$$\underline{u}\times\underline{v} = a_1b_2\,\underline{k} - a_1c_2\,\underline{j} - b_1a_2\,\underline{k} + b_1c_2\,\underline{i} + c_1a_2\,\underline{j} - c_1b_2\,\underline{i}$$

$$\underline{u}\times\underline{v} = (b_1c_2 - b_2c_1)\underline{i} - (a_1c_2 - a_2c_1)\underline{j} + (a_1b_2 - a_2b_1)\underline{k}$$

$$\Rightarrow \underline{u}\times\underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Parallel Vectors:-

$\underline{u} \parallel \underline{v}$ iff $\underline{u}\times\underline{v} = \underline{0}$

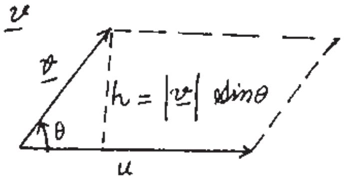
Note:- For every vector $\underline{u}$, we know that

$\underline{0}\times\underline{u} = \underline{0}$

∴ we say that zero vector is parallel to every vector.

Area of Parallelogram:-

Let $\underline{u}$ and $\underline{v}$ be along the adjacent sides of a $\parallel$gm.



and θ is the angle between them.

Then $|\underline{u}|$ and $|\underline{v}|$ are the length of adjacent sides of the $\parallel$gm.

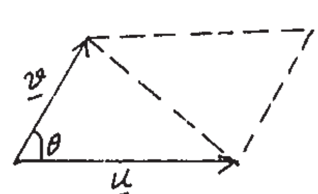
$$\text{Area of the } \parallel\text{gm} = (\text{base})\ (\text{height})$$
$$= |\underline{u}|\ (|\underline{v}|\sin\theta)$$
$$= |\underline{u}|\,|\underline{v}|\sin\theta$$
$$= |\underline{u}\times\underline{v}|$$

Area of Triangle:-

From fig.

$$\text{Area of triangle} = \frac{1}{2}(\text{area of } \parallel\text{gm})$$
$$= \frac{1}{2}|\underline{u}\times\underline{v}|$$

Where $\underline{u}$ and $\underline{v}$ are acting along the adjacent sides of the triangle.



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Example 1:-

Find a vector perpendicular to each of the vectors $\underline{a} = 2\underline{i} - \underline{j} + \underline{k}$ and $\underline{b} = 4\underline{i} + 2\underline{j} - \underline{k}$

Sol:- Required vector $= \underline{a}\times\underline{b}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ 4 & 2 & -1 \end{vmatrix}$$

$$= \underline{i}(+1-2) - \underline{j}(-2-4) + \underline{k}(4+4)$$
$$= -\underline{i} + 6\underline{j} + 8\underline{k} \quad \text{Ans.}$$

✱ EXERCISE 7.4 ✱

① (i) Given that

$\underline{a} = 2\underline{i} + \underline{j} - \underline{k}$ and $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Now

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\underline{a}\times\underline{b} = \underline{i}(1-1) - \underline{j}(2+1) + \underline{k}(-2-1)$$

$$\boxed{\underline{a}\times\underline{b} = 0\underline{i} - 3\underline{j} - 3\underline{k}}$$

$$\underline{b}\times\underline{a} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \underline{i}(1-1) - \underline{j}(-1-2) + \underline{k}(1+2)$$

$$\boxed{\underline{b}\times\underline{a} = 0\underline{i} + 3\underline{j} + 3\underline{k}}$$

$$\underline{a}\cdot(\underline{a}\times\underline{b}) = (2\underline{i}+\underline{j}-\underline{k})\cdot(0\underline{i}-3\underline{j}-3\underline{k})$$
$$= (2)(0)+(1)(-3)+(-1)(-3) = 0-3+3$$
$$= 0 \Rightarrow \underline{a} \perp \underline{a}\times\underline{b}$$

$$\underline{a}\cdot(\underline{b}\times\underline{a}) = (2\underline{i}+\underline{j}-\underline{k})\cdot(0\underline{i}+3\underline{j}+3\underline{k})$$
$$= (2)(0)+(1)(3)+(-1)(3)$$
$$= 0+3-3 = 0 \Rightarrow \underline{a} \perp \underline{b}\times\underline{a}$$

$$\underline{b}\cdot(\underline{a}\times\underline{b}) = (\underline{i}-\underline{j}+\underline{k})\cdot(0\underline{i}-3\underline{j}-3\underline{k})$$
$$= (1)(0)+(-1)(-3)+(1)(-3)$$
$$= 0+3-3 = 0 \Rightarrow \underline{b} \perp \underline{a}\times\underline{b}$$

$$\underline{b}\cdot(\underline{b}\times\underline{a}) = (\underline{i}-\underline{j}+\underline{k})\cdot(0\underline{i}+3\underline{j}+3\underline{k})$$
$$= (1)(0)+(-1)(3)+(1)(3)$$
$$= 0-3+3 = 0 \Rightarrow \underline{b} \perp \underline{b}\times\underline{a}$$

(ii) Given that

$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + 0\underline{k}$

$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + 0\underline{k}$

$$a \times b = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix}$$

$$\Rightarrow a \times b = i(0-0) - j(0-0) + k(-1-1)$$

$$\Rightarrow a \times b = 0i - 0j - 2k = -2k$$

$$b \times a = \begin{vmatrix} i & j & k \\ 1 & -1 & 0 \\ 1 & 1 & 0 \end{vmatrix} = i(0-0) - j(0-0) + k(1+1)$$

$$b \times a = 0i - 0j + 2k = 2k$$

$$a \cdot (a \times b) = (i + j + 0k) \cdot (0i - 0j - 2k)$$
$$= (1)(0) + (1)(0) + (0)(-2) = 0+0+0$$
$$= 0 \Rightarrow a \perp a \times b$$

$$a \cdot (b \times a) = (i + j + 0k) \cdot (0i - 0j + 2k)$$
$$= (1)(0) + (1)(0) + (0)(2) = 0+0+0$$
$$= 0 \Rightarrow a \perp b \times a$$

$$b \cdot (a \times b) = (i - j + 0k) \cdot (0i - 0j - 2k)$$
$$= (1)(0) + (-1)(0) + (0)(-2) = 0+0+0$$
$$= 0 \Rightarrow b \perp a \times b$$

$$b \cdot (b \times a) = (i - j + 0k) \cdot (0i - 0j + 2k)$$
$$= (1)(0) + (-1)(0) + (0)(2) = 0+0+0$$
$$= 0 \Rightarrow b \perp b \times a$$

(iii) Given that

$a = 3i - 2j + k$, $b = i + j + 0k$

$$a \times b = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = i(0-1) - j(0-1) + k(3+2)$$

$$\boxed{a \times b = -i + j + 5k}$$

$$b \times a = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$$b \times a = i(1-0) - j(1-0) + k(-2-3)$$

$$\boxed{b \times a = i - j - 5k}$$

$$a \cdot (a \times b) = (3i - 2j + k) \cdot (-i + j + 5k)$$
$$= (3)(-1) + (-2)(1) + (1)(5) = -3 - 2 + 5$$
$$= 0 \Rightarrow a \perp a \times b$$

$$b \cdot (a \times b) = (i + j + 0k) \cdot (-i + j + 5k)$$
$$= (1)(-1) + (1)(1) + (0)(5) = -1 + 1 + 0$$
$$= 0 \Rightarrow b \perp a \times b$$

$$b \cdot (b \times a) = (i + j + 0k) \cdot (i - j - 5k)$$
$$= (1)(1) + (1)(-1) + (0)(-5) = 1 - 1 + 0$$
$$= 0 \Rightarrow b \perp b \times a$$

$$a \cdot (b \times a) = (3i - 2j + k) \cdot (i - j - 5k)$$
$$= (3)(1) + (-2)(-1) + (1)(-5)$$
$$= 3 + 2 - 5 = 0 \Rightarrow a \perp b \times a$$

(iv) Given that

$a = -4i + j - 2k$ and

$b = 2i + j + k$

$$a \times b = \begin{vmatrix} i & j & k \\ -4 & 1 & -2 \\ 2 & 1 & 1 \end{vmatrix} = i(1+2) - j(-4+4) + k(-4-2)$$

$$\Rightarrow \boxed{a \times b = 3i - 0j - 6k}$$

$$b \times a = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 1 & -2 \end{vmatrix}$$

$$b \times a = i(-2-1) - j(-4+4) + k(2+4)$$

$$\boxed{b \times a = -3i - 0j + 6k}$$

$$a \cdot (a \times b) = (-4i + j - 2k) \cdot (3i - 0j - 6k)$$
$$= (-4)(3) + (1)(0) + (-2)(-6) = -12 + 0 + 12$$
$$= 0 \Rightarrow a \perp a \times b$$

$$a \cdot (b \times a) = (-4i + j - 2k) \cdot (-3i - 0j + 6k)$$
$$= (-4)(-3) + (1)(0) + (-2)(6) = 12 + 0 - 12$$
$$= 0 \Rightarrow a \perp b \times a$$

$$b \cdot (a \times b) = (2i + j + k) \cdot (3i - 0j - 6k)$$
$$= (2)(3) + (1)(0) + (1)(-6) = 6 + 0 - 6$$
$$= 0 \Rightarrow b \perp a \times b$$

$$b \cdot (b \times a) = (2i + j + k) \cdot (-3i - 0j + 6k)$$
$$= (2)(-3) + (1)(0) + (1)(6) = -6 + 0 + 6$$
$$= 0 \Rightarrow b \perp b \times a$$

To Find a Unit Vector perpendicular to The Plane Containing a and b and To find Sine of the angle between them.

We know that

$$a \times b = |a|\,|b| \sin\theta \; \hat{n} \quad \text{------} (1)$$

$$|a \times b| = |a|\,|b| \sin\theta \quad \text{——} (2) \quad \because |\hat{n}| = 1$$

Dividing (1) by (2), we get

$$\frac{a \times b}{|a \times b|} = \hat{n}$$

or $\boxed{\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}}$

From ② $\dfrac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|} = \sin\theta$ or

$$\boxed{\sin\theta = \frac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|}}$$

---

② (i) Given that

$\underline{a} = 2\underline{i} - 6\underline{j} - 3\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+36+9} = \sqrt{49} = 7$

$\underline{b} = 4\underline{i} + 3\underline{j} - \underline{k} \Rightarrow |\underline{b}| = \sqrt{16+9+1} = \sqrt{26}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = \underline{i}(6+9) - \underline{j}(-2+12) + \underline{k}(6+24)$$

$\underline{a} \times \underline{b} = 15\underline{i} - 10\underline{j} + 30\underline{k}$

$\Rightarrow |\underline{a} \times \underline{b}| = \sqrt{(15)^2 + (-10)^2 + (30)^2} = \sqrt{225+100+900} = \sqrt{1225} = 35$

$\because \hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$\Rightarrow \hat{n} = \dfrac{15\underline{i} - 10\underline{j} + 30\underline{k}}{35} = \dfrac{15}{35}\underline{i} - \dfrac{10}{35}\underline{j} + \dfrac{30}{35}\underline{k}$

$\Rightarrow \boxed{\hat{n} = \dfrac{3}{7}\underline{i} - \dfrac{2}{7}\underline{j} + \dfrac{6}{7}\underline{k}}$

If θ be the angle between $\underline{a}$ and $\underline{b}$, then $\sin\theta = \dfrac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|}$

$\Rightarrow \sin\theta = \dfrac{35}{(7)(\sqrt{26})} \Rightarrow \boxed{\sin\theta = \dfrac{5}{\sqrt{26}}}$ Ans

---

(ii) Given that

$\underline{a} = -\underline{i} - \underline{j} - \underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+1} = \sqrt{3}$

$\underline{b} = 2\underline{i} - 3\underline{j} + 4\underline{k} \Rightarrow |\underline{b}| = \sqrt{4+9+16} = \sqrt{29}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 2 & -3 & 4 \end{vmatrix} = \underline{i}(-4-3) - \underline{j}(-4+2) + \underline{k}(3+2)$$

$\Rightarrow \underline{a} \times \underline{b} = -7\underline{i} + 2\underline{j} + 5\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{(-7)^2 + (2)^2 + (5)^2} = \sqrt{49+4+25}$

$|\underline{a} \times \underline{b}| = \sqrt{78}$ Let $\hat{n}$ be the required unit vector

Then $\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}$

$= \dfrac{-7\underline{i} + 2\underline{j} + 5\underline{k}}{\sqrt{78}}$

$\hat{n} = -\dfrac{7}{\sqrt{78}}\underline{i} + \dfrac{2}{\sqrt{78}}\underline{j} + \dfrac{5}{\sqrt{78}}\underline{k}$ Ans.

22 Let θ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \dfrac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|} = \dfrac{\sqrt{78}}{\sqrt{3}\sqrt{29}} = \dfrac{\sqrt{78}}{\sqrt{3 \times 29}}$

$= \dfrac{\sqrt{78}}{\sqrt{87}} = \sqrt{\dfrac{78}{87}} = \sqrt{\dfrac{26}{29}}$ Ans

---

(iii) Given that

$\underline{a} = 2\underline{i} - 2\underline{j} + 4\underline{k} \Rightarrow |\underline{a}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

$\underline{b} = -\underline{i} + \underline{j} - 2\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+4} = \sqrt{6}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{vmatrix} = \underline{i}(4-4) - \underline{j}(-4+4) + \underline{k}(2-2)$$

$\underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} + 0\underline{k}$

$|\underline{a} \times \underline{b}| = \sqrt{0^2+0^2+0^2} = \sqrt{0} = 0$

Let $\hat{n}$ be the required unit vector, then $\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{0\underline{i} - 0\underline{j} + 0\underline{k}}{0}$

$\Rightarrow \hat{n}$ is arbitrary (not unique)

Let θ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

$\sin\theta = \dfrac{|\underline{a} \times \underline{b}|}{|\underline{a}|\,|\underline{b}|} = \dfrac{0}{(2\sqrt{6})(\sqrt{6})} = 0$ Ans

---

(iv) Given that

$\underline{a} = \underline{i} + \underline{j} = \underline{i} + \underline{j} + 0\underline{k} \Rightarrow |\underline{a}| = \sqrt{1+1+0} = \sqrt{2}$

$\underline{b} = \underline{i} - \underline{j} = \underline{i} - \underline{j} + 0\underline{k} \Rightarrow |\underline{b}| = \sqrt{1+1+0} = \sqrt{2}$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{vmatrix} = \underline{i}(0-0) - \underline{j}(0-0) + \underline{k}(-1-1)$$

$\Rightarrow \underline{a} \times \underline{b} = 0\underline{i} - 0\underline{j} - 2\underline{k}$

$\Rightarrow |\underline{a} \times \underline{b}| = \sqrt{0^2+0^2+(-2)^2} = \sqrt{4} = 2$

Let $\hat{n}$ be the required unit vector, then

$\hat{n} = \dfrac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|} = \dfrac{0\underline{i} - 0\underline{j} - 2\underline{k}}{2}$

$\hat{n} = \dfrac{0}{2}\underline{i} - \dfrac{0}{2}\underline{j} - \dfrac{2}{2}\underline{k} = -\underline{k}$ Ans.

Let θ be the angle between the vectors $\underline{a}$ and $\underline{b}$, then

$$\sin\theta = \frac{|a \times b|}{|a|\,|b|}$$

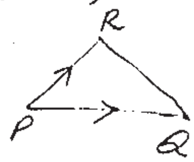
$$\Rightarrow \sin\theta = \frac{2}{\sqrt{2}\,\sqrt{2}} = \frac{2}{(\sqrt{2})^2} = \frac{2}{2} = 1 \quad \text{Ans}$$

---

③ (i) Given that

$P(0,0,0), Q(2,3,2), R(-1,1,4)$

Then



$\vec{PQ} = (2-0)i + (3-0)j + (2-0)k$

$\vec{PQ} = 2i + 3j + 2k$

$\vec{PR} = (-1-0)i + (1-0)j + (4-0)k = -i + j + 4k$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 3 & 2 \\ -1 & 1 & 4 \end{vmatrix}$$

$= i(12-2) - j(8+2) + k(2+3)$

$= 10i - 10j + 5k$

$|\vec{PQ} \times \vec{PR}| = \sqrt{(10)^2 + (-10)^2 + (5)^2}$

$= \sqrt{100 + 100 + 25} = \sqrt{225} = 15$

Now Area of $\Delta PQR = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2}(15) = \frac{15}{2}$ Square units.

(ii) Given that $P(1,-1,-1), Q(2,0,-1)$ and $R(0,2,1)$. Then

$\vec{PQ} = (2-1)i + (0+1)j + (-1+1)k$

$\Rightarrow \vec{PQ} = i + j + 0k$

$\vec{PR} = (0-1)i + (2+1)j + (1+1)k = -i + 3j + 2k$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$= i(2-0) - j(2-0) + k(3+1)$

$\vec{PQ} \times \vec{PR} = 2i - 2j + 4k$

$|\vec{PQ} \times \vec{PR}| = \sqrt{4+4+16} = \sqrt{24} = 2\sqrt{6}$

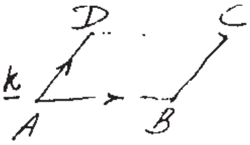
Now area of $\Delta PQR = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$= \frac{1}{2}(2\sqrt{6}) = \sqrt{6}$ square units.

---

④ (i) Given that $A(0,0,0), B(1,2,3)$

$C(2,-1,1), D(3,1,4)$



$\vec{AB} = (1-0)i + (2-0)j + (3-0)k$

$\vec{AB} = i + 2j + 3k$

$\vec{AD} = (3-0)i + (1-0)j + (4-0)k$

$\vec{AD} = 3i + j + 4k$

Now area of ||gm ABCD

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 3 & 1 & 4 \end{vmatrix}$$

$= i(8-3) - j(4-9) + k(1-6)$

$\vec{AB} \times \vec{AD} = 5i + 5j - 5k$

$|\vec{AB} \times \vec{AD}| = \sqrt{25+25+25} = \sqrt{75} = 5\sqrt{3}$

Area of ||gm ABCD $= |\vec{AB} \times \vec{AD}| = 5\sqrt{3}$ Ans

---

(ii) Given that $A(1,2,-1), B(4,2,-3)$

$C(6,-5,2)$ & $D(9,-5,0)$

$\vec{AB} = (4-1)i + (2-2)j + (-3+1)k = 3i + 0j - 2k$

$\vec{AD} = (9-1)i + (-5-2)j + (0+1)k = 8i - 7j + k$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 3 & 0 & -2 \\ 8 & -7 & 1 \end{vmatrix}$$

$= i(0-14) - j(3+16) + k(-21-0)$

$\vec{AB} \times \vec{AD} = -14i - 19j - 21k$

$\therefore$ Area of ||gm ABCD $= |\vec{AB} \times \vec{AD}|$

$= \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$

$= \sqrt{196 + 361 + 441} = \sqrt{998}$ Ans

(iii) Given that $A(-1,1,1), B(-1,2,2)$

$C(-3,4,-5)$ and $D(-3,5,-4)$

$\vec{AB} = (-1+1)i + (2-1)j + (2-1)k = 0i + j + k$

$\vec{AD} = (-3+1)i + (5-1)j + (-4-1)k = -2i + 4j - 5k$

$$\vec{AB} \times \vec{AD} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 4 & -5 \end{vmatrix}$$

$= i(-5-4) - j(0+2) + k(0+2)$

$\vec{AB} \times \vec{AD} = -9i - 2j + 2k$

$\therefore$ Area of ||gm ABCD $= |\vec{AB} \times \vec{AD}|$

$= \sqrt{81+4+4} = \sqrt{89}$ Ans

(5) (i) Given that

$u = 5i - j + k$

$v = j - 5k = 0i + j - 5k$

$w = -15i + 3j - 3k$

$\because w = -3(5i - j + k)$

$\Rightarrow w = -3u$

$\Rightarrow w \parallel u$ in opposite direction.

(ii) Given that

$u = i + 2j - k$, $v = -i + j + k$

$w = -\frac{\pi}{2} i - \pi j + \frac{\pi}{2} k$

$\because u \cdot v = (i + 2j - k)\cdot(-i + j + k)$

$= (1)(-1) + (2)(1) + (-1)(1)$

$= -1 + 2 - 1 = 0$

$\therefore u \perp v$

$\because v \cdot w = (-i + j + k)\cdot(-\frac{\pi}{2} i - \pi j + \frac{\pi}{2} k)$

$= (-1)(-\frac{\pi}{2}) + (1)(-\pi) + (1)(\frac{\pi}{2})$

$= \frac{\pi}{2} - \pi + \frac{\pi}{2} = \frac{\pi - 2\pi + \pi}{2} = \frac{0}{2}$

$\Rightarrow v \cdot w = 0$

$\therefore v \perp w$

Also $w = -\frac{\pi}{2} i - \pi j + \frac{\pi}{2} k$

$\Rightarrow w = -\frac{\pi}{2}(i + 2j - k)$

$\Rightarrow w = -\frac{\pi}{2} u$

$\Rightarrow w \parallel u$ in opposite direction.

---

(6) Prove that $a \times (b + c) + b \times (c + a) + c \times (a + b) = 0$

Sol:- $L.H.S. = a \times (b + c) + b \times (c + a) + c \times (a + b)$

$= a \times b + a \times c + b \times c + b \times a + c \times a + c \times b$

$= a \times b - c \times a + b \times c - a \times b + c \times a - b \times c$

$\because a \times c = -c \times a$

$b \times a = -a \times b$ and

$c \times b = -b \times c$

$\therefore L.H.S. = 0 = R.H.S.$

---

(7) If $a + b + c = 0$, then prove that

$a \times b = b \times c = c \times a$

Sol:- Given that

$a + b + c = 0$ —— (1)

Taking cross product of (1) with a we get

$a \times (a + b + c) = a \times 0$

$\Rightarrow a \times a + a \times b + a \times c = 0$

$\Rightarrow 0 + a \times b - c \times a = 0$ $\because a \times c = -c \times a$

$\Rightarrow a \times b = c \times a$ —— (2) & $a \times a = 0$

Taking cross product of (1) with b, we get

$b \times (a + b + c) = b \times 0$

$\Rightarrow b \times a + b \times b + b \times c = 0$

$\Rightarrow -a \times b + 0 + b \times c = 0$ $\because b \times a = -a \times b$

$\Rightarrow b \times c = a \times b$

$\Rightarrow a \times b = b \times c$ —— (3)

$\therefore$ From (2) & (3), we get

$a \times b = b \times c = c \times a$ (Proved)

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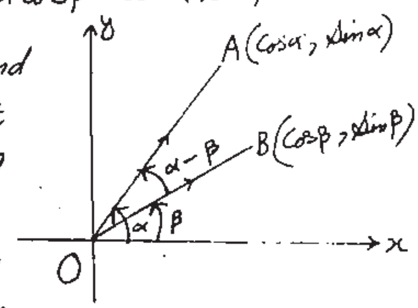
(8) Prove that

$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

Sol:- Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be the unit vectors making angles α and β with x-axis respectively. Then



$\overrightarrow{OA} = \cos\alpha\, i + \sin\alpha\, j$

$\overrightarrow{OB} = \cos\beta\, i + \sin\beta\, j$

$$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} i & j & k \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$|\overrightarrow{OB}|\,|\overrightarrow{OA}| \sin(\alpha - \beta)\, k = i(0-0) - j(0-0) + k(\cos\beta \sin\alpha - \sin\beta \cos\alpha)$

$(1)(1)\sin(\alpha - \beta) k = (\sin\alpha \cos\beta - \cos\alpha \sin\beta) k$

$\sin(\alpha - \beta)\, k = (\sin\alpha \cos\beta - \cos\alpha \sin\beta) k$

$\Rightarrow \sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

(Proved)

(9) $\underline{a} \times \underline{b} = \underline{0}$

$\Rightarrow \underline{a} \parallel \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$

or $\underline{a} = \underline{b} = \underline{0}$

Now $\underline{a} \cdot \underline{b} = 0$

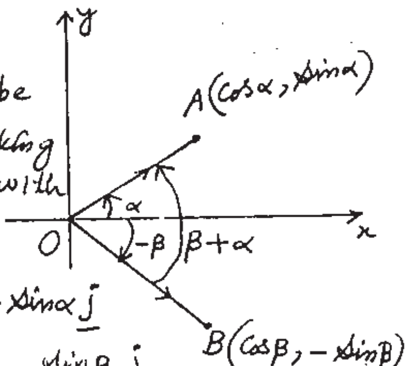
$\Rightarrow \underline{a} \perp \underline{b}$ or $\underline{a} = \underline{0}$ or $\underline{b} = \underline{0}$ or

$\underline{a} = \underline{b} = \underline{0}$

Example 3 :- Prove that

$\sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

Sol:-



Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ be unit vectors making angles α and β with x-axis resp.

Then $\overrightarrow{OA} = \cos\alpha\, \underline{i} + \sin\alpha\, \underline{j}$

$\overrightarrow{OB} = \cos\beta\, \underline{i} - \sin\beta\, \underline{j}$

$$\overrightarrow{OB} \times \overrightarrow{OA} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \cos\beta & -\sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix}$$

$$\Rightarrow |\overrightarrow{OB}|\,|\overrightarrow{OA}| \sin(\beta+\alpha)\, \underline{k} = \underline{i}\,(0-0) - \underline{j}\,(0-0) + \underline{k}\,(\sin\alpha \cos\beta + \cos\alpha \sin\beta)$$

$$\Rightarrow (1)\,(1) \sin(\alpha+\beta)\, \underline{k} = (\sin\alpha \cos\beta + \cos\alpha \sin\beta)\, \underline{k}$$

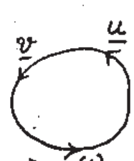
$$\Rightarrow \sin(\alpha+\beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

(Proved)

Scalar Triple Product of Vectors :-

The scalar triple product of vectors $\underline{u}$, $\underline{v}$ and $\underline{w}$ is defined by

$\underline{u} \cdot (\underline{v} \times \underline{w})$ or $(\underline{u} \times \underline{v}) \cdot \underline{w}$



Note that

$\underline{u} \cdot (\underline{v} \times \underline{w}) = \underline{v} \cdot (\underline{w} \times \underline{u}) = \underline{w} \cdot (\underline{u} \times \underline{v})$

This can be written as

$[\underline{u}\ \underline{v}\ \underline{w}] = [\underline{v}\ \underline{w}\ \underline{u}] = [\underline{w}\ \underline{u}\ \underline{v}]$

Component Form :-

Let $\underline{u} = a_1 \underline{i} + b_1 \underline{j} + c_1 \underline{k}$

$\underline{v} = a_2 \underline{i} + b_2 \underline{j} + c_2 \underline{k}$

$\underline{w} = a_3 \underline{i} + b_3 \underline{j} + c_3 \underline{k}$

To find $\underline{u} \cdot (\underline{v} \times \underline{w})$, we have

$$\underline{v} \times \underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\underline{v} \times \underline{w} = (b_2 c_3 - b_3 c_2)\, \underline{i} - (a_2 c_3 - a_3 c_2)\, \underline{j} + (a_2 b_3 - a_3 b_2)\, \underline{k}$$

Now

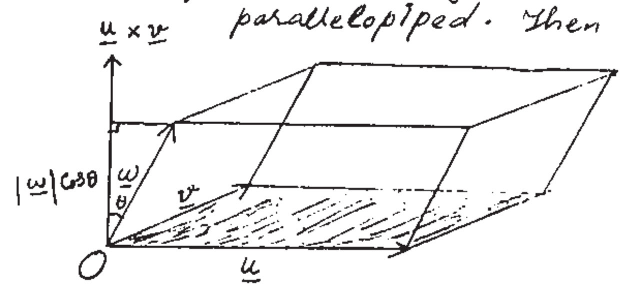
$$\underline{u} \cdot (\underline{v} \times \underline{w}) = a_1 (b_2 c_3 - b_3 c_2) - b_1 (a_2 c_3 - a_3 c_2) + c_1 (a_2 b_3 - a_3 b_2)$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(Determinant formula for scalar triple product of $\underline{u}$, $\underline{v}$ and $\underline{w}$)

The Volume of parallelopiped :-

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ are along coterminous edges of parallelopiped. Then



Area of $\parallel$opiped = (Area of $\parallel$gm) (height)

$= |\underline{u} \times \underline{v}|\ \ |\underline{w}| \cos\theta$

$= (\underline{u} \times \underline{v}) \cdot \underline{w} = [\underline{u}\ \underline{v}\ \underline{w}]$