Exercise 9.4

Prove the following identities:

1.

$$\tan\theta+\cot\theta=\csc\theta\sec\theta$$
$$L.H.S=\tan\theta+\cot\theta$$
$$L.H.S=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$
$$L.H.S=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$L.H.S=\frac{1}{\sin\theta\cos\theta}$$
$$L.H.S=\frac{1}{\sin\theta}\frac{1}{\cos\theta}$$
$$L.H.S=\csc\theta\sec\theta=R.H.S$$

2.

$$\sec\theta\csc\theta\sin\theta\cos\theta=1$$
$$L.H.S=\sec\theta\csc\theta\sin\theta\cos\theta$$
$$L.H.S=\frac{1}{\cos\theta}\frac{1}{\sin\theta}\sin\theta\cos\theta$$
$$L.H.S=1=R.H.S$$

3.

$$\cos\theta+\tan\theta\sin\theta=\sec\theta$$
$$L.H.S=\cos\theta+\tan\theta\sin\theta$$
$$L.H.S=\cos\theta+\frac{\sin\theta}{\cos\theta}\sin\theta$$
$$L.H.S=\frac{\cos^2\theta+\sin^2\theta}{\cos\theta}$$
$$L.H.S=\frac{1}{\cos\theta}$$
$$L.H.S=\sec\theta=R.H.S$$

4.

$$\csc\theta+\tan\theta\sec\theta=\csc\theta\sec^2\theta$$
$$L.H.S=\csc\theta+\tan\theta\sec\theta$$
$$L.H.S=\frac{1}{\sin\theta}+\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}$$
$$L.H.S=\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos^2\theta}$$
$$L.H.S=\frac{1}{\sin\theta\cos^2\theta}$$
$$L.H.S=\frac{1}{\sin\theta}\frac{1}{\cos^2\theta}$$
$$L.H.S=\csc\theta\sec^2\theta$$

5.

$$\sec^2\theta-\csc^2\theta=\tan^2\theta-\cot^2\theta$$
$$R.H.S=\tan^2\theta-\cot^2\theta$$
$$R.H.S=\frac{\sin^2\theta}{\cos^2\theta}-\frac{\cos^2\theta}{\sin^2\theta}$$
$$R.H.S=\frac{\sin^4\theta-\cos^4\theta}{\cos^2\theta\sin^2\theta}$$
$$R.H.S=\frac{(\sin^2\theta-\cos^2\theta)(\sin^2\theta+\cos^2\theta)}{\cos^2\theta\sin^2\theta}$$
$$R.H.S=\frac{(\sin^2\theta-\cos^2\theta)(1)}{\cos^2\theta\sin^2\theta}$$
$$R.H.S=\frac{\sin^2\theta}{\cos^2\theta\sin^2\theta}-\frac{\cos^2\theta}{\cos^2\theta\sin^2\theta}$$
$$R.H.S=\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}$$
$$R.H.S=\sec^2\theta-\csc^2\theta=L.H.S$$

6.

$$\cot^2\theta-\cos^2\theta=\cot^2\theta\cos^2\theta$$
$$L.H.S=\cot^2\theta-\cos^2\theta$$
$$L.H.S=\frac{\cos^2\theta}{\sin^2\theta}-\cos^2\theta$$
$$L.H.S=\frac{\cos^2\theta-\sin^2\theta\cos^2\theta}{\sin^2\theta}$$
$$L.H.S=\frac{\cos^2\theta(1-\sin^2\theta)}{\sin^2\theta}$$
$$L.H.S=\frac{\cos^2\theta}{\sin^2\theta}\cos^2\theta$$
$$L.H.S=\cot^2\theta\cos^2\theta=R.H.S$$

7.

$$(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=1$$
$$L.H.S=(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)$$
$$L.H.S=\sec^2\theta-\tan^2\theta=1=R.H.S$$

8.

$$2\cos^2\theta-1=1-2\sin^2\theta$$
$$L.H.S=2\cos^2\theta-1$$
$$L.H.S=2(1-\sin^2\theta)-1$$
$$L.H.S=2-2\sin^2\theta-1$$
$$L.H.S=1-2\sin^2\theta=R.H.S$$

9.

$$\cos^2\theta-\sin^2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$R.H.S = \frac{1-\frac{\sin^2\theta}{\cos^2\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}$$

$$R.H.S = \frac{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}$$

$$R.H.S = \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$$

$$R.H.S = \frac{\cos^2\theta-\sin^2\theta}{1}$$

$$R.H.S = \cos^2\theta-\sin^2\theta = L.H.S$$

10.

$$\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{\cot\theta-1}{\cot\theta+1}$$

$$R.H.S = \frac{\cot\theta-1}{\cot\theta+1}$$

$$R.H.S = \frac{\frac{\cos\theta}{\sin\theta}-1}{\frac{\cos\theta}{\sin\theta}+1}$$

$$R.H.S = \frac{\frac{\cos\theta-\sin\theta}{\sin\theta}}{\frac{\cos\theta+\sin\theta}{\sin\theta}}$$

$$R.H.S = \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = L.H.S$$

11.

$$\frac{\sin\theta}{1+\cos\theta}+\cot\theta = \cos ec\,\theta$$

$$L.H.S = \frac{\sin\theta}{1+\cos\theta}+\cot\theta$$

$$L.H.S = \frac{\sin\theta}{1+\cos\theta}+\frac{\cos\theta}{\sin\theta}$$

$$L.H.S = \frac{\sin^2\theta+\cos\theta(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$L.H.S = \frac{\sin^2\theta+\cos\theta+\cos^2\theta}{\sin\theta(1+\cos\theta)}$$

$$L.H.S = \frac{(\sin^2\theta+\cos^2\theta)+\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$L.H.S = \frac{1+\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$L.H.S = \frac{1}{\sin\theta}$$

$$L.H.S = \csc\theta = R.H.S$$

12.

$$\frac{\cot^2\theta-1}{1+\cot^2\theta} = 2\cos^2\theta-1$$

$$L.H.S = \frac{\cot^2\theta-1}{1+\cot^2\theta}$$

$$L.H.S = \frac{\frac{\cos^2\theta}{\sin^2\theta}-1}{1+\frac{\cos^2\theta}{\sin^2\theta}}$$

$$L.H.S = \frac{\frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\sin^2\theta}}$$

$$L.H.S = \frac{\cos^2\theta-\sin^2\theta}{\sin^2\theta+\cos^2\theta}$$

$$L.H.S = \frac{\cos^2\theta-(1-\cos^2\theta)}{1}$$

$$L.H.S = \cos^2\theta-1+\cos^2\theta$$

$$L.H.S = 2\cos^2\theta-1 = R.H.S$$

13.

$$\frac{1+\cos\theta}{1-\cos\theta} = (\csc\theta+\cot\theta)^2$$

$$R.H.S = (\csc\theta+\cot\theta)^2$$

$$R.H.S = \csc^2\theta+\cot^2\theta+2\csc\theta\cot\theta$$

$$R.H.S = \frac{1}{\sin^2\theta}+\frac{\cos^2\theta}{\sin^2\theta}+2\frac{1}{\sin\theta}\frac{\cos\theta}{\sin\theta}$$

$$R.H.S = \frac{1+\cos^2\theta+2\cos\theta}{\sin^2\theta}$$

$$R.H.S = \frac{\cos^2\theta+2\cos\theta+1}{1-\cos^2\theta}$$

$$R.H.S = \frac{(1+\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)}$$

$$R.H.S = \frac{1+\cos\theta}{1-\cos\theta} = L.H.S$$

14.

$$(\sec\theta-\tan\theta)^2 = \frac{1-\sin\theta}{1+\sin\theta}$$

$$L.H.S = (\sec\theta-\tan\theta)^2$$

$$L.H.S = \sec^2\theta+\tan^2\theta-2\sec\theta\tan\theta$$

$$L.H.S = \frac{1}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}-2\frac{1}{\cos\theta}\frac{\sin\theta}{\cos\theta}$$

$$L.H.S = \frac{1+\sin^2\theta-2\sin\theta}{\cos^2\theta}$$

$$L.H.S = \frac{(1-\sin\theta)^2}{1-\sin^2\theta}$$

$$L.H.S = \frac{(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$L.H.S = \frac{1-\sin\theta}{1+\sin\theta} = R.H.S$$

15.

$$\frac{2\tan\theta}{1+\tan^2\theta} = 2\sin\theta\cos\theta$$

$$L.H.S = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$L.H.S = \frac{2\tan\theta}{\sec^2\theta}$$

$$L.H.S = \frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}$$

$$L.H.S = 2\frac{\sin\theta}{\cos\theta}.\cos^2\theta$$

$$L.H.S = 2\sin\theta\cos\theta = R.H.S$$

16.

$$\frac{1-\sin\theta}{\cos\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$$L.H.S = \frac{1-\sin\theta}{\cos\theta}.\frac{1+\sin\theta}{1+\cos\theta}$$

$$L.H.S = \frac{1-\sin^2\theta}{\cos\theta(1+\sin\theta)}$$

$$L.H.S = \frac{\cos^2\theta}{\cos\theta(1+\sin\theta)}$$

$$L.H.S = \frac{\cos\theta}{1+\sin\theta} = R.H.S$$

17.

$$(\tan\theta+\cot\theta)^2 = \sec^2\theta\csc^2\theta$$

$$L.H.S = (\tan\theta+\cot\theta)^2$$

$$L.H.S = \left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^2$$

$$L.H.S = \left(\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)^2$$

$$L.H.S = \left(\frac{1}{\cos\theta\sin\theta}\right)^2$$

$$L.H.S = \frac{1}{\cos^2\theta\sin^2\theta}$$

$$L.H.S = \frac{1}{\cos^2\theta}\frac{1}{\sin^2\theta}$$

$$L.H.S = \sec^2\theta\csc^2\theta = R.H.S$$

18.

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \tan\theta+\sec\theta$$

$$L.H.S = \frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$$

$$L.H.S = \frac{\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}-\frac{1}{\cos\theta}+1}$$

$$L.H.S = \frac{\frac{\sin\theta+1-\cos\theta}{\cos\theta}}{\frac{\sin\theta-1+\cos\theta}{\cos\theta}}$$

$$L.H.S = \frac{\sin\theta+1-\cos\theta}{\sin\theta-1+\cos\theta}.\frac{\sin\theta+1+\cos\theta}{\sin\theta+1+\cos\theta}$$

$$L.H.S = \frac{(\sin\theta+1)-\cos\theta}{(\sin\theta+\cos\theta)-1}.\frac{(\sin\theta+1)+\cos\theta}{(\sin\theta+\cos\theta)+1}$$

$$L.H.S = \frac{(\sin\theta+1)^2-\cos^2\theta}{(\sin\theta+\cos\theta)^2-1}$$

$$L.H.S = \frac{\sin^2\theta+1+2\sin\theta-\cos^2\theta}{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}$$

$$L.H.S = \frac{\sin^2\theta+1+2\sin\theta-(1-\sin^2\theta)}{1+2\sin\theta\cos\theta-1}$$

$$L.H.S = \frac{\sin^2\theta+1+2\sin\theta-1+\sin^2\theta}{2\sin\theta\cos\theta}$$

$$L.H.S = \frac{2\sin^2\theta+2\sin\theta}{2\sin\theta\cos\theta}$$

$$L.H.S = \frac{2\sin\theta(\sin\theta+1)}{2\sin\theta\cos\theta}$$

$$L.H.S = \frac{\sin\theta+1}{\cos\theta}$$

$$L.H.S = \frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}$$

$$L.H.S = \tan\theta+\sec\theta = R.H.S$$

19.

$$\frac{1}{\csc\theta-\cot\theta}-\frac{1}{\sin\theta} = \frac{1}{\sin\theta}-\frac{1}{\csc\theta+\cot\theta}$$

$$L.H.S = \frac{1}{\csc\theta-\cot\theta}-\frac{1}{\sin\theta}$$

$$L.H.S = \frac{1}{\csc\theta-\cot\theta}.\frac{\csc\theta+\cot\theta}{\csc\theta+\cot\theta}-\frac{1}{\sin\theta}$$

$$L.H.S = \frac{\csc\theta+\cot\theta}{\csc^2\theta-\cot^2\theta}-\frac{1}{\sin\theta}$$

$$L.H.S = \frac{\csc\theta+\cot\theta}{1}-\frac{1}{\sin\theta}$$

$$L.H.S = \csc\theta+\cot\theta-\frac{1}{\sin\theta}$$

$$L.H.S = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}$$
$$L.H.S = \frac{1+\cos\theta-1}{\sin\theta}$$
$$L.H.S = \frac{\cos\theta}{\sin\theta}$$
$$L.H.S = \cot\theta \qquad (1)$$

Now,

$$R.H.S = \frac{1}{\sin\theta} - \frac{1}{\csc\theta+\cot\theta}$$
$$R.H.S = \frac{1}{\sin\theta} - \frac{1}{\csc\theta+\cot\theta}.\frac{\csc\theta-\cot\theta}{\csc\theta-\cot\theta}$$
$$R.H.S = \frac{1}{\sin\theta} - \frac{\csc\theta-\cot\theta}{\csc^2\theta-\cot^2\theta}$$
$$R.H.S = \frac{1}{\sin\theta} - \frac{\csc\theta-\cot\theta}{1}$$
$$R.H.S = \frac{1}{\sin\theta} - \csc\theta + \cot\theta$$
$$R.H.S = \frac{1}{\sin\theta} - \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$
$$R.H.S = \frac{1-1+\cos\theta}{\sin\theta}$$
$$R.H.S = \frac{\cos\theta}{\sin\theta}$$
$$R.H.S = \cot\theta$$

From (1) and (2),

$$L.H.S = R.H.S$$

20.

$$\sin^3\theta - \cos^3\theta = (\sin\theta-\cos\theta)(1+\sin\theta\cos\theta)$$
$$L.H.S = \sin^3\theta - \cos^3\theta$$
$$L.H.S = (\sin\theta)^3 - (\cos\theta)^3$$
$$L.H.S = (\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)$$
$$L.H.S = (\sin\theta-\cos\theta)(1+\sin\theta\cos\theta) = R.H.S$$

21.

$$\sin^6\theta - \cos^6\theta = (\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta)$$
$$L.H.S = \sin^6\theta - \cos^6\theta$$
$$L.H.S = (\sin^2\theta)^3 - (\cos^2\theta)^3$$
$$L.H.S = (\sin^2\theta-\cos^2\theta)(\sin^4\theta+\cos^4\theta+\sin^2\theta\cos^2\theta)$$
$$\left\{\begin{array}{c} \therefore \sin^2\theta+\cos^2\theta = 1 \\ \Rightarrow (\sin^2\theta+\cos^2\theta)^2 = (1)^2 \\ \Rightarrow \sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta = 1 \\ \Rightarrow \sin^4\theta+\cos^4 = 1-2\sin^2\theta\cos^2\theta \end{array}\right\}$$
$$L.H.S = (\sin^2\theta-\cos^2\theta)(1-2\sin^2\theta\cos^2\theta+\sin^2\theta\cos^2\theta)$$
$$L.H.S = (\sin^2\theta-\cos^2\theta)(1-\sin^2\theta\cos^2\theta) = R.H.S$$

22.

$$\sin^6\theta+\cos^6\theta = 1-3\sin^2\theta\cos^2\theta$$
$$L.H.S = \sin^6\theta+\cos^6\theta$$
$$L.H.S = (\sin^2\theta)^3+(\cos^2\theta)^3$$
$$L.H.S = (\sin^2\theta+\cos^2\theta)(\sin^4\theta+\cos^4\theta-\sin^2\theta\cos^2\theta)$$
$$L.H.S = 1.(\sin^4\theta+\cos^4\theta-\sin^2\theta\cos^2\theta)$$
$$\left\{\begin{array}{c} \therefore \sin^2\theta+\cos^2\theta = 1 \\ \Rightarrow (\sin^2\theta+\cos^2\theta)^2 = (1)^2 \\ \Rightarrow \sin^4\theta+\cos^4\theta+2\sin^2\theta\cos^2\theta = 1 \\ \Rightarrow \sin^4\theta+\cos^4 = 1-2\sin^2\theta\cos^2\theta \end{array}\right\}$$
$$L.H.S = 1-2\sin^2\theta\cos^2\theta-\sin^2\theta\cos^2\theta$$
$$L.H.S = 1-3\sin^2\theta\cos^2\theta) = R.H.S$$

23.

$$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta} = 2\sec^2\theta$$
$$L.H.S = \frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}$$
$$L.H.S = \frac{1-\sin\theta+1+\sin\theta}{(1+\sin\theta)(1-\sin\theta)}$$
$$L.H.S = \frac{2}{1-\sin^2\theta}$$
$$L.H.S = \frac{2}{\cos^2\theta}$$
$$L.H.S = 2.\frac{1}{\cos^2\theta}$$
$$L.H.S = 2\sec^2\theta = R.H.S$$

24.

$$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} = \frac{2}{1-2\sin^2\theta}$$
$$L.H.S = \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}+\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}$$
$$L.H.S = \frac{(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$
$$L.H.S = \frac{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta+\sin^2\theta-2\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}$$
$$L.H.S = \frac{1+1}{1-\sin^2\theta-\sin^2\theta}$$
$$L.H.S = \frac{2}{1-2\sin^2\theta} = R.H.S$$