Exercise 10.4

Question # 1

Express the following products as sums or differences:

(i) $2\sin 3\theta \cos\theta$ (ii) $2\cos 5\theta \cos 3\theta$ (iii) $\sin 5\theta \cos 2\theta$

(iv) $2\sin 7\theta \sin 2\theta$ (v) $\cos(x+y)\sin(x-y)$ (vi) $\cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)$

(vii) $\sin 12^\circ \sin 46^\circ$ (viii) $\sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)$

Solution

(i) Since $2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$

Put $\alpha = 3\theta$ and $\beta = \theta$

$$2\sin 3\theta\cos\theta = \sin(3\theta+\theta)+\sin(3\theta-\theta)$$
$$= \sin 4\theta + \sin 2\theta$$

(ii) Since $2\cos\alpha\cos\beta = \cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)$

Put $\alpha = 5\theta$ and $\beta = 3\theta$

$$2\cos 5\theta\cos 3\theta = \cos(5\theta+3\theta)-\cos(5\theta-3\theta)$$
$$= \cos 8\theta - \cos 2\theta$$

(iii) Since $2\sin\alpha\cos\beta = \sin(\alpha+\beta)+\sin(\alpha-\beta)$

Put $\alpha = 5\theta$ and $\beta = 2\theta$

$$2\sin 5\theta\cos 2\theta = \sin(5\theta+2\theta)+\sin(5\theta-2\theta)$$
$$= \sin 7\theta + \sin 3\theta$$
$$\Rightarrow \sin 5\theta\cos 2\theta = \frac{1}{2}\left(\sin 7\theta + \sin 3\theta\right)$$

(iv) Since $-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha = 7\theta$ and $\beta = 2\theta$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta)-\cos(\alpha-\beta)$$
$$-2\sin 7\theta\sin 2\theta = \cos(7\theta+2\theta)-\cos(7\theta-2\theta)$$
$$= \cos 9\theta - \cos 5\theta$$
$$2\sin 7\theta\sin 2\theta = \cos 5\theta - \cos 9\theta$$

(v) Since $2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta)$

Put $\alpha = x+y$, $\beta = x-y$

$$2\cos(x+y)\sin(x-y) = \sin(x+y+x-y)-\sin(x+y-x+y)$$
$$= \sin 2x - \sin 2y$$
$$\Rightarrow \cos(x+y)\sin(x-y) = \frac{1}{2}\left(\sin 2x - \sin 2y\right)$$

(vi) Since $2\cos\alpha\cos\beta = \cos\left(\alpha+\beta\right)+\cos\left(\alpha-\beta\right)$

Put $\alpha = 2x+30^\circ$ and $\beta = 2x-30^\circ$

$$2\cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)=\cos\left(2x+30^\circ+2x-30^\circ\right)+\cos\left(2x+30^\circ-2x+30^\circ\right)$$
$$=\cos\left(4x\right)+\cos\left(60^\circ\right)$$
$$\Rightarrow\ \cos\left(2x+30^\circ\right)\cos\left(2x-30^\circ\right)=\frac{1}{2}\left(\cos 4x+\cos 60^\circ\right)$$

(vii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha=12^\circ$ and $\beta=46^\circ$

$$-2\sin 12^\circ\sin 46^\circ=\cos(12+46)-\cos(12-46)$$
$$=\cos 58-\cos(-34)$$
$$=\cos 58-\cos 34 \qquad \because\ \cos(-\theta)=\cos\theta$$
$$\Rightarrow\ \sin 12^\circ\sin 46^\circ=-\frac{1}{2}(\cos 58-\cos 34)$$
$$=\frac{1}{2}(\cos 34-\cos 58)$$

(viii) Since $-2\sin\alpha\sin\beta=\cos(\alpha+\beta)-\cos(\alpha-\beta)$

Put $\alpha=x+45^\circ$ and $\beta=x-45^\circ$

$$-2\sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)=\cos\left\{(x+45^\circ)+(x-45^\circ)\right\}-\cos\left\{(x+45^\circ)-(x-45^\circ)\right\}$$
$$=\cos 2x-\cos 90^\circ$$
$$\Rightarrow\ \sin\left(x+45^\circ\right)\sin\left(x-45^\circ\right)=\cos 90^\circ-\frac{1}{2}\cos 2x$$

Question # 2

Express the following sum or difference as product:

(i) $\sin 5\theta+\sin 3\theta$ (ii) $\sin 8\theta-\sin 4\theta$ (iii) $\cos 6\theta+\cos 3\theta$

(iv) $\cos 7\theta-\cos\theta$ (v) $\cos 12^\circ+\cos 48^\circ$ (vi) $\sin(x+30^\circ)+\sin(x-30^\circ)$

Solution

(i) Since $\sin\alpha+\sin\beta=2\sin\left(\dfrac{\alpha+\beta}{2}\right)\cos\left(\dfrac{\alpha-\beta}{2}\right)$

Put $\alpha=5\theta$, $\beta=3\theta$

$$\sin 5\theta+\sin 3\theta=2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)$$
$$=2\sin\left(\frac{8\theta}{2}\right)\cos\left(\frac{2\theta}{2}\right)=2\sin 4\theta\ \cos\theta$$

(ii) Since $\sin\alpha-\sin\beta=2\cos\left(\dfrac{\alpha+\beta}{2}\right)\sin\left(\dfrac{\alpha-\beta}{2}\right)$

Put $\alpha=8\theta$, $\beta=4\theta$

$$\sin 8\theta-\sin 4\theta=2\cos\left(\frac{8\theta+4\theta}{2}\right)\sin\left(\frac{8\theta-4\theta}{2}\right)$$
$$=2\cos 6\theta\ \sin 2\theta$$

(iii) *Do yourself*

(iv) Since $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha = 7\theta$, $\beta = \theta$

$$\cos 7\theta - \cos\theta = -2\sin\left(\frac{7\theta+\theta}{2}\right)\sin\left(\frac{7\theta-\theta}{2}\right) = -2\sin\left(\frac{8\theta}{2}\right)\sin\left(\frac{6\theta}{2}\right)$$
$$= -2\sin 4\theta \sin 3\theta$$

(v) Since $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha = 12$, $\beta = 48$

$$\cos 12 + \cos 48 = 2\cos\left(\frac{12+48}{2}\right)\cos\left(\frac{12-48}{2}\right)$$
$$= 2\cos\left(\frac{60}{2}\right)\cos\left(\frac{-36}{2}\right) = 2\cos 30\cos(-18)$$
$$= 2\cos 30^\circ \cos 18^\circ \qquad \because\ \cos(-\theta) = \cos\theta$$

(vi) Since $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

Put $\alpha = x+30$, $\beta = x-30$

$$\sin(x+30) + \sin(x-30) = 2\sin\left(\frac{x+30+x-30}{2}\right)\cos\left(\frac{x+30-x+30}{2}\right)$$
$$= 2\sin\left(\frac{2x}{2}\right)\cos\left(\frac{60}{2}\right) = 2\sin x\cos 30$$

Question # 3

Prove the following identities:

(i) $\dfrac{\sin 3x - \sin x}{\cos x - \cos 3x} = \cot 2x$ (ii) $\dfrac{\sin 8x + \sin 2x}{\cos 8x + \cos 2x} = \tan 5x$

(iii) $\dfrac{\sin\alpha - \sin\beta}{\sin\alpha + \sin\beta} = \cot\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right)$

Solution

(i) L.H.S $= \dfrac{\sin 3x - \sin x}{\cos x - \cos 3x}$

$$= \frac{2\cos\left(\frac{3x+x}{2}\right)\sin\left(\frac{3x-x}{2}\right)}{-2\sin\left(\frac{x+3x}{2}\right)\sin\left(\frac{x-3x}{2}\right)} = \frac{\cos\left(\frac{4x}{2}\right)\sin\left(\frac{2x}{2}\right)}{-\sin\left(\frac{4x}{2}\right)\sin\left(\frac{-2x}{2}\right)}$$

$$=\frac{\cos(2x)\sin(x)}{+\sin(2x)\sin(x)} \quad =\cot 2x \quad = \text{R.H.S}$$

(ii) *Do yourself*

(iii) L.H.S $=\dfrac{\sin\alpha-\sin\beta}{\sin\alpha+\sin\beta}$

$$=\frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} =\cot\left(\frac{\alpha+\beta}{2}\right)\tan\left(\frac{\alpha-\beta}{2}\right)= \text{ R.H.S}$$

Question # 4

Prove that:

(i) $\cos 20^\circ+\cos 100^\circ+\cos 140^\circ=0$

(ii) $\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)=\frac{1}{2}\cos 2\theta$

(iii) $\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}=\tan 4\theta$

Solution

(i) L.H.S $=\cos 20^\circ+\cos 100^\circ+\cos 140^\circ$

$$=\left(\cos 100^\circ+\cos 20^\circ\right)+\cos 140^\circ$$

$$=2\cos\left(\frac{100+20}{2}\right)\cos\left(\frac{100-20}{2}\right)+\cos 140^\circ$$

$$=2\cos 60^\circ\cos 40^\circ+\cos 140^\circ \quad =2\left(\frac{1}{2}\right)\cos 40^\circ+\cos 140^\circ$$

$$=\cos 140^\circ+\cos 40^\circ \quad =2\cos\left(\frac{140+40}{2}\right)\cos\left(\frac{140-40}{2}\right)$$

$$=2\cos 90^\circ\cos 50^\circ \ =2(0)\cos 50^\circ \ =0 = \text{R.H.S}$$

(ii) L.H.S $=\sin\left(\frac{\pi}{4}-\theta\right)\sin\left(\frac{\pi}{4}+\theta\right)$

$$=\left(\sin\frac{\pi}{4}\cos\theta-\cos\frac{\pi}{4}\sin\theta\right)\left(\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right)$$

$$=\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$

$$=\left(\frac{1}{\sqrt{2}}\cos\theta\right)^2-\left(\frac{1}{\sqrt{2}}\sin\theta\right)^2 \quad =\frac{1}{2}\cos^2\theta-\frac{1}{2}\sin^2\theta$$

$$=\frac{1}{2}\left(\cos^2\theta-\sin^2\theta\right) \ =\frac{1}{2}\cos 2\theta \quad = \text{R.H.S}$$

(iii) L.H.S $=\dfrac{\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta}{\cos\theta+\cos 3\theta+\cos 5\theta+\cos 7\theta}$

$$=\frac{(\sin 7\theta+\sin\theta)+(\sin 5\theta+\sin 3\theta)}{(\cos 7\theta+\cos\theta)+(\cos 5\theta+\cos 3\theta)}$$

$$=\frac{2\sin\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\sin\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}{2\cos\left(\frac{7\theta+\theta}{2}\right)\cos\left(\frac{7\theta-\theta}{2}\right)+2\cos\left(\frac{5\theta+3\theta}{2}\right)\cos\left(\frac{5\theta-3\theta}{2}\right)}$$

$$=\frac{2\sin 4\theta\cos 3\theta+2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta+2\cos 4\theta\cos\theta}$$

$$=\frac{2\sin 4\theta(\cos 3\theta+\cos\theta)}{2\cos 4\theta(\cos 3\theta+\cos\theta)}\quad=\frac{\sin 4\theta}{\cos 4\theta}\quad=\tan 4\theta\quad=\text{R.H.S}$$

Question # 5

Prove that:

(i) $\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ=\frac{1}{16}$ (ii) $\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}=\frac{3}{16}$

Solution

(i) L.H.S $=\cos 20^\circ\cos 40^\circ\cos 60^\circ\cos 80^\circ$

$$=\cos 20^\circ\cos 40^\circ\left(\frac{1}{2}\right)\cos 80^\circ\quad=\frac{1}{2}\cos 80^\circ\cos 40^\circ\cos 20^\circ$$

$$=\frac{1}{4}\left(2\cos 80^\circ\cos 40^\circ\right)\cos 20^\circ\quad=\frac{1}{4}\left(\cos(80+40)+\cos(80-40)\right)\cos 20^\circ$$

$$=\frac{1}{4}\left(\cos 120^\circ+\cos 40^\circ\right)\cos 20^\circ\quad=\frac{1}{4}\left(-\frac{1}{2}+\cos 40^\circ\right)\cos 20^\circ$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{4}\cos 40^\circ\cos 20^\circ\quad=-\frac{1}{8}+\frac{1}{8}\left(2\cos 40^\circ\cos 20^\circ\right)$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}(\cos(40+20)+\cos(40-20))$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}(\cos 60+\cos 20)\quad=-\frac{1}{8}\cos 20^\circ+\frac{1}{8}\left(\frac{1}{2}+\cos 20\right)$$

$$=-\frac{1}{8}\cos 20^\circ+\frac{1}{16}+\frac{1}{8}\cos 20^\circ=\frac{1}{16}\quad=\text{R.H.S}$$

(ii) L.H.S $=\sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{\pi}{3}\sin\frac{4\pi}{9}$

$$=\sin\frac{180^\circ}{9}\sin\frac{2(180^\circ)}{9}\sin\frac{(180^\circ)}{3}\sin\frac{4(180^\circ)}{9}\qquad\because\ \pi=180^\circ$$

$$=\sin 20^\circ\sin 40^\circ\sin 60^\circ\sin 80^\circ\quad=\sin 20^\circ\sin 40^\circ\frac{\sqrt{3}}{2}\sin 80^\circ$$

$$=\frac{\sqrt{3}}{2}\sin 80^\circ\sin 40^\circ\sin 20^\circ\quad=-\frac{\sqrt{3}}{4}\left(-2\sin 80^\circ\sin 40^\circ\right)\sin 20^\circ$$

$$=-\frac{\sqrt{3}}{4}\left(\cos(80+40)-\cos(80-40)\right)\sin 20^\circ$$

$$=-\frac{\sqrt{3}}{4}\left(\cos 120^\circ-\cos 40^\circ\right)\sin 20^\circ \quad =-\frac{\sqrt{3}}{4}\left(-\frac{1}{2}-\cos 40^\circ\right)\sin 20^\circ$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{4}\cos 40^\circ\sin 20^\circ \;=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\left(2\cos 40^\circ\sin 20^\circ\right)$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\left(\sin(40+20)-\sin(40-20)\right)$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\left(\sin 60^\circ-\sin 20^\circ\right) \;=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{\sqrt{3}}{8}\left(\frac{\sqrt{3}}{2}-\sin 20^\circ\right)$$

$$=\frac{\sqrt{3}}{8}\sin 20^\circ+\frac{3}{16}-\frac{\sqrt{3}}{8}\sin 20^\circ \;=\frac{3}{16} \quad = \text{R.H.S}$$

(iii) *Do yourself as above*