Conjugate of a Complex Number.

For a complex number $z = x+iy$, the complex number $\overline{z} = \overline{x+iy} = x-iy$ is called conjugate of z.

Here z and $\overline{z}$ have same real part but their imaginary parts differ in sign, so z and $\overline{z}$ are called conjugate of each other.

Thus $5+4i$ and $5-4i$ are conjugate complex numbers.

Note: If $z = \overline{z}$, then z is called self conjugate.

Since for $a \in \mathbb{R}$

$\overline{a} = a$

$\Rightarrow$ Every real number is self conjugate.

Powers of i

$i^2 = -1 \qquad \because i = \sqrt{-1}$

$i^3 = (i^2) \cdot i = (-1) \cdot i = -i$

$i^4 = (i^2)^2 = (-1)^2 = 1$

$i^5 = (i^2)^2 \cdot i = (-1)^2 \cdot i = 1 \cdot i = i$

$i^6 = (i^2)^3 = (-1)^3 = -1$

$i^7 = (i^2)^3 \cdot i = (-1)^3 \cdot i = (-1)i = -i$

$i^8 = (i^2)^4 = (-1)^4 = 1$

and so on.

$\Rightarrow$ Powers of i should be expressed in terms of powers of i^2.

EXERCISE 1.2

(1) Verify the addition properties of complex numbers.

Solution:

i) Closure Property

For $(a,b), (c,d) \in \mathbb{C}$

$(a,b) + (c,d) = (a+c, b+d) \in \mathbb{C}$

$\because$ + is closed in $\mathbb{R}$

ii) Associative Property

For $(a,b), (c,d), (e,f) \in \mathbb{C}$

$[(a,b) + (c,d)] + (e,f) = (a+c, b+d) + (e,f)$

$= ((a+c)+e, (b+d)+f)$

$= (a+(c+e), b+(d+f))$

$\because$ '+' is associative in $\mathbb{R}$

$= (a,b) + (c+e, d+f)$

$= (a,b) + [(c,d) + (e,f)]$

iii) Additive Identity

$\forall (a,b) \in \mathbb{C}$ we have $(0,0) \in \mathbb{C}$ such that

$(a,b) + (0,0) = (a+0, b+0) = (a,b)$

$\Rightarrow (0,0)$ is additive identity in $\mathbb{C}$.

iv) Additive Inverse

$\forall (a,b) \in \mathbb{C}$ there is $(-a,-b) \in \mathbb{C}$ such that

$(a,b) + (-a,-b) = (a-a, b-b)$

$= (0,0)$

$\Rightarrow (a,b)$ and $(-a,-b)$ are additive inverse of each other.

v) Commutative Property

$\forall (a,b), (c,d) \in \mathbb{C}$

The set $C=\{x+iy/x,y\in R,\ i=\sqrt{-1}\}$ is called set of complex numbers.

Operations on Complex Numbers:

Let $z_1=(x_1,y_1)=x_1+iy_1$

$z_2=(x_2,y_2)=x_2+iy_2$

I) Addition:

$$z_1+z_2=(x_1,y_1)+(x_2,y_2)$$
$$=x_1+iy_1+x_2+iy_2$$
$$=x_1+x_2+iy_1+iy_2$$
$$=(x_1+x_2)+i(y_1+y_2)$$
$$=(x_1+x_2,\ y_1+y_2)$$

ii) Subtraction:

$$z_1-z_2=(x_1,y_1)-(x_2,y_2)$$
$$=(x_1+iy_1)-(x_2+iy_2)$$
$$=x_1+iy_1-x_2-iy_2$$
$$=x_1-x_2+iy_1-iy_2$$
$$=(x_1-x_2)+i(y_1-y_2)$$
$$=(x_1-x_2,\ y_1-y_2)$$

iii) Multiplication

$$z_1z_2=(x_1,y_1)\cdot(x_2,y_2)$$
$$=(x_1+iy_1)\cdot(x_2+iy_2)$$
$$=x_1x_2+ix_1y_2+ix_2y_1+i^2y_1y_2$$
$$=x_1x_2+ix_1y_2+ix_2y_1+(-1)y_1y_2$$
$$=x_1x_2+ix_1y_2+ix_2y_1-y_1y_2$$
$$=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)$$
$$=(x_1x_2-y_1y_2,\ x_1y_2+x_2y_1)$$

iv) Division:

$$\frac{z_1}{z_2}=\frac{(x_1,y_1)}{(x_2,y_2)}=\frac{x_1+iy_1}{x_2+iy_2}$$
$$=\frac{x_1+iy_1}{x_2+iy_2}\cdot\frac{x_2-iy_2}{x_2-iy_2}$$
$$=\frac{x_1x_2-ix_1y_2+ix_2y_1-i^2y_1y_2}{x_2^2-i^2y_2^2}$$
$$=\frac{x_1x_2-ix_1y_2+ix_2y_1-(-1)y_1y_2}{x_2^2-(-1)y_2^2}$$
$$=\frac{x_1x_2-ix_1y_2+ix_2y_1+y_1y_2}{x_2^2+y_2^2}$$
$$=\frac{x_1x_2+y_1y_2+ix_2y_1-ix_1y_2}{x_2^2+y_2^2}$$
$$=\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$$
$$=\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}\right)+i\left(\frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$$
$$=\left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2},\ \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\right)$$

v) Equality:

$$(x_1,y_1)=(x_2,y_2)\Leftrightarrow x_1=x_2,\ y_1=y_2$$

vi) Scalar Multiplication

If $z=(x,y)=x+iy$ be a complex number and k is any real constant then

$$kz=k(x,y)=k(x+iy)$$
$$=kx+iky$$
$$=(kx,ky)$$

Note that Every real number a can be written as $a=a+i.0$

$\therefore$ Set of real numbers is subset of set of complex numbers.

$(a,b)+(c,d)=(a+c, b+d)$

$=(c+a, d+b)$ $\because$ '+' is commutative in $\mathbb{R}$

$=(c,d)+(a,b)$

② Verify the multiplication properties of the complex numbers.

Solution:

i) Closure Property

$\forall (a,b),(c,d)\in C$

$(a,b)\cdot(c,d)=(ac-bd, ad+bc)\in C$

$\because$ 'x' is closed in $\mathbb{R}$

ii) Associative Property

For $(a,b),(c,d),(e,f)\in C$

Consider

$[(a,b)\cdot(c,d)]\cdot(e,f)=(ac-bd, ad+bc)\cdot(e,f)$

$=((ac-bd)e-(ad+bc)f, (ac-bd)f+(ad+bc)e)$

$=(ace-bde-adf-bcf, acf-bdf+ade+bce)$ ①

Now consider

$(a,b)\cdot[(c,d)\cdot(e,f)]=(a,b)\cdot[(ce-df, cf+de)]$

$=(a(ce-df)-b(cf+de), a(cf+de)+b(ce-df))$

$=(ace-adf-bcf-bde, acf+ade+bce-bdf)$ ②

$\therefore$ From ① & ②, we get

$[(a,b)\cdot(c,d)]\cdot(e,f)=(a,b)\cdot[(c,d)\cdot(e,f)]$

iii) Multiplicative Identity:

$\forall (a,b)\in C$ we have $(1,0)\in C$ such that

$(a,b)\cdot(1,0)=(a-0, 0+b)$

$=(a,b)$

$\Rightarrow (1,0)$ is multiplicative identity in C.

iv) Multiplicative Inverse.

$\forall (a,b)\in C$ there is

$$\frac{1}{(a,b)}=\frac{1}{a+ib}=\frac{1}{a+ib}\times\frac{a-ib}{a-ib}$$

$$=\frac{a-ib}{a^2-i^2b^2}=\frac{a-ib}{a^2-(-1)b^2}=\frac{a-ib}{a^2+b^2}$$

$$=\frac{a}{a^2+b^2}-i\frac{b}{a^2+b^2}\in C \text{ such that}$$

$$(a,b)\cdot\frac{1}{(a,b)}=1=1+i.0=(1,0)$$

$\Rightarrow (a,b)$ and $\frac{1}{(a,b)}$ are multiplicative inverse of each other.

v) Commutative Property

$\forall (a,b),(c,d)\in C$

$(a,b)\cdot(c,d)=(ac-bd, ad+bc)$

$=(ca-db, da+cb)$

$\because$ 'x' is commutative in $\mathbb{R}$

$=(c,d)\cdot(a,b)$

③ Verify the distributive law of complex numbers.

$(a,b)[(c,d)+(e,f)]=(a,b)(c,d)+(a,b)(e,f)$

L.H.S. $=(a,b)[(c,d)+(e,f)]$

$=(a,b)(c+e, d+f)$

$=(a(c+e)-b(d+f), a(d+f)+b(c+e))$

$=(ac+ae-bd-bf, ad+af+bc+be)$ ①

R.H.S. $=(a,b)\cdot(c,d)+(a,b)\cdot(e,f)$

$=(ac-bd, ad+bc)+(ae-bf, af+be)$

$=(ac-bd+ae-bf, ad+bc+af+be)$ ②

$\therefore$ From ① & ②

L.H.S. = R.H.S.

④ Simplify the following

i) $i^9 = (i^2)^4 \cdot i = (-1)^4 \cdot i = 1 \cdot i = i$ Ans

ii) $i^{14} = (i^2)^7 = (-1)^7 = -1$ Ans.

iii) $(-i)^{19} = [(-1)\cdot(i)]^{19} = (-1)^{19} \cdot i^{19} = -i^{19}$

$= -(i^2)^9 \cdot i = -(-1)^9 \cdot i$

$= -(-1)\ i = i$ Ans.

iv) $(-1)^{-\frac{21}{2}} = \dfrac{1}{(-1)^{+\frac{21}{2}}} = \dfrac{1}{[(-1)^{1/2}]^{21}} = \dfrac{1}{i^{21}}$

$= \dfrac{1}{(i^2)^{10} \cdot i} = \dfrac{1}{(-1)^{10} \cdot i} = \dfrac{1}{1 \cdot i}$

$= \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2} = \dfrac{i}{-1}$

$= -i$ Ans.

⑤ Write in terms of i

i) $\sqrt{-1}\, b = i b$ Ans.

ii) $\sqrt{-5} = \sqrt{(-1)(5)} = \sqrt{-1}\sqrt{5} = i\sqrt{5}$ Ans.

iii) $\sqrt{-\frac{16}{25}} = \sqrt{(-1)\cdot\frac{16}{25}} = \sqrt{-1}\sqrt{\frac{16}{25}} = i\frac{4}{5}$

$= \frac{4}{5} i$ Ans.

iv) $\sqrt{\dfrac{1}{-4}}$ Method I

$= \dfrac{\sqrt{1}}{\sqrt{-4}} = \dfrac{1}{\sqrt{(-1)(4)}} = \dfrac{1}{\sqrt{-1}\sqrt{4}} = \dfrac{1}{i \cdot 2}$

$= \dfrac{1}{2i} = \dfrac{1}{2i} \times \dfrac{i}{i} = \dfrac{i}{2i^2} = \dfrac{i}{2(-1)} = \dfrac{i}{-2}$

$= -\dfrac{i}{2}$ Ans

⑥ $(7, 9) + (3, -5) = (7+3, 9-5) = (10, 4)$ Ans.

⑦ $(8, -5) - (-7, 4) = (8+7, -5-4)$

$= (15, -9)$ Ans.

⑧ $(2, 6) \cdot (3, 7)$

$= (2\cdot3 - 6\cdot7,\ 2\cdot7 + 6\cdot3)$

$= (6-42,\ 14+18)$

$= (-36,\ 32)$ Ans.

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⑨ $(5, -4) \cdot (-3, -2)$

$= ((5)(-3) - (-4)(-2),\ (5)(-2) + (-4)(-3))$

$= (-15-8,\ -10+12) = (-23, 2)$ Ans.

⑩ $(0, 3) \cdot (0, 5)$

$= ((0)(0) - (3)(5),\ (0)(5) + (3)(0))$

$= (0-15,\ 0+0) = (-15, 0)$ Ans.

⑪ $(2, 6) \div (3, 7) = \dfrac{(2, 6)}{(3, 7)}$

$= \dfrac{2+6i}{3+7i} = \dfrac{2+6i}{3+7i} \times \dfrac{3-7i}{3-7i}$

$= \dfrac{6 - 14i + 18i - 42i^2}{(3)^2 - (7i)^2} = \dfrac{6 + 4i - 42(-1)}{9 - 49i^2}$

$= \dfrac{6+4i+42}{9-49(-1)} = \dfrac{48+4i}{9+49} = \dfrac{48+4i}{58}$

$= \dfrac{48}{58} + \dfrac{4}{58} i = \dfrac{24}{29} + \dfrac{2}{29} i$

$= \left(\dfrac{24}{29}, \dfrac{2}{29}\right)$ Ans.

⑫ $(5, -4) \div (-3, -8)$

$= \dfrac{(5, -4)}{(-3, -8)} = \dfrac{5-4i}{-3-8i} = \dfrac{5-4i}{-3-8i} \times \dfrac{-3+8i}{-3+8i}$

$= \dfrac{-15 + 40i + 12i - 32i^2}{(-3)^2 - (8i)^2}$

$= \dfrac{-15 + 52i - 32(-1)}{9 - 64i^2} = \dfrac{-15 + 52i + 32}{9 + 64}$

$= \dfrac{17+52i}{73} = \dfrac{17}{73} + \dfrac{52}{73} i$

$= \left(\dfrac{17}{73}, \dfrac{52}{73}\right)$ Ans.

⑬ Let the two conjugate complex numbers be

$z = x + iy$ and $\bar{z} = x - iy$ where $x, y \in \mathbb{R}$

Sum $= z + \bar{z}$

$= x + iy + x - iy$

$= 2x \in \mathbb{R} \quad \because x \in \mathbb{R}$

Product $= z\cdot\bar{z}$

$= (x+iy)(x-iy)$

$= x^2 - i^2y^2$

$= x^2-(-1)y^2 = x^2+y^2 \in \mathbb{R}$

$\because x, y \in \mathbb{R}$

(14) i) $(-4,7)$

Let $z=(-4,7)$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(-4,7)} = \frac{1}{-4+7i} = \frac{1}{-4+7i}\times\frac{-4-7i}{-4-7i}$

$= \frac{-4-7i}{(-4)^2-(7i)^2} = \frac{-4-7i}{16-49i^2}$

$= \frac{-4-7i}{16-49(-1)} = \frac{-4-7i}{16+49} = \frac{-4-7i}{65}$

$= -\frac{4}{65} - \frac{7}{65}i = \left(-\frac{4}{65}, -\frac{7}{65}\right)$ Ans.

ii) Let $z = (\sqrt{2}, -\sqrt{5})$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(\sqrt{2},-\sqrt{5})} = \frac{1}{\sqrt{2}-\sqrt{5}i}$

$= \frac{1}{\sqrt{2}-\sqrt{5}i}\times\frac{\sqrt{2}+\sqrt{5}i}{\sqrt{2}+\sqrt{5}i} = \frac{\sqrt{2}+\sqrt{5}i}{(\sqrt{2})^2-(\sqrt{5}i)^2}$

$= \frac{\sqrt{2}+\sqrt{5}i}{2-5i^2} = \frac{\sqrt{2}+\sqrt{5}i}{2-5(-1)} = \frac{\sqrt{2}+\sqrt{5}i}{2+5}$

$= \frac{\sqrt{2}+\sqrt{5}i}{7} = \frac{\sqrt{2}}{7}+\frac{\sqrt{5}}{7}i$

$= \left(\frac{\sqrt{2}}{7}, \frac{\sqrt{5}}{7}\right)$ Ans.

iii) Let $z = (1,0)$

Multiplicative inverse of $z = \frac{1}{z}$

$= \frac{1}{(1,0)} = \frac{1}{1+i.0} = \frac{1}{1+0} = \frac{1}{1} = 1$

$= 1+i.0 = (1,0)$ Ans.

(15) i) a^2+4b^2

$= a^2-(-1)4b^2 = a^2 - i^2 2^2 b^2$

$= (a)^2 - (i2b)^2$

$= (a+i2b)(a-i2b)$

$= (a+2bi)(a-2bi)$ Ans.

ii) $9a^2+16b^2$

$= 9a^2-(-1)16b^2$

$= (3a)^2 - i^2 4^2 b^2 = (3a)^2-(i4b)^2$

$= (3a+i4b)(3a-i4b)$

$= (3a+4bi)(3a-4bi)$ Ans.

iii) $3x^2+3y^2$

$= 3(x^2+y^2)$

$= 3[x^2-(-1)y^2]$

$= 3[(x)^2 - i^2y^2] = 3[(x)^2-(iy)^2]$

$= 3(x+iy)(x-iy)$ Ans.

(16) i) $\frac{2-7i}{4+5i} = \frac{2-7i}{4+5i}\times\frac{4-5i}{4-5i}$

$= \frac{8-10i-28i+35i^2}{(4)^2-(5i)^2}$

$= \frac{8-38i+35(-1)}{16-25i^2} = \frac{8-38i-35}{16+25}$

$= \frac{-27-38i}{41} = -\frac{27}{41} - \frac{38}{41}i$ Ans.

ii) $\frac{(-2+3i)^2}{1+i} = \frac{(-2)^2+(3i)^2+2(-2)(3i)}{1+i}$

$= \frac{4+9i^2-12i}{1+i} = \frac{4-9-12i}{1+i} = \frac{-5-12i}{1+i}$

$= \frac{-5-12i}{1+i}\times\frac{1-i}{1-i}$

$= \frac{-5+5i-12i+12i^2}{1^2-i^2} = \frac{-5-7i-12}{1-(-1)}$

$= \frac{-17-7i}{1+1} = \frac{-17-7i}{2} = -\frac{17}{2}-\frac{7}{2}i$ Ans

iii) $\frac{i}{1+i} = \frac{i}{1+i}\times\frac{1-i}{1-i} = \frac{i-i^2}{1^2-i^2}$

$= \frac{i-(-1)}{1-(-1)} = \frac{i+1}{2} = \frac{1+i}{2}$

$= \frac{1}{2}+\frac{i}{2}$ Ans.