1}{(x-1)^2 (x-2)} = \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{1}{x-2}$

(2) $\frac{x^2 + 7x + 11}{(x+2)^2 (x+3)}$

Solution:

Let $\frac{x^2 + 7x + 11}{(x+2)^2 (x+3)} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x+3}$

Multiplying both sides by $(x+2)^2 (x+3)$, we get

$x^2 + 7x + 11 = A(x+2)(x+3) + B(x+3) + C(x+2)^2$ ______ (1)

$x^2 + 7x + 11 = A(x^2 + 5x + 6) + B(x+3) + C(x^2 + 4x + 4)$

$x^2 + 7x + 11 = Ax^2 + 5Ax + 6A + Bx + 3B + Cx^2 + 4Cx + 4C$

$x^2 + 7x + 11 = Ax^2 + Cx^2 + 5Ax + Bx + 4Cx + 6A + 3B + 4C$ ______ (2)

To find A, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$(-3)^2 + 7(-3) + 11 = A(-3+2)(-3+3) + B(-3+3) + C(-3+2)^2$

$9 - 21 + 11 = A(-1)(0) + B(0) + C(-1)^2$

$20 - 21 = A(0) + B(0) + C(1)$

$-1 = C$

Or $\quad C = -1$

To find B, we put $(x+2)^2 = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$ in eq. (1), we get

$A \quad = 1$

$(-2)^2 + 7(-2) + 11 = A(-2+2)(-2+3) + B(-2+3) + C(-2+2)^2$

$4 - 14 + 11 = A(0)(1) + B(1) + C(0)^2$

$15 - 14 = A(0) + B(1) + C(0)$

Or $\quad 1 = B$

$\Rightarrow \quad B = 1$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$A + C = 1$

$A + (-1) = 1$

$$A - 1 = 1$$
$$A = 2$$

Thus required partial fractions are $\frac{2}{x+2}+\frac{1}{(x+2)^2}+\frac{-1}{x+3}$

Hence, $\frac{x^2+7x+11}{(x+2)^2(x+3)}=\frac{2}{x+2}+\frac{1}{(x+2)^2}+\frac{-1}{x+3}$

(3) $\frac{9}{(x-1)(x+2)^2}$

Solution:

Let $\frac{9}{(x-1)(x+2)^2}=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$

Multiplying both sides by $(x-1)(x+2)^2$, we get

$$9 = A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad \text{\_\_\_\_\_\_ (1)}$$
$$9 = A(x^2+4x+4) + B(x^2+x-2) + C(x-1)$$
$$9 = Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C$$
$$9 = Ax^2 + Bx^2 + 4Ax + Bx + Cx + 4A - 2B - C \qquad \text{\_\_\_\_\_\_ (2)}$$

To find C, we put $x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$9 = A(1+2)^2 + B(1-1)(1+2) + C(1-1)$$
$$9 = A(3)^2 + B(0)(3) + C(0)$$
$$9 = A(9) + B(0) + C(0)$$
$$9 = 9A$$

Or $9A = 9$

Dividing the both sides by '9', we get

To find C, we put $(x+2)^2 = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$ in eq. (1), we get

$$9 = A(-2+2)^2 + B(-2-1)(-2+2) + C(-2-1)$$
$$9 = A(0)^2 + C(-3)(0) + C(-3)$$
$$9 = A(0) + C(0) + C(-3)$$
$$9 = -3C$$

Or $-3C = 9$

Dividing both sides by '−3', we get

$\therefore \quad C = -3$

To find B, equating coefficient of x^2 on both sides of eq. (2), we get

$$A + B = 0$$
$$1 + B = 0$$

$$B = -1$$

Thus required partial fractions are $\dfrac{1}{x-1} + \dfrac{-1}{x+2} + \dfrac{-3}{(x+2)^2}$

Hence, $\dfrac{9}{(x-1)(x+2)^2} = \dfrac{1}{x-1} + \dfrac{-1}{x+2} + \dfrac{-3}{(x+2)^2}$

(4) $\dfrac{x^4+1}{x^2(x-1)}$

Solution:

$$\frac{x^4+1}{x^2(x-1)} = \frac{x^4+1}{x^3-x^2}$$

By long division, we have

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2 & x^4+1 \\ & \pm x^4 \quad \mp x^2 \\ \hline & x^3+1 \\ & \pm x^3 \quad \mp x^2 \\ \hline & x^2+1 \end{array}$$

$$\frac{x^4+1}{x^2(x-1)} = x+1+\frac{x^2+1}{x^2(x-1)}$$

Let $\dfrac{x^2+1}{x^2(x-1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1}$

Multiplying both sides by $x^2(x-1)$,, we get

$$x^2+1 = A(x+1)+B(x-1)+Cx^2 \qquad \text{\_\_\_\_\_\_\_\_ (1)}$$

$$x^2+1 = Ax^2-Ax+Bx-B+Cx^2$$

$$x^2+1 = Ax^2+Cx^2-Ax+Bx-B \qquad \text{\_\_\_\_\_\_\_\_ (2)}$$

To find C, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$(1)^2+1 = A(1)(1-1)+B(1-1)+C(1)^2$$

$$1+1 = A(1)(0)+B(0)+C(1)$$

$$2 = A(0)+B(0)+C(1)$$

$$2 = C$$

Or $$C = 2$$

To find B, we put $x^2 = 0 \Rightarrow x = 0$ in eq. (1), we get

$$(0)^2+1 = A(0)(0-1)+B(0-1)+C(0)^2$$

$$1 = A(0)(-1)+B(-1)+C(0)$$

$$1 = -B$$

Or $\quad B = -1$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$$A + C = 1$$
$$A + 2 = 1$$
$$A = 1 - 2$$
$$A = -1$$

Thus required partial fractions are $\frac{-1}{x} + \frac{-1}{x^2} + \frac{2}{x-1}$

Hence, $\frac{x^4+1}{x^2(x-1)} = x + 1 - \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x-1}$

(5) $\frac{7x+4}{(3x+2)(x+1)^2}$

Solution:

Let $\frac{7x+4}{(3x+2)(x+1)^2} = \frac{A}{3x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

Multiplying both sides by $(3x+2)(x+1)^2$, we get

$$7x + 4 = A(x+1)^2 + B(3x+2)(x+1) + C(3x+2) \qquad \text{\_\_\_\_\_\_ (1)}$$

$$7x + 4 = A(x^2+2x+1) + B(3x^2+5x+2) + C(3x+2)$$

$$7x + 4 = Ax^2 + 2Ax + A + 3Bx^2 + 5Bx + 2B + 3Cx + 2C$$

$$7x + 4 = Ax^2 + 3Bx^2 + 2Ax + 5Bx + 3(x + A + 2B + 2C) \qquad \text{\_\_\_\_\_\_ (2)}$$

To find A, we put $3x + 2 = 0 \Rightarrow 3x = -2 \Rightarrow x = \frac{-2}{3}$ in eq. (1), we get

$$7\left(-\frac{2}{3}\right) + 4 = A\left(-\frac{2}{3}+1\right)^2 + B\left(3\left(-\frac{2}{3}\right)+2\right)\left(-\frac{2}{3}+1\right) + C\left(3\left(-\frac{2}{3}\right)+2\right)$$

$$-\frac{14}{3} + 4 = A\left(\frac{1}{3}\right)^2 + B(-2+2)\left(\frac{1}{3}\right) + C(-2+2)$$

$$-\frac{2}{3} = A\left(\frac{1}{9}\right)^2 + B(0)\left(\frac{1}{3}\right) + C(0)$$

$$-\frac{2}{3} = A\left(\frac{1}{9}\right)^2 + B(0) + C(0)$$

$$-\frac{2}{3} = \frac{1}{9}A$$

Or $\quad \frac{1}{9}A = -\frac{2}{3}$

$$A = -\frac{2}{3} \times \frac{9}{1}$$
$$A = -6$$

To find C, we put $(x+1)^2 = 0 \Rightarrow x+1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$7(-1)+4 = A(-1+1)^2 + B(3(-1)+2)(-1+1) + C(3(-1)+2)$$
$$-7+4 = A(0)^2 + B(-3+2)(0) + C(-3+2)$$
$$-3 = A(0) + B(-1)(0) + C(-1)$$
$$-3 = A(0) + B(0) + C(-1)$$
$$-3 = -C$$

Or $\quad -C = -3$

$\Rightarrow \quad C = 3$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$$A + 3B = 0$$
$$-6 + 3B = 0$$
$$3B = 6$$
$$\Rightarrow \quad B = 2$$

Thus required partial fractions are $\frac{-6}{3x+2} + \frac{2}{x+1} + \frac{3}{(x+1)^2}$

Hence, $\frac{7x+4}{(3x+2)(x+1)^2} = \frac{-6}{3x+2} + \frac{2}{x+1} + \frac{3}{(x+1)^2}$

(6) $\frac{1}{(x-1)^2(x+1)}$

Solution:

Let $\frac{1}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$

Multiplying both sides by $(x-1)^2(x+1)$, we get

$$1 = A(x-1)(x+1) + B(x+1) + C(x-1)^2 \qquad \text{———— (1)}$$
$$1 = A(x^2-1) + B(x+1) + C(x^2-2x+1)$$
$$1 = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$$
$$1 = Ax^2 + Cx^2 + Bx - 2Cx - A + B + C \qquad \text{———— (2)}$$

To find C, we put $x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$1 = A(-1-1)(-1+1) + B(-1+1) + C(-1-1)^2$$

$$1 = A(-2)(0) + B(0) + C(-2)^2$$

$$1 = A(0) + B(0) + C(4)$$

$$1 = 4C$$

Or $4C = 1$

$\Rightarrow$ $C = \frac{1}{4}$

To find B, we put $(x-1)^2 = 0 \Rightarrow x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$1 = A(1-1)(1+1) + B(1+1) + C(1-1)^2$$

$$1 = A(0)(2) + B(2) + C(0)^2$$

$$1 = A(0) + B(2) + C(0)$$

$$1 = 2B$$

Or $2B = 1$

$\Rightarrow$ $B = \frac{1}{2}$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$$A + C = 0$$

$$A + \frac{1}{4} = 0$$

$$A = -\frac{1}{4}$$

Thus required partial fractions are $\frac{-1/4}{x-1} + \frac{1/2}{(x-1)^2} + \frac{1/4}{x+1}$

Hence, $\frac{1}{(x-2)^2(x+1)} = -\frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} + \frac{1}{4(x+1)}$

(7) $\frac{3x^2 + 15x + 16}{(x+2)^2}$

Solution:

$$\begin{array}{r|l} & \quad\quad\quad\quad 3 \\ x^2 + 4x + 4 & 3x^2 + 15x + 16 \\ & \underline{\pm 3x^2 \pm 12x \pm 12} \\ & \quad 3x + 4 \end{array}$$

$$\frac{3x^2 + 15x + 16}{x^2 + 4x + 4} = 3 + \frac{3x + 4}{(x+2)^2}$$

Let $\dfrac{3x+4}{(x+2)^2}=\dfrac{A}{x+2}+\dfrac{B}{(x+2)^2}$

Multiplying both sides by $(x+1)^2$, we get

$$3x+4=A(x+2)+B \qquad \text{\_\_\_\_\_\_\_\_ (1)}$$

$$3x+4=Ax+2A+B \qquad \text{\_\_\_\_\_\_\_\_ (2)}$$

To find B, we put $(x+2)^2=0 \Rightarrow x+2=0 \Rightarrow x=-2$ in eq. (1), we get

$$3(-2)+4=A(-2+2)+B$$

$$-6+4=A(0)+B$$

$$-2=B$$

Or $\quad B=-2$

To find A, equating coefficient of x on both sides of eq. (2), we get

$$A = 3$$

Thus required partial fractions are $\dfrac{3}{x+2}+\dfrac{-2}{(x+2)^2}$

Hence, $\dfrac{3x^2+15x+16}{(x+2)^2}=3+\dfrac{3}{x+2}-\dfrac{2}{(x+2)^2}$

(8) $\dfrac{1}{(x^2-1)(x+1)}$

Solution:

Let $\dfrac{1}{(x^2-1)(x+1)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

Multiplying both sides by $(x-1)(x+1)^2$, we get

$$1=A(x+1)^2+B(x-1)(x+1)+C(x-1) \qquad \text{\_\_\_\_\_\_\_\_ (1)}$$

$$1=A(x^2+2x+1)+B(x^2-1)+C(x-1)$$

$$1=Ax^2+2Ax+A+Bx^2-B+Cx-C$$

$$1=Ax^2+Bx^2+2Ax+Cx+A-B-C \qquad \text{\_\_\_\_\_\_\_\_ (2)}$$

To find B, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$1=A(1+1)^2+B(1-1)(1+1)+C(1-1)$$

$$1=A(2)^2+B(0)(2)+C(0)$$

$$1=A(4)+B(0)+C(0)$$

$$1=4A$$

Or $\quad 4A=1$

$\Rightarrow \quad A = \frac{1}{4}$

To find B, we put $(x+1)^2 = 0 \Rightarrow x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$1 = A(-1+1)^2 + B(-1-1)(-1+1) + C(-1-1)$$

$$1 = A(0)^2 + B(-2)(0) + C(-2)$$

$$1 = A(0) + B(0) + C(-2)$$

$$1 = -2C$$

Or $\quad -2C = 1$

$\Rightarrow \quad C = -\frac{1}{2}$

To find A, equating coefficient of x^2 on both sides of eq. (2), we get

$$A + B = 0$$

$$\frac{1}{4} + B = 0$$

$$B = -\frac{1}{4}$$

Thus required partial fractions are $\frac{1/4}{x-1} + \frac{-1/4}{x+1} + \frac{-1/2}{(x+1)^2}$

Hence, $\frac{1}{(x-1)(x+1)^2} = \frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{1}{2(x+1)^2}$

Resolution of fraction when D (x) consists of non-repeated irreducible quadratic factors.

Rule III:

If a quadratic factor $(ax^2 + bx + c)$ with $a \neq 0$ occur once as a factor of $D(x)$, the partial fraction is of the form $\frac{Ax+B}{(ax^2+bx+c)}$ where A and B are constants to be found.

SOLVED EXERCISE 4.3

Resolve into partial fractions.

(1) $\frac{3x-11}{(x+3)(x^2+1)}$

Solution:

$$\frac{3x-11}{(x+3)(x^2+1)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+1}$$