Exercise 8.3

Binomial Theorem when n is negative or fraction:

When n is negative or fraction and $|x|<1$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Where the general term of binomial expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\ldots\big(n-(r-1)\big)}{r!}x^r$$

Question # 1

Expand the following upto 4 times, taking the values of x such that the expansion in each case is valid.

(i) $(1-x)^{\frac{1}{2}}$ (ii) $(1+2x)^{-1}$ (iii) $(1+x)^{-\frac{1}{3}}$ (iv) $(4-3x)^{\frac{1}{2}}$

(v) $(8-2x)^{-1}$ (vi) $(2-3x)^{-2}$ (vii) $\dfrac{(1-x)^{-1}}{(1+x)^2}$ (viii) $\dfrac{\sqrt{(1+2x)}}{(1-x)}$

(ix) $\dfrac{(4+2x)^{\frac{1}{2}}}{(2-x)}$ (x) $(1+x-2x^2)^{\frac{1}{2}}$ (xi) $(1-2x+3x^2)^{\frac{1}{2}}$

Solution

(i) $$(1-x)^{\frac{1}{2}} = 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \ldots$$

$$= 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}(-x^3) + \ldots$$

$$= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \ldots$$

(ii) *Do yourself as above*

(iii) *Do yourself as above*

(iv) $$(4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3x}{4}\right)\right]^{\frac{1}{2}} = (4)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}} = 2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$$

$$= 2\left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \ldots\right]$$

$$=2\left[1-\frac{3x}{8}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{9x^2}{16}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(-\frac{27x^3}{64}\right)+\ldots\right]$$

$$=2\left[1-\frac{3x}{8}-\frac{1}{8}\left(\frac{9x^2}{16}\right)-\frac{1}{16}\left(\frac{27x^3}{64}\right)+\ldots\right]$$

$$=2\left[1-\frac{3x}{8}-\frac{9x^2}{128}-\frac{27x^3}{1024}+\ldots\right]$$

$$=2-\frac{3x}{4}-\frac{9x^2}{64}-\frac{27x^3}{512}+\ldots$$

(v) $(8-2x)^{\frac{1}{2}}=(8)^{-1}\left(1-\frac{2x}{8}\right)^{-1}=\frac{1}{8}\left(1-\frac{x}{4}\right)^{-1}$ *Now do yourself*

(vi) *Do yourself*

(vii) $\frac{(1-x)^{-1}}{(1+x)^2}=(1-x)^{-1}(1+x)^{-2}$

$$=\left(1+(-1)(-x)+\frac{(-1)(-1-1)}{2!}(-x)^2+\frac{(-1)(-1-1)(-1-2)}{3!}(-x)^3+\ldots\right)$$

$$\times\left(1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\ldots\right)$$

$$=\left(1+x+\frac{(-1)(-2)}{2}(x^2)+\frac{(-1)(-2)(-3)}{3\cdot 2}(-x^3)+\ldots\right)$$

$$\times\left(1-2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(x)^3+\ldots\right)$$

$$=\left(1+x+x^2+x^3+\ldots\ldots\ldots\ldots\right)\times\left(1-2x+3x^2-4x^3+\ldots\right)$$

$$=1+(x-2x)+(x^2-2x^2+3x^2)+(x^3-2x^3+3x^3-4x^3)+\ldots$$

$$=1-x+2x^2-2x^3+\ldots$$

(viii) *Do yourself as above*

(ix) $\frac{(4+2x)^{\frac{1}{2}}}{2-x}=(4+2x)^{\frac{1}{2}}(2-x)^{-1}=(4)^{\frac{1}{2}}\left(1+\frac{2x}{4}\right)^{\frac{1}{2}}(2)^{-1}\left(1-\frac{x}{2}\right)^{-1}$

$$=(4)^{\frac{1}{2}}\left(1+\frac{x}{2}\right)^{\frac{1}{2}}(2)^{-1}\left(1-\frac{x}{2}\right)^{-1}=2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\frac{1}{2}\left(1-\frac{x}{2}\right)^{-1}=\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{x}{2}\right)^{\frac{1}{2}}\left(1-\frac{x}{2}\right)^{-1}$$

$$=\left(1+\frac{1}{2}\left(\frac{x}{2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3+\ldots\right)$$

$$\times\left(1+(-1)\left(-\frac{x}{2}\right)+\frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3+\ldots\right)$$

$$=\left(1+\frac{x}{4}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x^2}{4}\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}\left(\frac{x^3}{8}\right)+\ldots\right)$$

$$\times\left(1+\frac{x}{2}+\frac{(-1)(-2)}{2}\left(\frac{x^2}{4}\right)+\frac{(-1)(-2)(-3)}{3\cdot 2}\left(-\frac{x^3}{8}\right)+\ldots\right)$$

$$=\left(1+\frac{x}{4}-\frac{x^2}{32}+\frac{x^3}{128}+\ldots\right)\times\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right)$$

$$=1+\left(\frac{x}{4}+\frac{x}{2}\right)+\left(-\frac{x^2}{32}+\frac{x^2}{8}+\frac{x^2}{4}\right)+\left(\frac{x^3}{128}-\frac{x^3}{64}+\frac{x^3}{16}+\frac{x^3}{8}\right)+\ldots$$

$$=1+\frac{3x}{4}+\frac{11x^2}{32}+\frac{23x^3}{128}+\ldots$$

(x) $\left(1+x-2x^2\right)^{\frac{1}{2}}=\left(1+(x-2x^2)\right)^{\frac{1}{2}}$

$$=1+\frac{1}{2}(x-2x^2)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x-2x^2)^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x-2x^2)^3+\ldots$$

$$=1+\frac{1}{2}(x-2x^2)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x^2-4x^3+4x^4)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2}$$

$$\left(x^3+3(x)^2(-2x^2)+3(x)(-2x^2)^2-(2x^2)^3\right)+\ldots$$

$$=1+\frac{1}{2}(x-2x^2)-\frac{1}{8}(x^2-4x^3+4x^4)+\frac{1}{16}\left(x^3-6x^4+12x^5-8x^6\right)+\ldots$$

$$=1+\frac{1}{2}x-\frac{2}{2}x^2-\frac{1}{8}x^2-\frac{4}{8}x^3+\frac{4}{8}x^4+\frac{1}{16}x^3-\frac{6}{16}x^4+\frac{12}{16}x^5-\frac{8}{16}x^6+\ldots$$

$$=1+\frac{1}{2}x-x^2-\frac{1}{8}x^2-\frac{1}{2}x^3+\frac{1}{2}x^4+\frac{1}{16}x^3-\frac{3}{8}x^4+\frac{3}{4}x^5-\frac{1}{8}x^6+\ldots$$

$$=1+\frac{1}{2}x-\frac{9}{8}x^2-\frac{9}{16}x^3+\ldots$$

(xi) *Do yourself as above*

Question # 2

Use the Binomial theorem find the value of the following to three places of decimials.

(i) $\sqrt{99}$ (ii) $(0.98)^{\frac{1}{2}}$ (iii) $(1.03)^{\frac{1}{3}}$ (iv) $\sqrt[3]{65}$

(v) $\sqrt[4]{17}$ (vi) $\sqrt[5]{31}$ (vii) $\dfrac{1}{\sqrt[3]{998}}$ (viii) $\dfrac{1}{\sqrt[5]{252}}$

(ix) $\dfrac{\sqrt{7}}{\sqrt{8}}$ (x) $(0.998)^{-\frac{1}{3}}$ (xi) $\dfrac{1}{\sqrt[6]{486}}$ (xii) $(1280)^{\frac{1}{4}}$

Solution

(i)
$$\sqrt{99}=(99)^{\frac{1}{2}}=(100-1)^{\frac{1}{2}}=(100)^{\frac{1}{2}}\left(1-\frac{1}{100}\right)^{\frac{1}{2}}$$
$$=10\left(1+\frac{1}{2}\left(-\frac{1}{100}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{100}\right)^{2}+\ldots\right)$$
$$=10\left(1-\frac{1}{200}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{10000}\right)+\ldots\right)$$
$$=10\left(1-0.005-\frac{1}{8}(0.0001)+\ldots\right)$$
$$=10(1-0.005-0.0000125+\ldots)$$
$$\approx 10(0.9949875)=9.949875$$
$$\approx 9.950$$

(ii) $(0.98)^{\frac{1}{2}}=(1-0.02)^{\frac{1}{2}}$ *Now do yourself*

(iii) $(1.03)^{\frac{1}{3}}=(1+0.03)^{\frac{1}{3}}$ *Now do yourself*

(iv) $\sqrt[3]{65}=(65)^{\frac{1}{3}}=(64-1)^{\frac{1}{3}}=(64)^{\frac{1}{3}}\left(1-\frac{1}{64}\right)^{\frac{1}{3}}$ *Now do yourself*

(v) $\sqrt[4]{17}=(17)^{\frac{1}{4}}=(16-1)^{\frac{1}{4}}=(16)^{\frac{1}{4}}\left(1-\frac{1}{16}\right)^{\frac{1}{4}}$ *Now do yourself*

(vi) $\sqrt[5]{31}=(31)^{\frac{1}{5}}=(32-1)^{\frac{1}{5}}=(32)^{\frac{1}{5}}\left(1-\frac{1}{32}\right)^{\frac{1}{5}}$ *Now do yourself*

(vii)

$$\frac{1}{\sqrt[3]{998}}=\frac{1}{(998)^{\frac{1}{3}}}=(998)^{-\frac{1}{3}}=(1000-2)^{-\frac{1}{3}}=(1000)^{-\frac{1}{3}}\left(1-\frac{2}{1000}\right)^{-\frac{1}{3}}$$

$$=\left(10^3\right)^{-\frac{1}{3}}\left(1-\frac{1}{500}\right)^{-\frac{1}{3}}$$

$$=\left(\frac{1}{10}\right)\left(1+\left(-\frac{1}{3}\right)\left(-\frac{1}{500}\right)+\frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!}\left(-\frac{1}{500}\right)^2+\ldots\right)$$

$$=\left(\frac{1}{10}\right)\left(1+\left(\frac{1}{1500}\right)+\frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2}\left(\frac{1}{250000}\right)+\ldots\right)$$

$$=\left(\frac{1}{10}\right)\left(1+(0.0006667)+\frac{2}{9}(0.000004)+\ldots\right)$$

$$=\left(\frac{1}{10}\right)(1+0.0006667+0.00000089+\ldots)$$

$$\approx\left(\frac{1}{10}\right)(1.00066759)=0.100066759\quad\approx 0.100$$ *Answer*

(viii)

$$\frac{1}{\sqrt[5]{252}}=\frac{1}{(252)^{\frac{1}{5}}}=(252)^{-\frac{1}{5}}=(243+9)^{-\frac{1}{5}}=(243)^{-\frac{1}{5}}\left(1+\frac{9}{243}\right)^{-\frac{1}{5}}$$

$$=\left(3^5\right)^{-\frac{1}{5}}\left(1+\frac{1}{27}\right)^{-\frac{1}{5}}$$ *Now do yourself as above*

(ix)

$$\frac{\sqrt{7}}{\sqrt{8}}=\sqrt{\frac{7}{8}}=\left(\frac{7}{8}\right)^{\frac{1}{2}}=\left(1-\frac{1}{8}\right)^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(-\frac{1}{8}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{8}\right)^2+\ldots$$

$$=1-\frac{1}{16}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{64}\right)+\ldots$$

$$=1-\frac{1}{16}-\frac{1}{8}\left(\frac{1}{64}\right)+\ldots$$

$$=1-\frac{1}{16}-\frac{1}{512}+\ldots$$

$$=1-0.0625-0.00195+\ldots$$

$$\approx 0.93555\quad\approx 0.936$$ *Answer*

(x) $(0.998)^{-\frac{1}{3}}=(1-0.002)^{-\frac{1}{3}}$ *Now do yourself as above*

(xi) $\dfrac{1}{\sqrt[6]{486}}=\dfrac{1}{(486)^{\frac{1}{6}}}=(486)^{-\frac{1}{6}}=(729-243)^{-\frac{1}{6}}=(729)^{-\frac{1}{6}}\left(1-\dfrac{243}{729}\right)^{-\frac{1}{6}}$

$=\left(3^{6}\right)^{-\frac{1}{6}}\left(1-\dfrac{1}{3}\right)^{-\frac{1}{6}}$ *Now do yourself*

(xii) $(1280)^{\frac{1}{4}}=(1296-16)^{\frac{1}{4}}=(1296)^{\frac{1}{4}}\left(1-\dfrac{16}{1296}\right)^{\frac{1}{4}}=\left(6^{4}\right)^{\frac{1}{4}}\left(1-\dfrac{1}{81}\right)^{\frac{1}{4}}$

Now do yourself

Question # 3

Find the coefficient of x^n in the expansion of

(i) $\dfrac{(1+x^2)}{(1+x)^2}$ (ii) $\dfrac{(1+x)^2}{(1-x)^2}$ (iii) $\dfrac{(1+x)^3}{(1-x)^2}$

(iv) $\dfrac{(1+x)^2}{(1-x)^3}$ (v) $\left(1-x+x^2-x^3+\dots\right)^2$

Solution

(i) $\dfrac{(1+x^2)}{(1+x)^2}=\left(1+x^2\right)(1+x)^{-2}$

$$=\left(1+x^2\right)\left(1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\dots\right)$$

$$=\left(1+x^2\right)\left(1-2x+\frac{(-2)(-3)}{2}x^2+\frac{(-2)(-3)(-4)}{3\cdot 2}x^3+\dots\right)$$

$$=\left(1+x^2\right)\left(1-2x+3x^2-4x^3+\dots\right)$$

$$=\left(1+x^2\right)\left(1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^3+\dots\right)$$

Following in this way we can write

$$\frac{(1+x^2)}{(1+x)^2}=\left(1+x^2\right)(1+(-1)2x+(-1)^2 3x^2+(-1)^3 4x^3+\dots+(-1)^{n-2}(n-1)x^{n-2}+$$

$$(-1)^{n-1}(n)x^{n-1}+(-1)^n(n+1)x^n+\dots)$$

So taking only terms involving x^n we get

$$(-1)^n(n+1)x^n+(-1)^{n-2}(n-1)x^n$$

$$=(-1)^n(n+1)x^n+(-1)^n(-1)^{-2}(n-1)x^n$$

$$=(-1)^n(n+1)x^n+(-1)^n(n-1)x^n \qquad \because\ (-1)^{-2}=1$$

$$=(n+1+n-1)(-1)^n x^n=(2n)(-1)^n x^n$$

Thus the coefficient of term involving x^n is $(2n)(-1)^n$

(ii)

Hint:

After solving you will get

$$\frac{(1+x^2)}{(1-x)^2}=(1+x^2)(1+2x+3x^2+4x^3+\ldots+(n-1)x^{n-2}+(n)x^{n-1}+(n+1)x^n+\ldots)$$

Do yourself as above

(iii) $\dfrac{(1+x)^3}{(1-x)^2}=(1+x)^3(1-x)^{-2}$

$$=(1+x)^3\left(1+(-2)(-x)+\frac{(-2)(-2-1)}{2!}(-x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3+\ldots\right)$$

$$=(1+x)^3\left(1+2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(-x^3)+\ldots\right)$$

$$=(1+3x+3x^2+x^3)(1+2x+3x^2+4x^3+\ldots)$$

Following in this way we can write

$$\frac{(1+x)^3}{(1-x)^2}=(1+3x+3x^2+x^3)(1+2x+3x^2+4x^3+\ldots+(n-2)x^{n-3}+(n-1)x^{n-2}$$
$$+(n)x^{n-1}+(n+1)x^n+\ldots)$$

So taking only terms involving x^n we have term

$$(n+1)x^n+3(n)x^n+3(n-1)x^n+(n-2)x^n$$
$$=((n+1)+3(n)+3(n-1)+(n-2))x^n$$
$$=(n+1+3n+3n-3+n-2)x^n$$
$$=(8n-4)x^n$$

Thus the coefficient of term involving x^n is $(8n-4)$.

(iv) $\dfrac{(1+x)^2}{(1-x)^3}=(1+x)^2(1-x)^{-3}$

$$=(1+x)^2\left(1+(-3)(-x)+\frac{(-3)(-3-1)}{2!}(-x)^2+\frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3+\ldots\ldots\right)$$

$$=(1+x)^2\left(1+(-3)(-x)+\frac{(-3)(-4)}{2}(-x)^2+\frac{(-3)(-4)(-5)}{3\cdot 2}(-x)^3+\ldots\ldots\right)$$

$$=(1+2x+x^2)\left(1+3x+\frac{(3)(4)}{2}(x^2)+\frac{(4)(5)}{2}(x^3)+\ldots\ldots\right)$$

$$=(1+2x+x^2)\left(1+\frac{(2)(3)}{2}x+\frac{(3)(4)}{2}x^2+\frac{(4)(5)}{2}x^3+\ldots\ldots\right)$$

Following in this way we can write

$$\frac{(1+x)^2}{(1-x)^3}=\left(1+2x+x^2\right)\left(1+\frac{(2)(3)}{2}x+\frac{(3)(4)}{2}x^2+\frac{(4)(5)}{2}x^3+\dots\dots\dots\dots\dots\right.$$

$$\left.+\frac{(n-1)(n)}{2}x^{n-2}+\frac{(n)(n+1)}{2}x^{n-1}+\frac{(n+1)(n+2)}{2}x^n+\dots\dots\right)$$

So taking only terms involving x^n we have term

$$\frac{(n+1)(n+2)}{2}x^n+2\frac{(n)(n+1)}{2}x^n+\frac{(n-1)(n)}{2}x^n$$

$$=\left((n+1)(n+2)+2(n)(n+1)+(n-1)(n)\right)\frac{x^n}{2}$$

$$=\left(n^2+n+2n+2+2n^2+2n+n^2-n\right)\frac{x^n}{2}$$

$$=\left(4n^2+4n+2\right)\frac{x^n}{2}\quad=2\left(2n^2+2n+1\right)\frac{x^n}{2}$$

$$=\left(2n^2+2n+1\right)x^n$$

Thus the coefficient of term involving x^n is $\left(2n^2+2n+1\right)$.

(v) Since we know that

$$(1+x)^{-1}=1-x+x^2-x^3+\dots\dots\dots\dots$$

Therefore

$$\left(1-x+x^2-x^3+\dots\dots\dots\right)^2=\left((1+x)^{-1}\right)^2=(1+x)^{-2}$$

$$=1+(-2)(x)+\frac{(-2)(-2-1)}{2!}(x)^2+\frac{(-2)(-2-1)(-2-2)}{3!}(x)^3+\dots\dots$$

$$=1-2x+\frac{(-2)(-3)}{2}(x)^2+\frac{(-2)(-3)(-4)}{3\cdot 2}(x)^3+\dots\dots\dots\dots\dots$$

$$=1-2x+3x^2-4x^3+\dots\dots\dots\dots\dots$$

$$=1+(-1)2x+(-1)^2 3x^2(-1)^3 4x^3+\dots\dots\dots\dots\dots$$

Following in this way we can write

$$=1+(-1)2x+(-1)^2 3x^2(-1)^3 4x^3+\dots\dots\dots\dots\dots+(-1)^n(n+1)x^n+\dots\dots$$

So the term involving x^n = $(-1)^n(n+1)x^n$

And hence coefficient of term involving x^n is $(-1)^n(n+1)$

Question # 4

If x so small that its square and higher powers can be neglected, then show that

(i) $\dfrac{1-x}{\sqrt{1-x}}\approx=1-\dfrac{3}{2}x$ (ii) $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}}\approx 1+\dfrac{3}{2}x$

(iii) $\dfrac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x}\approx\dfrac{1}{4}-\dfrac{17}{384}x$ (iv) $\dfrac{\sqrt{4+x}}{(1-x)^3}\approx 2+\dfrac{25}{4}x$

(v) $\dfrac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{1}{4}}}{(8+5x)^{\frac{1}{3}}} \approx \left(1-\dfrac{5x}{6}\right)$ (vi) $\dfrac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}} \approx \dfrac{3}{2}-\dfrac{61}{48}x$

(vii) $\dfrac{\sqrt{4-x}+(8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \approx 2-\dfrac{1}{12}x$

Solution

(i)

$$\text{L.H.S} = \frac{1-x}{\sqrt{1-x}} = \frac{1-x}{(1-x)^{\frac{1}{2}}} = (1-x)^{1-\frac{1}{2}} = (1-x)^{\frac{1}{2}}$$

$$= 1+\left(\frac{1}{2}\right)(-x) + \text{ squares and higher power of } x.$$

$$= 1-\frac{3}{2}x = \text{R.H.S} \quad \text{Proved}$$

(ii) Since $\dfrac{\sqrt{1+2x}}{\sqrt{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

Now $(1+2x)^{\frac{1}{2}} = 1+\left(\dfrac{1}{2}\right)(2x) +$ squares and higher power of x.

$$\approx 1+x$$

Now $(1-x)^{-\frac{1}{2}} = 1+\left(-\dfrac{1}{2}\right)(-x) +$ squares and higher power of x.

$$\approx 1+\frac{1}{2}x$$

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx (1+x)\left(1+\frac{1}{2}x\right)$$

$$= 1+x+\frac{1}{2}x \qquad \text{ignoring term involving } x^2.$$

$$= 1+\frac{3}{2}x \quad \text{Proved.}$$

(iii) $\dfrac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x} = \left((9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}\right)(4+5x)^{-1}$

Now $(9+7x)^{\frac{1}{2}} = 9^{\frac{1}{2}}\left(1+\dfrac{7x}{9}\right)^{\frac{1}{2}}$

$$= (3^2)^{\frac{1}{2}}\left(1+\left(\frac{1}{2}\right)\left(\frac{7x}{9}\right) + \textit{squres and higher of } x.\right)$$

$$\approx 3\left(1+\frac{7x}{18}\right) = 3+3\left(\frac{7x}{18}\right) = 3+\frac{7x}{6}$$

$$(16+3x)^{\frac{1}{4}}=(16)^{\frac{1}{4}}\left(1+\frac{3x}{16}\right)^{\frac{1}{4}}$$

$$=(2^4)^{\frac{1}{4}}\left(1+\left(\frac{1}{4}\right)\left(\frac{3x}{16}\right)+\textit{square and higher power of } x\right)$$

$$\approx(2)\left(1+\frac{3x}{64}\right)=2+2\left(\frac{3x}{64}\right)=2+\frac{3x}{32}$$

$$(4+5x)^{-1}=4^{-1}\left(1+\frac{5}{4}x\right)^{-1}$$

$$=\frac{1}{4}\left(1+(-1)\left(\frac{5}{4}x\right)+\textit{squares and higher power of } x\right)$$

$$\approx\frac{1}{4}\left(1-\frac{5}{4}x\right)=\frac{1}{4}-\frac{5}{16}x$$

So $$\frac{(9+7x)^{\frac{1}{2}}-(16+3x)^{\frac{1}{4}}}{4+5x}\approx\left[\left(3+\frac{7x}{6}\right)-\left(2+\frac{3x}{32}\right)\right]\left(\frac{1}{4}-\frac{5}{16}x\right)$$

$$=\left[3+\frac{7x}{6}-2-\frac{3x}{32}\right]\left(\frac{1}{4}-\frac{5}{16}x\right)=\left(1+\frac{103}{96}x\right)\left(\frac{1}{4}-\frac{5}{16}x\right)$$

$$=\frac{1}{4}+\frac{103}{384}x-\frac{5}{16}x=\frac{1}{4}-\frac{17}{384}x \quad \text{Proved}$$

(iv) *Do yourself*

(v) $$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}}=(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}(8+5x)^{-\frac{1}{3}}$$

Now $(1+x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(x)+\textit{square and higher power of } x$

$$\approx 1+\frac{1}{2}x$$

$$(4-3x)^{\frac{3}{2}}=4^{\frac{3}{2}}\left(1-\frac{3}{4}x\right)^{\frac{3}{2}}$$

$$=(2^2)^{\frac{3}{2}}\left(1+\left(\frac{3}{2}\right)\left(-\frac{3}{4}x\right)+\textit{square and higher power of } x\right)$$

$$\approx(2)^3\left(1-\frac{9}{8}x\right)=8\left(1-\frac{9}{8}x\right)$$

$$(8+5x)^{-\frac{1}{3}}=(8)^{-\frac{1}{3}}\left(1+\frac{5}{8}x\right)^{-\frac{1}{3}}$$

$$=(2^3)^{-\frac{1}{3}}\left(1+\left(-\frac{1}{3}\right)\left(\frac{5}{8}x\right)+\textit{square and higher power of } x\right)$$

$$\approx (2)^{-1}\left(1-\frac{5}{24}x\right) = \frac{1}{2}\left(1-\frac{5}{24}x\right)$$

So $$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \approx \left(1+\frac{1}{2}x\right)8\left(1-\frac{9}{8}x\right)\frac{1}{2}\left(1-\frac{5}{24}x\right)$$

$$= \frac{8}{2}\left(1+\frac{1}{2}x\right)\left(1-\frac{9}{8}x-\frac{5}{24}x\right)$$

$$= 4\left(1+\frac{1}{2}x\right)\left(1-\frac{4}{3}x\right) = 4\left(1+\frac{1}{2}x-\frac{4}{3}x\right) = 4\left(1-\frac{5}{6}x\right) \quad \text{Proved}$$

(vi) *Do yourself as above*

(vii) *Same as Question #*4 (iii)

Question # 5

If x is so small that its cube and higher power can be neglected, then show that

(i) $\sqrt{1-x-2x^2} = 1-\frac{1}{2}x-\frac{9}{8}x^2$ (ii) $\sqrt{\frac{1+x}{1-x}} = 1+x+\frac{1}{2}x^2$

Solution

(i) $\sqrt{1-x-2x^2} = \left(1-(x+2x^2)\right)^{\frac{1}{2}}$

$$= 1+\left(\frac{1}{2}\right)\left(-(x+2x^2)\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-(x+2x^2)\right)^2 + \textit{cube \& higher power of } x.$$

$$\approx 1-\left(\frac{1}{2}\right)(x+2x^2)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x+2x^2)^2$$

$$\approx 1-\frac{1}{2}x-\frac{1}{2}(2x^2)-\frac{1}{8}x^2 = 1-\frac{1}{2}x-x^2-\frac{1}{8}x^2$$

$$= 1-\frac{1}{2}x-\frac{9}{8}x^2 \quad \text{Proved}$$

(ii)

$$\sqrt{\frac{1+x}{1-x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

Now

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)x+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \textit{cube \& higher power of } x.$$

$$\approx 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 = 1+\frac{1}{2}x-\frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1+\left(-\frac{1}{2}\right)(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \textit{cube \& higher power of } x.$$

$$\approx 1+\frac{1}{2}x+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 = 1+\frac{1}{2}x+\frac{3}{8}x^2$$

So

$$\sqrt{\frac{1+x}{1-x}} = \left(1+\frac{1}{2}x-\frac{1}{8}x^2\right)\left(1+\frac{1}{2}x+\frac{3}{8}x^2\right)$$

$$= 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{2}x+\frac{1}{4}x^2+\frac{3}{8}x^2 = 1+x+\frac{1}{2}x^2 \quad \text{Proved}$$

Question # 6

If x is very nearly equal 1, then prove that $px^p - qx^q = (p-q)x^{p+q}$

Solution

Since x is nearly equal to 1 so suppose $x = 1+h$,
where h is so small that its square and higher powers be neglected

L.H.S = $px^p - qx^q$

$$= p(1+h)^p - q(1+h)^q$$

$$= p(1+ph + \textit{square \& higher power of } x)$$

$$-q(1+qh + \textit{square \& higher power of } h)$$

$$\approx p(1+ph) - q(1+qh)$$

$$= p + p^2h - q - q^2h \text{ .... (i)}$$

Now R.H.S = $(p-q)x^{p+q}$

$$= (p-q)(1+h)^{p+q}$$

$$= (p-q)\left(1+(p+q)h + \textit{square \& higher power of } h\right)$$

$$\approx (p-q)\left(1+(p+q)h\right) = (p-q)\left(1+ph+qh\right)$$

$$= p + p^2h + pqh - q - pqh - q^2h$$

$$= p + p^2h - q - q^2h \text{ .... (ii)}$$

From (i) and (ii)

L.H.S $\approx$ R.H.S Proved

Question # 7

If $p-q$ is small when compared with p or q, show that

$$\frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} = \left(\frac{p+q}{2q}\right)^{\frac{1}{n}}.$$

Solution Since $p-q$ is small when compare

Therefore let $p-q = h \Rightarrow p = q+h$

$$\text{L.H.S} = \frac{(2n+1)p+(2n-1)q}{(2n-1)p+(2n+1)q} = \frac{(2n+1)(q+h)+(2n-1)q}{(2n-1)(q+h)+(2n+1)q}$$

$$= \frac{2nq+q+2nh+h+2nq-q}{2nq-q+2nh-h+2nq+q} = \frac{4nq+2nh+h}{4nq+2nh-h}$$

$$=\frac{4nq+2nh+h}{4nq\left(1+\frac{2nh-h}{4nq}\right)} \quad =\frac{4nq+2nh+h}{4nq}\left(1+\frac{2nh-h}{4nq}\right)^{-1}$$

$$=\frac{4nq+2nh+h}{4nq}\left(1+(-1)\left(\frac{2nh-h}{4nq}\right)+square\ \&\ higher\ \ power\ of\ \ x^2\right)$$

$$=\frac{4nq+2nh+h}{4nq}\left(1-\frac{2nh-h}{4nq}\right) \quad =\frac{4nq+2nh+h}{4nq}\left(\frac{4nq-2nh+h}{4nq}\right)$$

$$\approx\frac{16n^2q^2+8n^2hq+4nhq-8n^2hq+4nhq}{16n^2q^2}$$ ignoring squares of h

$$=\frac{16n^2q^2+8nhq}{16n^2q^2} \quad =\frac{16n^2q^2}{16n^2q^2}+\frac{8nhq}{16n^2q^2}$$

$$=1+\frac{h}{2nq} \text{ .... (i)}$$

Now R.H.S $=\left(\frac{p+q}{2q}\right)^{\frac{1}{n}}=\left(\frac{q+h+q}{2q}\right)^{\frac{1}{n}}$

$$=\left(\frac{2q+h}{2q}\right)^{\frac{1}{n}}=\left(\frac{2q}{2q}+\frac{h}{2q}\right)^{\frac{1}{n}}=\left(1+\frac{h}{2q}\right)^{\frac{1}{n}}$$

$$=1+\left(\frac{1}{n}\right)\left(\frac{h}{2q}\right)+square\ \&\ higher\ \ power\ of\ \ h\,.$$

$$\approx 1+\frac{h}{2nq} \text{ .... (ii)}$$

Form (i) and (ii)

L.H.S $\approx$ R.H.S Proved

Question # 8

Show that $\left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}}\approx\frac{8n}{9n-N}-\frac{n+N}{4n}$ where n and N are nearly equal.

Solution Since n and N are nearly equal therefore consider $N=n+h$, where h is so small that its squares and higher power be neglected.

L.H.S $=\left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}}=\left(\frac{n}{2(n+n+h)}\right)^{\frac{1}{2}}$

$$=\left(\frac{n}{2(2n+h)}\right)^{\frac{1}{2}}=\left(\frac{2(2n+h)}{n}\right)^{-\frac{1}{2}}=\left(\frac{4n+2h}{n}\right)^{-\frac{1}{2}}=\left(4+\frac{2h}{n}\right)^{-\frac{1}{2}}$$

$$=(4)^{-\frac{1}{2}}\left(1+\frac{2h}{4n}\right)^{-\frac{1}{2}}=(2^2)^{-\frac{1}{2}}\left(1+\frac{h}{2n}\right)^{-\frac{1}{2}}$$

$$=(2)^{-1}\left(1+\left(-\frac{1}{2}\right)\frac{h}{2n}+square\ \&\ higher\ power\ of\ h\right)$$

$$=\frac{1}{2}\left(1-\frac{h}{4n}\right)\quad=\frac{1}{2}-\frac{h}{8n}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now R.H.S $=\dfrac{8n}{9n-N}-\dfrac{n+N}{4n}$

$$=\frac{8n}{9n-(n+h)}-\frac{n+(n+h)}{4n}\quad=\frac{8n}{9n-n-h}-\frac{n+n+h}{4n}$$

$$=\frac{8n}{8n-h}-\frac{2n+h}{4n}\quad=\frac{8n}{8n\left(1-\frac{h}{8n}\right)}-\frac{2n+h}{4n}\quad=\left(1-\frac{h}{8n}\right)^{-1}-\frac{2n+h}{4n}$$

$$=\left(1+(-1)\left(-\frac{h}{8n}\right)+square\ \&\ higher\ power\ of\ h\right)-\left(\frac{2n}{4n}+\frac{h}{4n}\right)$$

$$=\left(1+\frac{h}{8n}\right)-\left(\frac{1}{2}+\frac{h}{4n}\right)\quad=1+\frac{h}{8n}-\frac{1}{2}-\frac{h}{4n}$$

$$=\frac{1}{2}-\frac{h}{8n}\quad\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii)

L.H.S = R.H.S Proved

Question # 9

Identify the following series as binomial expansion and find the sum in each case.

(i) $1-\dfrac{1}{2}\left(\dfrac{1}{4}\right)+\dfrac{1\cdot 3}{2!\cdot 4}\left(\dfrac{1}{4}\right)^2-\dfrac{1\cdot 3\cdot 5}{3!\cdot 8}\left(\dfrac{1}{4}\right)^3+\ldots$

(ii) $1-\dfrac{1}{2}\left(\dfrac{1}{2}\right)+\dfrac{1\cdot 3}{2\cdot 4}\left(\dfrac{1}{2}\right)^2-\dfrac{1\cdot 3\cdot 5}{2.4.6}\left(\dfrac{1}{2}\right)^3+\ldots$

(iii) $1+\dfrac{3}{4}+\dfrac{3\cdot 5}{4\cdot 8}+\dfrac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\ldots$

(iv) $1-\dfrac{1}{2}\left(\dfrac{1}{3}\right)+\dfrac{1\cdot 3}{2\cdot 4}\left(\dfrac{1}{3}\right)^2-\dfrac{1\cdot 3\cdot 5}{2.4.6}\left(\dfrac{1}{3}\right)^3+\ldots$

Solution

(i) $$1-\frac{1}{2}\left(\frac{1}{4}\right)+\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2-\frac{1\cdot 3\cdot 5}{3!\cdot 8}\left(\frac{1}{4}\right)^3+\ldots$$

Suppose the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies $\quad nx=-\dfrac{1}{2}\left(\dfrac{1}{4}\right)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2\quad\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) $\qquad nx=-\dfrac{1}{8} \Rightarrow x=-\dfrac{1}{8n}$ .... (iii)

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(-\frac{1}{8n}\right)^2=\frac{1\cdot 3}{2!\cdot 4}\left(\frac{1}{4}\right)^2$$

$$\Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{64n^2}\right)=\frac{3}{2\cdot 4}\left(\frac{1}{16}\right)$$

$$\Rightarrow \frac{(n-1)}{128n}=\frac{3}{128} \quad \Rightarrow (n-1)=\frac{3}{128}\cdot 128n \quad \Rightarrow n-1=3n$$

$$\Rightarrow n-3n=1 \quad \Rightarrow -2n=1 \quad \Rightarrow \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=-\frac{1}{8\left(-\frac{1}{2}\right)} \quad \Rightarrow \boxed{x=\frac{1}{4}}$$

So

$$(1+x)^n=\left(1+\frac{1}{4}\right)^{-\frac{1}{2}}=\left(\frac{5}{4}\right)^{-\frac{1}{2}}=\left(\frac{4}{5}\right)^{\frac{1}{2}}=\sqrt{\frac{4}{5}}$$

(ii) *Do yourself as above*

(iii) $\qquad 1+\dfrac{3}{4}+\dfrac{3\cdot 5}{4\cdot 8}+\dfrac{3\cdot 5\cdot 7}{4\cdot 8\cdot 12}+\dots$

Suppose the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

This implies $\qquad nx=\dfrac{3}{4}$ .... (i)

$$\frac{n(n-1)}{2!}x^2=\frac{3\cdot 5}{4\cdot 8} \text{ .... (ii)}$$

From (i) $\qquad nx=\dfrac{3}{4} \Rightarrow x=\dfrac{3}{4n}$ .... (iii)

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{3}{4n}\right)^2=\frac{3\cdot 5}{4\cdot 8} \quad \Rightarrow \frac{n(n-1)}{2}\left(\frac{9}{16n^2}\right)=\frac{15}{32}$$

$$\Rightarrow \frac{9(n-1)}{32n}=\frac{15}{32} \quad \Rightarrow 9(n-1)=\frac{15}{32}\cdot 32n \quad \Rightarrow 9n-9=15n$$

$$\Rightarrow 9n-15n=9 \quad \Rightarrow -6n=9 \quad \Rightarrow n=-\frac{9}{6} \quad \Rightarrow \boxed{n=-\frac{3}{2}}$$

Putting value of n in equation (iii)

$$x=-\frac{3}{4\left(-\frac{3}{2}\right)} \Rightarrow \boxed{x=-\frac{1}{2}}$$

So $(1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{3}{2}}=\left(\frac{1}{2}\right)^{-\frac{3}{2}}=(2)^{\frac{3}{2}}=\left(\sqrt{2}\right)^3=2\sqrt{2}$ Answer

(iv) *Do yourself as above*

Question # 10

Use binomial theorem to show that $1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots=\sqrt{2}$

Solution $1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots$

Suppose the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies

$$nx=\frac{1}{4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{4\cdot 8} \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) $nx=\frac{1}{4} \Rightarrow x=\frac{1}{4n} \quad \ldots\ldots\ldots\ldots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2=\frac{1\cdot 3}{4\cdot 8} \Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right)=\frac{3}{32}$$

$$\Rightarrow \frac{(n-1)}{32n}=\frac{3}{32} \Rightarrow (n-1)=\frac{3}{32}\cdot 32n \Rightarrow n-1=3n$$

$$\Rightarrow n-3n=1 \Rightarrow -2n=1 \Rightarrow \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x=-\frac{1}{2}}$$

So $(1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}=\left(\frac{1}{2}\right)^{-\frac{1}{2}}=(2)^{\frac{1}{2}}=\sqrt{2}$

Hence $1+\frac{1}{4}+\frac{1\cdot 3}{4\cdot 8}+\frac{1\cdot 3\cdot 5}{4\cdot 8\cdot 12}+\ldots=\sqrt{2}$ Proved

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Question # 11

If $y=\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$, then prove that $y^2+2y-2=0$.

Solution $$y=\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$$

Adding 1 on both sides

$$1+y=1+\frac{1}{3}+\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2+\frac{1\cdot 3\cdot 5}{3!}\left(\frac{1}{3}\right)^3+\ldots$$

Let the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies

$$nx=\frac{1}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2 \quad \ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) $$nx=\frac{1}{3} \Rightarrow x=\frac{1}{3n} \quad \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{3n}\right)^2=\frac{1\cdot 3}{2!}\left(\frac{1}{3}\right)^2$$

$$\Rightarrow \frac{n(n-1)}{2}\left(\frac{1}{9n^2}\right)=\frac{3}{2}\cdot\frac{1}{9}$$

$$\Rightarrow \frac{n-1}{18n}=\frac{1}{6} \Rightarrow n-1=\frac{1}{6}\cdot 18\,n$$

$$\Rightarrow n-1=3n \Rightarrow n-3n=1$$

$$\Rightarrow -2n=1 \Rightarrow \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{1}{3\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x=-\frac{2}{3}}$$

So $$(1+x)^n=\left(1-\frac{2}{3}\right)^{-\frac{1}{2}}=\left(\frac{1}{3}\right)^{-\frac{1}{2}}$$

$$=(3)^{\frac{1}{2}}=\sqrt{3}$$

This implies

$$1+y=\sqrt{3}$$

On squaring both sides

$$(1+y)^2=\left(\sqrt{3}\right)^2$$

$$\Rightarrow 1+2y+y^2=3 \Rightarrow 1+2y+y^2-3=0$$

$$\Rightarrow y^2+2y-2=0 \quad \text{Proved.}$$

Question # 12

If $2y=\frac{1}{2^2}+\frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6}+\dots$, then prove that $4y^2+4y-1=0$.

Solution $\quad 2y=\frac{1}{2^2}+\frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6}+\dots$

Adding 1 on both sides

$$1+2y=1+\frac{1}{2^2}+\frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}+\frac{1\cdot 3\cdot 5}{3!}\cdot\frac{1}{2^6}+\dots$$

Let the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots$$

This implies

$$nx=\frac{1}{2^2}\ \dots\dots\dots\dots\dots\dots\dots\dots\ \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}\ \dots\dots\dots\dots\ \text{(ii)}$$

From (i) $\quad nx=\frac{1}{4} \quad\Rightarrow\quad x=\frac{1}{4n}\ \dots\dots\dots\dots\dots\ \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{1}{4n}\right)^2=\frac{1\cdot 3}{2!}\cdot\frac{1}{2^4}$$

$$\Rightarrow\ \frac{n(n-1)}{2}\left(\frac{1}{16n^2}\right)=\frac{3}{2}\cdot\frac{1}{16}$$

$$\Rightarrow\ \frac{n-1}{n}=3 \quad\Rightarrow\quad n-1=3n$$

$$\Rightarrow\ n-3n=1 \quad\Rightarrow\quad -2n=1 \quad\Rightarrow\quad \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{1}{4\left(-\frac{1}{2}\right)} \quad\Rightarrow\quad \boxed{x=-\frac{1}{2}}$$

So

$$(1+x)^n=\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}$$

$$=\left(\frac{1}{2}\right)^{-\frac{1}{2}}=(2)^{\frac{1}{2}}\ =\sqrt{2}$$

This implies

$$1+2y=\sqrt{2}$$

On squaring both sides

$$(1+2y)^2=\left(\sqrt{2}\right)^2$$

$$\Rightarrow\ 1+4y+4y^2=4 \qquad \Rightarrow\ 1+4y+4y^2-2=0$$
$$\Rightarrow\ 4y^2+4y-1=0 \qquad \text{Proved}$$

Question # 13

If $y=\frac{2}{5}+\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{2}{5}\right)^3+\ldots$, then prove that $y^2+2y-4=0$.

Solution $\quad y=\frac{2}{5}+\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{2}{5}\right)^3+\ldots$

Adding 1 on both sides

$$1+y=1+\frac{2}{5}+\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2+\frac{1\cdot3\cdot5}{3!}\left(\frac{2}{5}\right)^3+\ldots$$

Let the given series be identical with

$$(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots$$

This implies

$$nx=\frac{2}{5} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2=\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2 \ \ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) $\quad nx=\frac{2}{5} \quad \Rightarrow\ x=\frac{2}{5n} \ \ldots\ldots\ldots\ldots\ldots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!}\left(\frac{2}{5n}\right)^2=\frac{1\cdot3}{2!}\left(\frac{2}{5}\right)^2$$

$$\Rightarrow\ \frac{n(n-1)}{2}\left(\frac{4}{25n^2}\right)=\frac{3}{2}\left(\frac{4}{25}\right)$$

$$\Rightarrow\ \frac{n-1}{n}=3 \quad \Rightarrow\ n-1=3n \ \Rightarrow\ n-3n=1$$

$$\Rightarrow\ -2n=1 \quad \Rightarrow\ \boxed{n=-\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x=\frac{2}{5\left(-\frac{1}{2}\right)} \ \Rightarrow\ \boxed{x=-\frac{4}{5}}$$

So $\quad (1+x)^n=\left(1-\frac{4}{5}\right)^{-\frac{1}{2}}=\left(\frac{1}{5}\right)^{-\frac{1}{2}}$

$$=(5)^{\frac{1}{2}}=\sqrt{5}$$

This implies

$$1+y=\sqrt{5}$$

On squaring both sides

$$(1+y)^2 = \left(\sqrt{5}\right)^2$$

$$\Rightarrow\ 1+2y+y^2=5 \quad \Rightarrow\ 1+2y+y^2-5=0$$

$$\Rightarrow\ y^2+2y-4=0 \quad \text{Proved.}$$