§ 3.2 **FIELD:** A non empty set F is called a field, if

(i) F is an abelian group under '+'

(ii) F – {0} an abelian group under multiplication

(iii) Right distributive law holdes in F

e.g. Set of real numbers R and set of complex numbers C are fields

EXERCISE 3.2

1. If $A = [a_{ij}]_{3\times 4}$, **show that (i)** $I_3 A = A$, **(ii)** $A I_4 = A$.

Solution. $I_3 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

$$= \begin{bmatrix} a_{11}+0+0 & a_{12}+0+0 & a_{13}+0+0 & a_{14}+0+0 \\ 0+a_{21}+0 & 0+a_{22}+0 & 0+a_{23}+0 & 0+a_{24}+0 \\ 0+0+a_{31} & 0+0+a_{32} & 0+0+a_{33} & 0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

(ii) $A I_4 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} a_{11}+0+0+0 & 0+a_{12}+0+0 & 0+0+a_{13}+0 & 0+0+0+a_{14} \\ a_{21}+0+0+0 & 0+a_{22}+0+0 & 0+0+a_{23}+0 & 0+0+0+a_{24} \\ a_{31}+0+0+0 & 0+a_{32}+0+0 & 0+0+a_{33}+0 & 0+0+0+a_{34} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = A$$

2. Find the inverse of the following matrices :

(i) $\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ (iii) $\begin{bmatrix} 2i & i \\ i & -i \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix}$

Solution. Note that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \Rightarrow |A| = ad - bc$;

$$\text{Adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \text{ and } A^{-1} = \frac{1}{|A|} \text{ adj. } A$$

(i) Let $A = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow |A| = 3-(-2) = 5$; Adj $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{ adj. } A = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 1/5 & 1/5 \\ -2/5 & 3/5 \end{bmatrix}$$

(ii) Let $B = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix} \Rightarrow |B| = -10+12 = 2$; Adj $B = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$

$$\therefore \quad B^{-1} = \frac{1}{|B|} \text{ adj. } B = \frac{1}{2} \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 5/2 & -3/2 \\ 4/2 & -2/2 \end{bmatrix} = \begin{bmatrix} 5/2 & -3/2 \\ 2 & -1 \end{bmatrix}$$

(iii) Let $C = \begin{bmatrix} 2i & i \\ i & -i \end{bmatrix} \Rightarrow |C| = -2i^2 - i^2 = -3i^2 = 3$; Adj $C = \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix}$

$$\therefore \quad C^{-1} = \frac{1}{|C|} \text{ adj. } C = \frac{1}{3} \begin{bmatrix} -i & -i \\ -i & 2i \end{bmatrix} = \begin{bmatrix} -i/3 & -i/3 \\ -i/3 & 2i/3 \end{bmatrix}$$

(iv) $D = \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \Rightarrow |D| = 6-6 = 0$, as $|D| = 0$ so D^{-1} does not exist.

3. Solve the following system of linear equations :

(i) $2x_1 - 3x_2 = 5$ (ii) $4x_1 + 3x_2 = 5$ (iii) $3x - 5y = 1$

$5x_1 + x_2 = 4$ $\quad$ $3x_1 - x_2 = 7$ $\quad$ $-2x + y = -3$

Solution. (i) Writing in matrix form, we get

$$\begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$\Rightarrow \quad AX = B \Rightarrow \quad X = A^{-1}B \quad \ldots (1)$

Here $\quad A = \begin{bmatrix} 2 & -3 \\ 5 & 1 \end{bmatrix} \quad \Rightarrow \quad |A| = \begin{vmatrix} 2 & -3 \\ 5 & 1 \end{vmatrix} = 2 + 15 = 17 \neq 0$

So A^{-1} exists $\therefore A^{-1} = \frac{1}{|A|}$ adj. $A = \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}$

From (1) $\quad X = A^{-1}B = \frac{1}{17}\begin{bmatrix} 1 & 3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 5 + 12 \\ -25 + 8 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 \\ -17 \end{bmatrix} = \begin{bmatrix} 17/17 \\ -17/17 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\therefore \quad \boxed{x_1 = 1, x_2 = -1}$

(ii) $4x_1 + 3x_2 = 5 \quad , \quad 3x_1 - x_2 = 7$

Writing in matrix form, we get

$$\begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$\Rightarrow \quad AX = B \Rightarrow \quad X = A^{-1}B \quad \ldots (1)$

Here $\quad A = \begin{bmatrix} 4 & 3 \\ 3 & -1 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 4 & 3 \\ 3 & -1 \end{vmatrix} = -4 - 9 = -13$

$$A^{-1} = \frac{1}{|A|} \text{ adj. } A = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}$$

From (1) $\quad X = A^{-1}B = \frac{1}{-13}\begin{bmatrix} -1 & -3 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -5-21 \\ -15 + 28 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-13}\begin{bmatrix} -26 \\ 13 \end{bmatrix} = \begin{bmatrix} -26/-13 \\ 13/-13 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\therefore \quad \boxed{x_1 = 2, x_2 = -1}$

(iii) $3x - 5y = 1, \quad -2x + y = -3$

Writing in matrix form, we get

$$\begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$\Rightarrow \quad AX = B \Rightarrow \quad X = A^{-1}B \quad \ldots \ (1)$

where $A = \begin{bmatrix} 3 & -5 \\ -2 & 1 \end{bmatrix} \Rightarrow \quad |A| = \begin{vmatrix} 3 & -5 \\ -2 & 1 \end{vmatrix} = 3 - 10 = -7$

$\therefore \quad A^{-1} = \dfrac{1}{|A|} \text{ adj. } A = \dfrac{1}{-7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}$

From (1) $\quad X = A^{-1}B = \dfrac{1}{-7}\begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \end{bmatrix} = \dfrac{1}{-7}\begin{bmatrix} 1-15 \\ 2-9 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{-7}\begin{bmatrix} -14 \\ -7 \end{bmatrix} = \begin{bmatrix} -14/-7 \\ -7/-7 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\therefore \quad \boxed{x = 2, y = 1}$

4. **If** $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$ **and** $C = \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$

Find (i) A − B (ii) B − A (iii) (A − B) − C (iv) A − (B − C).

Solution. (i) $A - B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1-2 & -1-1 & 2+1 \\ 3-1 & 2-3 & 5-4 \\ -1+1 & 0-2 & 4-1 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

(ii) $B - A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2-1 & 1+1 & -1-2 \\ 1-3 & 3-2 & 4-5 \\ -1+1 & 2-0 & 1-4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 1 & -1 \\ 0 & 2 & -3 \end{bmatrix}$$

(iii) $(A - B) - C = \begin{bmatrix} -1 & -2 & 3 \\ 2 & -1 & 1 \\ 0 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -1-1 & -2-3 & 3+2 \\ 2+1 & -1-2 & 1-0 \\ 0-3 & -2-4 & 3+1 \end{bmatrix} = \begin{bmatrix} -2 & -5 & 5 \\ 3 & -3 & 1 \\ -3 & -6 & 4 \end{bmatrix}$$

(iv) Now $B - C = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 3 & 4 \\ -1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & 0 \\ 3 & 4 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2-1 & 1-3 & -1+2 \\ 1+1 & 3-2 & 4-0 \\ -1-3 & 2-4 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 4 \\ -4 & -2 & 2 \end{bmatrix}$$

$\therefore\ A - (B - C) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 5 \\ -1 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 4 \\ -4 & -2 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & -1+2 & 2-1 \\ 3-2 & 2-1 & 5-4 \\ -1+4 & 0+2 & 4-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}$$

5. **If** $A = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}$, $B = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix}$ **and** $C = \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$, **then show that**

(i) $(AB)C = A(BC)$ **(ii)** $(A+B)C = AC + BC$.

Solution. (i) $AB = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \cdot \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} = \begin{bmatrix} -i^2 + 4i^2 & i + 2i^2 \\ -i - 2i^2 & 1 - i^2 \end{bmatrix}$

$= \begin{bmatrix} -3 & i-2 \\ 2-i & 2 \end{bmatrix}$ { by using $i^2 = -1$ }

$\therefore\ (AB)C = \begin{bmatrix} -3 & i-2 \\ 2-i & 2 \end{bmatrix} \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$

$= \begin{bmatrix} -6i + 2i - i^2 & 3 - 2i + i^2 \\ 4i - 2i^2 - 2i & -2 + i + 2i \end{bmatrix} = \begin{bmatrix} 1-4i & 2-2i \\ 2+2i & -2+3i \end{bmatrix}$ ...(1)

Again, $BC = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \cdot \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} = \begin{bmatrix} -2i^2 - i & i + i \\ 4i^2 - i^2 & -2i + i^2 \end{bmatrix}$

$$= \begin{bmatrix} 2-i & 2i \\ -3 & -1-2i \end{bmatrix} \qquad \{\text{ by using } i^2 = -1\}$$

$$\therefore\ A(BC) = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix}\begin{bmatrix} 2-i & 2i \\ -3 & -1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 2i-i^2-6i & 2i^2-2i-4i^2 \\ 2-i+3i & 2i+i+2i^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1-4i & 2-2i \\ 2+2i & -2+3i \end{bmatrix} \qquad \{\text{ by using } i^2 = -1\} \qquad \ldots(2)$$

From (1) and (2), we have $\quad (AB)C = A(BC)$.

(ii). $(A+B)C = AC + BC$.

Now $\quad A+B = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} = \begin{bmatrix} 0 & 1+2i \\ 1+2i & 0 \end{bmatrix}$

$$(A+B)C = \begin{bmatrix} 0 & 1+2i \\ 1+2i & 0 \end{bmatrix}\begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix}$$

$$= \begin{bmatrix} 0-i-2i^2 & 0+i+2i^2 \\ 2i+4i^2+0 & -1-2i+0 \end{bmatrix} = \begin{bmatrix} -i-2i^2 & i+2i^2 \\ 2i+4i^2 & -1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 2-i & i-2 \\ 2i-4 & -1-2i \end{bmatrix} \quad \ldots\ (1) \quad \{\text{ by using } i^2 = -1\}$$

Also $\quad AC = \begin{bmatrix} i & 2i \\ 1 & -i \end{bmatrix} \cdot \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} = \begin{bmatrix} 2i^2-2i^2 & -i+2i^2 \\ 2i+i^2 & -1-i^2 \end{bmatrix}$

$$= \begin{bmatrix} 0 & -2-i \\ -1+2i & 0 \end{bmatrix} \qquad \{\text{ by using } i^2 = -1\}$$

and $\quad BC = \begin{bmatrix} -i & 1 \\ 2i & i \end{bmatrix} \cdot \begin{bmatrix} 2i & -1 \\ -i & i \end{bmatrix} = \begin{bmatrix} -2i^2-i & i+i \\ 4i^2-i^2 & -2i+i^2 \end{bmatrix}$

$$= \begin{bmatrix} 2-i & 2i \\ -3 & -1-2i \end{bmatrix} \qquad \{\text{ by using } i^2 = -1\}$$

$$\therefore\ AC + BC = \begin{bmatrix} 0 & -2-i \\ -1+2i & 0 \end{bmatrix} + \begin{bmatrix} 2-i & 2i \\ -3 & -1-2i \end{bmatrix}$$

$$= \begin{bmatrix} 2-i & -2+i \\ -4+2i & -1-2i \end{bmatrix} \qquad \ldots\ (2)$$

From (1) and (2), we have $\quad (A+B)C = AC + BC$

6. **If A and B are square matrices of the same order, then explain why, in general**

(i) $(A+B)^2 \neq A^2+2AB+B^2$ (ii) $(A-B)^2 \neq A^2-2AB+B^2$

(iii) $(A+B)(A-B) \neq A^2-B^2$.

Solution.

(i) Now $(A+B)^2 = (A+B)(A+B) = A.A + A.B + B.A + B.B$

$= A^2 + AB + BA + B^2 \neq A^2 + AB + AB + B^2$

$(A+B)^2 \neq A^2 + 2AB + B^2$ since, in general, $BA \neq AB$

(ii) Now $(A-B)^2 = (A-B)(A-B) = A.A - A.B - B.A + B.B$

$= A^2 - AB - BA + B^2 \neq A^2 - AB - AB + B^2$

$(A-B)^2 \neq A^2 - 2AB + B^2$. since, in general, $BA \neq AB$

(iii) $(A+B)(A-B) = A.A - A.B + B.A - B.B$

$\neq A^2 - AB + AB + B^2$

$(A+B)(A-B) \neq A^2 - B^2$ since, in general, $BA \neq AB$.

7. **If** $A = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$, **then find** AA^t **and** A^tA.

Solution. Now $A^t = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$

$\therefore \quad AA^t = \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix}$

$= \begin{bmatrix} 4+1+9+0 & 2+0+12-0 & -6-5+6+0 \\ 2+0+12+0 & 1+0+16+4 & -3+0+8+2 \\ -6-5+6-0 & -3+0+8+2 & 9+25+4+1 \end{bmatrix} = \begin{bmatrix} 14 & 14 & -5 \\ 14 & 21 & 7 \\ -5 & 7 & 39 \end{bmatrix}$

Also $A^tA = \begin{bmatrix} 2 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & 4 & 2 \\ 0 & -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 3 & 0 \\ 1 & 0 & 4 & -2 \\ -3 & 5 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 4+1+9 & -2+0-15 & 6+4-6 & 0-2+3 \\ -2+0-15 & 1+0+25 & -3+0+10 & 0+0-5 \\ 6+4-6 & -3+0+10 & 9+16+4 & 0-8-2 \\ 0-2+3 & 0+0-5 & 0-8-2 & 0+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -17 & 4 & 1 \\ -17 & 26 & 7 & -5 \\ 4 & 7 & 29 & -10 \\ 1 & -5 & -10 & 5 \end{bmatrix}$$

8. Solve the following matrix equations for X :

(i) $3X - 2A = B$ if $A = \begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$.

Solution. (i) $3X = 2A + B = 2\begin{bmatrix} 2 & 3 & -2 \\ -1 & 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix}$

$$3X = \begin{bmatrix} 4 & 6 & -4 \\ -2 & 2 & 10 \end{bmatrix} + \begin{bmatrix} 2 & -3 & 1 \\ 5 & 4 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{3}\begin{bmatrix} 6 & 3 & -3 \\ 3 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

(ii) (ii) $2X - 3A = B$ if $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

$$2X = 3A + B = 3\begin{bmatrix} 1 & -1 & 2 \\ -2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$$

$$2X = \begin{bmatrix} 3 & -3 & 6 \\ -6 & 12 & 15 \end{bmatrix} + \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix}$$

$$\therefore X = \frac{1}{2}\begin{bmatrix} 6 & -4 & 6 \\ -2 & 14 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 3 \\ -1 & 7 & 8 \end{bmatrix}$$

9. Solve the following matrix equations for A :

(i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A - \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix}$

Solution. (i) $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} + \begin{bmatrix} -1 & -4 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2-1 & 3-4 \\ -1+3 & -2+6 \end{bmatrix}$

i.e. $$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$

gives $$\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

or $$\begin{bmatrix} 4a+3c & 4b+3d \\ 2a+2c & 2b+2d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$$

$\Rightarrow \quad 4a+3c = 1 \quad \ldots \quad (1), \qquad 4b+3d = -1 \quad \ldots \quad (2)$

$\qquad 2a+2c = 2 \quad \ldots \quad (3), \qquad 2b+2d = 4 \quad \ldots \quad (4)$

Eqns. (1) and (3) give $\quad 4a+3c = 1 \quad \ldots \quad (1)$ and $\quad a+c = 1 \quad \ldots \quad (3)$

To find a and c, put $c = 1-a$ in (3), then $4a+3(1-a) = 1 \quad \Rightarrow \quad a = -2$

Then $\quad c = 1-a = 1-(-2) = 1+2 = 3$

Eqns. (2) and (4) give

$\qquad 4b+3d = -1 \quad \ldots \quad (2)$ and $b+d = 2 \quad \ldots \quad (4)$

To find b and d, put $d = 2-b$ from (4) in (2),then

$$4b+3(2-b) = -1 \quad \Rightarrow \quad 4b-3b = -1-6 \quad \Rightarrow \quad b = -7$$

Then $\quad d \quad 2-b \quad 2-(-7) = 2+7 = 9$

Hence $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 3 & 9 \end{bmatrix}$

(ii) $$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix}$$

$$A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\therefore A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 3a+4b & a+2b \\ 3c+4d & c+2d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$$

$\Rightarrow \quad 3a+4b = 1 \quad \ldots \quad (1), \qquad a+2b = 2 \quad \ldots \quad (2)$

$\qquad 3c+4d = 2 \quad \ldots \quad (3), \qquad c+2d = 6 \quad \ldots \quad (4)$

To find a and b, put $a = 2-2b$ from (2) in (1), then

$$3(2-2b)+4b = 1 \quad \Rightarrow \quad 6-6b+4b = 1 \quad \Rightarrow \quad -2b = -5 \quad \Rightarrow \quad b = \frac{5}{2}$$

Then $a = 2-2b = 2-2.\left(\frac{5}{2}\right) = 2-5 = -3$

To find c and d, put $c = 6-2d$ from (4) in (3), then

$3(6-2d)+4d = 2 \quad \Rightarrow \quad 18-6d+4d = 2 \quad \Rightarrow \quad -2d = -16 \quad \Rightarrow \quad d = 8$

Then $c = 6-2d = 6-2(8) = 6-16 = -10$

Hence $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -3 & 5/2 \\ -10 & 8 \end{bmatrix}$