EXERCISE 6.4

Question # 1(i)

$$y^2 = 8x$$

Here $4a = 8 \quad \Rightarrow \quad a = 2$

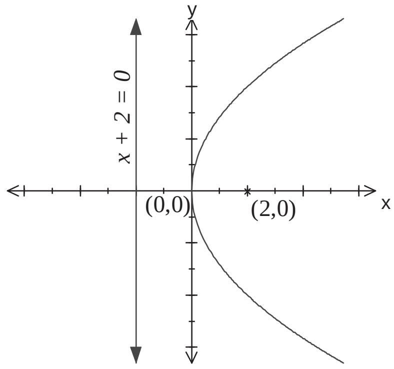
Vertex: $O(0,0)$

The axis of parabola is along x-axis and opening of parabola is to the right side.

Focus: $(a,0) = (2,0)$

Directrix: $x + a = 0$

$\Rightarrow \quad x + 2 = 0$



Question # 1(ii)

$$x^2 = -16y$$

Here $4a = 16 \quad \Rightarrow \quad a = 4$

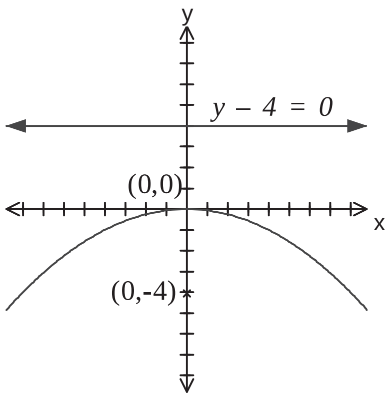
Vertex: $O(0,0)$

The axis of parabola is along $y-$axis and opening of the parabola is downward.

So Focus: $F(0,-a) = F(0,-4)$

Directrix: $y - a = 0$

$\Rightarrow \quad y - 4 = 0$



Question # 1(iii)

$$x^2 = 5y$$

Here $4a = 5 \quad \Rightarrow \quad a = \frac{5}{4}$

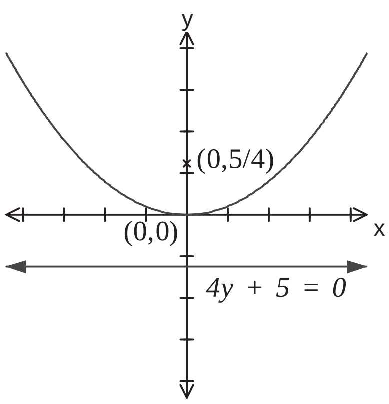
And vertex: $O(0,0)$

The axis of the parabola is along y-axis and opening of the parabola is upward.

Focus: $F(0,a) = F\left(0,\frac{5}{4}\right)$

Directrix: $y + a = 0 \quad \Rightarrow \quad y + \frac{5}{4} = 0$

$\Rightarrow \quad 4y + 5 = 0$



Question # 1(iv)

$$y^2 = -12x$$

Here $4a = 12 \quad \Rightarrow \quad a = 3$

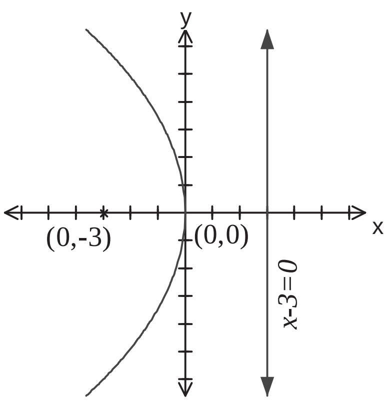
And vertex: $O(0,0)$

The axis of the parabola is along x-axis and opening of the parabola is to the left side.

Focus: $F(-a,0) = (-3,0)$

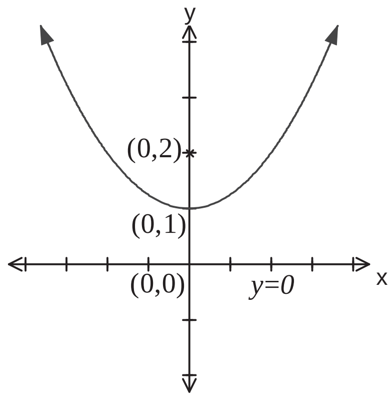
Directrix: $x - a = 0$

$\Rightarrow \quad x - 3 = 0$



Question # 1(v)

$$x^2 = 4(y-1) \quad \ldots\ldots\ldots\ldots (i)$$



Put $X = x$, $Y = y-1$

$\Rightarrow \; X^2 = 4Y \quad \ldots\ldots\ldots\ldots (ii)$

Here $4a = 4 \Rightarrow a = 1$

And vertex of parabola (ii) is $O(0,0)$ with axis of parabola is along $Y-$axis open upward.

$\because$ Vertex: $O(0,0)$

$\Rightarrow \; X = 0$, $Y = 0$

$\Rightarrow \; x = 0$, $y-1=0 \Rightarrow y=1$

$\Rightarrow \; (0,1)$ is vertex of parabola (i)

Now focus: $F(0,a) = F(0,1)$

$\Rightarrow \; X = 0$, $Y = 1$

$\Rightarrow \; x = 0$, $y-1 = 1$

$y = 1+1 \Rightarrow y = 2$

$\Rightarrow \; (0,2)$ is focus of parabola (i)

Directrix of parabola (ii) is

$Y+a = 0 \Rightarrow Y+1 = 0$

$\Rightarrow \; y-1+1 = 0$

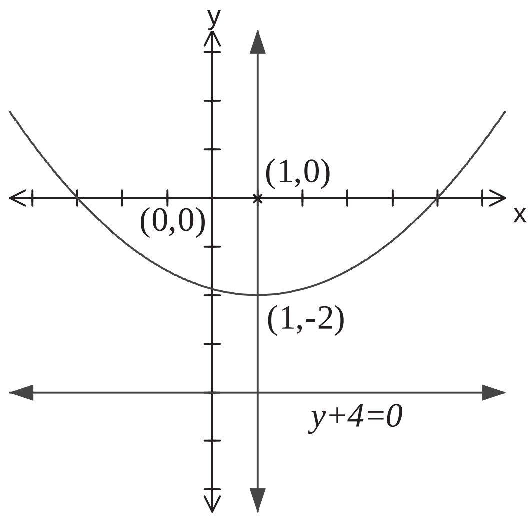
$\Rightarrow \; y = 0$ is directrix of parabola (i)

Question # 1(vi)

Do yourself

Question # 1(vii)

$$(x-1)^2 = 8(y+2) \quad \ldots\ldots\ldots\ldots (i)$$



Put $X = x-1$, $Y = y+2$ in (i)

$X^2 = 8Y \quad \ldots\ldots\ldots\ldots (ii)$

Here $4a = 8 \Rightarrow a = 2$

Axis of parabola is along Y-axis open upward with vertex of $(0,0)$

$\Rightarrow \; X = 0$, $Y = 0$

$\Rightarrow \; x-1 = 0$, $y+2 = 0$

$\Rightarrow \; x = 2$, $y = -2$

$\Rightarrow \; (1,-2)$ is vertex of parabola (i)

Focus of (ii) is $(0,a) = (0,2)$

$\Rightarrow \; X = 0$, $Y = 2$

$\Rightarrow \; x-1 = 0$, $y+2 = 2$

$\Rightarrow \; x = 1$, $y = 0$

$\Rightarrow \; (1,0)$ is the focus of given parabola (i)

Directrix of (ii)

$Y+a = 0 \Rightarrow Y+2=0$

$\Rightarrow \; y+2+2 = 0 \Rightarrow y+4 = 0$ is directrix of given parabola.

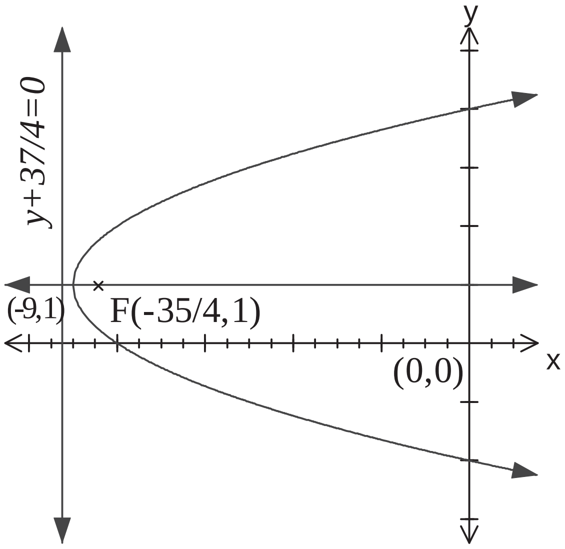
Question # 1(viii)

$$y = 6x^2 - 1$$

$$\Rightarrow \quad 6x^2 = y+1 \quad \Rightarrow \quad x^2 = \frac{1}{6}(y+1)$$

Now try yourself

Question # 1(ix)



$$x+8-y^2+2y = 0$$

$$\Rightarrow \quad y^2-2y = x+8$$

$$\Rightarrow \quad y^2-2y+1 = x+8+1$$

$$\Rightarrow \quad (y-1)^2 = x+9$$

Put $X = x+9$, $Y = y-1$

$$Y^2 = X$$

Here $4a = 1 \quad \Rightarrow \quad a = \frac{1}{4}$

The axis of parabola is along x-axis and it is opening to the right side.

Vertex of parabola (*ii*) is $(0,0)$

$$\Rightarrow \quad X = 0 \quad , \quad Y = 0$$

$$\Rightarrow \quad x+9 = 0 \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = -9 \quad , \quad y = 1$$

$\Rightarrow \quad (-9,1)$ is vertex of the parabola (*i*)

Focus: $(a,0) = \left(\frac{1}{4},0\right)$

$$X = \frac{1}{4} \quad , \quad Y = 0$$

$$\Rightarrow \quad x+9 = \frac{1}{4} \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = \frac{1}{4}-9 \quad , \quad y-1 = 0$$

$$\Rightarrow \quad x = -\frac{35}{4} \quad , \quad y = 1$$

$\Rightarrow \quad \left(-\frac{35}{4},1\right)$ is focus of parabola (*i*)

Directrix of parabola (*ii*)

$$X + a = 0$$

$$\Rightarrow \quad X + \frac{1}{4} = 0$$

$$\Rightarrow \quad x+9+\frac{1}{4} = 0$$

$\Rightarrow \quad x+\frac{37}{4} = 0$ is directrix of parabola (*i*)

Question # 1(x)

Do yourself as above

Question # 2(i)

Focus: $F(-3,1)$

Directrix: $x = 3$ i.e. $x-3 = 0$

Let $P(x,y)$ be any point on parabola then by definition

$$|PF| = \perp\text{ar distance of } P(x,y) \text{ from directrix}$$

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-3|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \sqrt{x^2+6x+9+y^2-2y+1} = |x-3|$$

On squaring

$$\Rightarrow x^2+6x+9+y^2-2y+1 = x^2-6x+9$$

$$\Rightarrow x^2+6x+9+y^2-2y+1-x^2+6x-9 = 0$$

$$\Rightarrow y^2+12x-2y+1 = 0$$

is required equation of parabola.

Question # 2(ii)

Do yourself as above.

Question # 2(iii)

Focus: $F(-3,1)$

Directrix: $x-2y-3 = 0$

Let $P(x,y)$ be any point on parabola, then by definition of parabola

$$|PF| = \perp\text{ar distance of } P(x,y) \text{ from directrix.}$$

$$\Rightarrow \sqrt{(x+3)^2+(y-1)^2} = \frac{|x-2y-3|}{\sqrt{(1)^2+(-2)^2}}$$

$$\Rightarrow \sqrt{x^2+6x+9+y^2-2y+1} = \frac{|x-2y-3|}{\sqrt{5}}$$

$$\Rightarrow \sqrt{5}\sqrt{x^2+6x+9+y^2-2y+1} = |x-2y-3|$$

On squaring

$$\Rightarrow 5\left(x^2+6x+9+y^2-2y+1\right) = x^2+4y^2+9-4xy+12y-6x$$

$$\Rightarrow 5x^2+30x+45+5y^2-10y+5-x^2-4y^2-9+4xy-12y+6x = 0$$

$$\Rightarrow 5x^2+y^2+36x-22y+4xy+41 = 0$$

is required equation

Question # 2(iv)

Given: Focus $(1,2)$, Vertex $(3,2)$

Focus and vertex implies that axis of parabola is parallel to x-axis and opening to left side. Therefore eq. of parabola with vertex $(3,2)$

$$(y-2)^2 = -4a(x-3) \text{ ………… } (i)$$

Now $a =$ Distance between focus and vertex

$$= \sqrt{(3-1)^2+(2-2)^2} = \sqrt{4+0} = 2$$

Putting in (i)

$$(y-2)^2 = -4(2)(x-3) \quad \Rightarrow \quad y^2-4y+4 = -8x+24$$

$$\Rightarrow y^2-4y+4+8x-20 = 0 \quad \Rightarrow \quad y^2-4y+8x-20 = 0 \text{ is req. eq.}$$

Question # 2(v)

Do yourself

Question # 2(vii)

Directrix: $y = 3$ i.e. $y - 3 = 0$

Vertex $(2,2)$

Since axis of parabola is parallel to y-axis (because directrix is parallel to x-axis). And opening is downward.

So equation of parabola with vertex $(h,k) = (2,2)$

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-2)^2 = -4a(y-2)$$

Now a = Distance of vertex $(2,2)$ from directrix

$$= \frac{|2-3|}{\sqrt{(0)^2+(1)^2}} = \frac{|-1|}{1} = 1$$

Putting in (i)

$$(x-2)^2 = -4(1)(y-2)$$

$$\Rightarrow x^2 - 4x + 4 = 4y - 8 \quad \Rightarrow x^2 - 4x + 4 - 4y + 8 = 0$$

$$\Rightarrow x^2 - 4x - 4y + 12 = 0 \quad \text{is req. eq.}$$

Question # 2(viii)

Directrix: $y = 1$

Latusractum $= 4a = 8 \quad \Rightarrow a = 2$

$\because$ Parabola is open downward

$\therefore$ Consider vertex $= (h,-1)$

And equation of parabola

$$(x-h)^2 = -4a(y-k)$$

$$\Rightarrow (x-h)^2 = -4(2)(y+1) \quad \Rightarrow x^2 - 2hx + h^2 = -8y - 8$$

$$\Rightarrow x^2 - 2hx + 8y + h^2 + 8 = 0 \quad \text{is req. eq.}$$

Question # 2(ix)

Axis of parabola: $y = 0$

Let vertex is (h,k)

$\because$ it lies on x-axis $\therefore$ $k = 0$

Now equation of parabola with vertex $(h,0)$

$$(y-0)^2 = 4a(x-h)$$

$$\Rightarrow y^2 = 4a(x-h) \quad \ldots\ldots\ldots\ldots (i)$$

$\because$ $(2,1)$ lies on parabola (i)

$$\therefore (1)^2 = 4a(2-h)$$

$$\Rightarrow 1 = 4a(2-h) \quad \ldots\ldots\ldots\ldots (ii)$$

Also $(11,-2)$ lies on parabola (i)

$$(-2)^2 = 4a(11-h)$$

$$\Rightarrow 4 = 4a(11-h)$$

$$\Rightarrow 1 = a(11-h) \quad \ldots\ldots\ldots\ldots\ldots (iii)$$

Dividing (i) & (ii)

$$\frac{1}{1} = \frac{4a(2-h)}{a(11-h)}$$

$$\Rightarrow 1 = \frac{4(2-h)}{(11-h)} \qquad \Rightarrow 11-h = 8-4h$$

$$\Rightarrow 4h-h = 8-11 \qquad \Rightarrow 3h = -3 \qquad \Rightarrow h = -1$$

Putting in (ii)

$$1 = 4a(2-(-1))$$

$$\Rightarrow 1 = 4a(3) \qquad \Rightarrow 1 = 12a \qquad \Rightarrow a = \frac{1}{12}$$

Using in (i)

$$y^2 = 4\left(\frac{1}{12}\right)(x-(-1)) \qquad \Rightarrow 3y^2 = x+1$$

$$\Rightarrow 3y^2 - x - 1 = 0$$

is the required equation.

Question # 3

i) When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$y = h$ (parallel to x-axis)

i.e. $y-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|y-h|}{\sqrt{(0)^2+(1)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|y-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = y^2-2hy+h^2 \qquad \Rightarrow x^2+y^2-y^2+2hy-h^2 = 0$$

$\Rightarrow x^2+2hy-h^2 = 0$ is req. equation.

ii) When directrix is parallel to y-axis.

When directrix is parallel to x-axis

Suppose $F(0,0)$ be focus and equation of directrix be

$x = h$ (parallel to y-axis)

i.e. $x-h = 0$

Now let $P(x,y)$ be any point on parabola the by definition of parabola

$|PF| = \perp$ar distance of $P(x,y)$ from directrix

$$\Rightarrow \sqrt{(x-0)^2+(y-0)^2} = \frac{|x-h|}{\sqrt{(1)^2+(0)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \frac{|x-h|}{1}$$

On squaring

$$\Rightarrow x^2+y^2 = x^2-2hx+h^2 \qquad \Rightarrow x^2+y^2-x^2+2hx-h^2 = 0$$

$\Rightarrow y^2+2hx-h^2 = 0$ is req. equation

Question # 4

Focus: $F\left(a\cos\alpha, a\sin\alpha\right)$ **Correction*

Directrix: $x\cos\alpha + y\sin\alpha + a = 0$ **Correction*

Let $P\left(x,y\right)$ be any point on parabola then by definition of parabola

$$\left|PF\right| = \perp\text{ar distance of } P\left(x,y\right) \text{ from directrix}$$

$$\Rightarrow \sqrt{\left(x-a\cos\alpha\right)^2+\left(y-a\sin\alpha\right)^2} = \frac{\left|x\cos\alpha+y\sin\alpha+a\right|}{\sqrt{\cos^2\alpha+\sin^2\alpha}}$$

On squaring

$$\left(x-a\cos\alpha\right)^2+\left(y-\sin\alpha\right)^2 = \frac{\left|x\cos\alpha+y\sin\alpha+a\right|^2}{1}$$

$$\Rightarrow x^2-2ax\cos\alpha+a^2\cos^2\alpha+y^2-2ay\sin\alpha+a^2\sin^2\alpha$$
$$= x^2\cos^2\alpha+y^2\sin^2\alpha+a^2+2ax\cos\alpha+2ay\sin\alpha+2xy\sin\alpha\cos\alpha$$

$$\Rightarrow x^2-x^2\cos\alpha+y^2-y^2\sin\alpha+a^2\left(\cos^2\alpha+\sin^2\alpha\right)$$
$$= a^2+2ax\cos\alpha+2ay\sin\alpha+2xy\sin\alpha\cos\alpha+2ax\cos\alpha+2ay\sin\alpha$$

$$\Rightarrow x^2\left(1-\cos^2\alpha\right)+y^2\left(1-\sin^2\alpha\right)+a^2\left(1\right)-a^2-2xy\sin\alpha\cos\alpha$$
$$= 4ax\cos\alpha+4ay\sin\alpha$$

$$\Rightarrow x^2\cos^2\alpha+y^2\sin^2\alpha-2xy\sin\alpha\cos\alpha = 4a\left(x\cos\alpha+y\sin\alpha\right)$$

$$\Rightarrow \left(x\sin\alpha-y\cos\alpha\right)^2 = 4a\left(x\cos\alpha+y\sin\alpha\right)$$

is equation of parabola which is given.

Question # 5

Consider equation of parabola

$$y^2 = 4ax$$

$$\Rightarrow y\cdot y = 4a\cdot x$$

$$\Rightarrow \frac{4a}{y} = \frac{y}{x}$$

$$\Rightarrow \frac{\textit{latus ractum}}{\textit{ordinate}} = \frac{\textit{ordinate}}{\textit{abscissa}}$$

$\Rightarrow$ ordinate is mean proportional between latus rectum and abscissa.

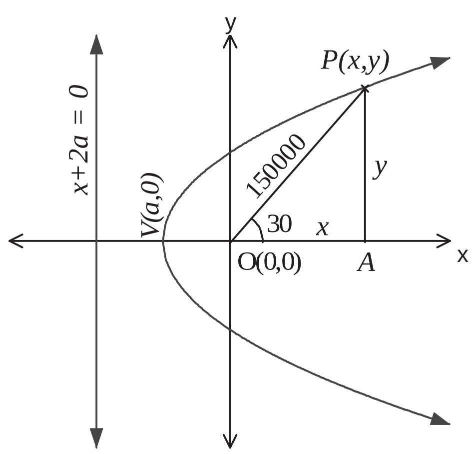
Question # 6

Suppose earth be at focus which is origin and $V\left(-a,0\right)$ be vertex of parabola.

Then directrix of parabola;

$$x = -2a$$

$$\Rightarrow x+2a = 0$$



Let comet be at a point $P\left(x,y\right)$ then by definition of parabola

$$\left|PF\right| = \perp\text{ar distance of } P\left(x,y\right) \text{ from directrix}$$

$$\Rightarrow \sqrt{\left(x-0\right)^2+\left(y-0\right)^2} = \frac{\left|x+2a\right|}{\sqrt{\left(1\right)^2+\left(0\right)^2}}$$

$$\Rightarrow \sqrt{x^2+y^2} = \left|x+2a\right|$$

On squaring

$$x^2+y^2 = \left(x+2a\right)^2 \quad \ldots\ldots\ldots.. (i)$$

Also by Pythagoras theorem in $\triangle$ABC

$$|OA|^2 + |AP|^2 = |OP|^2$$

$$\Rightarrow \quad x^2 + y^2 = (150000)^2 \quad \ldots\ldots\ldots\ldots..\ (ii)$$

Comparing (i) and (ii)

$$(x+2a)^2 = (150000)^2$$

$$\Rightarrow \quad x + 2a = \pm 150000 \quad \ldots\ldots\ldots\ldots\ldots\ (iii)$$

Now from right triangle OAP

$$\cos 30^\circ = \frac{|OA|}{|OP|} \qquad \Rightarrow \quad \frac{\sqrt{3}}{2} = \frac{x}{150000}$$

$$\Rightarrow \quad x = \frac{\sqrt{3}}{2}(150000)$$

Using in (iii)

$$\frac{\sqrt{3}}{2}(150000) + 2a = \pm 150000$$

$$\Rightarrow \quad 2a = \pm 150000 - \frac{\sqrt{3}}{2}(150000) \qquad \Rightarrow \quad 2a = \pm 150000 - \sqrt{3}(75000)$$

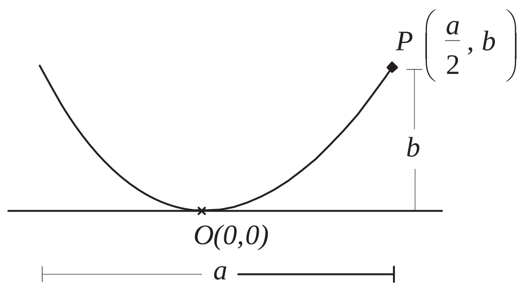
$$\Rightarrow \quad 2a = 75000\left(\pm 2 - \sqrt{3}\right) \qquad \Rightarrow \quad a = 37500\left(\pm 2 - \sqrt{3}\right)$$

Since a is shortest distance and can't be negative

Therefore $\quad a = 37500\left(2 - \sqrt{3}\right) Km$

Question # 7

Consider equation of parabola with vertex $O(0,0)$



$$x^2 = 4a'y \quad \ldots\ldots\ldots\ldots\ (i)$$

Since $P\left(\dfrac{a}{2}, b\right)$ lies on parabola

$$\left(\frac{a}{2}\right)^2 = 4a'(b) \quad \Rightarrow \quad a' = \frac{a^2}{16b}$$

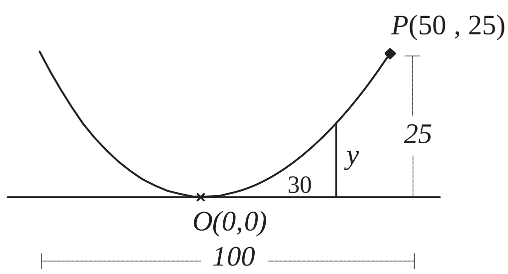
Putting in (i)

$$x^2 = 4\left(\frac{a^2}{16b}\right)y$$

$$\Rightarrow \quad x^2 = \frac{a^2}{4b}y \quad \text{is required equation.}$$

Question # 8

Suppose equation of parabola with vertex $(0,0)$



$$x^2 = 4ay \quad \ldots\ldots\ldots\ldots\ldots\ (i)$$

From figure, we see that $P(50,25)$ lies on parabola.

$$(50)^2 = 4a(25)$$

$$\Rightarrow \quad 2500 = 100a \qquad \Rightarrow \quad a = 25$$

Putting in (i)

$$x^2 = 4(25)y \quad \Rightarrow \quad x^2 = 100y$$

When $x = 30$

$$(30)^2 = 100y$$
$$\Rightarrow \quad y = \frac{900}{100} \quad \Rightarrow \quad y = 9$$
Hence the required height $= 9m$

Question # 9

Suppose the parabola

$$y^2 = 4ax \text{ .... } (i)$$

Let $P(x_1, y_1)$ be any point on parabola, then

$$y_1^2 = 4ax_1 \text{ ....} (ii)$$

Now differentiating (i) w.r.t x

$$\frac{d}{dx}y^2 = \frac{d}{dx}4ax \quad \Rightarrow \quad 2y\frac{dy}{dx} = 4a \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

Slope of tangent at $(x_1, y_1) = m_1 = \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{2a}{y_1}$

Now slope of $PS = m_2 = \dfrac{y_1 - 0}{x_1 - a}$

$$\Rightarrow \quad m_2 = \frac{y_1}{x_1 - a}$$

Now slope of line parallel to axis of parabola $= m_3 = 0$

(because axis of parabola is along x-axis)

Let θ_1 be angle between tangent and line parallel to axis of parabola, then

$$\tan\theta_1 = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{\frac{2a}{y_1} - 0}{1 + \left(\frac{2a}{y_1}\right)(0)} = \frac{2a/y_1}{1}$$

$$\Rightarrow \quad \tan\theta_1 = \frac{2a}{y_1} \text{ .... } (iii)$$

Let θ_2 be angle between tangent and PS, then

$$\tan\theta_2 = \frac{m_2 - m_1}{1 + m_2 m_1}$$

$$= \frac{\frac{y_1}{x_1 - a} - \frac{2a}{y_1}}{1 + \left(\frac{y_1}{x_1 - a}\right)\left(\frac{2a}{y_1}\right)} = \frac{\frac{y_1^2 - 2a(x_1 - a)}{y_1(x_1 - a)}}{\frac{x_1 - a + 2a}{x_1 - a}} = \frac{y_1^2 - 2ax_1 + 2a^2}{y_1(x_1 + a)}$$

$$= \frac{4ax_1 - 2ax_1 + 2a^2}{y_1(x_1 + a)} \qquad \text{from } (ii)$$

$$= \frac{2ax_1 + 2a^2}{y_1(x_1 + a)} = \frac{2a(x_1 + a)}{y_1(x_1 + a)}$$

$$\Rightarrow \quad \tan\theta_2 = \frac{2a}{y_1} \text{ .... } (iv)$$

From (iii) and (iv)

$$\tan\theta_1 = \tan\theta_2 \qquad \Rightarrow \quad \theta_1 = \theta_2$$

as required
