EXERCISE 3.5

1. **Solve the following systems of linear equations by Cramer's Rule :**

Solution (i) $2x + 2y + z = 3,\ 3x - 2y - 2z = 1,\ 5x + y - 3z = 2$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 3 & -2 & -2 \\ 5 & 1 & -3 \end{vmatrix} \begin{aligned} &= 2(6+2) - 2(-9+10) + 1(3+10) \\ &= 16 - 2 + 13 = 27 \neq 0 \end{aligned}$$

$\therefore$ Solution is possible.

$$\therefore\ x = \frac{\begin{vmatrix} 3 & 2 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & -3 \end{vmatrix}}{27} = \frac{3(6+2) - 2(-3+4) + 1(1+4)}{27} = \frac{24 - 2 + 5}{27} = \frac{27}{27} = 1$$

$$y = \frac{\begin{vmatrix} 2 & 3 & 1 \\ 3 & 1 & -2 \\ 5 & 2 & -3 \end{vmatrix}}{27} = \frac{2(-3+4) - 3(-9+10) + 1(6-5)}{27} = \frac{2 - 3 + 1}{27} = \frac{0}{27} = 0$$

$$z = \frac{\begin{vmatrix} 2 & 2 & 3 \\ 3 & -2 & 1 \\ 5 & 1 & 2 \end{vmatrix}}{27} = \frac{2(-4-1) - 2(6-5) + 3(3+10)}{27} = \frac{-10 - 2 + 39}{27} = \frac{27}{27} = 1$$

1(ii). $2x_1 - x_2 + x_3 = 5,\ 4x_1 + 2x_2 + 3x_3 = 8,\ 3x_1 - 4x_2 - x_3 = 3$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 4 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix} \begin{aligned} &= 2(-2+12) + 1(-4-9) + 1(-16-6) \\ &= 20 - 13 - 22 = -15 \neq 0 \end{aligned}$$

$\therefore$ Solution is possible.

$$\therefore\ x_1 = \frac{\begin{vmatrix} 5 & -1 & 1 \\ 8 & 2 & 3 \\ 3 & -4 & -1 \end{vmatrix}}{-15} = \frac{5(-2+12) + 1(-8-9) + 1(-32-6)}{-15} = \frac{50 - 17 - 38}{-15} = \frac{1}{3}$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5 & 1 \\ 4 & 8 & 3 \\ 3 & 3 & -1 \end{vmatrix}}{-15} = \frac{2(-8-9) - 5(-4-9) + 1(12-24)}{-15} = \frac{-34 + 65 - 12}{-15} = -\frac{19}{15}$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 5 \\ 4 & 2 & 8 \\ 3 & -4 & 3 \end{vmatrix}}{-15} = \frac{2(6+32)+1(12-24)+5(-16-6)}{-15} = \frac{76-12-110}{-15} = \frac{46}{15}$$

1(iii) $2x_1 - x_2 + x_3 = 8, x_1 + 2x_2 + 2x_3 = 6, x_1 - 2x_2 - x_3 = 1$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix} \begin{aligned} &= 2(-2+4)+1(-1-2)+1(-2-2) \\ &= 4-3-4 = -3 \neq 0 \\ &\therefore \text{ Solution is possible.} \end{aligned}$$

$$\therefore x_1 = \frac{\begin{vmatrix} 8 & -1 & 1 \\ 6 & 2 & 2 \\ 1 & -2 & -1 \end{vmatrix}}{-3} = \frac{8(-2+4)+1(-6-2)+1(-12-2)}{-3} = \frac{16-8-14}{-3} = 2$$

$$x_2 = \frac{\begin{vmatrix} 2 & 8 & 1 \\ 1 & 6 & 2 \\ 1 & 1 & -1 \end{vmatrix}}{-3} = \frac{2(-6-2)-8(-1-2)+1(1-6)}{-3} = \frac{-16+24-5}{-3} = -1$$

$$x_3 = \frac{\begin{vmatrix} 2 & -1 & 8 \\ 1 & 2 & 6 \\ 1 & -2 & 1 \end{vmatrix}}{-3} = \frac{2(2+12)+1(1-6)+8(-2-2)}{-3} = \frac{28-5-32}{-3} = 3$$

2. Use matrices to solve the following systems :

Solution (i) $x - 2y + z = -1,\ 3x + y - 2z = 4,\ y - z = 1$

$$\text{Here } |A| = \begin{vmatrix} 1 & -2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} \begin{aligned} &= 1(-1+2)-3(2-1) \\ &= 1-3 = -2 \neq 0 \text{ (Exp. by } C_1) \\ &\therefore \text{ Solution is possible.} \end{aligned}$$

$$\text{adj } A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} (-1+2) & -(-3+0) & (3-0) \\ -(2-1) & (-1-0) & -(1+0) \\ (4-1) & -(-2-3) & (1+6) \end{bmatrix}^t$$

$$= \begin{bmatrix} 1 & 3 & 3 \\ -1 & -1 & -1 \\ 3 & 5 & 7 \end{bmatrix}^t = \begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{-2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 1 & -1 & 3 \\ 3 & -1 & 5 \\ 3 & -1 & 7 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -1-4+3 \\ -3-4+5 \\ -3-4+7 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x = 1, y = 1, z = 0$$

Solution (ii) $2x_1 + x_2 + 3x_3 = 3, \ x_1 + x_2 - 2x_3 = 0, \ -3x_1 - x_2 + 2x_3 = -4$

Here $|A| = \begin{vmatrix} 2 & 1 & -3 \\ 1 & 1 & -2 \\ -3 & -1 & 2 \end{vmatrix}$ $= 2(2-2) - 1(2-6) - 3(-1+3)$
$= 0 + 4 - 6 = -2 \neq 0$

$\therefore$ Solution is possible.

$$\operatorname{adj} A = \begin{bmatrix} \begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ -3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ -3 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -3 & -1 \end{vmatrix} \\ \begin{vmatrix} 1 & -3 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} 0 & 4 & 2 \\ 1 & -5 & -1 \\ 1 & 1 & 1 \end{bmatrix}^t = \begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{-2}\begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \; = \frac{1}{-2}\begin{bmatrix} 0 & 1 & 1 \\ 4 & -5 & 1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 0+0-4 \\ 12-0-4 \\ 6-0-4 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} 0+0-4 \\ 12-0-4 \\ 6-0-4 \end{bmatrix} = \frac{1}{-2}\begin{bmatrix} -4 \\ 16 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 2,\; x_2 = -4,\; x_3 = -1.$$

Solution (iii) $x + y = 2,\; 2x - z = 1,\; 2y - 3z = -1$

Here $|A| = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ 0 & 2 & -3 \end{vmatrix}$ $= 1(0+2) - 1(-6-0) + 0$
$= 2 + 6 = 8 \neq 0$
$\therefore$ Solution is possible.

$$\text{adj}\,A = \begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 2 & -3 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 0 & -3 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & -3 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \end{bmatrix}^t$$

$$= \begin{bmatrix} (0+2) & -(-6+0) & (4-0) \\ -(-3-0) & (-3-0) & (-2-0) \\ (-1-0) & -(-1+0) & (0-2) \end{bmatrix}^t = \begin{bmatrix} 2 & 6 & 4 \\ 3 & -3 & -2 \\ -1 & 1 & -2 \end{bmatrix}^t$$

$$= \begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$\therefore \; A^{-1} = \frac{1}{|A|}\,\text{adj}\,A = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 2 & 3 & -1 \\ 6 & -3 & 1 \\ 4 & -2 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4+3+1 \\ 12-3-1 \\ 8-2+2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 1,\ y = 1,\ z = 1.$

3. Solve the following systems by reducing their augmented matrices to the echelon form and the reduced echelon forms.

Solution. (i). $x_1 - 2x_2 - 2x_3 = -1, \quad 2x_1 + 3x_2 + x_3 = 1, \quad 5x_1 - 4x_2 - 3x_3 = 1$

(a) : Using Echelon Form :

Here A_b denotes the augmented matrix.

$$A_b = \begin{bmatrix} 1 & -2 & -2 & -1 \\ 2 & 3 & 1 & 1 \\ 5 & -4 & -3 & 1 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 7 & 5 & 3 \\ 0 & 6 & 7 & 6 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 6 & 7 & 6 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - R_3 \end{matrix} \underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 19 & 24 \end{bmatrix} \begin{matrix} \text{by} \\ R_3 - 6R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} \begin{matrix} \text{by} \\ \frac{1}{19}R_3 \end{matrix} \quad \ldots \text{(I)}$$

Matrix in equation (I) is in Echelon Form.

Now, $R_1 \Rightarrow \; x_1 - 2x_2 - 2x_3 = -1 \qquad \ldots(1),$

$R_2 \Rightarrow \; 0 + x_2 - 2x_3 = -3 \qquad \ldots(2),$

$$R_3 \Rightarrow \quad 0 + 0 + x_3 \quad = \frac{24}{19} \qquad \ldots(3),$$

From backward substitution : (2) $\Rightarrow x_2 = 2x_3 - 3 = \frac{2(24)}{19} - 3 = \frac{48-57}{19} = -\frac{9}{19}$

and (1) $\Rightarrow x_1 = 2x_2 + 2x_3 - 1 = 2\left(-\frac{9}{19}\right) + 2\left(\frac{24}{10}\right) - 1 = -\frac{18}{19} + \frac{48}{19} - 1$

$$\Rightarrow x_1 = \frac{-18+48-19}{19} = \frac{11}{19}$$

(b) : Using Reduced Echelon Form :

Matrix in (I) can further be changed to *Reduced Echelon Form* :

$$\underset{\sim}{R}\begin{bmatrix} 1 & -2 & -2 & -1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} \underset{\sim}{R}\begin{bmatrix} 1 & 0 & -6 & -7 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} (\text{ by } R_1 + 2R_2)$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 11/19 \\ 0 & 1 & 0 & -9/19 \\ 0 & 0 & 1 & 24/19 \end{bmatrix} \begin{matrix} \text{by} \\ R_1 + 6R_3 \\ R_2 + 2R_3 \end{matrix} \quad \ldots \text{(II)}$$

From equation (II) :

Now, $R_1 \Rightarrow x_1 + 0 + 0 \quad = \frac{11}{19} \quad \Rightarrow \quad x_1 = \frac{11}{19}$

$R_1 \Rightarrow 0 + x_2 + 0 \quad = -\frac{9}{19} \quad \Rightarrow \quad x_2 = -\frac{9}{19}$

$R_1 \Rightarrow 0 + 0 + x_3 \quad = \frac{24}{19} \quad \Rightarrow \quad x_3 = \frac{24}{19}$

Solution. (ii). $x + 2y + z = 2, \; 2x + y + 2z = -1, \; 2x + 3y - z = 9$

(a): Using Echelon Form :

$$A_b = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & -1 \\ 2 & 3 & -1 & 9 \end{bmatrix} \underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -3 & 0 & -5 \\ 0 & -1 & -3 & 5 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & -1 & -3 & 5 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 4R_3 \to R_2 \end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 9 & -20 \end{bmatrix} \begin{matrix}\text{by} \\ R_3 + R_2 \to R_3\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \begin{matrix}\text{by} \\ \frac{1}{9}R_3\end{matrix} \qquad \ldots \text{(I)}$$

Matrix in equation (I) is in Echelon Form.

Now, $R_1 \Rightarrow \; x + 2y + z = 2 \qquad \ldots(1),$

$R_2 \Rightarrow \; 0 + y + 12z = -25 \qquad \ldots(2),$

$R_3 \Rightarrow \; 0 + 0 + z = -\frac{20}{9} \qquad \ldots(3),$

Solving by backward substitution, using $z = -20/9$**,weget**

$$(2) \Rightarrow \quad y + 12z = -25 \;\Rightarrow\; y = -25 - 12z = -25 - 12\left(-\frac{20}{9}\right)$$

$$y = -25 + \frac{80}{3} = \frac{-75+80}{3} = \frac{5}{3}$$

$$(1) \Rightarrow \quad x + 2y + z = 2 \;\Rightarrow\; x = 2 - 2y - z = 2 - 2\times\frac{5}{3} - \left(-\frac{20}{9}\right)$$

$$x = 2 - \frac{10}{3} + \frac{20}{9} = \frac{18-30+20}{9} = \frac{8}{9}$$

Hence $x = \frac{8}{9}$, $y = \frac{5}{3}$, $z = -\frac{20}{9}$.

(b): Using Reduced Echelon Form:

Equation (I) can further be reduced to Reduced echelon form :

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -23 & 52 \\ 0 & 1 & 12 & -25 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \begin{matrix}\text{by} \\ R_1 - 2R_2\end{matrix}$$

$$\underset{\sim}{R}\begin{bmatrix} 1 & 0 & 0 & 8/9 \\ 0 & 1 & 0 & 15/9 \\ 0 & 0 & 1 & -20/9 \end{bmatrix} \begin{matrix}\text{by} \\ R_1 + 23R_3 \\ R_2 - 12R_3\end{matrix} \qquad \ldots \text{(II)}$$

Matrix in Equation (II) is in reduced echelon form giving directly

$$x = \frac{8}{9}, \quad y = \frac{15}{9} = \frac{5}{3}, \quad z = -\frac{20}{9}.$$

(a): Using Echelon Form :

$$Ab = \begin{bmatrix} 1 & 4 & 2 & 2 \\ 2 & 1 & -2 & 9 \\ 3 & 2 & -2 & 12 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & -7 & -6 & 5 \\ 0 & -10 & -8 & 6 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & -10 & -8 & 6 \end{bmatrix} \begin{matrix} \text{by} \\ -\frac{1}{7}R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 4/7 & -8/7 \end{bmatrix} \begin{matrix} \text{by} \\ R_3 + 10R_2 \end{matrix} \cdots \quad \text{(I)}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \begin{matrix} \text{by} \\ \frac{7}{4}R_3 \end{matrix} \quad \cdots \quad \text{(I)}$$

Matrix in equation (I) is in Echelon Form.

$$R_1 \Longrightarrow \; x_1 + 4x_2 + 2x_3 = 2 \qquad \ldots(1),$$

$$R_2 \Longrightarrow \; 0 + x_2 + \frac{6}{7}x_3 = -\frac{5}{7} \quad \ldots(2),$$

$$R_3 \Longrightarrow \; 0 + 0 + x_3 = -2 \quad \ldots(3),$$

Solving by backward substitution,weget

(3) $\Longrightarrow \; x_3 = -2$

(2) $\Longrightarrow \; x_2 + \frac{6}{7}x_3 = -\frac{5}{7} \;\Longrightarrow\; x_2 = -\frac{5}{7} - \frac{6}{7}x_3 = -\frac{5}{7} - \frac{6}{7}(-2)$

$\Longrightarrow \; x_2 = -\frac{5}{7} + \frac{12}{7} = \frac{7}{7} = 1$

(1) $\Longrightarrow \; x_1 + 4x_2 + 2x_3 = 2 \;\Longrightarrow\; x_1 = 2 - 4x_2 - 2x_3$

$\Longrightarrow \; x_1 = 2 - 4(1) - 2(-2) = 2 - 4 + 4 = 2$

Hence $\quad x_1 = 2, \quad x_2 = 1, \quad x_3 = -2.$

(b): Using Reduced Echelon Form :

Equation (I) can further be reduced to Reduced echelon form :

$$\begin{bmatrix} 1 & 4 & 2 & 2 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 0 & -10/7 & 34/7 \\ 0 & 1 & 6/7 & -5/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_1 - 4R_2 \\ R_3 + 10R_2 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 0 & 0 & 14/7 \\ 0 & 1 & 0 & 7/7 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_1 + \frac{10}{7}R_3 \\ R_2 - \frac{6}{7}R_3 \end{matrix} \quad \ldots \text{(II)}$$

Matrix in equation (II) is in reduced echelon form giving directly

$$x_1 = \frac{14}{7} = 2, \quad x_2 = \frac{7}{7} = 1, \quad x_3 = -2.$$

4. Solve the following systems of homogeneous linear equations :

Note. In case of homogeneous system, it has non-zero solution only if the determinant of coefficients is zero, i.e., matrix of coefficients is singular, i.e., its rank is less that the number of variables. That is why we use the matrix of coefficients only for echelon form.

(i) $x + 2y - 2z = 0, \quad 2x + y + 5z = 0, \quad 5x + 4y + 8z = 0$

Solution (i). $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 5 \\ 5 & 4 & 8 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -3 & 9 \\ 0 & -6 & 18 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 5R_1 \end{matrix}$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \left(\begin{matrix} \text{by} \\ -\frac{1}{3}R_2, \frac{1}{6}R_3 \end{matrix} \right)$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_3 - R_2 \end{matrix} \quad \ldots \text{(I)}$$

Thus rank of A = 2 < 3 (No. of variables). Therefore, (I) gives

$R_1 \Rightarrow x + 2y - 2z = 0 \ \ldots \ \text{(i)},$

$R_2 \Rightarrow \quad y - 3z = 0 \ \ldots \ \text{(ii)},$

(iii) $\Rightarrow z = t$, then (ii) $\Rightarrow y = 3z = 3t$.

and (i) gives $x + 6t - 2t = 0 \Rightarrow x = - 4t$.

Thus, S.S. = { t , 3t, 4t } where t is any real number.

REMARKS.-**1.** If t = 1, then solution is { 1, 3 , − 4 }, if t = 2, then S.S. is { 2, 6, − 8 }, i.e., we can find infinity of solutions by giving different values to t.

In fact, when the number of equations is less than the number of variables, we have to give some arbitrary value of a variable and get the value of other/s in terms of that value.

2. If we are given a set of three homogeneous equations they will have {0,0,0} as obvious solution. This solution is called *Trivial Solution*. And a solution of the system other than zero solution is called *Non-TRivial Solution*.

3. In such cases, we will use parametric values to free variables. Free variales are those variales which are in the last alphabatics, e.g., out of x, y, z, the free variable is z, out of xy, the free variable is y and out of x_1, x_2, x_3 the free ariable is x_3 and paraeters are ; t, s, u, v, w etc.

Solution (ii). $x_1 + 4x_2 + 2x_3 = 0,\ 2x_1 + x_2 - 3x_3 = 0,\ 3x_1 + 2x_2 - 4x_3 = 0$

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \left(\begin{matrix} \text{by} \\ -\frac{1}{7}R_2 , -\frac{1}{10}R_3 \end{matrix} \right)$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_3 - R_2 \end{matrix} \qquad \ldots(I)$$

Thus rank of A = 2 < 3 (No. of variables). Therefore, (I) gives

$R_1 \Rightarrow \quad x_1 + 4x_2 + 2x_3 = 0 \qquad \ldots \quad (i)$,

$R_2 \Rightarrow \quad x_2 + x_3 = 0 \quad \ldots$ (ii),

$R_3 \Rightarrow \quad 0x_3 = 0 \quad \ldots$ (iii)

(iii) $\Rightarrow \quad x_3 = t$, then

(ii) $\Rightarrow \quad x_2 = -x_3 = -t$

and (i) gives

$x_1 - 4t + 2t = 0 \quad \Rightarrow \quad x_1 = 2t$.

Thus, S.S. = { 2t , − t, t } where t is any real number.

Solution (iii). $x_1 - 2x_2 - x_3 = 0$, $x_1 + x_2 + 5x_3 = 0$, $2x_1 - x_2 + 4x_3 = 0$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 1 & 1 & 5 \\ 2 & -1 & 4 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 6 \\ 0 & 3 & 6 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \left(\begin{matrix} \text{by} \\ \frac{1}{3}R_2 , \frac{1}{3}R_3 \end{matrix} \right)$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_3 - R_2 \end{matrix} \quad \ldots(\mathrm{I})$$

Thus rank of A = 2 < 3 (No. of variables). Therefore, (I) gives

$x_1 - 2x_2 - x_3 = 0$ .. (i), $x_2 + 2x_3 = 0$ ...(ii), $0x_3 = 0 \ldots$ (iii)

(iii) $\Rightarrow \quad x_3 = t$, then (ii) gives $x_2 = -2x_3 = -2t$ and (i) gives

$x_1 + 4t - t = 0 \quad \Rightarrow \quad x_1 = -3t$.

Thus, S.S. = { − 3t , − 2t, t } where t is any real number.

5. Find the value of λ for which the following systems have a non-trivial solution. Also solve the system for the value of λ.

Solution. (i) $x + y + z = 0$, $2x + y - \lambda z = 0$, $x + 2y - 2z = 0$

The system has non-trivial solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -\lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$1(-2 + 2\lambda) - 1(-4 + \lambda) + 1(4 - 1) = 0$

$$\Rightarrow \quad -2+2\lambda+4-\lambda+3 = 0$$

$$\lambda+5 = 0 \quad \Rightarrow \quad \lambda = -5$$

For the non-trivial solution (using $\lambda = -5$), we have

$$A_b = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 5 \\ 1 & 2 & -2 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 3 \\ 0 & 1 & -3 \end{bmatrix} \begin{matrix} \text{by} \\ R_2-2R_1 \\ R_3-R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} (\text{ by } -R_2)$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{by} \\ R_3-R_2 \end{matrix} \quad \ldots(I)$$

(I) $\Rightarrow \quad x+y+z=0,\ y-3z=0,\ 0z=0 \quad \Rightarrow \quad z=t.$

$y = 3z = 3t$, and $x+3t+t=0$, i.e., $x=-4t$

Thus, S.S. = { $-4t$, $3t$, t } where t is any real number.

Solution. (ii). $x_1+4x_2+\lambda x_3 = 0.\ 2x_1+x_2-3x_3 = 0,\ 3x_1+\lambda x_2-4x_3 = 0$

The systen has non-trivial solution if

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -3 \\ 3 & \lambda & -4 \end{vmatrix} = 0$$

$$1(-4+3\lambda)-4(-8+9)+\lambda(2\lambda-3)=0$$

or $\quad -4+3\lambda-4+2\lambda^2-3\lambda=0$

or $\quad 2\lambda^2-8=0 \Rightarrow \lambda^2=4 \Rightarrow \lambda=\pm 2$

<u>When $\lambda = 2$:</u>

$$A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 1 & -3 \\ 3 & 2 & -4 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & -7 & -7 \\ 0 & -10 & -10 \end{bmatrix} \begin{matrix} \text{by} \\ R_2-2R_1 \\ R_3-3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & -10 & -10 \end{bmatrix} (\text{ by } -R_2) \underset{\sim}{R} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{by, } R_3-R_2)\ldots(I)$$

$R_1 \Rightarrow \quad x_1 + 4x_2 + 2x_3 = 0 \quad \ldots \quad$ (i) ,

$R_2 \Rightarrow \quad x_3 = 0 \quad \ldots$ (ii),

Let $x_2 = t$, then

(i) $\Rightarrow \quad x_1 + 4t + 0 = 0 \Rightarrow \quad x_1 = -4t$

Thus, S.S. = { −4t , t, 0 } where t is any real number.

<u>When $\lambda = -2$:</u>

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 1 & -3 \\ 3 & -2 & -4 \end{bmatrix} \underset{\sim}{R} \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & -14 & 2 \end{bmatrix} \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & -7 & 1 \end{bmatrix} (\text{by } -R_2) \underset{\sim}{R} \begin{bmatrix} 1 & 4 & -2 \\ 0 & -7 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{by, } R_3 - R_2) \ldots (I)$$

$R_1 \Rightarrow \quad x_1 + 4x_2 - 2x_3 = 0 \quad \ldots \quad$ (i) ,

$R_2 \Rightarrow \quad -7x_2 + x_3 = 0 \quad \ldots$ (ii),

Let $x_2 = t$, then

(ii) $\Rightarrow \quad -7x_2 + t = 0 \Rightarrow \quad x_2 = \frac{t}{7}$

(i) $\Rightarrow \quad x_1 + 4.\frac{t}{7} - 2t = 0 \quad \Rightarrow \quad x_1 = 2t - \frac{4t}{7} = \frac{10t}{7}$

Thus, S.S. = $\left\{ \left(\frac{10t}{7}, \frac{t}{7}, t \right) \right\}$ where t is any real number.

$= \left\{ \left(10t, t, 7t \right) \right\}$ Multiplying by 7 .

For $\lambda = -2$, (A) reads :

$$x_1 + 4x_2 - 2x_3 = 0, \; x_2 + \frac{3-4}{7} x_3 = 0, \; 0x_3 = 0 \quad \Rightarrow \quad x_3 = k.$$

$$x_2 = -\frac{1}{7} x_3 = -\frac{1}{7} k, \text{ and } x_1 - \frac{4}{7} k - 2k = 0, \text{ i.e., } x_1 = -\frac{18}{7} k$$

6. Find the value of λ for which the following system does not possess a unique solution. Also solve the system for the value of λ.

$$x_1 + 4x_2 + \lambda x_3 = 2, \; 2x_1 + x_2 - 2x_3 = 11, \; 3x_1 + 2x_2 - 2x_3 = 16.$$

Solution.

The system does not possess a unique solution if

$$\begin{vmatrix} 1 & 4 & \lambda \\ 2 & 1 & -2 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$$1(-2+4) - 4(-4+6) + \lambda(4-3) = 0$$

$$2 - 8 + \lambda = 0 \Rightarrow \lambda = 6$$

For the solution put $\lambda = 6$ in the system, then solution is given by

$$A = \begin{bmatrix} 1 & 4 & 6 & 2 \\ 2 & 1 & -2 & 11 \\ 3 & 2 & -2 & 16 \end{bmatrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & -7 & -14 & 7 \\ 0 & -10 & -20 & 10 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \left(\begin{matrix} \text{by } -\frac{1}{7} R_2 \\ -\frac{1}{10} R_3 \end{matrix} \right)$$

$$\underset{\sim}{R} \begin{bmatrix} 1 & 4 & 6 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \text{by} \\ R_3 - R_2 \\ \rightarrow R_3 \end{matrix} \quad \ldots(I)$$

$R_1 \Rightarrow \quad x_1 + 4x_2 + 6x_3 = 2, \quad \ldots$ (i),

$R_2 \Rightarrow \quad x_2 + 2x_3 = -1 \quad \ldots$ (ii),

Let $\quad x_3 = t$

(ii) $\Rightarrow x_2 + 2t = -1 \Rightarrow x_2 = -1 - 2t$

(i) $\Rightarrow x_1 + 4(-1 - 2t) + 6t = 2,$

$\Rightarrow x_1 - 4 - 8t + 6t = 2,$

$\Rightarrow x_1 = 6 + 2t ,$

Thus, S.S. = $\{ 6 + 2t , -1 - 2t , t \}$ where t is any real number.