Exercise 10.2

Question # 1

Prove that

(i) $\sin(180^\circ+\theta)=-\sin\theta$ (ii) $\cos(180^\circ+\theta)=-\cos\theta$

(iii) $\tan\left(270^\circ-\theta\right)=\cot\theta$ (iv) $\cos(\theta-180^\circ)=-\cos\theta$

(v) $\cos(270^\circ+\theta)=\sin\theta$ (vi) $\sin(\theta+270^\circ)=-\cos\theta$

(vii) $\tan\left(180^\circ+\theta\right)=\tan\theta$ (viii) $\cos(360^\circ-\theta)=\cos\theta$

Solution

(i) L.H.S = $\sin(180+\theta)=\sin 180\cos\theta+\cos 180\sin\theta$

$$=\sin\left(0\right)\cos\theta+(-1)\sin\theta\ =0-\sin\theta\ =-\sin\theta\ =\text{R.H.S}$$

(ii) *Do youself*

(iii) L.H.S = $\tan\left(270^\circ-\theta\right)=\dfrac{\tan 270^\circ-\tan\theta}{1+\tan 270^\circ\tan\theta}$

$$=\frac{\tan 270^\circ\left(1-\dfrac{\tan\theta}{\tan 270^\circ}\right)}{\tan 270^\circ\left(\dfrac{1}{\tan 270^\circ}+\tan\theta\right)}=\frac{\left(1-\dfrac{\tan\theta}{\infty}\right)}{\left(\dfrac{1}{\infty}+\tan\theta\right)}$$

$$=\frac{(1-0)}{(0+\tan\theta)}=\frac{1}{\tan\theta}=\cot\theta\ =\text{R.H.S}$$

Remaining do yourself.

Question # 2

Find the values of the following:

(i) $\sin 15^\circ$ (ii) $\cos 15^\circ$ (iii) $\tan 15^\circ$

Solution

(i) Since $15=45-30$

So $\sin 15^\circ=\sin(45-30)=\sin 45^\circ\cos 30^\circ-\cos 45^\circ\sin 30^\circ$

$$=\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)\ =\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\ =\frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) Since $15=45-30$

So $\cos 15^\circ=\cos(45-30)=\cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$

$$=\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\ =\frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) Since $15=45-30$

So $\tan 15^\circ=\tan(45-30)=\dfrac{\tan 45^\circ-\tan 30^\circ}{1+\tan 45^\circ\tan 30^\circ}$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)} \quad =\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} \quad =\frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

For (iv), (v) and (vi), we have hint:

Hint: Use 105 = 60 + 45 in these questions

Question # 3

Prove that:

(i) $\sin(45^\circ+\alpha)=\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha)$ (ii) $\cos(45^\circ+\alpha)=\frac{1}{\sqrt{2}}(\cos\alpha-\sin\alpha)$

Solution

(i) L.H.S $=\sin(45+\alpha)$

$$=\sin 45^\circ\cos\alpha+\cos 45^\circ\sin\alpha=\left(\frac{1}{\sqrt{2}}\cos\alpha+\frac{1}{\sqrt{2}}\sin\alpha)\right)$$

$$=\frac{1}{\sqrt{2}}(\cos\alpha+\sin\alpha) \quad =\frac{1}{\sqrt{2}}(\sin\alpha+\cos\alpha) \quad = \text{R.H.S}$$

(ii) *Do yourself as above*

Question # 4

Prove that:

(i) $\tan(45+A)\tan(45-A)=1$ (ii) $\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)=0$

(iii) $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)=\cos\theta$ (iv) $\dfrac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}=\tan\frac{\theta}{2}$

(v) $\dfrac{1-\tan\theta\tan\varphi}{1+\tan\theta\tan\varphi}=\dfrac{\cos(\theta+\varphi)}{\cos(\theta-\varphi)}$

Solution

(i) L.H.S $=\tan(45+A)\tan(45-A)$

$$=\left(\frac{\tan 45^\circ+\tan A}{1-\tan 45^\circ\tan A}\right)\left(\frac{\tan 45^\circ-\tan A}{1+\tan 45^\circ\tan A}\right)$$

$$=\left(\frac{1+\tan A}{1-(1)\tan A}\right)\left(\frac{1-\tan A}{1+(1)\tan A}\right) \quad =\left(\frac{1+\tan A}{1-\tan A}\right)\left(\frac{1-\tan A}{1+\tan A}\right) \quad =1 = \text{R.H.S}$$

(ii) L.H.S $=\tan\left(\frac{\pi}{4}-\theta\right)+\tan\left(\frac{3\pi}{4}+\theta\right)$

$$=\left(\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\tan\theta}\right)+\left(\frac{\tan\frac{3\pi}{4}+\tan\theta}{1-\tan\frac{3\pi}{4}\tan\theta}\right)$$

$$=\left(\frac{1-\tan\theta}{1+(1)\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1-(-1)\tan\theta}\right)$$

$$=\left(\frac{1-\tan\theta}{1+\tan\theta}\right)+\left(\frac{-1+\tan\theta}{1+\tan\theta}\right)$$

$$=\frac{1-\tan\theta-1+\tan\theta}{1+\tan\theta}\quad=\frac{0}{1+\tan\theta}\quad=0=\text{R.H.S}$$

(iii) $\text{L.H.S} = \sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{3}\right)$

$$=\left(\sin\theta\cos\frac{\pi}{6}+\cos\theta\sin\frac{\pi}{6}\right)+\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$$

$$=\left(\sin\theta\,\frac{\sqrt{3}}{2}+\cos\theta\,\frac{1}{2}\right)+\left(\cos\theta\,\frac{1}{2}-\sin\theta\,\frac{\sqrt{3}}{2}\right)$$

$$=\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta+\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\quad=\cos\theta\;=\;R.H.S.$$

(iv) $\text{L.H.S} = \dfrac{\sin\theta-\cos\theta\tan\frac{\theta}{2}}{\cos\theta+\sin\theta\tan\frac{\theta}{2}}$

$$=\frac{\sin\theta-\cos\theta\,\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\cos\theta+\sin\theta\,\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}\quad=\frac{\dfrac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}{\dfrac{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}}$$

$$=\frac{\sin\theta\cos\frac{\theta}{2}-\cos\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}+\sin\theta\sin\frac{\theta}{2}}=\frac{\sin\left(\theta-\frac{\theta}{2}\right)}{\cos\left(\theta-\frac{\theta}{2}\right)}=\frac{\sin\left(\theta/2\right)}{\cos\left(\theta/2\right)}=\tan\frac{\theta}{2}=\text{R.H.S}$$

(v) $\text{L.H.S} = \dfrac{1-\tan\theta\tan\varphi}{1+\tan\theta\tan\varphi}$

$$=\frac{1-\dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\varphi}{\cos\varphi}}{1+\dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\varphi}{\cos\varphi}}=\frac{\dfrac{\cos\theta\cos\varphi-\sin\theta\sin\varphi}{\cos\theta\cos\varphi}}{\dfrac{\cos\theta\cos\varphi+\sin\theta\sin\varphi}{\cos\theta\cos\varphi}}$$

$$=\frac{\cos\theta\cos\varphi-\sin\theta\sin\varphi}{\cos\theta\cos\varphi+\sin\theta\sin\varphi}\quad=\frac{\cos(\theta+\varphi)}{\cos(\theta-\varphi)}=\text{R.H.S}$$

Question # 5

Show that: $\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\alpha-\sin^2\beta=\cos^2\beta-\sin^2\alpha$

Solution

$$\begin{aligned}
\text{L.H.S} &= \cos(\alpha+\beta)\cos(\alpha-\beta)\\
&=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)(\cos\alpha\cos\beta+\sin\alpha\sin\beta)\\
&=\left((\cos\alpha\cos\beta)^2-(\sin\alpha\sin\beta)^2\right)=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta\\
&=\cos^2\alpha\left(1-\sin^2\beta\right)-\left(1-\cos^2\alpha\right)\sin^2\beta\\
&=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta\\
&=\cos^2\alpha-\sin^2\beta \quad\ldots\ldots\ldots\ldots\ldots\ (i)\\
&=(1-\sin^2\alpha)-(1-\cos^2\beta)\\
&=1-\sin^2\alpha-1+\cos^2\beta\\
&=\cos^2\beta-\sin^2\alpha \quad\ldots\ldots\ldots\ldots\ldots\ (ii)
\end{aligned}$$

Question # 6 *Do yourself as above*
Hint: Just open the formulas

Question # 7

Show that

(i) $\cot(\alpha+\beta)=\dfrac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}$ (ii) $\cot(\alpha-\beta)=\dfrac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

(iii) $\dfrac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\dfrac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}$

Solution

(i)
$$\text{L.H.S} = \cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}=\frac{1}{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

$$=\frac{1-\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}=\frac{\tan\alpha\tan\beta\left(\dfrac{1}{\tan\alpha\tan\beta}-1\right)}{\tan\alpha\tan\beta\left(\dfrac{1}{\tan\beta}+\dfrac{1}{\tan\alpha}\right)}$$

$$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}=\frac{\cot\alpha\cot\beta-1}{\cot\alpha+\cot\beta}=\text{R.H.S}$$

(ii)

Do yourself as above

(iii)

$$\text{L.H.S} = \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}=\frac{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha}-\dfrac{\sin\beta}{\cos\beta}}=\frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}=\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\text{R.H.S}$$

Question # 8

If $\sin\alpha=\frac{4}{5}$ and $\cos\alpha=\frac{40}{41}$ where $0<\alpha<\frac{\pi}{2}$ and $0<\beta<\frac{\pi}{2}$.

Show that $\sin(\alpha-\beta)=\frac{133}{205}$

Solution Since $\sin\alpha=\frac{4}{5}$; $0<\alpha<\frac{\pi}{2}$

$\cos\alpha=\frac{40}{41}$; $0<\beta<\frac{\pi}{2}$

Now

$$\cos^2\alpha=1-\sin^2\alpha$$

$$\Rightarrow\cos\alpha=\pm\sqrt{1-\sin^2\alpha}$$

Since terminal ray of α is in the first quadrant so value of cos is +ive

$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$

$$\Rightarrow\cos\alpha=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}\quad\Rightarrow\boxed{\cos\alpha=\frac{3}{5}}$$

Also

$$\sin^2\beta=1-\cos^2\beta\qquad\Rightarrow\sin\beta=\pm\sqrt{1-\cos^2\beta}$$

Since terminal ray of β is in the first quadrant so value of sin is +ive

$$\sin\beta=\sqrt{1-\cos^2\beta}$$

$$\Rightarrow\sin\beta=\sqrt{1-\left(\frac{40}{41}\right)^2}=\sqrt{1-\frac{1600}{1681}}=\sqrt{\frac{81}{1681}}\qquad\Rightarrow\boxed{\sin\beta=\frac{9}{41}}$$

Now

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$=\left(\frac{4}{5}\right)\left(\frac{40}{41}\right)-\left(\frac{3}{5}\right)\left(\frac{9}{41}\right)=\frac{160}{205}-\frac{27}{205}=\frac{133}{205}$$

i.e. $\sin(\alpha-\beta)=\frac{133}{205}$ *Proved*

Question # 9

If $\sin\alpha=\frac{4}{5}$ and $\sin\beta=\frac{12}{13}$ where $\frac{\pi}{2}<\alpha<\pi$ and $\frac{\pi}{2}<\beta<\pi$. Find

(i) $\sin(\alpha+\beta)$ (ii) $\cos(\alpha+\beta)$ (iii) $\tan(\alpha+\beta)$

(iv) $\sin(\alpha-\beta)$ (v) $\cos(\alpha-\beta)$ (vi) $\tan(\alpha-\beta)$

In which quadrant do the terminal sides of the angles of measures $(\alpha+\beta)$ and $(\alpha-\beta)$ lie?

Solution

Since $\sin\alpha = \dfrac{4}{5}$; $\dfrac{\pi}{2} < \alpha < \pi$

$\sin\beta = \dfrac{12}{13}$; $\dfrac{\pi}{2} < \beta < \pi$

Since $\cos^2\alpha = 1 - \sin^2\alpha \Rightarrow \cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

As terminal ray of α lies in the IInd quadrant so value of cos is –ive

$$\cos\alpha = -\sqrt{1-\sin^2\alpha}$$

$$\Rightarrow \cos\alpha = -\sqrt{1-\left(\frac{4}{5}\right)^2} = -\sqrt{1-\frac{16}{25}} = -\sqrt{\frac{9}{25}} \quad \Rightarrow \boxed{\cos\alpha = -\frac{3}{5}}$$

Now

$$\cos^2\beta = 1 - \sin^2\beta$$

$$\Rightarrow \cos\beta = \pm\sqrt{1-\sin^2\beta}$$

As terminal ray of β lies in the IInd quadrant so value of cos is –ive

$$\cos\beta = -\sqrt{1-\sin^2\beta}$$

$$\Rightarrow \cos\beta = -\sqrt{1-\left(\frac{12}{13}\right)^2} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{25}{169}} \quad \Rightarrow \boxed{\cos\beta = -\frac{5}{13}}$$

(i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

(ii) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

(iii) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{-56/65}{-33/65} = \dfrac{56}{33}$

(iv), (v) & (vi) *Do yourself as above*

Since $\sin(\alpha+\beta)$ is –ive so terminal are of $\alpha+\beta$ is in IIIrd or IVth quadrant and $\cos(\alpha+\beta)$ is –ive so terminal are of $\alpha+\beta$ is in IInd or IIIrd quadrant therefore terminal ray of $\alpha+\beta$ lies in the IIIrd quadrant.

Similarly after solving (*iv*), (*v*) & (*vi*) find quadrant for $\alpha-\beta$ yourself.

Question # 10

Find $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, given that

(i) $\tan\alpha=\frac{3}{4}$, $\sin\beta=\frac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the I quadrant.

(ii) $\tan\alpha=-\frac{15}{8}$, $\sin\beta=-\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

Solution

(i) Since $\tan\alpha=\frac{3}{4}$

As $\tan\alpha$ is +ive and terminal arm of α in not in the Ist quad. Therefor it lies in IIIrd quad.

Now

$$\sec^2\alpha=1+\tan^2\alpha$$

$$\Rightarrow \sec\alpha=\pm\sqrt{1+\tan^2\alpha}$$

Since terminal arm of α is in the IIIrd quad., therefor value of sec is –ive

$$\sec\alpha=-\sqrt{1+\tan^2\alpha}$$

$$\Rightarrow \sec\alpha=-\sqrt{1+\left(\frac{3}{4}\right)^2}=-\sqrt{1+\frac{9}{16}}=-\sqrt{\frac{25}{16}} \qquad \Rightarrow \sec x=-\frac{5}{4}$$

Now $\cos\alpha=\frac{1}{\sec\alpha}=\frac{1}{-5/4} \qquad \Rightarrow \boxed{\cos\alpha=-\frac{4}{5}}$

Now $\frac{\sin\alpha}{\cos\alpha}=\tan\alpha \qquad \Rightarrow \sin\alpha=\tan\alpha\cos\alpha$

$$\Rightarrow \sin\alpha=\left(\frac{3}{4}\right)\left(-\frac{4}{5}\right) \qquad \Rightarrow \boxed{\sin\alpha=-\frac{3}{5}}$$

Since $\cos\beta=\frac{5}{13}$

As $\cos\beta$ is +ive and terminal arm of β is not in the Ist quad., therefore it lies in IVth quad.

Now $\sin^2\beta=1-\cos^2\beta$

$$\Rightarrow \sin\beta=\pm\sqrt{1-\cos^2\beta}$$

Since terminal ray of β is in the fourth quadrant so value of sin is –ive

$$\sin\beta=-\sqrt{1-\cos^2\beta}$$

$$\Rightarrow \sin\beta=-\sqrt{1-\left(\frac{5}{13}\right)^2}=-\sqrt{1-\frac{25}{169}}=-\sqrt{\frac{144}{169}} \qquad \Rightarrow \boxed{\sin\beta=-\frac{12}{13}}$$

Now $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(-\frac{3}{5}\right)\left(\frac{5}{13}\right)+\left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right)=-\frac{3}{13}+\frac{48}{65} \qquad \Rightarrow \boxed{\sin(\alpha+\beta)=\frac{33}{65}}$$

And $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(-\frac{4}{5}\right)\left(\frac{5}{13}\right)-\left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right)=-\frac{4}{13}-\frac{36}{65}\quad\Rightarrow\quad\boxed{\cos(\alpha+\beta)=-\frac{56}{65}}$$

(ii)

Do yourself as above

Question # 11

Prove that: $\dfrac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ}=\tan 37^\circ$

Solution

R.H.S $=\tan 37^\circ=\tan(45-8)$ $\qquad\because\ 37=45-8$

$$=\frac{\tan 45^\circ-\tan 8^\circ}{1+\tan 45^\circ\tan 8^\circ}=\frac{1-\tan 8^\circ}{1+(1)\tan 8^\circ}$$

$$=\frac{1-\frac{\sin 8^\circ}{\cos 8^\circ}}{1+\frac{\sin 8^\circ}{\cos 8^\circ}}=\frac{\frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ+\sin 8^\circ}{\cos 8^\circ}}=\frac{\cos 8^\circ-\sin 8^\circ}{\cos 8^\circ+\sin 8^\circ}=\text{L.H.S}$$

Question # 12

If α,β,γ are the angles of a tringle ABC, show that

$$\cot\frac{\beta}{2}+\cot\frac{\alpha}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Solution

Since α,β and γ are angles of triangle therefore

$$\alpha+\beta+\gamma=180\quad\Rightarrow\alpha+\beta=180-\gamma$$

$$\Rightarrow\frac{\alpha+\beta}{2}=\frac{180-\gamma}{2}\quad\Rightarrow\frac{\alpha}{2}+\frac{\beta}{2}=90-\frac{\gamma}{2}$$

Now $\quad\tan\left(\dfrac{\alpha}{2}+\dfrac{\beta}{2}\right)=\tan\left(90-\dfrac{\gamma}{2}\right)$

$$\Rightarrow\frac{\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\cot\frac{\gamma}{2}\qquad\because\ \tan\left(90-\frac{\gamma}{2}\right)=\cot\frac{\gamma}{2}$$

$$\Rightarrow\frac{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\left(\frac{1}{\tan\frac{\beta}{2}}+\frac{1}{\tan\frac{\alpha}{2}}\right)}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\left(\frac{1}{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}-1\right)}=\cot\frac{\gamma}{2}\qquad\Rightarrow\frac{\cot\frac{\beta}{2}+\cot\frac{\alpha}{2}}{\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1}=\cot\frac{\gamma}{2}$$

$$\Rightarrow\cot\frac{\beta}{2}+\cot\frac{\alpha}{2}=\cot\frac{\gamma}{2}\left(\cot\frac{\alpha}{2}\cot\frac{\beta}{2}-1\right)$$

$$\Rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}-\cot\frac{\gamma}{2}$$
$$\Rightarrow \cot\frac{\beta}{2}+\cot\frac{\alpha}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$$

Question # 13

If $\alpha+\beta+\gamma=180^\circ$, show that
$\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$

Solution

Since α,β and γ are angles of triangle therefore

$$\alpha+\beta+\gamma=180 \qquad \Rightarrow \alpha+\beta=180-\gamma$$

Now $\tan(\alpha+\beta)=\tan(180-\gamma)$

$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan\left(2(90)-\gamma\right)$$
$$\Rightarrow \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=-\tan\gamma$$
$$\Rightarrow \tan\alpha+\tan\beta=-\tan\gamma(1-\tan\alpha\tan\beta)$$
$$\Rightarrow \tan\alpha+\tan\beta=-\tan\gamma+\tan\alpha\tan\beta\tan\gamma$$
$$\Rightarrow \tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$$

Dividing through out by $\tan\alpha\tan\beta\tan\gamma$

$$\frac{\tan\alpha}{\tan\alpha\tan\beta\tan\gamma}+\frac{\tan\beta}{\tan\alpha\tan\beta\tan\gamma}+\frac{\tan\gamma}{\tan\alpha\tan\beta\tan\gamma}=\frac{\tan\alpha\tan\beta\tan\gamma}{\tan\alpha\tan\beta\tan\gamma}$$
$$\Rightarrow \cot\beta\cot\gamma+\cot\alpha\cot\gamma+\cot\alpha\cot\beta=1$$
$$\Rightarrow \cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1$$

Question # 14

Express the following in the form $r\sin(\theta+\phi)$ or $r\sin(\theta-\phi)$, where terminal sides of the angles of measure θ and ϕ are in the first quadrant:

(i) $12\sin\theta+5\cos\theta$ (ii) $3\sin\theta-4\cos\theta$ (iii) $\sin\theta-\cos\theta$
(iv) $5\sin\theta-4\cos\theta$ (v) $\sin\theta+\cos\theta$

Solution

(i) $12\sin\theta+5\cos\theta$

Let $12=r\cos\varphi$ and $5=r\sin\varphi$

Squaring and adding

$$(12)^2+(5)^2=r^2\cos^2\varphi+r^2\sin^2\varphi$$
$$\Rightarrow 144+25=r^2\left(\cos^2\varphi+\sin^2\varphi\right)$$
$$\Rightarrow 169=r^2(1)$$
$$\Rightarrow r=\sqrt{169}=13$$

$$\frac{5}{12}=\frac{r\sin\varphi}{r\cos\varphi}$$
$$\frac{5}{12}=\tan\varphi$$
$$\Rightarrow \varphi=\tan^{-1}\frac{5}{12}$$

Now

$$12\sin\theta+5\cos\theta=r\cos\varphi\sin\theta+r\sin\varphi\cos\theta$$
$$=r\left(\cos\varphi\sin\theta+\sin\varphi\cos\theta\right)$$

$$= r\sin(\theta+\varphi) \qquad \text{where } r=13 \text{ and } \varphi=\tan^{-1}\frac{5}{12}$$

(ii) $3\sin\theta - 4\cos\theta$

Let $3 = r\cos\varphi$ and $-4 = r\sin\varphi$

Squaring and adding

$$(3)^2 + (-4)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$

$$\Rightarrow 9+16 = r^2\left(\cos^2\varphi+\sin^2\varphi\right)$$

$$\Rightarrow 25 = r^2(1)$$

$$\Rightarrow r=\sqrt{25}=5$$

$$\frac{-4}{3}=\frac{r\sin\varphi}{r\cos\varphi}$$

$$-\frac{4}{3}=\tan\varphi$$

$$\Rightarrow \varphi=\tan^{-1}\left(-\frac{4}{3}\right)$$

$$3\sin\theta-4\cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= r(\cos\varphi\sin\theta+\sin\varphi\cos\theta)$$
$$= r\sin(\theta+\varphi) \qquad \text{where } r=5 \text{ and } \varphi=\tan^{-1}\left(-\frac{4}{3}\right)$$

(iii) $\sin\theta-\cos\theta$

Let $1 = r\cos\varphi$ and $-1 = r\sin\varphi$

Squaring and adding

$$(1)^2 + (-1)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$

$$\Rightarrow 1+1 = r^2\left(\cos^2\varphi+\sin^2\varphi\right)$$

$$\Rightarrow 2 = r^2(1)$$

$$\Rightarrow r=\sqrt{2}$$

$$\frac{-1}{1}=\frac{r\sin\varphi}{r\cos\varphi}$$

$$-1=\tan\varphi$$

$$\Rightarrow \varphi=\tan^{-1}(-1)$$

Now

$$\sin\theta-\cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= r(\cos\varphi\sin\theta+\sin\varphi\cos\theta)$$
$$= r\sin(\theta+\varphi) \qquad \text{where } r=\sqrt{2} \text{ and } \varphi=\tan^{-1}(-1)$$

(iv) $5\sin\theta-4\cos\theta$

Let $5 = r\cos\varphi$ and $-4 = r\sin\varphi$

Squaring and adding

$$(5)^2 + (-4)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$

$$\Rightarrow 25+16 = r^2\left(\cos^2\varphi+\sin^2\varphi\right)$$

$$\Rightarrow 41 = r^2(1)$$

$$\Rightarrow r=\sqrt{41}$$

$$\frac{-4}{5}=\frac{r\sin\varphi}{r\cos\varphi}$$

$$-\frac{4}{5}=\tan\varphi$$

$$\Rightarrow \varphi=\tan^{-1}\left(-\frac{4}{5}\right)$$

Now

$$5\sin\theta-4\cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$$
$$= r(\cos\varphi\sin\theta+\sin\varphi\cos\theta)$$

$$= r\sin(\theta + \varphi) \qquad \text{where } r = \sqrt{41} \text{ and } \varphi = \tan^{-1}\left(-\frac{4}{5}\right)$$

(v) $\sin\theta + \cos\theta$

Let $1 = r\cos\varphi$ and $1 = r\sin\varphi$

Squaring and adding

$$(1)^2 + (1)^2 = r^2\cos^2\varphi + r^2\sin^2\varphi$$

$$\Rightarrow 1 + 1 = r^2\left(\cos^2\varphi + \sin^2\varphi\right)$$

$$\Rightarrow 2 = r^2(1)$$

$$\Rightarrow r = \sqrt{2}$$

$$\frac{1}{1} = \frac{r\sin\varphi}{r\cos\varphi}$$

$$1 = \tan\varphi$$

$$\Rightarrow \varphi = \tan^{-1}(1)$$

Now $\sin\theta + \cos\theta = r\cos\varphi\sin\theta + r\sin\varphi\cos\theta$

$$= r\left(\cos\varphi\sin\theta + \sin\varphi\cos\theta\right)$$

$$= r\sin(\theta + \varphi) \qquad \text{where } r = \sqrt{2} \text{ and } \varphi = \tan^{-1}(1)$$

(vi) *Do yourself as above*