

EXERCISE 6.2

Equation of tangent and normal to the circle

Consider an equation of circle

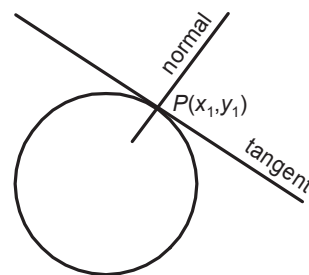
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then equation of tangent at (x_1, y_1) is

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

The equation of normal at (x_1, y_1) is

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$



(See proof at page 257)

Question # 1

$$x^2 + y^2 = 25$$

Differentiating w.r.t. x

$$2x + 2y \frac{dy}{dx} = 0 \Rightarrow 2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

At $(4, 3)$:

$$\text{Slope of tangent at } (4, 3) = m = \left. \frac{dy}{dx} \right|_{(4,3)} = -\frac{4}{3}$$

Now equation of tangent at $(4, 3)$ having slope $-\frac{4}{3}$

$$\begin{aligned} y - 3 &= -\frac{4}{3}(x - 4) &\Rightarrow 3y - 9 &= -4x + 16 \\ \Rightarrow 4x - 16 + 3y - 9 &= 0 &\Rightarrow \boxed{4x + 3y - 25 = 0} \end{aligned}$$

Since normal is \perp to tangent therefore

$$\text{Slope of normal at } (4, 3) = -\frac{1}{m} = -\frac{1}{-4/3} = \frac{3}{4}$$

Now equation of normal at $(4, 3)$ having slope $\frac{3}{4}$

$$\begin{aligned} y - 3 &= \frac{3}{4}(x - 4) \\ \Rightarrow 4y - 12 &= 3x - 12 &\Rightarrow 3x - 12 - 4y + 12 &= 0 \\ \Rightarrow \boxed{3x - 4y = 0} \end{aligned}$$

At $(5\cos\theta, 5\sin\theta)$

$$\text{Slope of tangent at } (5\cos\theta, 5\sin\theta) = m = \left. \frac{dy}{dx} \right|_{(5\cos\theta, 5\sin\theta)} = -\frac{5\cos\theta}{5\sin\theta} = -\frac{\cos\theta}{\sin\theta}$$

Now equation of tangent at $(5\cos\theta, 5\sin\theta)$ having slope $-\frac{\cos\theta}{\sin\theta}$

$$\begin{aligned} y - 5\sin\theta &= -\frac{\cos\theta}{\sin\theta}(x - 5\cos\theta) \\ \Rightarrow y\sin\theta - 5\sin^2\theta &= -x\cos\theta + 5\cos^2\theta \\ \Rightarrow x\cos\theta + y\sin\theta &= 5\sin^2\theta + 5\cos^2\theta \\ \Rightarrow x\cos\theta + y\sin\theta &= 5(\sin^2\theta + \cos^2\theta) \\ \Rightarrow x\cos\theta + y\sin\theta &= 5(1) \\ \Rightarrow \boxed{x\cos\theta + y\sin\theta = 5} \end{aligned}$$

Since normal are \perp ar to tangent therefore

$$\text{Slope of normal} = -\frac{1}{m} = \frac{\sin \theta}{\cos \theta}$$

Now equation of normal at $(5\cos\theta, 5\sin\theta)$ having slope $\frac{\sin \theta}{\cos \theta}$

$$\begin{aligned} y - 5\sin\theta &= \frac{\sin\theta}{\cos\theta}(x - 5\cos\theta) \\ \Rightarrow y\cos\theta - 5\sin\theta\cos\theta &= x\sin\theta - 5\sin\theta\cos\theta \\ \Rightarrow x\sin\theta - 5\sin\theta\cos\theta - y\cos\theta + 5\sin\theta\cos\theta &= 0 \\ \Rightarrow \boxed{x\sin\theta - y\cos\theta = 0} \end{aligned}$$

Question # 1(ii) [Alternative Method]

$$\begin{aligned} 3x^2 + 3y^2 + 5x - 13y + 2 &= 0 \\ \Rightarrow x^2 + y^2 + \frac{5}{3}x - \frac{13}{3}y + \frac{2}{3} &= 0 \end{aligned}$$

Comparing it with general equation of circle

$$\begin{aligned} 2g &= \frac{5}{3}, \quad 2f = -\frac{13}{3}, \quad c = \frac{2}{3} \\ \Rightarrow g &= \frac{5}{6}, \quad f = -\frac{13}{6} \end{aligned}$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

$$\text{Here } (x_1, y_1) = \left(1, \frac{10}{3}\right)$$

$$\begin{aligned} \Rightarrow 1 \cdot x + \frac{10}{3} \cdot y + \frac{5}{6}(x+1) - \frac{13}{6}\left(y + \frac{10}{3}\right) + \frac{2}{3} &= 0 \\ \Rightarrow x + \frac{10}{3}y + \frac{5}{6}x + \frac{5}{6} - \frac{13}{6}y + \frac{130}{18} + \frac{2}{3} &= 0 \\ \Rightarrow \frac{11}{6}x + \frac{7}{6}y + \frac{157}{18} = 0 \Rightarrow 33x + 21y + 157 = 0 \end{aligned}$$

is the required tangent.

Now equation of normal at (x_1, y_1)

$$(y - y_1)(x_1 + g) = (x - x_1)(y_1 + f)$$

$$\text{Here } (x_1, y_1) = \left(1, \frac{10}{3}\right)$$

$$\begin{aligned} \Rightarrow \left(y - \frac{10}{3}\right)\left(1 + \frac{5}{6}\right) &= (x - 1)\left(\frac{10}{3} - \frac{13}{6}\right) \\ \Rightarrow \left(y - \frac{10}{3}\right)\left(\frac{11}{6}\right) &= (x - 1)\left(\frac{7}{6}\right) \Rightarrow 11y - \frac{110}{3} = 7x - 7 \\ \Rightarrow 7x - 7 - 11y + \frac{110}{3} &= 0 \Rightarrow 7x - 11y + \frac{89}{3} = 0 \\ \Rightarrow 21x - 33y + 89 &= 0 \end{aligned}$$

is required equation of normal.

Question # 2

$$4x^2 + 4y^2 - 16x + 24y - 117 = 0 \dots\dots\dots (i)$$

Since abscissa = -4 , so putting $x = -4$ in given eq.

$$4(-4)^2 + 4y^2 - 16(-4) + 24y - 117 = 0$$

$$\Rightarrow 64 + 4y^2 + 64 + 24y - 117 = 0$$

$$\Rightarrow 4y^2 + 24y + 11 = 0$$

$$\Rightarrow y = \frac{-24 \pm \sqrt{(24)^2 - 4(4)(11)}}{2(4)}$$

$$= \frac{-24 \pm \sqrt{576 - 176}}{8} = \frac{-24 \pm \sqrt{400}}{8}$$

$$\Rightarrow y = \frac{-24 \pm 20}{8}$$

$$\Rightarrow y = \frac{-24 + 20}{8} \quad \text{or} \quad y = \frac{-24 - 20}{8}$$

$$\Rightarrow y = -\frac{1}{2} \quad \text{or} \quad y = -\frac{11}{2}$$

So we have points $\left(-4, -\frac{1}{2}\right)$ & $\left(-4, -\frac{11}{2}\right)$

$$(i) \Rightarrow x^2 + y^2 - 4x + 6y - \frac{117}{4} = 0$$

Comparing it with general equation of circle

$$2g = -4 \quad , \quad 2f = 6 \quad , \quad c = -\frac{117}{4}$$

$$\Rightarrow g = -2 \quad , \quad f = 3$$

Now equation of tangent at (x_1, y_1)

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

$$\text{For } (x_1, y_1) = \left(-4, -\frac{1}{2}\right)$$

Solve yourself as Q # 1(ii)

$$\text{For } (x_1, y_1) = \left(-4, -\frac{11}{2}\right)$$

Solve yourself as Q # 1(ii)

Position of the point with a circle

Consider the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

The point $P(x_1, y_1)$ lies on the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

The point $P(x_1, y_1)$ lies outside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$$

And the point $P(x_1, y_1)$ lies inside the circle if

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c < 0$$

Question # 3(i)

$$x^2 + y^2 = 81$$

$$\Rightarrow x^2 + y^2 - 81 = 0 \dots\dots\dots (i)$$

To check the position of point $(5, 6)$, Putting $x = 5$ & $y = 6$ on L.H.S of (i)

$$\begin{aligned} (5)^2 + (6)^2 - 81 &= 25 + 36 - 81 \\ &= -20 < 0 \end{aligned}$$

Hence $(5, 6)$ lies inside the circle.

Question # 3(ii)

$$2x^2 + 2y^2 + 12x - 8y + 1 = 0$$

$$\Rightarrow x^2 + y^2 + 6x - 4y + \frac{1}{2} = 0 \dots\dots\dots (i)$$

To check the position of point put $x=5$ & $y=6$ on L.H.S of (i)

$$\begin{aligned} (5)^2 + (6)^2 + 6(5) - 4(6) + \frac{1}{2} \\ = 25 + 36 + 30 - 24 + \frac{1}{2} \\ = \frac{135}{2} > 0 \end{aligned}$$

Hence (5,6) lies outside the circle.

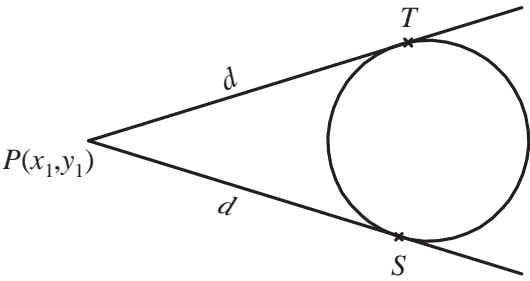
Length of tangent to the circle

Consider equation of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

If d denotes length of tangent from point $P(x_1, y_1)$ to the circle then

$$d = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$



Question # 4

$$5x^2 + 5y^2 - 10x + 15y - 131 = 0$$

$$\Rightarrow x^2 + y^2 - 2x + 3y - \frac{131}{5} = 0$$

Now length of tangent from point $P(x_1, y_1)$ is

$$d = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

For $(x_1, y_1) = (-5, 4)$

$$\begin{aligned} d &= \sqrt{(-5)^2 + (4)^2 - 2(-5) + 3(4) - \frac{131}{5}} \\ &= \sqrt{25 + 16 + 10 + 12 - \frac{131}{5}} \\ &= \sqrt{\frac{184}{5}} = 2\sqrt{\frac{46}{5}} \text{ units} \end{aligned}$$

Question # 5

$$2x + 3y = 13 \dots\dots\dots (i)$$

$$x^2 + y^2 = 13 \dots\dots\dots (ii)$$

From (i)

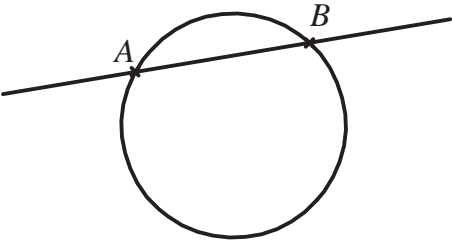
$$2x = 13 - 3y$$

$$\Rightarrow x = \frac{13 - 3y}{2} \dots\dots\dots (iii)$$

Putting in (ii)

$$\left(\frac{13 - 3y}{2}\right)^2 + y^2 = 13$$

$$\Rightarrow \frac{169 - 78y + 9y^2}{4} + y^2 = 13 \Rightarrow \frac{169 - 78y + 9y^2 + 4y^2}{4} = 13$$



$$\begin{aligned} \Rightarrow 13y^2 - 78y + 169 &= 104 \Rightarrow 13y^2 - 78y + 169 - 104 = 0 \\ \Rightarrow 13y^2 - 78y + 65 &= 0 \Rightarrow y^2 - 6y + 5 = 0 \\ \Rightarrow y^2 - 5y - y + 5 &= 0 \Rightarrow y(y-5) - 1(y-5) = 0 \\ \Rightarrow (y-5)(y-1) &= 0 \\ \Rightarrow y = 5 \text{ or } y = 1 \end{aligned}$$

Putting in (iii)

$$\begin{array}{l|l} x = \frac{13-3(5)}{2} & x = \frac{13-3(1)}{2} \\ = -1 & = 5 \end{array}$$

$\Rightarrow (-1, 5)$ and $(5, 1)$ are end points of chord intercepted.

$$\begin{aligned} \text{So length of chord} &= \sqrt{(5+1)^2 + (1-5)^2} \\ &= \sqrt{36+16} = \sqrt{52} = 2\sqrt{13} \end{aligned}$$

Question # 6

$$x^2 + y^2 - 2x - 2y - 39 = 0 \dots\dots\dots (i)$$

$$x + 2y = 6 \dots\dots\dots (ii)$$

Just solve (i) & (ii) to get the points

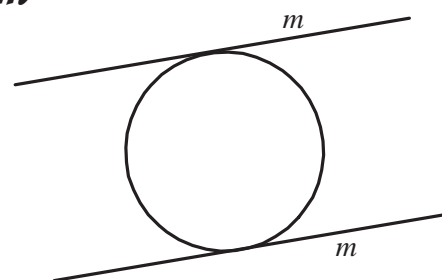
Equation of tangent to the circle having slope m

Consider an equation of circle

$$x^2 + y^2 = a^2$$

Then equations of tangents parallel to the line having slope m are

$$y = mx \pm a\sqrt{1+m^2}$$



Question # 7

$$x^2 + y^2 = 2$$

Centre of circle is at origin with radius $a = \sqrt{2}$

i) Let $l: x - 2y + 1 = 0$

$$\text{Slope of } l = m = -\frac{1}{-2} = \frac{1}{2}$$

Since required tangent is parallel to l

$$\therefore \text{Slope of tangent} = m = \frac{1}{2}$$

Now equations of tangents are

$$y = mx \pm a\sqrt{1+m^2}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\left(\frac{1}{2}\right)^2} \Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{1+\frac{1}{4}}$$

$$\Rightarrow y = \frac{1}{2}x \pm \sqrt{2}\sqrt{\frac{5}{4}} \Rightarrow y = \frac{1}{2}x \pm \sqrt{\frac{10}{4}} \Rightarrow y = \frac{1}{2}x \pm \frac{\sqrt{10}}{2}$$

$$\Rightarrow 2y = x \pm \sqrt{10}$$

$$\Rightarrow x - 2y \pm \sqrt{10} = 0 \text{ are the req. tangents.}$$

ii) Do yourself as above

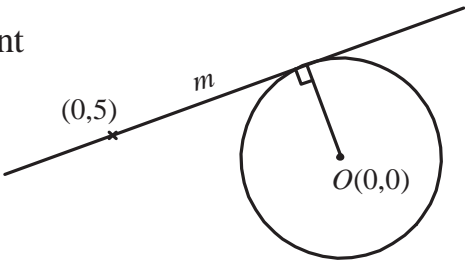
Question # 8(i)

$$x^2 + y^2 = 16$$

\Rightarrow radius of circle is 4 with centre $O(0,0)$

Let slope of required tangent be m , then eq. of tangent passing through $(0,5)$

$$\begin{aligned} y-5 &= m(x-0) \\ \Rightarrow y-5 &= mx \\ \Rightarrow mx-y+5 &= 0 \dots\dots\dots (i) \end{aligned}$$



Now since (i) is tangent to circle, therefore

Radius of circle = \perp ar distance of (i) from centre $O(0,0)$

$$\begin{aligned} \Rightarrow 4 &= \frac{|m(0)-0+5|}{\sqrt{m^2+(-1)^2}} \\ \Rightarrow 4 &= \frac{|5|}{\sqrt{m^2+1}} \quad \Rightarrow 4\sqrt{m^2+1} = |5| \end{aligned}$$

On squaring

$$\begin{aligned} \left(4\sqrt{1+m^2}\right)^2 &= |5|^2 \\ \Rightarrow 16(m^2+1) &= 25 \quad \Rightarrow 16m^2+16 = 25 \\ \Rightarrow 16m^2 &= 25-16 \quad \Rightarrow 16m^2 = 9 \\ \Rightarrow m^2 &= \frac{9}{16} \quad \Rightarrow m = \pm \frac{3}{4} \end{aligned}$$

When $m = \frac{3}{4}$, putting in (i)

$$\begin{aligned} \frac{3}{4}x - y + 5 &= 0 \\ \Rightarrow 3x - 4y + 20 &= 0 \end{aligned}$$

When $m = -\frac{3}{4}$, putting in (i)

$$\begin{aligned} -\frac{3}{4}x - y + 5 &= 0 \\ \Rightarrow 3x + 4y - 20 &= 0 \end{aligned}$$

Question # 8(ii)

$$x^2 + y^2 + 4x + 2y = 0$$

Comparing it with general equation of circle

We have $2g = 4$, $2f = 2$, $c = 0$

$$\Rightarrow g = 2 \quad , \quad f = 1$$

Centre $C(-g,-f) = C(-2,-1)$

$$\text{Radius} = \sqrt{g^2 + f^2 - c} = \sqrt{(2)^2 + (1)^2 - 0} = \sqrt{4+1} = \sqrt{5}$$

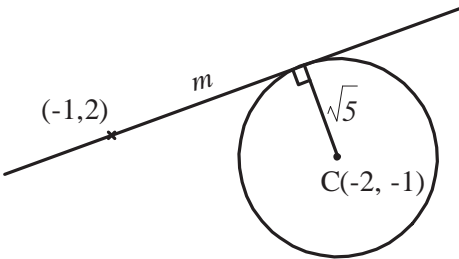
Let m be a slope of required tangent, so equation of tangent passing thorough $(-1,2)$

$$\begin{aligned} y-2 &= m(x+1) \\ \Rightarrow y-2 &= mx+m \\ \Rightarrow mx-y+(m+2) &= 0 \dots\dots\dots (i) \end{aligned}$$

\because (i) is tangent of circle,

$\therefore \perp$ ar distance of tangent from centre $(-2,-1) =$ Radius of circle

$$\begin{aligned} \Rightarrow \frac{|m(-2)-(-1)+(m+2)|}{\sqrt{(m)^2+(-1)^2}} &= \sqrt{5} \\ \Rightarrow \frac{|-2m+1+m+2|}{\sqrt{m^2+1}} &= \sqrt{5} \end{aligned}$$



$$\Rightarrow |-m+3| = \sqrt{5} \cdot \sqrt{m^2+1}$$

On squaring

$$m^2 - 6m + 9 = 5(m^2 + 1)$$

$$\Rightarrow 5m^2 + 5 - m^2 + 6m - 9 = 0 \Rightarrow 4m^2 + 6m - 4 = 0$$

$$\Rightarrow 2m^2 + 3m - 2 = 0 \Rightarrow 2m^2 + 4m - m - 2 = 0$$

$$\Rightarrow 2m(m+2) - 1(m+2) = 0 \Rightarrow (m+2)(2m-1) = 0$$

$$\Rightarrow m+2 = 0 \quad \text{or} \quad 2m-1 = 0$$

$$\Rightarrow m = -2 \quad \text{or} \quad m = \frac{1}{2}$$

Putting value of m in (i)

$$-2x - y + (-2+2) = 0$$

$$\Rightarrow -2x - y + 0 = 0$$

$$\Rightarrow 2x + y = 0$$

$$\frac{1}{2}x - y + \left(\frac{1}{2} + 2\right) = 0$$

$$\Rightarrow \frac{1}{2}x - y + \frac{5}{2} = 0$$

$$\Rightarrow x - 2y + 5 = 0$$

Question # 8(iii)

$$(x+1)^2 + (y-2)^2 = 26$$

$$\Rightarrow (x - (-1))^2 + (y - 2)^2 = (\sqrt{26})^2$$

Centre of circle is $(-1, 2)$ and radius $\sqrt{26}$

Now do yourself as above.

Note: To find point of contact, solve equation of tangent and circle.

Question # 9

Incomplete statement

Question # 10

Given: $2x^2 + 2y^2 - 8x + 12y + 21 = 0$

$$\Rightarrow x^2 + y^2 - 4x + 6y + \frac{21}{2} = 0$$

Comparing it with general equation of circle

$$2g = -4, \quad 2f = 6, \quad c = \frac{21}{2}$$

$$\Rightarrow g = -2, \quad f = 3$$

Let the point of contact of two tangent be $P(x_1, y_1)$ and $Q(x_2, y_2)$

Eq. of tangent at $P(x_1, y_1)$

$$x_1x + y_1y + g(x + x_1) + f(y + y_1) + c = 0$$

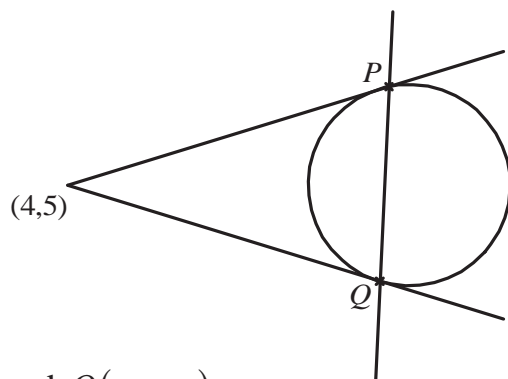
$$\Rightarrow x_1x + y_1y - 2(x + x_1) + 3(y + y_1) + \frac{21}{2} = 0$$

$$\Rightarrow x_1x + y_1y - 2x - 2x_1 + 3y + 3y_1 + \frac{21}{2} = 0$$

Since tangent is drawn from $(4, 5)$, therefore

$$\Rightarrow x_1(4) + y_1(5) - 2(4) - 2x_1 + 3(5) + 3y_1 + \frac{21}{2} = 0$$

$$\Rightarrow 4x_1 + 5y_1 - 8 - 2x_1 + 15 + 3y_1 + \frac{21}{2} = 0$$



$$\Rightarrow 2x_1 + 8y_1 + \frac{35}{3} = 0$$

$$\Rightarrow 6x_1 + 24y_1 + 35 = 0 \dots\dots\dots (i)$$

Similarly equation of tangent passing through $Q(x_2, y_2)$ and $(4,5)$ gives

$$6x_2 + 24y_2 + 35 = 0 \dots\dots\dots (ii)$$

Eqs. (i) and (ii) show that both points P & Q lies on the line

$$6x + 24y + 35 = 0$$

So it is the required equation of chord of contact.
