EXERCISE 7.1

① (i) $P(2,3)$, $Q(6,-2)$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = [6,-2]-[2,3]$

$= [6-2, -2-3]$

$= [4,-5] = 4\underline{i} - 5\underline{j}$ Ans.

Method II

$\overrightarrow{PQ} = (6-2)\underline{i} + (-2-3)\underline{j}$

$\overrightarrow{PQ} = 4\underline{i} - 5\underline{j}$ Ans.

(ii) $P(0,5)$, $Q(-1,-6)$

$\overrightarrow{PQ} = (-1-0)\underline{i} + (-6-5)\underline{j}$

$\overrightarrow{PQ} = -\underline{i} - 11\underline{j}$ Ans.

② (i) Given that $\underline{u} = 2\underline{i} - 7\underline{j}$

$|\underline{u}| = \sqrt{(2)^2+(-7)^2} = \sqrt{4+49} = \sqrt{53}$ Ans.

(ii) $\underline{u} = \underline{i} + \underline{j}$

$|\underline{u}| = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2}$ Ans.

(iii) $\underline{u} = [3,-4]$

$|\underline{u}| = \sqrt{(3)^2+(-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$ Ans.

③ Given that $\underline{u} = 2\underline{i} - 7\underline{j}$, $\underline{v} = \underline{i} - 6\underline{j}$, $\underline{w} = -\underline{i} + \underline{j}$

(i) $\underline{u} + \underline{v} - \underline{w} = (2\underline{i} - 7\underline{j}) + (\underline{i} - 6\underline{j}) - (-\underline{i} + \underline{j})$

$= 2\underline{i} - 7\underline{j} + \underline{i} - 6\underline{j} + \underline{i} - \underline{j}$

$= 4\underline{i} - 14\underline{j}$ Ans.

(ii) $2\underline{u} - 3\underline{v} + 4\underline{w}$

$= 2(2\underline{i} - 7\underline{j}) - 3(\underline{i} - 6\underline{j}) + 4(-\underline{i} + \underline{j})$

$= 4\underline{i} - 14\underline{j} - 3\underline{i} + 18\underline{j} - 4\underline{i} + 4\underline{j}$

$= -3\underline{i} + 8\underline{j}$ Ans.

(iii) $\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} + \frac{1}{2}\underline{w}$

$= \frac{1}{2}(2\underline{i} - 7\underline{j}) + \frac{1}{2}(\underline{i} - 6\underline{j}) + \frac{1}{2}(-\underline{i} + \underline{j})$

$= \underline{i} - \frac{7}{2}\underline{j} + \frac{1}{2}\underline{i} - 3\underline{j} - \frac{1}{2}\underline{i} + \frac{1}{2}\underline{j}$

$= \left(1 + \frac{1}{2} - \frac{1}{2}\right)\underline{i} + \left(-\frac{7}{2} - 3 + \frac{1}{2}\right)\underline{j}$

$= \underline{i} - 6\underline{j}$ Ans.

④ Given that $A(1,-1)$, $B(2,0)$, $C(-1,3)$ and $D(-2,2)$

$\overrightarrow{AB} + \overrightarrow{CD} = (2-1)\underline{i} + (0+1)\underline{j} + (-2+1)\underline{i} + (2-3)\underline{j}$

$= \underline{i} + \underline{j} - \underline{i} - \underline{j} = 0\underline{i} + 0\underline{j} = \underline{0}$

⑤ Given that $\overrightarrow{AB} = 4\underline{i} - 2\underline{j}$

$B(-2,5)$, $O(0,0)$

$\because \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\Rightarrow \overrightarrow{AB} = \overrightarrow{OB} + \overrightarrow{AO} \Rightarrow \overrightarrow{AB} - \overrightarrow{OB} = \overrightarrow{AO}$

$\Rightarrow \overrightarrow{AO} = \overrightarrow{AB} - \overrightarrow{OB}$

$= (4\underline{i} - 2\underline{j}) - (-2\underline{i} + 5\underline{j})$

$= 4\underline{i} - 2\underline{j} + 2\underline{i} - 5\underline{j}$

$\overrightarrow{AO} = 6\underline{i} - 7\underline{j}$ Ans.

⑥ (i) Given that $\underline{v} = 2\underline{i} - \underline{j}$

$|\underline{v}| = \sqrt{(2)^2+(-1)^2} = \sqrt{4+1} = \sqrt{5}$

Let $\hat{\underline{v}}$ be a unit vector along $\underline{v}$

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{2\underline{i} - \underline{j}}{\sqrt{5}} = \frac{2}{\sqrt{5}}\underline{i} - \frac{1}{\sqrt{5}}\underline{j}$ Ans.

(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

$|\underline{v}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{\frac{1+3}{4}}$

$|\underline{v}| = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$

Let $\hat{\underline{v}}$ be a unit vector along $\underline{v}$, then

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{\frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}}{1} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$ Ans.

(iii) $\underline{v} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$

$|\underline{v}| = \sqrt{\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4} + \frac{1}{4}} = \sqrt{\frac{4}{4}} = 1$

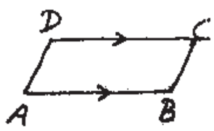
Let $\hat{\underline{v}}$ be a unit vector along $\underline{v}$, then

$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|} = \frac{-\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}}{1} = -\frac{\sqrt{3}}{2}\underline{i} - \frac{1}{2}\underline{j}$ Ans.

⑦ Given that $A(2,-4)$, $B(4,0)$, $C(1,6)$

Let $D(x,y)$ be the required point.

(i) Given that $ABCD$ is a ||gm.



$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow (4-2)\underline{i} + (0+4)\underline{j} = (1-x)\underline{i} + (6-y)\underline{j}$

$\Rightarrow 2\underline{i} + 4\underline{j} = (1-x)\underline{i} + (6-y)\underline{j}$

$\Rightarrow 2 = 1-x \quad \& \quad 4 = 6-y$

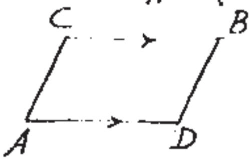
$\Rightarrow x = 1-2 \qquad y = 6-4$

$\Rightarrow \boxed{x=-1} \qquad \boxed{y=2}$

$\therefore D(-1,2)$ Ans.

(ii) Given that $ADBC$ is a $\|gm$

$\therefore \overrightarrow{AD} = \overrightarrow{CB}$



$\Rightarrow (x-2)\underline{i} + (y+4)\underline{j} = (4-1)\underline{i} + (0-6)\underline{j}$

$\Rightarrow (x-2)\underline{i} + (y+4)\underline{j} = 3\underline{i} + (-6)\underline{j}$

$\Rightarrow x-2 = 3 \quad \& \quad y+4 = -6$

$\Rightarrow x = 3+2 \quad \& \quad y = -6-4$

$\Rightarrow x = 5 \quad \& \quad y = -10$

$\therefore D(5,-10)$ Ans.

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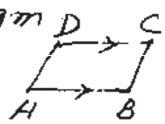
⑧ (i) Given that $B(4,1)$, $C(-2,3)$ & $D(-8,0)$

Let $A(x,y)$ be the required point.

Given that $ABCD$ is a $\|gm$



$\therefore \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow (4-x)\underline{i} + (1-y)\underline{j} = (-2+8)\underline{i} + (3-0)\underline{j}$

$\Rightarrow (4-x)\underline{i} + (1-y)\underline{j} = 6\underline{i} + 3\underline{j}$

$\Rightarrow 4-x = 6 \quad \& \quad 1-y = 3$

$\Rightarrow -x = 6-4, \quad -y = 3-1$

$\Rightarrow -x = 2, \quad -y = 2$

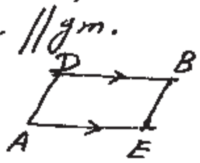
$\Rightarrow x = -2, \quad y = -2$

$\therefore A(-2,-2)$

(ii) Given that $B(4,1)$, $C(-2,3)$ and $D(-8,0)$. $A(-2,-2)$

Let $E(x,y)$ be the required point.

Given that $AEBD$ is a $\|gm$.



$\therefore \overrightarrow{AE} = \overrightarrow{DB}$

$\Rightarrow (x+2)\underline{i} + (y+2)\underline{j} = (4+8)\underline{i} + (1-0)\underline{j}$

$\Rightarrow (x+2)\underline{i} + (y+2)\underline{j} = 12\underline{i} + \underline{j}$

$\Rightarrow x+2 = 12 \quad \& \quad y+2 = 1$

$\Rightarrow x = 10, \quad y = -1$

$\therefore E(10,-1)$ Ans.

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⑨ Given that

$O(0,0)$, $A(-3,7)$, $B(1,0)$

Also $\overrightarrow{OP} = \overrightarrow{AB}$

$\Rightarrow$ Let $P(x,y)$ be the required point.

$\because \overrightarrow{OP} = \overrightarrow{AB}$

$\Rightarrow (x-0)\underline{i} + (y-0)\underline{j} = (1+3)\underline{i} + (0-7)\underline{j}$

$\Rightarrow x\underline{i} + y\underline{j} = 4\underline{i} + (-7)\underline{j}$

$\Rightarrow x = 4$ and $y = -7$

$\therefore P(4,-7)$ is the required point.

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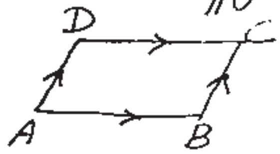
⑩ Given that $A(0,0)$, $B(a,0)$, $C(b,c)$ and $D(b-a,c)$

To prove that $ABCD$ is a $\|gm$.



$\overrightarrow{AB} = (a-0)\underline{i} + (0-0)\underline{j}$

$\boxed{\overrightarrow{AB} = a\underline{i}}$

$\overrightarrow{DC} = (b-b+a)\underline{i} + (c-c)\underline{j}$

$\Rightarrow \boxed{\overrightarrow{DC} = a\underline{i}}$

$\overrightarrow{AD} = (b-a-0)\underline{i} + (c-0)\underline{j}$

$\Rightarrow \boxed{\overrightarrow{AD} = (b-a)\underline{i} + c\underline{j}}$

$\overrightarrow{BC} = (b-a)\underline{i} + (c-0)\underline{j}$

$\Rightarrow \boxed{\overrightarrow{BC} = (b-a)\underline{i} + c\underline{j}}$

we see that

$\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$

$\therefore ABCD$ is a $\|gm$.

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⑪ Given that $B(1,2)$, $C(-2,5)$ and $D(4,11)$. and $\overrightarrow{AB} = \overrightarrow{CD}$

Let $A(x,y)$.

$\because \overrightarrow{AB} = \overrightarrow{CD}$

$\therefore (1-x)\underline{i} + (2-y)\underline{j} = (4+2)\underline{i} + (11-5)\underline{j}$

$\Rightarrow (1-x)\underline{i} + (2-y)\underline{j} = 6\underline{i} + 6\underline{j}$

$\Rightarrow 1-x = 6 \quad \& \quad 2-y = 6$

$\Rightarrow -x = 6-1 \qquad -y = 6-2$

$\Rightarrow -x = 5 \qquad -y = 4$

$\Rightarrow x = -5 \qquad y = -4$

$\therefore A(-5,-4)$ Ans.

(12) (i) Given that

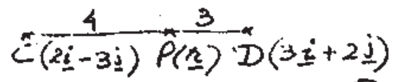
p.v. of $C = 2\underline{i} - 3\underline{j}$

p.v. of $D = 3\underline{i} + 2\underline{j}$

Let p.v. of $P = \underline{r}$

Let P divides CD in the ratio $4:3$

Then $\underline{r} = \dfrac{4(3\underline{i}+2\underline{j})+3(2\underline{i}-3\underline{j})}{4+3}$



$\underline{r} = \dfrac{12\underline{i}+8\underline{j}+6\underline{i}-9\underline{j}}{7}$

$\underline{r} = \dfrac{18\underline{i} - \underline{j}}{7}$

$\underline{r} = \dfrac{18}{7}\underline{i} - \dfrac{1}{7}\underline{j}$ Ans.

(ii) Given that

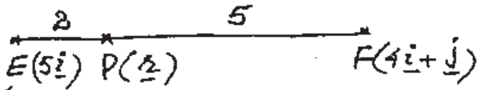
p.v. of $E = 5\underline{i}$

p.v. of $F = 4\underline{i} + \underline{j}$

Let p.v. of $P = \underline{r}$

Let P divides EF in the ratio $2:5$

Then

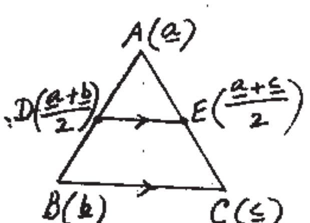


$\underline{r} = \dfrac{2(4\underline{i}+\underline{j})+5(5\underline{i})}{2+5}$

$\underline{r} = \dfrac{8\underline{i}+2\underline{j}+25\underline{i}}{7} = \dfrac{33\underline{i}+2\underline{j}}{7}$

$\underline{r} = \dfrac{33}{7}\underline{i} + \dfrac{2}{7}\underline{j}$ Ans.

(14) Let ABC be the triangle in which



$\vec{OA} = \underline{a}$

$\vec{OB} = \underline{b}$

$\vec{OC} = \underline{c}$, where O is the origin.

Let D & E be the mid points of AB and AC respectively. Then

p.v. of $D = \vec{OD} = \dfrac{\underline{a}+\underline{b}}{2}$ and

p.v. of $E = \vec{OE} = \dfrac{\underline{a}+\underline{c}}{2}$

To prove that

$\vec{DE} \parallel \vec{BC}$ and $|\vec{DE}| = \frac{1}{2}|\vec{BC}|$

Now $\vec{DE} = \vec{OE} - \vec{OD} = \dfrac{\underline{a}+\underline{c}}{2} - \dfrac{\underline{a}+\underline{b}}{2}$

$\vec{DE} = \dfrac{\underline{a}+\underline{c}-\underline{a}-\underline{b}}{2} = \dfrac{\underline{c}-\underline{b}}{2}$

$\vec{DE} = \dfrac{\underline{c}-\underline{b}}{2}$ —— ①

$\vec{BC} = \vec{OC} - \vec{OB}$

$\Rightarrow \vec{BC} = \underline{c} - \underline{b}$ —— ②

Using ② in ①, we get

$\vec{DE} = \dfrac{\vec{BC}}{2} \Rightarrow \vec{DE} = \frac{1}{2}\vec{BC}$

This shows that

$\vec{DE} \parallel \vec{BC}$ and $|\vec{DE}| = \frac{1}{2}|\vec{BC}|$

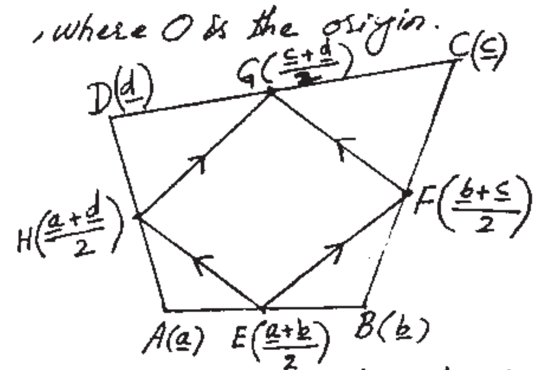
(Proved)

(15) Let $ABCD$ be the quadrilateral in which $\vec{OA} = \underline{a}$

$\vec{OB} = \underline{b}$

$\vec{OC} = \underline{c}$

$\vec{OD} = \underline{d}$, where O is the origin.



Let E, F, G and H be the mid points of the sides AB, BC, CD and AD respectively. Then $\vec{OE} = \dfrac{\underline{a}+\underline{b}}{2}$

$\vec{OF} = \dfrac{\underline{b}+\underline{c}}{2}$, $\vec{OG} = \dfrac{\underline{c}+\underline{d}}{2}$, $\vec{OH} = \dfrac{\underline{a}+\underline{d}}{2}$

To prove that $EFGH$ is a ||gm.

$\vec{EF} = \vec{OF} - \vec{OE} = \dfrac{\underline{b}+\underline{c}}{2} - \dfrac{\underline{a}+\underline{b}}{2} = \dfrac{\underline{b}+\underline{c}-\underline{a}-\underline{b}}{2}$

$\boxed{\vec{EF} = \dfrac{\underline{c}-\underline{a}}{2}}$ —— ①

$\vec{HG} = \vec{OG} - \vec{OH} = \dfrac{\underline{c}+\underline{d}}{2} - \dfrac{\underline{a}+\underline{d}}{2} = \dfrac{\underline{c}+\underline{d}-\underline{a}-\underline{d}}{2}$

$\boxed{\vec{HG} = \dfrac{\underline{c}-\underline{a}}{2}}$ —— ②

$\vec{EH} = \vec{OH} - \vec{OE} = \dfrac{\underline{a}+\underline{d}}{2} - \dfrac{\underline{a}+\underline{b}}{2} = \dfrac{\underline{a}+\underline{d}-\underline{a}-\underline{b}}{2}$

$\boxed{\vec{EH} = \dfrac{\underline{d}-\underline{b}}{2}}$ —— ③

$\vec{FG} = \vec{OG} - \vec{OF} = \dfrac{\underline{c}+\underline{d}}{2} - \dfrac{\underline{b}+\underline{c}}{2} = \dfrac{\underline{c}+\underline{d}-\underline{b}-\underline{c}}{2}$

$\boxed{\vec{FG} = \dfrac{\underline{d}-\underline{b}}{2}}$ —— ④

$\therefore$ From ①, ②, ③ & ④, we get

$\vec{EF} = \vec{HG}$ and $\vec{EH} = \vec{FG}$

$\therefore EFGH$ is a ||gm. (Proved)

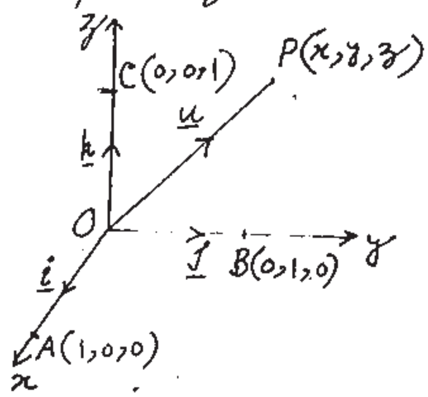
Vectors in Space:-

Given a point $P(x,y,z)$ in space there is a unique vector $\underline{u}$ in the space such that

$$\overrightarrow{OP} = \underline{u} = [x, y, z] = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$$

$$|\underline{u}| = \sqrt{x^2 + y^2 + z^2}$$

$\underline{i} = [1,0,0]$, $\underline{j} = [0,1,0]$, $\underline{k} = [0,0,1]$ are unit vectors along x-axis, y-axis and z-axis respectively.



Properties of vectors:-

(i) Commutative Property:-

$$\underline{u} + \underline{v} = \underline{v} + \underline{u}$$

(ii) Associative Property:-

$$(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$$

(iii) Inverse for vector addition:-

$$\underline{u} + (-\underline{u}) = \underline{u} - \underline{u} = \underline{0}$$

(iv) Distributive Property:-

$$a(\underline{v} + \underline{w}) = a\underline{v} + a\underline{w} \text{ for } a \in \mathbb{R}$$

(v) Scalar Multiplication.

$$a(b\underline{u}) = (ab)\underline{u} \quad \forall\, a, b \in \mathbb{R}$$

Distance Between Two Points in Space:-

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be the two points in space such that $\overrightarrow{OP_1} = [x_1, y_1, z_1]$ and $\overrightarrow{OP_2} = [x_2, y_2, z_2]$

Then $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = [x_2, y_2, z_2] - [x_1, y_1, z_1]$

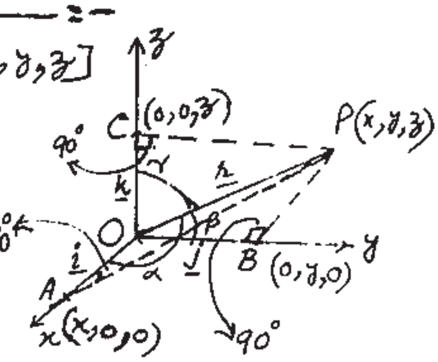
$$\overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

$\therefore$ Distance between P_1 and $P_2 = |\overrightarrow{P_1P_2}| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$

(Distance Formula)

Direction Angles and direction Cosines of a vector:-

Let $\overrightarrow{OP} = \underline{r} = [x, y, z]$ be a non-zero vector.



Let $\underline{r}$ makes angles α, β and γ with x-axis, y-axis and z-axis respectively. Such that $0 \le \alpha \le \pi$, $0 \le \beta \le \pi$ and $0 \le \gamma \le \pi$. Then

(i) the angles α, β and γ are called direction angles and

(ii) the numbers $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called direction cosines of $\underline{r}$.

Q. Prove that

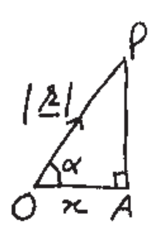
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Proof:- Let $\overrightarrow{OP} = \underline{r} = [x, y, z]$ be a non-zero vector. Let $\underline{r}$ makes angles α, β and γ with x-axis, y-axis and z-axis respectively.

To prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

From the right $\triangle OAP$



$$\cos\alpha = \frac{OA}{OP} = \frac{x}{|\underline{r}|}$$

Similarly

$$\cos\beta = \frac{y}{|\underline{r}|} \text{ and } \cos\gamma = \frac{z}{|\underline{r}|}$$

where $|\underline{r}| = \sqrt{x^2 + y^2 + z^2}$

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \dfrac{x^2}{|\underline{r}|^2} + \dfrac{y^2}{|\underline{r}|^2} + \dfrac{z^2}{|\underline{r}|^2}$

$$= \frac{x^2 + y^2 + z^2}{|\underline{r}|^2} = \frac{|\underline{r}|^2}{|\underline{r}|^2} = 1$$

(proved)