Exercise 5.1

Resolving the following into partial fractions:

Question # 1

$$\frac{1}{x^2-1}$$

Solution

$$\frac{1}{x^2-1}=\frac{1}{(x-1)(x+1)}$$

Now suppose

$$\frac{1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

Multiplying both sides by $(x-1)(x+1)$ we get

$$1=A(x+1)+B(x-1)\ \ldots\ldots\ldots\ldots\ldots(\text{i})$$

Put $x-1=0 \Rightarrow x=1$ in equation (i)

$$1=A(1+1)+B(0) \Rightarrow 1=2A+0 \Rightarrow \boxed{A=\frac{1}{2}}$$

Now put $x+1=0 \Rightarrow x=-1$ in equation (i)

$$1=A(0)+B(-1-1) \Rightarrow 1=0-2B \Rightarrow \boxed{B=-\frac{1}{2}}$$

Hence

$$\frac{1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

$$=\frac{1}{2(x-1)}-\frac{1}{2(x+1)} \quad \text{Answer}$$

Question # 2

$$\frac{x^2+1}{(x+1)(x-1)}$$

Solution

$$\begin{array}{r|l} & \ \ 1 \\ \hline x^2-1 & x^2+1 \\ & \underline{\overset{-}{x^2}\ \overset{+}{-1}} \\ & \ \ 2 \end{array}$$

$$\frac{x^2+1}{(x+1)(x-1)}=\frac{x^2+1}{x^2-1}$$

$$=2+\frac{1}{x^2-1} \quad =2+\frac{1}{(x+1)(x-1)}$$

Now consider

$$\frac{2}{(x+1)(x-1)}=\frac{A}{x+1}+\frac{B}{x-1}$$

Multiplying both sides by $(x+1)(x-1)$

$$2 = A(x-1) + B(x+1) \quad \text{...............(i)}$$

Put $x+1=0 \Rightarrow x=-1$ in equation (i)

$$2 = A(-1-1) + B(0) \quad \Rightarrow 2 = -2A + 0 \Rightarrow \boxed{A=-1}$$

Now put $x-1=0 \Rightarrow x=1$ in equation (i)

$$2 = A(0) + B(1+1) \quad \Rightarrow 2 = 0 + 2B \quad \Rightarrow \boxed{B=1}$$

So $$\frac{2}{(x+1)(x-1)} = \frac{-1}{x+1} + \frac{1}{x-1}$$

Hence

$$\frac{x^2+1}{(x+1)(x-1)} = 2 + \frac{-1}{(x+1)} + \frac{1}{(x-1)}$$

$$= 2 - \frac{1}{(x+1)} + \frac{1}{(x-1)} \qquad \textit{Answer}$$

Question # 3

$$\frac{2x+1}{(x-1)(x+2)(x+3)}$$

Solution

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x+3}$$

Multiplying both side by $(x-1)(x+2)(x+3)$

$$2x+1 = A(x+2)(x+3) + B(x-1)(x+3) + C(x-1)(x+2) \quad \text{............(i)}$$

Put $x-1=0 \Rightarrow x=1$ in equation (i)

$$2(1)+1 = A(1+2)(1+3) + B(0) + C(0)$$

$$3 = A(3)(4) + 0 + 0 \quad \Rightarrow 3 = 12A \quad \Rightarrow \frac{3}{12} = A \quad \Rightarrow \boxed{A = \frac{1}{4}}$$

Now put $x+2=0 \Rightarrow x=-2$ in equation (i)

$$2(-2)+1 = A(0) + B(-2-1)(-2+3) + C(0)$$

$$-4+1 = 0 + B(-3)(1) + 0 \quad \Rightarrow -3 = -3B \quad \Rightarrow \boxed{B = 1}$$

Now put $x+3=0 \Rightarrow x=-3$ in equation (i)

$$2(-3)+1 = A(0) + B(0) + C(-3-1)(-3+2)$$

$$-6+1 = 0 + 0 + C(-4)(-1) \quad \Rightarrow -5 = 4C \quad \Rightarrow \boxed{C = -\frac{5}{4}}$$

So

$$\frac{2x+1}{(x-1)(x+2)(x+3)} = \frac{1/4}{x-1} + \frac{1}{x+2} + \frac{-5/4}{x+3}$$

$$= \frac{1}{4(x-1)} + \frac{1}{x+2} - \frac{5}{4(x+3)} \qquad \textit{Answer}$$

Question # 4

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)}$$

$$\because \; x^2+7x+10 = x^2+5x+2x+10$$
$$= x(x+5)+2(x+5)$$
$$= (x+5)(x+2)$$

Solution

$$\frac{3x^2-4x-5}{(x-2)(x^2+7x+10)} = \frac{3x^2-4x-5}{(x-2)(x+5)(x+2)}$$

Now resolving into partial fraction.

$$\frac{3x^2-4x-5}{(x-2)(x+5)(x+2)} = \frac{A}{x-2}+\frac{B}{x+5}+\frac{C}{x+2}$$

$$\left[\begin{array}{l} \textit{Do yourself. You will get} \\ A=-\dfrac{1}{28},\; B=\dfrac{30}{7},\; C=-\dfrac{5}{4} \end{array}\right]$$

Question # 5

$$\frac{1}{(x-1)(2x-1)(3x-1)}$$

Solution

$$\frac{1}{(x-1)(2x-1)(3x-1)} = \frac{A}{x-1}+\frac{B}{2x-1}+\frac{C}{3x-1}$$

Multiplying both side by $(x-1)(2x-1)(3x-1)$.

$$1 = A(2x-1)(3x-1)+B(x-1)(3x-1)+C(2x-1)(3x-1) \text{ ....(i)}$$

Put $x-1=0 \;\Rightarrow x=1$ in equation (i)

$$1 = A(2(1)-1)(3(1)-1)+B(0)+C(0) \quad \Rightarrow 1 = A(1)(2)+0+0$$

$$\Rightarrow 1 = 2A \quad \Rightarrow \boxed{A=\frac{1}{2}}$$

Put $2x-1=0 \;\Rightarrow 2x=1 \Rightarrow x=\frac{1}{2}$ in equation (i)

$$1 = A(0)+B\left(\frac{1}{2}-1\right)\left(3\left(\frac{1}{2}\right)-1\right)+C(0) \quad \Rightarrow 1 = 0+B\left(-\frac{1}{2}\right)\left(\frac{1}{2}\right)+0$$

$$\Rightarrow 1 = -\frac{1}{4}B \quad \Rightarrow \boxed{B=-4}$$

Put $3x-1=0 \;\Rightarrow 3x=1 \Rightarrow x=\frac{1}{3}$ in equation (i)

$$1 = A(0)+B(0)+C\left(\frac{1}{3}-1\right)\left(2\left(\frac{1}{3}\right)-1\right) \quad \Rightarrow 1 = 0+0+C\left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right)$$

$$\Rightarrow 1 = \frac{2}{9}C \quad \Rightarrow \boxed{C=\frac{9}{2}}$$

Hence $\qquad \dfrac{1}{(x-1)(2x-1)(3x-1)} = \dfrac{\frac{1}{2}}{x-1} + \dfrac{-4}{2x-1} + \dfrac{\frac{9}{2}}{3x-1}$

$$= \frac{1}{2(x-1)} - \frac{4}{2x-1} + \frac{9}{2(3x-1)} \quad \textit{Answer}$$

Question # 6

$$\frac{x}{(x-a)(x-b)(x-c)}$$

Solution

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

Multiplying both sides by $(x-a)(x-b)(x-c)$.

$$x = A(x-b)(x-c) + B(x-a)(x-c) + C(x-a)(x-b) \text{ ....(i)}$$

Put $x-a=0 \Rightarrow x=a$ in equation (i)

$$a = A(a-b)(a-c) + B(0) + C(0)$$

$$\Rightarrow a = A(a-b)(a-c) + 0 + 0 \qquad \Rightarrow \boxed{A = \frac{a}{(a-b)(a-c)}}$$

Now put $x-b=0 \Rightarrow x=b$ in equation (i)

$$a = A(0) + B(b-a)(b-c) + C(0)$$

$$\Rightarrow a = 0 + B(b-a)(b-c) + 0 \qquad \Rightarrow \boxed{B = \frac{b}{(b-a)(b-c)}}$$ Now put

$x-c=0 \Rightarrow x=c$ in equation (i)

$$c = A(0) + B(0) + C(c-a)(c-b)$$

$$\Rightarrow c = 0 + 0 + C(c-a)(c-b) \qquad \Rightarrow \boxed{B = \frac{c}{(c-a)(c-b)}}$$

So

$$\frac{x}{(x-a)(x-b)(x-c)} = \frac{a/(a-b)(a-c)}{x-a} + \frac{b/(b-a)(b-c)}{x-b} + \frac{c/(c-a)(c-b)}{x-c}$$

$$= \frac{a}{(a-b)(a-c)(x-a)} + \frac{b}{(b-a)(b-c)(x-b)} + \frac{c}{(c-a)(c-b)(x-c)}$$

Answer

Question # 7

$$\frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1}$$

Solution

$$\frac{6x^3 + 5x^2 - 7}{2x^2 - x - 1}$$

$$=3x+4+\frac{7x-3}{2x^2-x-1}$$
$$=3x+4+\frac{7x-3}{2x^2-2x+x-1}$$
$$=3x+4+\frac{7x-3}{2x(x-1)+1(x-1)}$$
$$=3x+4+\frac{7x-3}{(x-1)(2x+1)}$$

$$\begin{array}{r|l} & 3x+4 \\ \hline 2x^2-x-1 & 6x^3+5x^2-7 \\ & \underline{\overset{-}{6x^3}\overset{+}{-}3x^2\overset{+}{-}3x} \\ & 8x^2+3x-7 \\ & \underline{\overset{-}{8x^2}\overset{+}{-}4x\overset{+}{-}4} \\ & 7x-3 \end{array}$$

Now Consider
$$\frac{7x-3}{(x-1)(2x+1)}=\frac{A}{x-1}+\frac{B}{2x+1}$$
$$\left[\begin{array}{c} \textit{Find value of } A \,\&\, B \textit{ yourself} \\ \textit{You will get } A=4/3 \textit{ and } B=13/3 \end{array}\right]$$

so
$$\frac{7x-3}{(x-1)(2x+1)}=\frac{4/3}{x-1}+\frac{13/3}{2x+1}=\frac{4}{3(x-1)}+\frac{13}{3(2x+1)}$$

Hence
$$\frac{6x^3+5x^2-7}{2x^2-x-1}=3x+4+\frac{4}{3(x-1)}+\frac{13}{3(2x+1)} \qquad \textit{Answer}$$

Question # 8

$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$$

$$\begin{array}{r|l} & 1 \\ \hline 2x^3+x^2-3x & 2x^3+x^2-5x+3 \\ & \underline{\overset{-}{2x^3}\overset{-}{+}x^2\overset{+}{-}3x} \\ & -2x+3 \end{array}$$

Solution
$$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}$$
$$=1+\frac{-2x+3}{2x^3+x^2-3x}$$
$$=1+\frac{-2x+3}{x(2x^2+x-3)} \qquad =1+\frac{-2x+3}{x(2x^2+3x-2x-3)}$$
$$=1+\frac{-2x+3}{x\big(x(2x+3)-1(2x+3)\big)} \qquad =1+\frac{-2x+3}{x(2x+3)(x-1)}$$

Now consider
$$\frac{3-2x}{x(2x+3)(x-1)}=\frac{A}{x}+\frac{B}{2x+3}+\frac{C}{x-1}$$
$$\Rightarrow 3-2x=A(2x+3)(x-1)+Bx(x-1)+Cx(2x+3) \text{ .... (i)}$$
Put $x=0$ in equation (i)
$$3-2(0)=A\big(2(0)+3\big)\big((0)-1\big)+B(0)+C(0) \qquad \Rightarrow 3-0=A\big(0+3\big)\big(-1\big)+0+0$$
$$\Rightarrow 3=-3A \qquad \Rightarrow \boxed{A=-1}$$

Now put $2x+3=0 \Rightarrow 2x=-3 \Rightarrow x=-\frac{3}{2}$ in equation (i)

$$3-2\left(-\frac{3}{2}\right)=A(0)+B\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)+C(0) \quad \Rightarrow 3+3=0+B\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)+0$$

$$\Rightarrow 6=\frac{15}{4}B \quad \Rightarrow B=(6)\left(\frac{4}{15}\right) \quad \Rightarrow \boxed{B=\frac{8}{5}}$$

Now put $x-1=0 \Rightarrow x=1$ in equation (i)

$$3-2(1)=A(0)+B(0)+C(1)(2(1)+3) \Rightarrow 1=0+0+5C \quad \Rightarrow \boxed{C=\frac{1}{5}}$$

So $$\frac{3-2x}{x(2x+3)(x-1)}=\frac{-1}{x}+\frac{8/5}{2x+3}+\frac{1/5}{x-1}=-\frac{1}{x}+\frac{8}{5(2x+3)}+\frac{1}{5(x-1)}$$

Hence $$\frac{2x^3+x^2-5x+3}{2x^3+x^2-3x}=1-\frac{1}{x}+\frac{8}{5(2x+3)}+\frac{1}{5(x-1)} \qquad \textit{Answer}$$

Question # 9

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}$$

Solution

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}=\frac{(x-1)(x^2-3x-5x+15)}{(x-2)(x^2-4x-6x+24)}$$

$$=\frac{(x-1)(x^2-8x+15)}{(x-2)(x^2-10x+24)} \quad =\frac{x^3-8x^2+15x-x^2+8x-15}{x^3-10x^2+24x-2x^2+20x-48}$$

$$=\frac{x^3-9x^2+23x-15}{x^3-12x^2+44x-48}$$

$$=1+\frac{3x^2-21x+33}{x^3-12x^2+44x-48}$$

$$=1+\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}$$

$$\begin{array}{r|l} & 1 \\ \hline x^3-12x^2+44x-48 & x^3-9x^2+23x-15 \\ & \underline{\underset{-}{x^3}\underset{+}{-12x^2}\underset{-}{+44x}\underset{+}{-48}} \\ & 3x^2-21x+33 \end{array}$$

Now Suppose

$$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}=\frac{A}{x-2}+\frac{B}{x-4}+\frac{C}{x-6}$$

$$\left[\begin{array}{c} \textit{Find value of } A, B \textit{ and } C \textit{ yourself} \\ \textit{You will get } A=3/8,\ B=3/4,\ C=15/8 \end{array}\right]$$

So $$\frac{3x^2-21x+33}{(x-2)(x-4)(x-6)}=\frac{3/8}{x-2}+\frac{3/4}{x-4}+\frac{15/8}{x-6}$$

$$=\frac{3}{8(x-2)}+\frac{3}{4(x-4)}+\frac{15}{8(x-6)}$$

Hence

$$\frac{(x-1)(x-3)(x-5)}{(x-2)(x-4)(x-6)}=1+\frac{3}{8(x-2)}+\frac{3}{4(x-4)}+\frac{15}{8(x-6)} \quad \textit{Answer}$$

Question # 10

$$\frac{1}{(1-ax)(1-bx)(1-cx)}$$

Solution

$$\frac{1}{(1-ax)(1-bx)(1-cx)}=\frac{A}{1-ax}+\frac{B}{1-bx}+\frac{C}{1-cx}$$

Multiplying both sides by $(1-ax)(1-bx)(1-cx)$.

$$1=A(1-bx)(1-cx)+B(1-ax)(1-cx)+C(1-ax)(1-bx) \text{ ....(i)}$$

Put $1-ax=0 \Rightarrow ax=1 \Rightarrow x=\frac{1}{a}$ in equation (i).

$$1=A\left(1-b\cdot\frac{1}{a}\right)\left(1-c\cdot\frac{1}{a}\right)+B(0)+C(0) \qquad \Rightarrow 1=A\left(1-\frac{b}{a}\right)\left(1-\frac{c}{a}\right)+0+0$$

$$\Rightarrow 1=A\left(\frac{a-b}{a}\right)\left(\frac{a-c}{a}\right) \qquad \Rightarrow 1=A\frac{(a-b)(a-c)}{a^2} \qquad \Rightarrow \boxed{A=\frac{a^2}{(a-b)(a-c)}}$$

$$\left[\begin{array}{c}\textit{Find value of } B \,\&\, C \textit{ yourself as } A.\\ \textit{You will get } B=\frac{b^2}{(b-a)(b-c)},\ C=\frac{c^2}{(c-a)(c-b)}\end{array}\right]$$

Hence $$\frac{1}{(1-ax)(1-bx)(1-cx)}=\frac{\frac{a^2}{(a-b)(a-c)}}{1-ax}+\frac{\frac{b^2}{(b-a)(b-c)}}{1-bx}+\frac{\frac{c^2}{(c-a)(c-b)}}{1-cx}$$

$$=\frac{a^2}{(a-b)(a-c)(1-ax)}+\frac{b^2}{(b-a)(b-c)(1-bx)}+\frac{c^2}{(c-a)(c-b)(1-cx)}$$

Answer

Question # 11

$$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$$

Solution $$\frac{x^2+a^2}{(x^2+b^2)(x^2+c^2)(x^2+d^2)}$$

Put $y=x^2$ in above.

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}$$

Now consider

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)}=\frac{A}{y+b^2}+\frac{B}{y+c^2}+\frac{C}{y+d^2}$$

$$\Rightarrow \quad y+a^2=A(y+c^2)(y+d^2)+B(y+b^2)(y+d^2)+C(y+b^2)(y+c^2) \text{ ....(i)}$$

Put $y+b^2=0 \Rightarrow y=-b^2$ in equation (i)

$$-b^2+a^2 = A(-b^2+c^2)(-b^2+d^2)+B(0)+C(0)$$

$$\Rightarrow a^2-b^2 = A(c^2-b^2)(d^2-b^2)+0+0 \quad \Rightarrow \boxed{A=\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}}$$

Now put $y+c^2=0 \Rightarrow y=-c^2$ in equation (i)

$$-c^2+a^2 = A(0)+B(-c^2+b^2)(-b^2+d^2)+C(0)$$

$$\Rightarrow a^2-c^2 = 0+B(b^2-c^2)(d^2-c^2)+0 \quad \Rightarrow \boxed{B=\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)}}$$

ow put $y+d^2=0 \Rightarrow y=-d^2$ in equation (i)

$$-d^2+a^2 = A(0)+B(0)+C(-d^2+b^2)(-d^2+c^2)$$

$$\Rightarrow a^2-d^2 = 0+0+C(b^2-d^2)(c^2-d^2) \quad \Rightarrow \boxed{C=\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)}}$$

Hence

$$\frac{y+a^2}{(y+b^2)(y+c^2)(y+d^2)} = \frac{\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)}}{y+b^2}+\frac{\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)}}{y+c^2}+\frac{\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)}}{y+d^2}$$

$$=\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(y+b^2)}+\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(y+c^2)}+\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(y+d^2)}$$

Since $y=x^2$

$$=\frac{a^2-b^2}{(c^2-b^2)(d^2-b^2)(x^2+b^2)}+\frac{a^2-c^2}{(b^2-c^2)(d^2-c^2)(x^2+c^2)}+\frac{a^2-d^2}{(b^2-d^2)(c^2-d^2)(x^2+d^2)}$$