EXERCISE 4.4

Question # 1

(i) Let x be a cube root of 8 then

> Review:
> $\omega = \dfrac{-1+\sqrt{-3}}{2}$
> $\omega^2 = \dfrac{-1-\sqrt{-3}}{2}$

$$x = (8)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = 8$$

$$\Rightarrow \quad x^3 - 8 = 0 \qquad \Rightarrow \quad (x)^3 - (2)^3 = 0$$

$$\Rightarrow \quad (x-2)(x^2 + 2x + 4) = 0$$

$$\Rightarrow \quad x - 2 = 0 \quad \text{or} \quad x^2 + 2x + 4 = 0$$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2} = 2\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad x = 2\left(\frac{-1+\sqrt{-3}}{2}\right) \quad \text{or} \quad x = 2\left(\frac{-1-\sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad x = 2\omega \qquad \text{or} \qquad x = 2\omega^2$$

Hence cube root of 8 are $2, 2\omega$ and $2\omega^2$.

(ii) Hint

Considering x as a cube root of -8 and Solving as above you will get the following values of x

$$x = -2 \ , \quad x = \frac{2+2\sqrt{-3}}{2} \quad , \quad x = \frac{2-2\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = -2\left(\frac{-1-\sqrt{-3}}{2}\right) \quad , \quad x = -2\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$\Rightarrow \quad x = -2\omega^2 \quad , \quad x = -2\omega$$

Hence cube root of -8 are $-2, -2\omega$ and $-2\omega^2$.

(iii) *Do yourself as (iv) below.*

(iv) Let x be a cube root of -27 then

$$x = (-27)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = -27$$

$$\Rightarrow \quad x^3 + 27 = 0 \qquad \Rightarrow \quad (x)^3 + (3)^3 = 0$$

$$\Rightarrow \quad (x+3)(x^2 - 3x + 9) = 0$$

$$\Rightarrow \quad x + 3 = 0 \quad \text{or} \quad x^2 - 3x + 9 = 0$$

$$\Rightarrow \quad x = -3 \quad \text{or} \quad x = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{9-36}}{2} = \frac{3 \pm \sqrt{-27}}{2} = \frac{3 \pm 3\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = \frac{3+3\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{3-3\sqrt{-3}}{2}$$

$$\Rightarrow \quad x=-3\left(\frac{-1-\sqrt{-3}}{2}\right) \quad \text{or} \quad x=-3\left(\frac{-1+\sqrt{-3}}{2}\right)$$
$$\Rightarrow \quad x=-3\omega^2 \qquad \text{or} \qquad x=-3\omega$$

Hence cube root of -27 are $-3,-3\omega$ and $-3\omega^2$.

(v) Let x be a cube root of 64 then

$$x=(64)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3=64$$
$$\Rightarrow \quad x^3-64=0 \qquad \Rightarrow \quad (x)^3-(4)^3=0$$
$$\Rightarrow \quad (x-4)(x^2+4x+16)=0$$
$$\Rightarrow \quad x-4=0 \quad \text{or} \quad x^2+4x+16=0$$
$$\Rightarrow \quad x=4 \quad \text{or} \quad x=\frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$
$$=\frac{-4\pm\sqrt{16-64}}{2}=\frac{-4\pm\sqrt{-48}}{2}$$
$$=\frac{-4\pm 4\sqrt{-3}}{2} \qquad \because \ 48=16\times 3$$
$$\Rightarrow \quad x=\frac{-4+4\sqrt{-3}}{2} \quad \text{or} \quad x=\frac{-4-4\sqrt{-3}}{2}$$
$$\Rightarrow \quad x=4\left(\frac{-1+\sqrt{-3}}{2}\right) \quad \text{or} \quad x=4\left(\frac{-1-\sqrt{-3}}{2}\right)$$
$$\Rightarrow \quad x=4\omega \qquad \text{or} \qquad x=4\omega^2$$

Hence cube root of 64 are $4,\ 4\omega$ and $4\omega^2$.

Question # 2

(i) $$\left(1+\omega-\omega^2\right)^8=\left(1+\omega+w^2-2w^2\right)^8$$
$$=\left(0-2w^2\right)^8 \qquad \because \ 1+\omega+\omega^2=0$$
$$=(-2)^8(w^2)^8=256\,\omega^{16}$$
$$=256\,\omega^{15}\cdot\omega=256\left(\omega^3\right)^5\cdot\omega$$
$$=256\,(1)^5\cdot\omega=256\,\omega \quad \text{Answer} \qquad \because \ \omega^3=1$$

(ii) $$\omega^{28}+w^{29}+1=\omega^{27}\cdot\omega+\omega^{27}\cdot\omega^2+1$$
$$=\left(\omega^3\right)^9\cdot\omega+\left(\omega^3\right)^9\cdot\omega^2+1$$
$$=(1)^9\cdot\omega+(1)^9\cdot\omega^2+1 \qquad \because \ \omega^3=1$$
$$=\omega+\omega^2+1$$
$$=0 \quad \text{Answer} \qquad \because \ 1+\omega+\omega^2=0$$

(iii) $$\left(1+\omega-\omega^2\right)\left(1-\omega+\omega^2\right)$$
$$=\left(1+\omega+\omega^2-2\omega^2\right)\left(1+\omega+\omega^2-2\omega\right) \qquad \because \ 1+\omega+\omega^2=0$$
$$=\left(0-2\omega^2\right)(0-2\omega)=\left(-2\omega^2\right)(-2\omega)$$
$$=4\,\omega^3=4(1)=4 \quad \text{Answer} \qquad \because \ \omega^3=1$$

(iv) $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^7 + \left(\dfrac{-1-\sqrt{-3}}{2}\right)^7$ $\quad \because \omega = \dfrac{-1+\sqrt{-3}}{2}, \quad \omega^2 = \dfrac{-1-\sqrt{-3}}{2}$

$= \omega^7 + (\omega^2)^7$

$= \omega^7 + \omega^{14}$

$= \omega^6 \cdot \omega + \omega^{12} \cdot \omega^2 = \left(\omega^3\right)^3 \cdot \omega + \left(\omega^3\right)^4 \cdot \omega^2$

$= (1)^3 \cdot \omega + (1)^4 \cdot \omega^2$

$= \omega + \omega^2 = -1$ Answer $\quad \because 1+\omega+\omega^2 = 0$

(v) $\left(-1+\sqrt{-3}\right)^5 + \left(-1-\sqrt{-3}\right)^5$

$= \left(2 \cdot \dfrac{-1+\sqrt{-3}}{2}\right)^5 + \left(2 \cdot \dfrac{-1-\sqrt{-3}}{2}\right)^5$ $\quad \because \omega = \dfrac{-1+\sqrt{-3}}{2}, \quad \omega^2 = \dfrac{-1-\sqrt{-3}}{2}$

$= (2 \cdot \omega)^5 + \left(2 \cdot \omega^2\right)^5$

$= 32\,\omega^5 + 32\,\omega^{10} = 32\,\omega^3 \cdot \omega^2 + 32\,\omega^9 \cdot \omega^1$

$= 32\,(1) \cdot \omega^2 + 32\,(1) \cdot \omega$ $\quad \because \omega^9 = \left(\omega^3\right)^3 = (1)^3 = 1$

$= 32(\omega + \omega^2)$

$= 32(-1) = -32$ $\quad \because 1+\omega+\omega^2 = 0$

Question # 3

(i) R.H.S$=(x-y)(x-\omega y)\left(x-\omega^2 y\right)$

$=(x-y)\left[x\left(x-\omega^2 y\right) - \omega y\left(x-\omega^2 y\right)\right]$

$=(x-y)\left[x^2 - \omega^2 xy - \omega xy + \omega^3 y^2\right]$

$=(x-y)\left[x^2 - \left(\omega^2+\omega\right)xy + (1)y^2\right]$ $\quad \because \omega^3 = 1$

$=(x-y)\left[x^2 - (-1)\,xy + y^2\right]$ $\quad \because 1+\omega+\omega^2 = 0$

$=(x-y)\left[x^2 + xy + y^2\right]$ $\quad \therefore \omega+\omega^2 = -1$

$=x^3 - y^3 = \text{L.H.S}$

(ii) R.H.S$=(x+y+z)\left(x+\omega y+\omega^2 z\right)\left(x+\omega^2 y+\omega z\right)$

$=(x+y+z)[x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz$
$+\omega^2 xz + \omega^4 yz + \omega^3 z^2]$

$=(x+y+z)[x^2 + (\omega^2+\omega)xy + (\omega+\omega^2)xz + (\omega^2+\omega^4)yz$
$+\omega^3 y^2 + \omega^3 z^2]$

$=(x+y+z)[x^2 + (-1)xy + (-1)xz + (\omega^2+\omega)yz + (1)y^2 + (1)z^2]$

$\because \omega^4 = \omega \;\;\&\;\; \omega+\omega^2 = -1$

$=(x+y+z)[x^2 + y^2 + z^2 - xy + (-1)yz - xz]$

$=(x+y+z)[x^2 + y^2 + z^2 - xy - yz - xz]$

$= x^3 + y^3 + z^3 - 3xyz = \text{L.H.S}$

(iii) L.H.S$=(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$............$2n$ factors

$= [(1+\omega)(1+\omega^2)][(1+\omega^4)(1+\omega^8)]$............ n factors

$= [(1+\omega)(1+\omega^2)][(1+\omega^3\cdot\omega)(1+\omega^6\cdot\omega^2)]$............ n factors

$= [(1+\omega)(1+\omega^2)][(1+\omega^3\cdot\omega)(1+(\omega^3)^2\cdot\omega^2)]$............ n factors

$$= [(1+\omega)(1+\omega^2)][(1+1\cdot\omega)(1+(1)^2\cdot\omega^2)]\text{............ } n \text{ factors}$$
$$= [(1+\omega)(1+\omega^2)][(1+\omega)(1+\omega^2)]\text{............ } n \text{ factors}$$
$$= \left[(1+\omega)(1+\omega^2)\right]^n = \left[1+\omega+\omega^2+\omega^3\right]^n$$
$$= [0+1]^n \qquad \because\ 1+\omega+\omega^2 = 0\ ,\ \omega^3 = 1$$
$$= [1]^n \quad = 1 \quad = \text{R.H.S}$$

Question # 4 (i)

Let $\quad x^2 + x + 1 = 0$ ....(i)

Since ω is root of (i) therefore

$$\omega^2 + \omega + 1 = 0\text{...........}(ii)$$

To prove ω^2 is root of (i)

Consider $\quad \left(\omega^2\right)^2 + \omega^2 + 1 = \omega^4 + 2\omega^2 + 1 - \omega^2$

$$= \left(\omega^2+1\right)^2 - \omega^2 = \left(\omega^2+1+\omega\right)\left(\omega^2+1-\omega\right)$$
$$= (0)\left(\omega^2+1-\omega\right) \qquad \text{from } (i)$$

$\Rightarrow \quad \left(\omega^2\right)^2 + \omega^2 + 1 = 0$ ....(iii)

$\Rightarrow \quad \omega^2$ is the root of the equation (i).

Now subtracting (ii) from (iii)

$$\begin{array}{rrrr} \left(\omega^2\right)^2 + \omega^2 & & +1 & = 0 \\ \omega^2 & +\omega & +1 & = 0 \\ - & - & - & \\ \hline \omega^4 & -\omega & & = 0 \end{array}$$

$\Rightarrow \quad \omega\left(\omega^3 - 1\right) = 0$

$\Rightarrow \quad \omega^3 - 1 = 0 \qquad$ as $\quad \omega \neq 0$

$\Rightarrow \quad \boxed{\omega^3 = 1}$

Question # 5

Let x be a cube root of -1 then

$$x = (-1)^{\frac{1}{3}} \qquad \Rightarrow \quad x^3 = -1$$

$\Rightarrow \quad x^3 + 1 = 0 \qquad \Rightarrow \quad (x)^3 + (1)^3 = 0$

$\Rightarrow \quad (x+1)(x^2 - x + 1) = 0$

$\Rightarrow \quad x + 1 = 0 \quad$ or $\quad x^2 - x + 1 = 0$

$\Rightarrow \quad x = -1 \qquad$ or $\quad x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$$= \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2}$$

$$\Rightarrow \quad x = \frac{1+\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{1-\sqrt{-3}}{2}$$

$$\Rightarrow \quad x = \frac{1+\sqrt{3}i}{2} \quad \text{or} \quad x = \frac{1-\sqrt{3}i}{2}$$

Hence complex cube root of -1 are $\dfrac{1+\sqrt{3}i}{2}$ and $\dfrac{1-\sqrt{3}i}{2}$.

Question # 6

Since 2ω and $2\omega^2$ are roots of required equation, therefore

$$(x-2\omega)\left(x-2\omega^2\right)=0$$

$$\Rightarrow\ x^2-2\omega x-2\omega^2x+4\omega^3=0$$

$$\Rightarrow\ x^2-2x(\omega+\omega^2)+4(1)=0 \qquad \because\ \omega^3=1$$

$$\Rightarrow\ x^2-2x(-1)+4=0 \qquad \because\ 1+\omega+\omega^2=0$$

$$\Rightarrow\ x^2+2x+4=0$$

is the required equation.

Question # 7

(i) Let x be a fourth root of 16 then

$$x=(16)^{\frac{1}{4}} \qquad \Rightarrow\ x^4=16$$

$$\Rightarrow\ x^4-16=0 \qquad \Rightarrow\ \left(x^2\right)^2-(4)^2=0$$

$$\Rightarrow\ \left(x^2+4\right)\left(x^2-4\right)=0$$

$$\Rightarrow\ x^2+4=0 \quad \text{or} \quad x^2-4=0$$

$$\Rightarrow\ x^2=-4 \quad \text{or} \quad x^2=4$$

$$\Rightarrow\ x=\pm\sqrt{-4} \quad \text{or} \quad x=\pm\sqrt{4}$$

$$\Rightarrow\ x=\pm 2i \quad \text{or} \quad x=\pm 2$$

Hence the four fourth root of 16 are $2,-2,2i,-2i$.

(ii) *Do yourself as above.* ***Hint:*** $81=(9)^2$

(iii) Let x be a fourth root of 625 then

$$x=(625)^{\frac{1}{4}} \qquad \Rightarrow\ x^4=625$$

$$\Rightarrow\ x^4-625=0 \qquad \Rightarrow\ \left(x^2\right)^2-(25)^2=0$$

$$\Rightarrow\ \left(x^2+25\right)\left(x^2-25\right)=0$$

$$\Rightarrow\ x^2+25=0 \quad \text{or} \quad x^2-25=0$$

$$\Rightarrow\ x^2=-25 \quad \text{or} \quad x^2=25$$

$$\Rightarrow\ x=\pm\sqrt{-25} \quad \text{or} \quad x=\pm\sqrt{25}$$

$$\Rightarrow\ x=\pm 5i \quad \text{or} \quad x=\pm 5$$

Hence the four fourth root of 625 are $5,-5,5i,-5i$.

Question # 8

(i) $2x^4-32=0$

$$\Rightarrow\ 2(x^4-16)=0 \qquad \Rightarrow\ x^4-16=0$$

Now do you as in Question # 7 (i)

(ii) $3y^5-243y=0$

$$\Rightarrow\ 3y\left(y^4-81\right)=0$$

$$\Rightarrow\ 3y=0 \quad \text{or} \quad y^4-81=0$$

$$\Rightarrow \quad y = 0 \quad \text{or} \quad \left(y^2\right)^2 - (9)^2 = 0$$
$$\Rightarrow \quad \left(y^2 + 9\right)\left(y^2 - 9\right) = 0$$
$$\Rightarrow \quad y^2 + 9 = 0 \quad \text{or} \quad y^2 - 9 = 0$$
$$\Rightarrow \quad y^2 = -9 \quad \text{or} \quad y^2 = 9$$
$$\Rightarrow \quad y = \pm\sqrt{-9} \quad \text{or} \quad y = \pm\sqrt{9}$$
$$\Rightarrow \quad y = \pm 3i \quad \text{or} \quad y = \pm 3$$

Hence S.Set $= \left\{0, \pm 3, \pm\, 3i\right\}$

(iii)

$$x^3 + x^2 + x + 1 = 0$$
$$\Rightarrow \quad x^2(x+1) + 1(x+1) = 0$$
$$\Rightarrow \quad (x+1)(x^2+1) = 0$$
$$\Rightarrow \quad x + 1 = 0 \quad \text{or} \quad x^2 + 1 = 0$$
$$\Rightarrow \quad x = -1 \quad \text{or} \quad x^2 = -1 \quad \Rightarrow \quad x = \pm i$$

Hence S.Set $= \{-1, \pm i\}$

(iv)

$$5x^5 - 5x = 0$$
$$\Rightarrow \quad 5x(x^4 - 1) = 0$$
$$\Rightarrow \quad 5x = 0 \quad \text{or} \quad x^4 - 1 = 0$$
$$\Rightarrow \quad x = 0 \quad \text{or} \quad \left(x^2\right)^2 - (1)^2 = 0$$
$$\Rightarrow \quad \left(x^2 + 1\right)\left(x^2 - 1\right) = 0$$
$$\Rightarrow \quad x^2 + 1 = 0 \quad \text{or} \quad x^2 - 1 = 0$$
$$\Rightarrow \quad x^2 = -1 \quad \text{or} \quad x^2 = 1$$
$$\Rightarrow \quad x = \pm i \quad \text{or} \quad x = \pm 1$$

Hence S.Set $= \left\{0, \pm 1, \pm i\right\}$