Exercise 3.5

Q#1:

$$\int \frac{3x+1}{x^2-x-6}\,dx$$

$$\int \frac{3x+1}{x^2-3x+2x-6}\,dx$$

$$\int \frac{3x+1}{x(x-3)+2(x-3)}\,dx$$

$$\int \frac{3x+1}{(x-3)(x+2)}$$

Changing into partial fraction.

$$\frac{3x+1}{(x-3)(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x+2)}$$

$$3x+1 = A(x+2) + B(x-3)$$

$$= Ax + 2A + Bx - 3B$$

$$3x+1 = (A+B)x + (2A-3B)$$

By comparison we obtain

$$3 = A+B \quad \text{—— i)}$$

$$1 = 2A-3B \quad \text{—— ii)}$$

$$2(i) - ii) \Rightarrow \quad 2A + 2B = 6$$

$$\underline{\cancel{2A} \underset{+}{-} 3B = \underset{-}{1}}$$

$$5B = 5$$

$$\Rightarrow B = 1$$

Put value of B in i) we obtain

$$\Rightarrow A + 1 = 3$$

$$\Rightarrow A = 3 - 1$$

$$\Rightarrow A = 2$$

So $$\frac{3x+1}{(x-3)(x+2)} = \frac{2}{(x-3)} + \frac{1}{(x+2)}$$

$$\int \frac{3x+1}{(x-3)(x+2)}\,dx = \int \left(\frac{2}{(x-3)} + \frac{1}{(x+2)} \right) dx$$

$$= \int \frac{2}{(x-3)}\,dx + \int \frac{1}{(x+2)}\,dx$$

$$= 2\int \frac{1}{(x-3)}\,dx + \int \frac{1}{(x+2)}\,dx$$

$$= 2\ln|x-3| + \ln|x+2| + C$$

(2) $$\int \frac{5x+8}{(x+3)(2x-1)}\,dx$$

Solve as previous question.

(3) $$\int \frac{x^2+3x-34}{x^2+2x-15}\,dx$$

$$\begin{array}{r|l} & 1 \\ x^2+2x-15 & x^2+3x-34 \\ & \cancel{x^2} \pm 2x \mp 15 \\ \hline & x-19 \end{array}$$

$$\int \left(1 + \frac{(x-19)}{(x^2+2x-15)}\right)dx$$

$$\int 1\,dx + \int \frac{(x-19)}{(x^2+2x-15)}\,dx$$

$$x + \int \frac{(x-19)}{x^2+5x-3x-15}\,dx$$

$$x + \int \frac{(x-19)}{x(x+5)-3(x+5)}\,dx$$

$$x + \int \frac{(x-19)}{(x+5)(x-3)}\,dx \quad \text{——— i)}$$

Now solve $\int \frac{(x-19)}{(x+5)(x-3)}\,dx$

as previous question and then obtained answer write in i)

④ $\int \frac{(a-b)x}{(x-a)(x-b)}\,dx \quad (a>b)$

Solve this question as question # 1 is solved.

⑤ $\int \frac{3-x}{1-x-6x^2}\,dx$

$$\int \frac{3-x}{-6x^2-x+1}\,dx$$

$$\int \frac{3-x}{-6x^2-3x+2x+1}$$

$$\int \frac{3-x}{-3x(2x+1)+1(2x+1)}\,dx$$

$$\int \frac{3-x}{(2x+1)(1-3x)}\,dx$$

Now solve it according to previous question

⑥ $\int \frac{2x}{x^2-a^2}\,dx \quad (x>a)$

$$\int \frac{2x}{(x+a)(x-a)}\,dx$$

Solve this question according to previous question.

⑦ $\int \frac{1}{6x^2+5x-4}\,dx$

$$\int \frac{1}{6x^2+8x-3x-4}\,dx$$

$$\int \frac{1}{2x(3x+4)-1(3x+4)}\,dx$$

$$\int \frac{1}{(3x+4)(2x-1)}\,dx$$

Solve this as previous questions are solved.

(8) $$\int \frac{2x^3-3x^2-x-7}{2x^2-3x-2}\,dx$$

$$\begin{array}{r|l} & \quad x \\ 2x^2-3x-2 & 2x^3-3x^2-x-7 \\ & \underline{2x^3 \mp 3x^2 \mp 2x} \\ & \qquad\qquad x-7 \end{array}$$

$$\int\left[x+\frac{(x-7)}{(2x^2-3x-2)}\right]dx$$

$$\int x\,dx+\int\frac{(x-7)}{(2x^2-3x-2)}\,dx$$

$$\frac{x^2}{2}+\int\frac{(x-7)}{2x^2-4x+x-2}\,dx$$

$$\frac{x^2}{2}+\int\frac{(x-7)}{2x(x-2)+1(x-2)}\,dx$$

$$\frac{x^2}{2}+\int\frac{(x-7)}{(x-2)(2x+1)}\,dx \quad \text{— i)}$$

Solve $\int \frac{(x-7)}{(x-2)(2x+1)}\,dx$ according to previous question and put answer in i)

Here we introduce another method to solve question. choice is youres.

(9) $$\int \frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}\,dx$$

changing into partial fraction

$$\frac{3x^2-12x+11}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

$$3x^2-12x+11=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2) \quad \text{— i}$$

Put $x=1$ in i)

$$3-12+11=A(1-2)(1-3)+B(1-1)(1-3)+C(1-1)(1-2)$$

$$-9+11=A(-1)(-2)$$

$$2 = 2A$$
$$1 = A$$
$$\Rightarrow A = 1$$

Now put $x = 2$ in i)

$$3(2)^2 - 12(2) + 11 = B(2-1)(2-3)$$
$$3(4) - 12(2) + 11 = B(1)(-1)$$
$$12 - 24 + 11 = -B$$
$$-1 = -B$$
$$1 = B$$
$$\Rightarrow B = 1$$

Now put $x = 3$ in i)

$$3(3)^2 - 12(3) + 11 = C(3-1)(3-2)$$
$$3(9) - 12(3) + 11 = C(2)(1)$$
$$27 - 36 + 11 = 2C$$
$$2 = 2C$$
$$\Rightarrow C = 1$$

So

$$\frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)} = \frac{1}{(x-1)} + \frac{1}{(x-2)} + \frac{1}{(x-3)}$$

$$\int \frac{3x^2 - 12x + 11}{(x-1)(x-2)(x-3)}\,dx = \int \left[\frac{1}{(x-1)} + \frac{1}{(x+2)} + \frac{1}{(x-3)}\right] dx$$

$$= \int \frac{1}{(x-1)}\,dx + \int \frac{1}{(x+2)}\,dx + \int \frac{1}{(x-3)}\,dx$$

$$= \ln|x-1| + \ln|x+2| + \ln|x-3| + c$$

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(10) $\displaystyle\int \frac{2x-1}{x(x-1)(x-3)}\,dx$

Solve according to previous question.

(11) $$\int \frac{5x^2+9x+6}{(x^2-1)(2x+3)}\,dx$$

$$\int \frac{5x^2+9x+6}{(x+1)(x-1)(2x+3)}\,dx$$

Solve according to previous question.

(12) $$\int \frac{4+7x}{(1+x)^2(2+3x)}\,dx$$

changing into partial fraction

$$\frac{4+7x}{(1+x)^2(2+3x)} = \frac{A}{(2+3x)} + \frac{B}{(1+x)} + \frac{C}{(1+x)^2}$$

multiply by $(1+x)^2(2+3x)$ in both sides we obtain

$$4+7x = A(1+x)^2 + B(1+x) + C(2+3x) \quad \text{— i}$$

Put $x = -\frac{2}{3}$ in i)

$$4+7\left(-\frac{2}{3}\right) = A\left(1-\frac{2}{3}\right)^2$$

$$4-\frac{14}{3} = A\left(\frac{1}{3}\right)^2$$

$$-\frac{2}{3} = A\left(\frac{1}{9}\right)$$

$$A = -\frac{2}{3}\times 9$$

$$A = -6$$

Now

Put $x = -1$ in i)

$$4+7(-1) = C(2+3(-1))$$

$$4-7 = C(2-3)$$

$$-3 = -C$$

$$3 = C$$

$$\Rightarrow C = 3$$

Equating the Coefficient of x^2 on both sides of i)

$$A + 3B = 0 \qquad \text{where } A = -6$$
$$-6 + 3B = 0$$
$$3B = 6$$
$$B = 2$$

So

$$\frac{4+7x}{(2+3x)(1+x^2)} = \frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}$$

$$\int \frac{4+7x}{(2+3x)(1+x)^2}dx = \int\left[\frac{-6}{2+3x} + \frac{2}{1+x} + \frac{3}{(1+x)^2}\right]dx$$

$$= \int \frac{-6}{2+3x}dx + \int \frac{2}{1+x}dx + \int \frac{3}{(1+x)^2}dx$$

$$= -2\int \frac{3}{2+3x}dx + 2\int \frac{1}{1+x}dx + 3\int (1+x)^{-2}dx$$

$$= -2\ln|2+3x| + 2\ln|1+x| + 3\frac{(1+x)^{-2+1}}{-2+1} + C$$

$$= -2\ln|2+3x| + 2\ln|1+x| - \frac{3}{(1+x)} + C$$

(13) $\int \frac{2x^2}{(x-1)(x+1)}dx$

Solve as question # 12 solved.

(14) $\int \frac{1}{(x-1)(x+1)^2}dx$

Solve as question # 12 is solved.

(15) $\int \frac{x+4}{x^3-3x^2+4}dx$

Factorizing x^3-3x^2+4 by synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$(x+1)(x^2-4x+4)$

$(x+1)(x-2)^2$

it can be written as

$$\int \frac{x+4}{(x^3-3x^2+4)} dx = \int \frac{x+4}{(x+1)(x-2)^2} dx$$

Now solve it by previous method.

(16) $\int \frac{x^3-6x^2+25}{(x+1)^2(x-2)^2} dx$

this question is also solved by previous method.

(17) $\int \frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} dx$

changing into partial fraction

$$\frac{x^3+22x^2+14x-17}{(x-3)(x+2)^3} = \frac{A}{(x-3)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} + \frac{D}{(x+2)^3}$$

Now solve it by previous method.

(18) $\int \frac{x-2}{(x+1)(x^2+1)} dx$

changing into partial fraction:

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$$

$$x-2 = A(x^2+1) + (Bx+C)(x+1) \quad \text{--- i)}$$

Put $x = -1$ in i)

$$-1-2 = A[(1)^2+1]$$

$$-3 = A(1+1)$$

$$-3 = 2A$$

From equation i)

$$x-2 = A(x^2+1) + (Bx+C)(x+1)$$

$$= A(x^2+1) + Bx^2 + Bx + Cx + C$$

Equating coefficient of like powers of x

$$\Rightarrow \quad A + B = 0 \quad \text{—— I}$$

$$B + C = 1 \quad \text{—— II}$$

Put $A = -\frac{3}{2}$ in I

$$-\frac{3}{2} + B = 0$$

$$\Rightarrow \quad B = \frac{3}{2}$$

Put the value of B in II

$$\frac{3}{2} + C = 1$$

$$C = 1 - \frac{3}{2}$$

$$C = -\frac{1}{2}$$

So

$$\frac{x-2}{(x+1)(x^2+1)} = \frac{-\frac{3}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

$$= -\frac{3}{2(x+1)} + \frac{(3x-1)}{2(x^2+1)}$$

$$\int \frac{x-2}{(x+1(x^2+1)} dx = \int \left[\frac{-3}{2(x+1)} + \frac{(3x-1)}{2(x^2+1)} \right] dx$$

$$= \int \frac{-3}{2(x+1)} dx + \int \frac{(3x-1)}{2(x^2+1)} dx$$

$$= -\frac{3}{2} \int \frac{1}{(x+1)} dx + \frac{1}{4} \int \frac{2(3x-1)}{(x^2+1)} dx$$

$$= -\frac{3}{2}\ln|x+1| + \frac{1}{4}\int \frac{(6x-2)}{(x^2+1)}\,dx$$

$$= -\frac{3}{2}\ln|x+1| + \frac{1}{4}\left[\int \frac{6x}{(x^2+1)}\,dx - \int \frac{2}{(x^2+1)}\,dx\right]$$

$$= -\frac{3}{2}\ln|x+1| + \frac{1}{4}\left[3\int \frac{2x}{(x^2+1)}\,dx - 2\int \frac{1}{(x^2+1)}\,dx\right]$$

$$= -\frac{3}{2}\ln|x+1| + \frac{1}{4}\left[3\ln|x^2+1| - 2\tan^{-1}x + C\right]$$

$$= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\ln|x^2+1| - \frac{2}{4}\tan^{-1}x + \frac{1}{4}C$$

$$= -\frac{3}{2}\ln|x+1| + \frac{3}{4}\ln|x^2+1| - \frac{1}{2}\tan^{-1}x + C'$$

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(19) $\displaystyle\int \frac{x}{(x-1)(x^2+1)}\,dx$ Solve from

Solved question.

(20) $\displaystyle\int \frac{9x-7}{(x+3)(x^2+1)}\,dx$

Solve this question according to the previous questions are solved.

(21) $\displaystyle\int \frac{1+4x}{(x-3)(x^2+4)}\,dx$

solve it according to previous questions.

(22) $\int \frac{12}{x^3+8}\,dx$

$\int \frac{12}{(x)^3+(2)^3}\,dx$

$\int \frac{12}{(x+2)(x^2-2x+4)}\,dx$

Change into partial fraction.

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{A}{(x+2)} + \frac{Bx+C}{(x^2-2x+4)}$$

multiply by $(x+2)(x^2-2x+4)$ in both sides.

$12 = A(x^2-2x+4) + (Bx+c)(x+2)$ — i)

Put $x=-2$ in i)

$12 = A[(-2)^2-2(-2)+4]$

$12 = A\,12$

$1 = A$

$\Rightarrow A = 1$

Equating coefficient of like power of x from i)

$A+B=0$ — ii)

$-2A+2B+C=0$ — iii)

Put $A=1$

ii) $\Rightarrow$ $1+B=0$

$\Rightarrow$ $B=-1$

iii) $\Rightarrow$ $-2(1)+2(-1)+C=0$

$-2-2+C=0$

$C=4$

So

$$\frac{12}{(x+2)(x^2-2x+4)} = \frac{1}{(x+2)} + \frac{(-x+4)}{x^2-2x+4}$$

$$= \frac{1}{(x+2)} - \frac{(x-4)}{x^2-2x+4}$$

$$\int \frac{12}{(x+2)(x^2-2x+4)}\,dx = \int\left[\frac{1}{x+2} - \frac{(x-4)}{x^2-2x+4}\right]dx$$

$$= \int \frac{1}{x+2}\,dx - \int \frac{(x-4)}{x^2-2x+4}\,dx$$

$$= \ln|x+2| - \frac{1}{2}\int \frac{2(x-4)}{x^2-2x+4}\,dx$$

$$= \ln|x+2| - \frac{1}{2}\left[\int \frac{2x-8}{x^2-4x+4}\right]dx$$

$$= \ln|x+2| - \frac{1}{2}\left[\int \frac{2x-2-6}{x^2-2x+4}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\int \frac{(2x-2)}{x^2-2x+4}\,dx - \int \frac{6}{x^2-2x+4}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\ln|x^2-2x+4| - 6\int \frac{1}{(x-2)^2}\,dx\right]$$

$$= \ln|x+2| - \frac{1}{2}\left[\ln|x^2-2x+4| + 6\,\frac{1}{(x-2)} + C\right]$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{6}{2(x-2)} + \frac{C}{2}$$

$$= \ln|x+2| - \frac{1}{2}\ln|x^2-2x+4| + \frac{3}{(x-2)} + C'$$

(23) $$\int \frac{9x+6}{x^3-8}\,dx$$

$$\int \frac{9x+6}{(x)^3-(2)^3}$$

$$\int \frac{9x+6}{(x-2)(x^2+2x+4)}\,dx$$

Solve according previous question.

(24) $$\int \frac{2x^2+5x+3}{(x-1)^2(x^2+4)}\,dx$$

change into Partial fraction

$$\frac{2x^2+5x+3}{(x-1)^2(x^2+4)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+4)}$$

Now solve it according to previous experience.

(25) $$\int \frac{2x^2-x-7}{(x+2)^2(x^2+x+1)}\,dx$$

solve this question according to previous experience.

(26) $$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}\,dx$$

change into partial fraction.

$$\frac{3x+1}{(4x^2+1)(x^2-x+1)} = \frac{(Ax+B)}{(4x^2+1)} + \frac{(Cx+D)}{(x^2-x+1)}$$

multiply by $(4x^2+1)(x^2-x+1)$ in both sides.

$$3x+1 = (Ax+B)(x^2-x+1) + (Cx+D)(4x^2+1)$$

$$= Ax^3 - Ax^2 + Ax + Bx^2 - Bx + B + 4Cx^3 + Cx + 4Dx^2 + D$$

$$= (A+4C)x^3 + (-A+B+4D)x^2 + (A-B+C)x + B+D$$

Equating coefficient of like powers of x

$$A + 4C = 0 \quad \text{——— I}$$

$$-A + B + 4D = 0 \quad \text{——— II}$$

$$A - B + C = 3 \quad \text{——— III}$$

$$B + D = 1 \quad \text{——— IV}$$

adding II and III

$$4D + C = 3$$

$$\Rightarrow \quad C = 3 - 4D \quad \text{——— V}$$

I) $\Rightarrow$ $A = -4C$

$\Rightarrow$ $A = -4(3-4D)$

$A = -12 + 16D$ ——— VI

IV $\Rightarrow$ $B = 1 - D$ ——— VII

Put value of A, B & C in III

$$-12 + 16D - 1 + D + 3 - 4D = 3$$

$$13D - 10 = 3$$

$$13D = 13$$

$$D = 1$$

$\text{VII} \Rightarrow \quad B = 1 - D$

$$B = 1 - 1$$

$$B = 0$$

$\text{V} \Rightarrow \quad C = 3 - 4D$

$$C = 3 - 4(1)$$

$$= 3 - 4$$

$$C = -1$$

$\text{VI} \Rightarrow \quad A = -12 + 16D$

$$= -12 + 16(1)$$

$$= -12 + 16$$

$$A = 4$$

So $\dfrac{3x+1}{(4x^2+1)(x^2-x+1)} = \dfrac{4x+0}{4x^2+1} + \dfrac{(-x+1)}{x^2-x+1}$

$$= \frac{4x}{4x^2+1} - \frac{(x-1)}{(x^2-x+1)}$$

$$\int \frac{3x+1}{(4x^2+1)(x^2-x+1)}\,dx = \int\left[\frac{4x}{4x^2+1} - \frac{(x-1)}{(x^2-x+1)}\right]dx$$

$$= \int \frac{4x}{4x^2+1}\,dx - \int \frac{(x-1)}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\int \frac{8x}{4x^2+1}\,dx - \frac{1}{2}\int \frac{2(x-1)}{x^2-x+1}\,dx$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\int \frac{2x-2}{x^2-x+1}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\int \frac{(2x-1)-1}{x^2-x+1}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\int \frac{(2x-1)}{x^2-x+1}\,dx - \int \frac{1}{x^2-x+1}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\ln|x^2-x+1| - \int \frac{1}{x^2-x+1+\frac{1}{4}-\frac{1}{4}}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\ln|x^2-x+1| - \int \frac{1}{(x^2-x+\frac{1}{4})+(1-\frac{1}{4})}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\ln|x^2-x+1| - \int \frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\,dx\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\ln|x^2-x+1| - \frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\frac{(x-\frac{1}{2})}{\frac{\sqrt{3}}{2}}\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\left[\ln|x^2-x+1| - \frac{2}{\sqrt{3}}\tan^{-1}\frac{(2x-1)}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\ln|4x^2+1| - \frac{1}{2}\ln|x^2-x+1| + \frac{1}{\sqrt{3}}\tan^{-1}\frac{(2x-1)}{\sqrt{3}}$$

(27) $$\int \frac{4x+1}{(x^2+4)(x^2+4x+5)}\,dx$$

Solve this according to previous question.

(28) $$\int \frac{6a^2}{(x^2+a^2)(x^2+4a^2)}\,dx$$

Solve this question as question # 26 is solved.

(29) $$\int \frac{-\ }{\ \ 1}\,dx$$

$$\int \frac{\ }{(x\ \ \ \ )}\ \ \frac{\ }{\ }\,dx$$

Now solve . sum by previous sum.

(30) $$\int \frac{3x-\ }{(x^2-x+2)(\ \ \ +x+2)}\,dx$$

Solve this sum with the help of previous sum.

(31) $$\int \frac{3x^3\ \ \ \ ^2+9x+5}{(x^2+\ \ +1)(x^2+2x+3)}\,dx$$

Solve this sum with the help of previous sum.