EXERCISE 3.6

Definite Integral

Let $\int f(x)dx = \varphi(x)+c$

Then $\int_a^b f(x)\,dx = \left|\varphi(x)\right|_a^b$ *or* $[\varphi(x)]_a^b$

$= \varphi(b)-\varphi(a)$

Also

- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

where $a<c<b$

Question # 1

$$\int_1^2 (x^2+1)dx = \int_1^2 x^2\,dx + \int_1^2 dx$$

$$= \left|\frac{x^3}{3}\right|_1^2 + |x|_1^2 = \left(\frac{2^3}{3}-\frac{1^3}{3}\right)+(2-1)$$

$$= \frac{8}{3}-\frac{1}{3}+1 = \frac{10}{3}$$

Question # 2

$$\int_{-1}^1 \left(x^{\frac{1}{3}}+1\right)dx = \int_{-1}^1 x^{\frac{1}{3}}dx + \int_{-1}^1 dx$$

$$= \left|\frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\right|_{-1}^1 + |x|_{-1}^1 = \left|\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right|_{-1}^1 + (1-(-1))$$

$$= \frac{3}{4}\left((1)^{\frac{4}{3}}-(-1)^{\frac{4}{3}}\right)+(1+1)$$

$$= \frac{3}{4}(1-1)+2 = 2$$

Question # 3

$$\int_{-2}^0 \frac{1}{(2x-1)^2}dx = \int_{-2}^0 (2x-1)^{-2}\,dx$$

$$= \left|\frac{(2x-1)^{-2+1}}{(-2+1)\cdot 2}\right|_{-2}^0 = \left|\frac{(2x-1)^{-1}}{(-1)\ 2}\right|_{-2}^0$$

$$= \frac{(2(0)-1)^{-1}}{-2} - \frac{(2(-2)-1)^{-1}}{-2}$$

$$= \frac{(0-1)^{-1}}{-2} - \frac{(-4-1)^{-1}}{-2}$$

$$= \frac{(-1)^{-1}}{-2} - \frac{(-5)^{-1}}{-2}$$

$$= \frac{1}{(-2)(-1)} - \frac{1}{(-2)(-5)}$$

$$= \frac{1}{2}-\frac{1}{10} = \frac{2}{5}$$

Question # 4

$$\int_{-6}^2 \sqrt{3-x}\,dx = \int_{-6}^2 (3-x)^{\frac{1}{2}}\,dx$$

$$= \left|\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)}\right|_{-6}^2 = \left|\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)}\right|_{-6}^2$$

$$= -\frac{2}{3}\left|(3-x)^{\frac{3}{2}}\right|_{-6}^2$$

$$= -\frac{2}{3}\left((3-2)^{\frac{3}{2}}-(3+6)^{\frac{3}{2}}\right)$$

$$= -\frac{2}{3}\left((1)^{\frac{3}{2}}-(9)^{\frac{3}{2}}\right) = -\frac{2}{3}(1-8) = \frac{14}{3}$$

Question # 5

$$\int_1^{\sqrt{5}} \sqrt{(2t-1)^3}\,dt = \int_1^{\sqrt{5}} (2t-1)^{\frac{3}{2}}\,dt$$

$$= \left|\frac{(2t-1)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot 2}\right|_1^{\sqrt{5}} = \left|\frac{(2t-1)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot 2}\right|_1^{\sqrt{5}}$$

$$= \left|\frac{(2t-1)^{\frac{5}{2}}}{5}\right|_1^{\sqrt{5}} = \frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5} - \frac{(2(1)-1)^{\frac{5}{2}}}{5}$$

$$= \frac{(2\sqrt{5}-1)^{\frac{5}{2}}}{5} - \frac{1}{5}$$

$$= \frac{\sqrt{(2\sqrt{5}-1)^5}}{5} - \frac{1}{5} \qquad \textit{Ans.}$$

Question # 6

$$\int_2^{\sqrt{5}} x\sqrt{x^2-1}\,dx = \int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\cdot x\,dx$$

$$= \frac{1}{2}\int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\cdot 2x\,dx$$

$$= \frac{1}{2}\int_2^{\sqrt{5}} (x^2-1)^{\frac{1}{2}}\cdot \frac{d}{dx}(x^2-1)\,dx$$

$$= \frac{1}{2}\left|\frac{(x^2-1)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_2^{\sqrt{5}} = \frac{1}{2}\left|\frac{(x^2-1)^{\frac{3}{2}}}{\frac{3}{2}}\right|_2^{\sqrt{5}}$$

$$= \frac{1}{2}\cdot\frac{2}{3}\left[\left(\left(\sqrt{5}\right)^2-1\right)^{\frac{3}{2}} - \left((2)^2-1\right)^{\frac{3}{2}}\right]$$

$$= \frac{1}{3}\left[(5-1)^{\frac{3}{2}}-(4-1)^{\frac{3}{2}}\right] = \frac{1}{3}\left[(4)^{\frac{3}{2}}-(3)^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}\left[\left(2^2\right)^{\frac{3}{2}}-(3)^{1+\frac{1}{2}}\right] \quad =\frac{1}{3}\left[(2)^3-3(3)^{\frac{1}{2}}\right]$$
$$=\frac{1}{3}\left[8-3\sqrt{3}\right]$$

Question # 7

$$\int_1^2 \frac{x}{x^2+2}dx \quad =\frac{1}{2}\int_1^2 \frac{2x}{x^2+2}dx$$
$$=\frac{1}{2}\int_1^2 \frac{\frac{d}{dx}\left(x^2+2\right)}{x^2+2}\,dx \quad =\frac{1}{2}\left|\ln\left|x^2+2\right|\right|_1^2$$
$$=\frac{1}{2}\left(\ln\left|2^2+2\right|-\ln\left|1^2+2\right|\right)$$
$$=\frac{1}{2}\left(\ln 6-\ln 3\right)$$
$$=\frac{1}{2}\ln\left(\frac{6}{3}\right) \quad =\frac{1}{2}\ln 2$$

Question # 8

$$\int_2^3\left(x-\frac{1}{x}\right)^2\,dx \quad =\int_2^3\left(x^2+\frac{1}{x^2}-2\right)dx$$
$$=\int_2^3 x^2\,dx+\int_2^3 x^{-2}\,dx-2\int_2^3 dx$$

Now do yourself

Question # 9

$$\int_{-1}^{1}\left(x+\frac{1}{2}\right)\sqrt{x^2+x+1}\,dx$$
$$=\int_{-1}^{1}\left(\frac{2x+1}{2}\right)\left(x^2+x+1\right)^{\frac{1}{2}}\,dx$$
$$=\frac{1}{2}\int_{-1}^{1}\left(x^2+x+1\right)^{\frac{1}{2}}(2x+1)\,dx$$
$$=\frac{1}{2}\int_{-1}^{1}\left(x^2+x+1\right)^{\frac{1}{2}}\frac{d}{dx}(2x+1)\,dx$$
$$=\frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_{-1}^{1}$$

NOTE

$$(3)^{\frac{3}{2}}=(3)^{1+\frac{1}{2}} =3^1\cdot 3^{\frac{1}{2}}=3\sqrt{3}$$

$$=\frac{1}{2}\left|\frac{\left(x^2+x+1\right)^{\frac{3}{2}}}{\frac{3}{2}}\right|_{-1}^{1} \quad =\frac{1}{3}\left|\left(x^2+x+1\right)^{\frac{3}{2}}\right|_{-1}^{1}$$
$$=\frac{1}{3}\left[\left((1)^2+(1)+1\right)^{\frac{3}{2}}-\left((-1)^2+(-1)+1\right)^{\frac{3}{2}}\right]$$
$$=\frac{1}{3}\left[(1+1+1)^{\frac{3}{2}}-(1-1+1)^{\frac{3}{2}}\right]$$
$$=\frac{1}{3}\left[(3)^{\frac{3}{2}}-(1)^{\frac{3}{2}}\right] \quad =\frac{1}{3}\left[3\sqrt{3}-1\right]$$
$$=\sqrt{3}-\frac{1}{3}$$

Question # 10

$$\int_0^3\frac{dx}{x^2+9} \quad =\int_0^3\frac{dx}{x^2+3^2}$$
$$=\left|\frac{1}{3}Tan^{-1}\frac{x}{3}\right|_0^3$$
$$=\frac{1}{3}Tan^{-1}\left(\frac{3}{3}\right)-\frac{1}{3}Tan^{-1}\left(\frac{0}{3}\right)$$
$$=\frac{1}{3}Tan^{-1}(1)-\frac{1}{3}Tan^{-1}(0)$$
$$=\frac{1}{3}\left(\frac{\pi}{4}\right)-\frac{1}{3}(0) \quad =\frac{\pi}{12}$$

Question # 11

Let $I=\int_1^2\ln x\,dx \quad =\int_1^2\ln x\cdot 1\,dx$

Integrating by parts

$$I=\left|\ln x\cdot x\right|_1^2-\int_1^2 x\cdot\frac{1}{x}dx$$
$$=\left|x\ln x\right|_1^2-\int_1^2 dx$$
$$=\left(2\cdot\ln 2-1\cdot\ln 1\right)-\left|x\right|_1^2$$
$$=\left(2\cdot\ln 2-1\cdot(0)\right)-(2-1)$$
$$=\left(2\cdot\ln 2-0\right)-1=2\ln 2-1$$

Question # 12

$$\int_0^2\left(e^{\frac{x}{2}}-e^{-\frac{x}{2}}\right)\,dx$$
$$=\int_0^2 e^{\frac{x}{2}}\,dx-\int_0^2 e^{-\frac{x}{2}}\,dx$$
$$=\left|\frac{e^{\frac{x}{2}}}{1/2}\right|_0^2-\left|\frac{e^{-\frac{x}{2}}}{-1/2}\right|_0^2 \quad = 2\left|e^{\frac{x}{2}}\right|_0^2+2\left|e^{-\frac{x}{2}}\right|_0^2$$
$$= 2\left(e^{\frac{2}{2}}-e^{\frac{0}{2}}\right)+2\left(e^{-\frac{2}{2}}-e^{-\frac{0}{2}}\right)$$
$$= 2\left(e^1-e^0\right)+2\left(e^{-1}-e^0\right)$$
$$= 2\left(e-1+\frac{1}{e}-1\right) \quad = 2\left(e+\frac{1}{e}-2\right)$$
$$= 2\left(\frac{e^2+1-2e}{e}\right) \quad = 2\frac{(e-1)^2}{e}$$

Question # 13

Let $I=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{\cos 2\theta+1}d\theta$

$$=\int_0^{\pi/4}\frac{\cos\theta+\sin\theta}{2\cos^2\theta}d\theta \qquad \because\ \cos^2\theta=\frac{1+\cos 2\theta}{2}$$
$$=\int_0^{\pi/4}\left(\frac{\cos\theta}{2\cos^2\theta}++\frac{\sin\theta}{2\cos^2\theta}\right)d\theta$$

$$=\int_0^{\pi/4}\frac{1}{2\cos\theta}\,d\theta+\int_0^{\pi/4}\frac{\sin\theta}{2\cos\theta\cdot\cos\theta}\,d\theta$$

$$=\frac{1}{2}\int_0^{\pi/4}\sec\theta\,d\theta+\frac{1}{2}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta$$

$$=\frac{1}{2}\Big|\ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}+\frac{1}{2}\Big|\sec\theta\Big|_0^{\pi/4}$$

$$=\frac{1}{2}\left(\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|-\ln|\sec(0)+\tan(0)|\right)+\frac{1}{2}\left(\sec\frac{\pi}{4}-\sec(0)\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-\ln|1+0|\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|-0\right)+\frac{1}{2}\left(\sqrt{2}-1\right)$$

$$=\frac{1}{2}\left(\ln\left|\sqrt{2}+1\right|+\sqrt{2}-1\right)\qquad \textit{Ans.}$$

Question # 14

$$\int_0^{\pi/6}\cos^3\theta\,d\theta\;=\int_0^{\pi/6}\cos^2\theta\cdot\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$

$$=\int_0^{\pi/6}\cos\theta\,d\theta-\int_0^{\pi/6}\sin^2\theta\cos\theta\,d\theta$$

$$=\Big|\sin\theta\Big|_0^{\pi/6}-\int_0^{\pi/6}\sin^2\theta\frac{d}{d\theta}\sin\theta\,d\theta$$

$$=\left(\sin\frac{\pi}{6}-\sin(0)\right)-\left|\frac{\sin^3\theta}{3}\right|_0^{\pi/6}$$

$$=\left(\frac{1}{2}-0\right)-\frac{1}{3}\left(\sin^3\frac{\pi}{6}-\sin^3(0)\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\left(\frac{1}{2}\right)^2-(0)^3\right)$$

$$=\frac{1}{2}-\frac{1}{3}\left(\frac{1}{8}\right)\quad=\frac{1}{2}-\frac{1}{24}\quad=\frac{11}{24}$$

Question # 15

$$\int_{\pi/6}^{\pi/4}\cos^2\theta\cdot\cot^2\theta\,d\theta$$

$$=\int_{\pi/6}^{\pi/4}\cos^2\theta\left(\operatorname{cosec}^2\theta-1\right)d\theta$$

$$=\int_{\pi/6}^{\pi/4}\left(\cos^2\theta\operatorname{cosec}^2\theta-\cos^2\theta\right)d\theta$$

$$=\int_{\pi/6}^{\pi/4}\left(\cos^2\theta\frac{1}{\sin^2\theta}-\cos^2\theta\right)d\theta$$

$$=\int_{\pi/6}^{\pi/4}\cot^2\theta\,d\theta-\int_{\pi/6}^{\pi/4}\cos^2\theta\,d\theta$$

$$=\int_{\pi/6}^{\pi/4}\left(\operatorname{cosec}^2\theta-1\right)d\theta-\int_{\pi/6}^{\pi/4}\left(\frac{1+\cos\theta}{2}\right)d\theta$$

$$=\int_{\pi/6}^{\pi/4}\csc^2\theta\,d\theta-\int_{\pi/6}^{\pi/4}d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4}d\theta-\frac{1}{2}\int_{\pi/6}^{\pi/4}\cos2\theta d\theta$$

$$=\Big|-\cot\theta\Big|_{\pi/6}^{\pi/4}-\frac{3}{2}\int_{\pi/6}^{\pi/4}d\theta-\frac{1}{2}\left|\frac{\sin2\theta}{2}\right|_{\pi/6}^{\pi/4}$$

$$=\left(-\cot\frac{\pi}{4}+\cot\frac{\pi}{6}\right)-\frac{3}{2}\Big|\theta\Big|_{\pi/6}^{\pi/4}-\frac{1}{2}\left(\frac{\sin2\left(\pi/4\right)}{2}-\frac{\sin2\left(\pi/6\right)}{2}\right)$$

$$=\left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{4}-\frac{\pi}{6}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}/2}{2}\right)$$

$$=\left(-1+\sqrt{3}\right)-\frac{3}{2}\left(\frac{\pi}{12}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right)$$

$$=-1+\sqrt{3}-\frac{\pi}{8}-\frac{1}{4}+\frac{\sqrt{3}}{8}\quad=-\frac{5}{4}+\frac{9}{8}\sqrt{3}-\frac{\pi}{8}$$

$$=\frac{9\sqrt{3}-10-\pi}{8}$$

Question # 16

$$\int_0^{\pi/4}\cos^4t\,dt\quad=\int_0^{\pi/4}\left(\cos^2t\right)^2dt$$

$$=\int_0^{\pi/4}\left(\frac{1+\cos2t}{2}\right)^2dt$$

$$=\int_0^{\pi/4}\left(\frac{1+2\cos2t+\cos^2 2t}{4}\right)dt$$

$$=\frac{1}{4}\int_0^{\pi/4}\left(1+2\cos2t+\cos^2 2t\right)dt$$

$$=\frac{1}{4}\int_0^{\pi/4}\left(1+2\cos2t+\frac{1+\cos4t}{2}\right)dt$$

$$=\frac{1}{4}\int_0^{\pi/4}\left(\frac{2+4\cos2t+1+\cos4t}{2}\right)dt$$

$$=\frac{1}{8}\int_0^{\pi/4}\left(3+4\cos2t+\cos4t\right)dt$$

$$=\frac{1}{8}\left|3t+4\frac{\sin2t}{2}+\frac{\sin4t}{4}\right|_0^{\pi/4}$$

$$=\frac{1}{8}\left(3\left(\frac{\pi}{4}\right)+2\sin 2\left(\pi/4\right)+\frac{\sin 4\left(\pi/4\right)}{4}\right.$$
$$\left.-3(0)-2\sin 2(0)-\frac{\sin 4(0)}{4}\right)$$
$$=\frac{1}{8}\left(\frac{3\pi}{4}+2+\frac{0}{4}-0-0-\frac{0}{4}\right) \quad =\frac{1}{8}\left(\frac{3\pi}{4}+2\right)$$
$$=\frac{1}{8}\left(\frac{3\pi+8}{4}\right) \quad =\frac{3\pi+8}{32}$$

Question # 17

Let $I=\int_0^{\pi/3}\cos^2\theta\sin\theta\ d\theta$

Put $t=\cos\theta \quad \Rightarrow \quad dt=-\sin\theta\ d\theta$

$\Rightarrow \ -dt=\sin\theta\ d\theta$

When $\theta=0$ then $t=1$

And when $\theta=\pi/3$ then $t=1/2$

So $I=\int_1^{1/2}t^2(-dt)$

$$=-\int_1^{1/2}t^2\,dt \quad =-\left|\frac{t^3}{3}\right|_1^{1/2}$$
$$=-\left(\frac{\left(1/2\right)^3}{3}-\frac{(1)^3}{3}\right) \quad =-\left(\frac{1/8}{3}-\frac{1}{3}\right)$$
$$=-\left(\frac{1}{24}-\frac{1}{3}\right) \quad =-\left(-\frac{7}{24}\right) \quad =\frac{7}{24}$$

Question # 18

$$\int_0^{\pi/4}\left(1+\cos^2\theta\right)\tan^2\theta\ d\theta$$
$$=\int_0^{\pi/4}\left(1+\cos^2\theta\right)\frac{\sin^2\theta}{\cos^2\theta}\ d\theta$$
$$=\int_0^{\pi/4}\left(\frac{\sin^2\theta}{\cos^2\theta}+\sin^2\theta\right)d\theta$$
$$=\int_0^{\pi/4}\left(\tan^2\theta+\sin^2\theta\right)d\theta$$
$$=\int_0^{\pi/4}\left(\sec^2\theta-1+\frac{1-\cos 2\theta}{2}\right)d\theta$$
$$=\int_0^{\pi/4}\left(\frac{2\sec^2\theta-2+1-\cos 2\theta}{2}\right)d\theta$$
$$=\frac{1}{2}\int_0^{\pi/4}\left(2\sec^2\theta-1-\cos 2\theta\right)d\theta$$
$$=\frac{1}{2}\left|2\tan\theta-\theta-\frac{\sin 2\theta}{2}\right|_0^{\pi/4}$$
$$=\frac{1}{2}\left(2\tan\frac{\pi}{4}-\frac{\pi}{4}-\frac{\sin 2\left(\pi/4\right)}{2}\right.$$
$$\left.-2\tan(0)+0+\frac{\sin 2(0)}{2}\right)$$
$$=\frac{1}{2}\left(2(1)-\frac{\pi}{4}-\frac{1}{2}-2(0)+0+0\right)$$
$$=\frac{1}{2}\left(\frac{3}{2}-\frac{\pi}{4}\right) \quad =\frac{1}{2}\left(\frac{6-\pi}{4}\right) \quad =\frac{6-\pi}{8}$$

Question # 19

Let $I=\int_0^{\pi/4}\frac{\sec\theta}{\sin\theta+\cos\theta}\,d\theta$

$$=\int_0^{\pi/4}\frac{\sec\theta}{\cos\theta\left(\frac{\sin\theta}{\cos\theta}+1\right)}\,d\theta$$
$$=\int_0^{\pi/4}\frac{\sec^2\theta}{\left(\tan\theta+1\right)}\,d\theta$$

Put $t=\tan\theta+1 \quad \Rightarrow \quad dt=\sec^2\theta\,d\theta$

When $x=0$ then $t=1$

Also when $x=\frac{\pi}{4}$ then $t=2$

So $I=\int_1^2\frac{dt}{t}$

$$=\left|\ln t\right|_1^2$$
$$=\ln 2-\ln 1 \quad =\ln 2-0 \quad =\ln 2$$

Review

If $f(x)=\begin{cases} g(x) & : \quad a\le x\le b \\ h(x) & : \quad b\le x\le c\end{cases}$

Then

$$\int_a^c f(x)\,dx=\int_a^b g(x)+\int_b^c h(x)$$

Question # 20

Let $I=\int_{-1}^{5}|x-3|\,dx$

Since

$$|x-3| =\begin{cases} x-3 & if \quad x-3\ge 0 \ \Rightarrow \ x\ge 3 \\ -(x-3) & if \quad x-3<0 \ \Rightarrow \ x<3\end{cases}$$

So $\int_{-1}^{5}|x-3|\,dx=\int_{-1}^{3}\left[-(x-3)\right]dx+\int_3^5(x-3)\,dx$

$$=-\int_{-1}^{3}(x-3)\,dx+\int_3^5(x-3)\,dx$$
$$=-\left|\frac{(x-3)^2}{2}\right|_{-1}^{3}+\left|\frac{(x-3)^2}{2}\right|_3^5$$
$$=-\left(\frac{(3-3)^2}{2}-\frac{(-1-3)^2}{2}\right)+\left(\frac{(5-3)^2}{2}-\frac{(3-3)^2}{2}\right)$$

$$= -\left(\frac{0}{2}-\frac{16}{2}\right)+\left(\frac{4}{2}-\frac{0}{2}\right) \quad = 8+2 \quad = 10$$

Question # 21

Let $I = \int_{1/8}^{1} \frac{\left(x^{1/3}+2\right)^2}{x^{2/3}}\, dx$

$$= \int_{1/8}^{1} \left(x^{1/3}+2\right)^2 x^{-2/3}\, dx$$

Put $t = x^{\frac{1}{3}} + 2$

$\Rightarrow \; dt = \frac{1}{3}x^{-\frac{2}{3}}dx \quad \Rightarrow \; 3\,dt = x^{-\frac{2}{3}}dx$

When $x = \frac{1}{8}$ then $t = \frac{5}{2}$

And when $x = 1$ then $t = 3$

So $I = \int_{5/2}^{3} (t)^2\, 3dt \quad = 3\left|\frac{t^3}{3}\right|_{5/2}^{3}$

$$= 3\left(\frac{3^3}{3}-\frac{\left(5/2\right)^3}{3}\right) \quad = 3\left(\frac{27}{3}-\frac{125/8}{3}\right)$$

$$= 3\left(\frac{27}{3}-\frac{125}{24}\right) \quad = 3\left(\frac{91}{24}\right) \quad = \frac{91}{8}$$

Question # 22

$$\int_1^3 \frac{x^2-2}{x+1}dx$$

$$\begin{array}{r|l} & x-1 \\ \hline x+1 & x^2-2 \\ & \underline{\mp x^2 \mp x} \\ & -x-2 \\ & \underline{\mp x \mp 1} \\ & -1 \end{array}$$

$$= \int_1^3 \left(x-1-\frac{1}{x+1}\right) dx$$

$$= \int_1^3 x\,dx - \int_1^3 dx - \int_1^3 \frac{dx}{x+1}$$

$$= \left|\frac{x^2}{2}\right|_1^3 - \left|\,x\,\right|_1^3 - \left|\ln\left|x+1\right|\right|_1^3$$

$$= \left(\frac{3^2}{2}-\frac{1^2}{2}\right)-(3-1)-\left(\ln|3+1|-\ln|1+1|\right)$$

$$= \left(\frac{9}{2}-\frac{1}{2}\right)-(2)-(\ln 4-\ln 2)$$

$$= 4-2-\ln\frac{4}{2} \quad = 2-\ln 2$$

Question # 23

$$\int_2^3 \frac{3x^2-2x+1}{(x-1)\left(x^2+1\right)}\, dx$$

$$= \int_2^3 \frac{3x^2-2x+1}{x^3-x^2+x-1}\, dx$$

$$= \int_2^3 \frac{\frac{d}{dx}\left(x^3-x^2+x-1\right)}{x^3-x^2+x-1}\, dx$$

$$= \left|\ln\left|x^3-x^2+x-1\right|\right|_2^3$$

$$= \ln\left|3^3-3^2+3-1\right| - \ln\left|2^3-2^2+2-1\right|$$

$$= \ln\left|27-9+3-1\right| - \ln\left|8-4+2-1\right|$$

$$= \ln 20 - \ln 5 \quad = \ln\frac{20}{5} \quad = \ln 4$$

Question # 24

$$\int_0^{\pi/4} \frac{\sin x - 1}{\cos^2 x}\, dx \quad = \int_0^{\pi/4}\left(\frac{\sin x}{\cos^2 x}-\frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\pi/4}\left(\frac{\sin x}{\cos x\cdot\cos x}-\frac{1}{\cos^2 x}\right) dx$$

$$= \int_0^{\pi/4}\left(\sec x\tan x - \sec^2 x\right) dx$$

$$= \left|\sec x - \tan x\right|_0^{\pi/4}$$

$$= \left(\sec\frac{\pi}{4}-\tan\frac{\pi}{4}\right)-\left(\sec(0)-\tan(0)\right)$$

$$= \sqrt{2}-1-1+0 \quad = \sqrt{2}-2$$

Question # 25

Let $I = \int_0^{\pi/4} \frac{1}{1+\sin x}\, dx$

$$= \int_0^{\pi/4} \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\, dx$$

$$= \int_0^{\pi/4} \frac{1-\sin x}{1-\sin^2 x}\, dx \quad = \int_0^{\pi/4} \frac{1-\sin x}{\cos^2 x}\, dx$$

Now same as Question # 24

Question # 26

Let $I = \int_0^1 \frac{3x}{\sqrt{4-3x}}\, dx$

Put $t = 4-3x \quad \Rightarrow \; 3x = 4-t$

Also $dt = -3dx \quad \Rightarrow \; -\frac{1}{3}dt = dx$

When $x = 0$ then $t = 4$

And when $x = 1$ then $t = 1$

So $I = \int_4^1 \frac{4-t}{\sqrt{t}}\left(-\frac{1}{3}dt\right)$

$$= -\frac{1}{3}\int_4^1\left(\frac{4}{t^{1/2}}-\frac{t}{t^{1/2}}\right)dt$$

$$= +\frac{1}{3}\int_1^4\left(4t^{-\frac{1}{2}}-t^{\frac{1}{2}}\right)dt$$

Now do yourself

Question # 27

Let $I = \int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin x\left(2+\sin x\right)}\, dx$

Put $t = \sin x \quad \Rightarrow \; dt = \cos x\, dx$

When $x=\frac{\pi}{6}$ then $t=\frac{1}{2}$

When $x=\frac{\pi}{2}$ then $t=1$

So $I=\int_{1/2}^{1}\frac{dt}{t(2+t)}$

Now consider

$$\frac{1}{t(2+t)}=\frac{A}{t}+\frac{B}{2+t}$$

$\Rightarrow\ 1=A(2+t)+Bt$ …… (i)

Put $t=0$ in (i)

$1=A(2+0)+B(0)\ \Rightarrow\ 1=2A\ \Rightarrow\ A=\frac{1}{2}$

Put $2+t=0\Rightarrow\ t=-2$ in eq. (i)

$1=0+B(-2)\ \Rightarrow\ 1=-2B\ \Rightarrow\ B=-\frac{1}{2}$

So $\frac{1}{t(2+t)}=\frac{1/2}{t}+\frac{-1/2}{2+t}$

$$\Rightarrow\ \int_{1/2}^{1}\frac{1}{t(2+t)}dt=\int_{1/2}^{1}\frac{1/2}{t}\,dt+\int_{1/2}^{1}\frac{-1/2}{2+t}\,dt$$

$$=\frac{1}{2}\int_{1/2}^{1}\frac{1}{t}\,dt-\frac{1}{2}\int_{1/2}^{1}\frac{1}{2+t}\,dt$$

$$=\frac{1}{2}\Big|\ln|t|\Big|_{1/2}^{1}-\frac{1}{2}\Big|\ln|2+t|\Big|_{1/2}^{1}$$

$$=\frac{1}{2}\left[\ln|1|-\ln\left|\frac{1}{2}\right|\right]-\frac{1}{2}\left[\ln|2+1|-\ln\left|2+\frac{1}{2}\right|\right]$$

$$=\frac{1}{2}\left[0-\ln\frac{1}{2}\right]-\frac{1}{2}\left[\ln 3-\ln\frac{5}{2}\right]$$

$$=\frac{1}{2}\left[-\ln\frac{1}{2}-\ln 3+\ln\frac{5}{2}\right]$$

$$=\frac{1}{2}\ln\left(\frac{5/2}{1/2\times 3}\right)\quad=\frac{1}{2}\ln\left(\frac{5}{3}\right)$$

Question # 28

Let $I=\int_{0}^{\pi/2}\frac{\sin x\,dx}{(1+\cos x)(2+\cos x)}$

Put $t=\cos x\ \Rightarrow\ dt=-\sin x\,dx$

$\Rightarrow\ -dt=\sin x\,dx$

When $x=0$ then $t=1$

And when $x=\frac{\pi}{2}$ then $t=0$

So $I=\int_{1}^{0}\frac{-dt}{(1+t)(2+t)}$

$$=-\int_{1}^{0}\frac{dt}{(1+t)(2+t)}\quad=\int_{0}^{1}\frac{dt}{(1+t)(2+t)}$$

Now consider

$$\frac{1}{(1+t)(2+t)}=\frac{A}{1+t}+\frac{B}{2+t}$$

$\Rightarrow\ 1=A(2+t)+B(1+t)$ …. (i)

Put $1+t=0\ \Rightarrow\ t=-1$ in (i)

$1=A(2-1)+0\ \Rightarrow\ A=1$

Put $2+t=0\ \Rightarrow\ t=-2$ in (i)

$1=0+B(1-2)\ \Rightarrow\ 1=-B$ i.e. $B=-1$

So

$$\frac{1}{(1+t)(2+t)}=\frac{1}{1+t}+\frac{-1}{2+t}$$

$$\int_{0}^{1}\frac{1}{(1+t)(2+t)}\,dt=\int_{0}^{1}\frac{1}{1+t}\,dt-\int_{0}^{1}\frac{1}{2+t}\,dt$$

$$=\Big|\ln|1+t|\Big|_{0}^{1}-\Big|\ln|2+t|\Big|_{0}^{1}$$

$$=\left(\ln|1+1|-\ln|1+0|\right)-\left(\ln|2+1|-\ln|2+0|\right)$$

$$=\ln 2-0-\ln 3+\ln 2$$

$$=\ln\left(\frac{2\times 2}{3}\right)\quad=\ln\left(\frac{4}{3}\right)$$