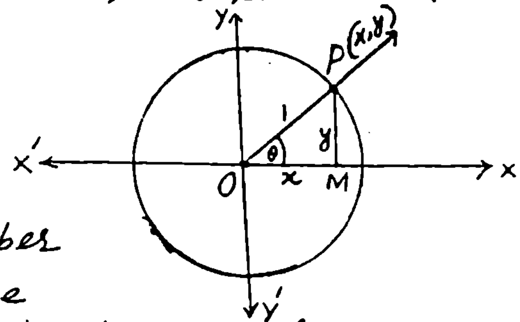
11 TRIGONOMETRIC FUNCTIONS AND THEIR GRAPHS III

Domains and Ranges of Sine and Cosine Functions

Let us consider a unit circle with centre at origin O.
Let $P(x,y)$ be any point on the circle such that $\angle XOP = \theta$ is in standard position. Then



$\sin\theta = \frac{y}{1} \Rightarrow \sin\theta = y$

$\cos\theta = \frac{x}{1} \Rightarrow \cos\theta = x$

$\Rightarrow$ corresponding to any real number θ, there is one and only one value of x and y i.e, one and only one value for each $\sin\theta$ and $\cos\theta$.

Hence $\sin\theta$ and $\cos\theta$ are functions of θ.

$\because$ $\sin\theta$ and $\cos\theta$ are defined for all $\theta \in \mathbb{R}$, the set of real numbers.

$\therefore$ Domain of $\sin\theta = \mathbb{R}$

Domain of $\cos\theta = \mathbb{R}$

To find the range, we have

Since $P(x,y)$ is a point on the unit circle with centre at O.

$\therefore -1 \le x \le 1$ and $-1 \le y \le 1$

$\Rightarrow -1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$

Domains and Ranges of Tangent and Cotangent Fuctions.

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From figure ; $\tan\theta = \frac{y}{x}$, $x \ne 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e., Y-axis)

$\Rightarrow \theta \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

$\Rightarrow \theta \ne (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

$\therefore$ Domain of $\tan\theta$ is $\theta \in \mathbb{R}$ but $\theta \ne (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

and rang of $\tan\theta = \mathbb{R}$

Now $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$, $y \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (i.e., X-axis).

$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow \theta \neq n\pi$; $n \in \mathbb{Z}$

$\therefore$ domain of $\cot\theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi$, $n \in \mathbb{Z}$.

and range of $\cot\theta$ is $\mathbb{R}$

Domains and Ranges of Secant and Cosecant Fuctions:

From fig. $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$, $x \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OY or OY' (i.e; Y-axis)

$\Rightarrow \theta \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm 5\frac{\pi}{2}, \ldots$

$\Rightarrow \theta \neq (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$

$\therefore$ Domain of $\sec\theta$ is $\theta \in \mathbb{R}$ but $\theta \neq (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$.

As $\sec\theta$ attains all real values except those between -1 and 1.

$\therefore$ Range of $\sec\theta = \mathbb{R} - \{x / -1 < x < 1\}$

Now $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{y}$, $y \neq 0$

$\Rightarrow$ terminal side $\overrightarrow{OP}$ should not coincide with OX or OX' (i.e., X-axis)

$\Rightarrow \theta \neq 0, \pm\pi, \pm 2\pi, \ldots$

$\Rightarrow \theta \neq n\pi$, $n \in \mathbb{Z}$

$\therefore$ Domain of $\operatorname{cosec}\theta$ is $\theta \in \mathbb{R}$ but $\theta \neq n\pi$, $n \in \mathbb{Z}$

As $\operatorname{cosec}\theta$ attains all real values except those between -1 and 1.

$\therefore$ Range of $\operatorname{cosec}\theta = \mathbb{R} - \{x \mid -1 < x < 1\}$

Now summarizing the above results in the form of a table as:

Function	Domain	Range
$y = \sin x$	$\mathbb{R}$	$-1 \le y \le 1$
$y = \cos x$	$\mathbb{R}$	$-1 \le y \le 1$
$y = \tan x$	$x \in \mathbb{R}$ but $x \ne (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$	$\mathbb{R}$
$y = \cot x$	$x \in \mathbb{R}$ but $x \ne n\pi$, $n \in \mathbb{Z}$	$\mathbb{R}$
$y = \sec x$	$x \in \mathbb{R}$ but $x \ne (2n+1)\frac{\pi}{2}$, $n \in \mathbb{Z}$	$\mathbb{R} - \{x \mid -1 < x < 1\}$
$y = \operatorname{cosec} x$	$x \in \mathbb{R}$ but $x \ne n\pi$, $n \in \mathbb{Z}$	$\mathbb{R} - \{x \mid -1 < x < 1\}$

Periodic Function

A function f is said to be periodic if for every x belonging to its domain D, there exists a positive number p such that $x + p \in D$ and

$$f(x+p) = f(x).$$

If p is the least positive number satisfying these conditions, then it is called the period of f.

Periodicity: All the six trigonometric functions repeat their values for each increase or decrease of 2π in θ. This behaviour of trigonometric functions is called periodicity.

Theorem

Sine is a periodic function and its period is 2π.

Proof: Let p be the period of sine. Then

$$\sin(\theta + p) = \sin\theta, \quad \forall\, \theta \in \mathbb{R} \quad (1)$$

putting $\theta = 0$ in (1), we get

$\sin(0+p) = \sin 0 \Rightarrow \sin p = 0$

$\Rightarrow p = \sin^{-1}(0)$

$\Rightarrow p = 0, \pi, 2\pi, \dots$

i) If $p = \pi$, then from (1)

$\sin(\theta + \pi) = \sin\theta$

$\Rightarrow -\sin\theta = \sin\theta$ (not true)

$\because \sin(\theta+\pi) = -\sin\theta$

$\therefore \pi$ is not the period of $\sin\theta$

ii) If $p = 2\pi$, then from (1)

$\sin(\theta + 2\pi) = \sin\theta$

$\Rightarrow \sin\theta = \sin\theta$ (true) $\quad \because \sin(\theta + 2\pi) = \sin\theta$

$\therefore 2\pi$ is the period of $\sin\theta$

Theorem

Tangent is a periodic function and its period is π.

Proof: Let p be the period of tan. Then

$$\tan(\theta + p) = \tan\theta, \quad \forall\, \theta \in \mathbb{R} \quad (1)$$

Putting $\theta = 0$ in (1), we get

$\tan(0 + p) = \tan 0$

$\Rightarrow \tan p = 0$

$\Rightarrow p = 0, \pi, 2\pi, 3\pi, \dots$

$p=0$ can't be the period of $\tan\theta$ $\because$ $p=0$ is not positive.

If $p=\pi$, then from ①

$\tan(\theta+\pi) = \tan\theta$

$\Rightarrow \tan\theta = \tan\theta$ (true)

$\therefore$ π is the period of $\tan\theta$

$\because$ it is the least +ve number for which $\tan(\theta+\pi) = \tan\theta$.

Similarly we can prove that

i) 2π is the period of $\cos\theta$

ii) 2π is the period of $\operatorname{cosec}\theta$

iii) 2π is the period of $\sec\theta$

iv) π is the period of $\cot\theta$.

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* EXERCISE 11.1 *

Find the periods of the following functions.

1) $\sin 3x = \sin(3x+2\pi)$

$= \sin 3\left(x+\frac{2\pi}{3}\right)$

$\therefore$ period of $\sin 3x = \frac{2\pi}{3}$ Ans.

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2) $\cos 2x = \cos(2x+2\pi)$

$= \cos 2(x+\pi)$

$\therefore$ period of $\cos 2x = \pi$ Ans.

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3) $\tan 4x = \tan(4x+\pi)$

$= \tan 4\left(x+\frac{\pi}{4}\right)$

$\therefore$ period of $\tan 4x = \frac{\pi}{4}$ Ans.

④ $\cot\frac{x}{2} = \cot\left(\frac{x}{2}+\pi\right)$

$= \cot\frac{1}{2}(x+2\pi)$

$\therefore$ period of $\cot\frac{x}{2} = 2\pi$ Ans.

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⑤ $\sin\frac{x}{3} = \sin\left(\frac{x}{3}+2\pi\right)$

$= \sin\frac{1}{3}(x+6\pi)$

$\therefore$ period of $\sin\frac{x}{3} = 6\pi$ Ans.

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⑥ $\operatorname{cosec}\frac{x}{4} = \operatorname{cosec}\left(\frac{x}{4}+2\pi\right)$

$= \operatorname{cosec}\frac{1}{4}(x+8\pi)$

$\therefore$ period of $\operatorname{cosec}\frac{x}{4} = 8\pi$ Ans.

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⑦ $\sin\frac{x}{5} = \sin\left(\frac{x}{5}+2\pi\right)$

$= \frac{1}{5}\sin(x+10\pi)$

$\therefore$ period of $\sin\frac{x}{5} = 10\pi$ Ans.

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⑧ $\cos\frac{x}{6} = \cos\left(\frac{x}{6}+2\pi\right)$

$= \cos\frac{1}{6}(x+12\pi)$

$\therefore$ period of $\cos\frac{x}{6} = 12\pi$

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⑨ $\tan\frac{x}{7} = \tan\left(\frac{x}{7}+\pi\right)$

$= \tan\frac{1}{7}(x+7\pi)$

$\therefore$ period of $\tan\frac{x}{7} = 7\pi$ Ans.

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⑩ $\cot 8x = \cot(8x+\pi)$

$= \cot 8\left(x+\frac{\pi}{8}\right)$

$\therefore$ period of $\cot 8x = \frac{\pi}{8}$ Ans.

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⑪ $\sec 9x = \sec(9x+2\pi)$

$= \sec 9\left(x+\frac{2\pi}{9}\right)$

$\therefore$ period of $\sec 9x = \frac{2\pi}{9}$ Ans.

(12) $\text{Cosec } 10x$

$= \text{Cosec}(10x + 2\pi)$

$= \text{Cosec } 10\left(x + \frac{2\pi}{10}\right)$

$= \text{Cosec } 10\left(x + \frac{\pi}{5}\right)$

$\therefore$ period of $\text{Cosec } 10x = \frac{\pi}{5}$ Ans.

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(13) $3\sin x = 3\sin(x + 2\pi)$

$\therefore$ period of $3\sin x = 2\pi$ Ans.

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(14) $2\cos x = 2\cos(x + 2\pi)$

$\therefore$ period of $2\cos x = 2\pi$ Ans.

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(15) $3\cos\frac{x}{5} = 3\cos\left(\frac{x}{5} + 2\pi\right)$

$= 3\cos\frac{1}{5}(x + 10\pi)$

$\therefore$ period of $3\cos\frac{x}{5} = 10\pi$ Ans.
