Remainder Theorem:-

If a polynomial $f(x)$ of degree $n \geq 1$, n is non negative integer is divided by $(x-a)$ till no x term exists in the remainder Then $f(a)$ is a remainder

PROOF: Suppose a polynomial $f(x)$ is divided by $(x-a)$ Then there exists a unique quotient $q(x)$ and a unique remainder R s.t

Dividend = Divisor × quotient + Remainder

$$f(x) = (x-a)\,q(x) + R$$

putting $x=a$ we get

$$f(a) = (a-a)\,q(a) + R = 0 + R = R$$

Factor Theorem:-

The polynomial $(x-a)$ is a factor of polynomial $f(x)$ if and only if $f(a)=0$ i-e $(x-a)$ is a factor of $f(x)$ if and only if $x=a$ is a root of polynomial equation $f(x)=0$

PROOF. Suppose $q(x)$ is The quotient and R is remainder when a polynomial $f(x)$ is divided by $(x-a)$.

Then by remainder Theorem

$$f(x) = (x-a)\,q(x) + R$$

Since $f(a)=0 \Rightarrow R=0$

$\therefore\ f(x) = (x-a)\,q(x)$

$(x-a)$ is a factor of $f(x)$

Conversely if $(x-a)$ is a factor of $f(x)$ Then

$R = f(a) = 0$

which proves the theorem

EXERCISE 4.5

Use The remainder Theorem To find the remainder

1. Let $f(x) = x^2+3x+7$

Sol. $x - a = x+1 \Rightarrow a=-1$

Remainder $= f(a)$ By Remainder Theorem

$R = f(-1) = (-1)^2 + 3(-1) + 7$
$= 1 - 3 + 7 = 5$

2. Let $f(x) = x^3 - x^2 + 5x + 4$

$x - a = x - 2 \Rightarrow a = 2$

Remainder $= f(a)$

$R = f(2) = (2)^3 - (2)^2 + 5(2) + 4$
$= 8 - 4 + 10 + 4 = 18$

3. Let $f(x) = 3x^4 + 4x^3 + x - 5$

$x - a = x + 1 \Rightarrow a = -1$

$R = f(a) = f(-1) = 3(-1)^4 + 4(-1)^3 - 1 - 5$
$= 3 - 4 - 1 - 5 = -7$

4. $f(x) = x^3 - 2x^2 + 3x + 3$

$x - a = x - 3 \Rightarrow a = 3$

$R = f(a) = f(3) = 3^3 - 2(3)^2 + 3(3) + 3$
$= 27 - 18 + 9 + 3 = 21$

Use factor Theorem .....

5. $x-1,\ x^2+4x-5$

Sol. Let $f(x) = x^2 + 4x - 5$

and $x - a = x - 1 \Rightarrow a = 1$

Remainder $= f(a) = f(1)$

$R = (1)^2 + 4(1) - 5 = 5 - 5 = 0$

Hence $(x-1)$ is a factor of $f(x)$ by factor theorem.

6. Let $f(x) = x^3 + x^2 - 7x + 1$

$x - a = x - 2 \Rightarrow a = 2$

$R = f(a) = f(2) = 2^3 + 2^2 - 7(2) + 1$
$= 8 + 4 - 14 + 1 = -1 \neq 0$

$\Rightarrow (x-2)$ is not a factor of $f(x)$

7. Let $f(w) = 2w^3 + w^2 - 4w + 7$

$w - a = w + 2 \Rightarrow a = -2$

Remainder $= f(w) = f(-2)$

$R = 2(-2)^3+(-2)^2-4(-2)+7$

$= -16+4+8+7 = 3 \neq 0$

$\Rightarrow (w+2)$ is not a factor of $f(w)$

8. Let $f(x) = x^n - a^n$

where n is +ve integer

$x-a = x-a \Rightarrow a = a$

$R = f(x) = f(a) = a^n - a^n = 0$

$\Rightarrow (x-a)$ is a factor of $f(x)$

9. Let $f(x) = x^n + a^n$

where n is odd integer

$x-a = x+a \Rightarrow a = -a$

$R = f(x) = f(-a) = (-a)^n + a^n$

$= -a^n + a^n = 0$

$\Rightarrow (x+a)$ is a factor of $f(x)$

10. Let $f(x) = x^4+2x^3+kx^2+3$

$K = ?$ $\quad R = 1$

$x-a = x-2 \Rightarrow a = 2$

$R = f(a) = f(2)$

$1 = (2)^4+2(2)^3+k(2)^2+3$

$1 = 16+16+4k+3$

$\Rightarrow 4K = -34 \Rightarrow K = -\frac{17}{2}$

11. $K = ?$ $\quad R = 14$

Sol. $f(x) = x^3+2x^2+Kx+4$

$x-a = x-2 \Rightarrow a=2$

$R = f(x) = f(a) = f(2)$

$14 = (2)^3+2(2)^2+2K+4$

$14 = 8+8+2K+4$

$\Rightarrow 2K = -6 \Rightarrow K = -3$

Use Synthetic division....

12. factorize the polynomial

Sol $f(x) = x^3+0x^2-7x+6$ $\quad x=2$

$$\begin{array}{r|rrrr}
2 & 1 & 0 & -7 & 6 \\
 & 0 & 2 & 4 & -6 \\
\hline
 & 1 & 2 & -3 & | \; 0
\end{array}$$

Quotient $= x^2+2x-3$ $\quad R=0$

$= x^2+3x - x-3$

$= x(x+3)-1(x+3)$

$= (x+3)(x-1)$

Hence $x^3-7x+6 = (x-2)(x+3)(x-1)$

13. Let $f(x) = x^3+0\cdot x^2-28x-48$ $\quad x=-4$

$$\begin{array}{r|rrrr}
-4 & 1 & 0 & -28 & -48 \\
 & 0 & -4 & 16 & 48 \\
\hline
 & 1 & -4 & -12 & | \; 0
\end{array}$$

Quotient $= x^2-4x-12$

$= x^2-6x+2x-12$

$= x(x-6)+2(x-6)$

$= (x-6)(x+2)$

Hence

$x^3-28x-48 = (x+4)(x-6)(x+2)$

14. Let $f(x) = 2x^4+7x^3-4x^2-27x-18$

$x = 2, \quad x = -3$

$$\begin{array}{r|rrrrr}
2 & 2 & 7 & -4 & -27 & -18 \\
 & 0 & 4 & 22 & 36 & 18 \\
\hline
-3 & 2 & 11 & 18 & 9 & | \; 0 \\
 & 0 & -6 & -15 & -9 & \\
\hline
 & 2 & 5 & 3 & | \; 0 &
\end{array}$$

Quotient $= 2x^2+5x+3$

$= 2x^2+2x+3x+3$

$= 2x(x+1)+3(x+1)$

$= (x+1)(2x+3)$

Hence $2x^4+7x^3-4x^2-27x-18$

$= (x-2)(x+3)(x+1)(2x+3)$

15. Use Synthetic division to find the values of p and q if $x+1$ and $x-2$ are the factors of $f(x) = x^3+px^2+qx+6$

Sol $x-a = x+1 \Rightarrow a = -1$

$x-a = x-2 \Rightarrow a = 2$

By Synthetic division

$$\begin{array}{r|rrrr}
-1 & 1 & p & q & 6 \\
 & 0 & -1 & -p+1 & -q+p-1 \\
\hline
2 & 1 & p-1 & q-p+1 & | \; -q+p+5 \\
 & 0 & 2 & 2p+2 & \\
\hline
 & 1 & p+1 & | \; q+p+3 &
\end{array}$$

Since $(x+1)$ and $(x-2)$ are the factors of $f(x)$

Then Remainder $= 0$

$$-q+p+5=0 \quad \text{---} (1)$$
$$q+p+3=0 \quad \text{---} (2)$$
$$2p+8=0 \Rightarrow p=-4$$

put $p=-4$ in (2)

$\Rightarrow q-4+3=0 \Rightarrow q=1$

$p=-4 \quad q=1$

16. Find the values of a and b if -2 and 2 are the roots of $f(x)=x^3-4x^2+ax+b$

Sol. $a=-2 \quad a=2$

-2	1	-4	a	b
	0	-2	12	$-2a-24$
2	1	-6	$a+12$	$b-2a-24$
	0	2	-8	
	1	-4	$a+4$	

Since $-2, 2$ are the roots of $f(x)$ then Remainder $=0$

$a+4=0 \Rightarrow a=-4$

$b-2a-24=0$

$\Rightarrow b-2(-4)-24=0$

$\Rightarrow b+8-24=0$

$\Rightarrow b-16=0 \Rightarrow b=16$

$a=-4 \quad b=16$

Relation Between Roots & Coefficients of Quadratic Eq

Quadratic eq $ax^2+bx+c=0$

formula $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Let α, β be the roots of the eq

$\alpha=\dfrac{-b+\sqrt{b^2-4ac}}{2a} \quad \beta=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$

$$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=-\frac{b}{a}=\text{Sum of roots}$$

$$\alpha\beta=\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{(-b)^2-(\sqrt{b^2-4ac})^2}{4a^2}$$

$$P=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

Product of roots $=\dfrac{c}{a}$

Formation of an Eq whose Roots are given:

If α, β be the roots of the quadratic equation

Then $x=\alpha \quad x=\beta$

$\Rightarrow x-\alpha=0 \quad x-\beta=0$

Equation becomes

$$(x-\alpha)(x-\beta)=0$$

$\Rightarrow x^2-\beta x-\alpha x+\alpha\beta=0$

$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$

$\Rightarrow x^2-Sx+P=0$