SOLVED EXERCISE 7.4

In Problems 1– 6, simplify each expression to a single trigonometric function.

1. $\dfrac{\sin^2 x}{\cos^2 x}$

Solution

$$\frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin^2 x}{\cos^2 x}\right)^2$$

$$= (\tan)^2 \qquad \because \frac{\sin x}{\cos x} = \tan x$$

$$= \tan^2 x$$

2. $\tan x \sin x \sec x$

Solution

$$\tan x \sin x \sec x = \frac{\sin x}{\cos x}.\sin x.\frac{1}{\cos x}$$

$$= \frac{\sin x.\sin x}{\cos x.\cos x}$$

$$= \frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin x}{\cos x}\right)^2$$

$$= (\tan x)^2 \qquad \because \frac{\sin x}{\cos x} = \tan x$$

$$= \tan^2 x$$

3. $\dfrac{\tan x}{\cos x}$

Solution

$$\frac{\tan x}{\cos x} = \frac{\tan x}{\cos x} = \frac{\sin x / \cos x}{1 / \cos x}$$

$$= \frac{\sin x}{\cos x}.\frac{\cos x}{1}$$

$$= \sin x$$

4. $1 - \cos^2 x$

Solution

$$1-\cos^2 x = 1-\left(1-\sin^2 x\right) \qquad \because \cos^2 x = 1-\sin^2 x$$
$$= 1-1+\sin^2 x$$
$$= \sin^2 x$$

5. $\sec^2 x - 1$

Solution

$$1-\cos^2 x \quad = \frac{1}{\cos^2 x} - 1 \qquad \because \sec x = \frac{1}{\cos x}$$
$$= \frac{1-\cos^2}{\cos^2 x}$$
$$= \frac{\sin^2 x}{\cos^2 x} \qquad \because 1-\cos^2 x = \sin^2 x$$
$$= \left(\frac{\sin x}{\cos x}\right)^2$$
$$= (\tan x)^2$$
$$= \tan^2 x$$

6. $\sec^2 x . \cot^2 x$

In problems 7 – 24, verify the identities.

Solution

$$\sin^2 x . \cot^2 x \quad = \sin^2 x . \frac{\cos^2 x}{\sin^2 x} \qquad \because \cot x = \frac{\cos x}{\sin x}$$
$$= \cos^2 x$$

In problem 7 – 24, verify the identities.

7. $(1-\sin\theta)(1+\sin\theta) = \cos^2\theta$

Solution

$$\text{L.H.S.} = (1-\sin\theta)(1+\sin\theta)$$
$$= 1-\sin^2\theta$$
$$= \cos^2\theta$$
$$= \text{R.H.S.}$$

Hence proved

8. $\dfrac{\sin\theta+\cos\theta}{\cos\theta} = 1+\tan\theta$

Solution

$$\text{L.H.S.} = \frac{\sin\theta+\cos\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\cos\theta}$$

$$= \tan\theta + 1 \qquad \because \frac{\sin\theta}{\cos\theta} = 1$$

$$= 1 + \tan\theta$$

$$= \text{R.H.S.}$$

Hence proved.

9. $(\tan\theta + \cot\theta)\tan\theta = \sec^2\theta$

Solution

$$\text{L.H.S.} = (\tan\theta + \cot\theta)\tan\theta$$

$$= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right) \cdot \frac{\sin\theta}{\cos\theta}$$

$$= \left(\frac{1}{\sin\theta\cos\theta}\right)\frac{\sin\theta}{\cos\theta} \qquad \because \sin^2\theta + \cos^2\theta = 1$$

$$= \frac{1}{\cos^2\theta}$$

$$= \sec^2\theta$$

$$= \text{R.H.S.}$$

Hence proved.

10. $(\cot\theta + \operatorname{cosec}\theta)(\tan\theta - \sin\theta) = \sec\theta - \cos\theta$

Solution

$$\text{L.H.S.} = (\cot\theta + \operatorname{cosec}\theta)(\tan\theta - \sin\theta)$$

$$= \left(\frac{\cot\theta}{\sin\theta} + \frac{1}{\sin\theta}\right)\left(\frac{\sin\theta}{\cos\theta} - \sin\theta\right)$$

$$= \left(\frac{\cos\theta + 1}{\sin\theta}\right)\left(\frac{\sin\theta - \sin\theta\cos\theta}{\cos\theta}\right)$$

$$= \left(\frac{1 + \cos\theta}{\sin\theta}\right)\left[\frac{\sin\theta(1 - \cos\theta)}{\cos\theta}\right]$$

$$= \frac{1 + \cos\theta(1 - \cos\theta)}{\cos\theta}$$

$$= \frac{1 - \cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \frac{\cos^2\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} - \cos\theta$$

$$= \sec\theta - \cos\theta$$

= R.H.S.

Hence proved.

11. $$\frac{\sin\theta + \cos\theta}{\tan^2\theta - 1} = \frac{\cos^2\theta}{\sin\theta - \cos\theta}$$

Solution

L.H.S. $$= \frac{\sin\theta + \cos\theta}{\tan^2\theta - 1}$$

$$= \frac{\sin\theta + \cos\theta}{\frac{\sin^1\theta}{\cos^2\theta} - 1}$$

$$= \frac{\sin\theta + \cos\theta}{\frac{\sin^2\theta - \cos^2\theta}{\cos^2\theta}}$$

$$= \frac{\cos^2\theta(\sin\theta + \cos\theta)}{\sin^2\theta - \cos^2\theta}$$

$$= \frac{\cos^2\theta(\sin\theta + \cos\theta)}{(\sin\theta - \cos\theta)(\sin\theta + \cos\theta)}$$

$$= \frac{\cos^2\theta}{\sin\theta - \cos\theta}$$

= R.H.S.

Hence proved.

12. $\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta = \operatorname{cosec}\theta$

Solution:

$$\begin{aligned}
\text{L.H.S.} &= \frac{\cos^2\theta}{\sin\theta} + \sin\theta \\
&= \frac{\cos^2\theta + \sin\theta}{\sin\theta} \\
&= \frac{1}{\sin\theta} \\
&= \operatorname{cosec}\theta \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

13. $\sec\theta - \cos\theta = \tan\theta\sin\theta$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \sec\theta - \cos\theta \\
&= \frac{1}{\cos\theta} - \cos\theta \\
&= \frac{1-\cos^2\theta}{\cos\theta} \\
&= \frac{\sin^2\theta}{\cos\theta} \\
&= \frac{\sin\theta}{\cos\theta}.\sin\theta \\
&= \tan\theta\sin\theta \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

14. $\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta = \sec\theta$

Solution

$$\begin{aligned}
\text{L.H.S.} &= \frac{\sin^2\theta}{\cos\theta} + \cos\theta \\
&= \frac{\sin^2\theta + \cos\theta}{\cos\theta} \\
&= \frac{1}{\cos\theta} \\
&= \sec\theta \\
&= \text{R.H.S.}
\end{aligned}$$

Hence proved.

15. $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= \tan\theta + \cot\theta \\
&= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta}\cdot\frac{1}{\cos\theta} \\
&= \operatorname{cosec}\theta \sec\theta \\
&= \text{R.H.S.}
\end{aligned}
$$

Hence proved.

16. $(\tan\theta + \cot\theta) = (\cos\theta + \sin\theta) = \sec\theta + \operatorname{cosec}\theta$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= (\tan\theta + \cot\theta)(\cos\theta + \sin\theta) \\
&= \left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)(\cos\theta + \sin\theta) \\
&= \left(\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}\right)(\cos\theta + \sin\theta) \\
&= \left(\frac{1}{\sin\theta\cos\theta}\right)(\cos\theta + \sin\theta) \\
&= \frac{\cos\theta + \sin\theta}{\sin\theta\cos\theta} \\
&= \frac{\cos\theta}{\sin\theta\cos\theta} + \frac{\sin\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta} + \frac{1}{\cos\theta} \\
&= \operatorname{cosec}\theta + \sec\theta \\
&= \text{R.H.S.}
\end{aligned}
$$

Hence proved.

17. $\sin\theta(\tan\theta + \cot\theta) = \sec\theta$

Solution

$$\text{L.H.S.} = \sin\theta(\tan\theta + \cot\theta)$$

$$= \sin\theta\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$$

$$= \sin\theta\left(\frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\right)$$

$$= \sin\theta\left(\frac{1}{\sin\theta\cos\theta}\right)$$

$$= \frac{1}{\cos\theta}$$

$$= \sec\theta$$

$$= \text{R.H.S.}$$

Hence proved.

18. $\dfrac{1+\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1+\cos\theta} = 2\operatorname{cosec}\theta$

Solution

$$\text{L.H.S.} = \frac{1+\cos\theta}{\sin\theta} + \frac{\sin\theta}{1+\cos\theta}$$

$$= \frac{(1+\cos\theta)^2 + (\sin\theta)^2}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{1 + 2\cos\theta + 1}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2 + 2\cos\theta}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2(1+\cos\theta)}{\sin\theta(1+\cos\theta)}$$

$$= \frac{2}{\sin\theta}$$

$$= 2\operatorname{cosec}\theta$$

$$= \text{R.H.S.}$$

Hence proved.

19. $\dfrac{1}{1-\cos\theta} + \dfrac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$

Solution

$$\text{L.H.S.} = \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$$

$$= \frac{1+\cos\theta+1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$$

$$= \frac{2}{1-\cos^2\theta}$$

$$= \frac{2}{\sin^2\theta}$$

$$= 2\operatorname{cosec}^2\theta$$

$$= \text{R.H.S.}$$

Hence proved.

20. $\dfrac{1+\sin\theta}{1-\sin\theta} - \dfrac{1-\sin\theta}{1+\sin\theta} = 4\tan\theta\sec\theta$

Solution

$$\text{L.H.S.} = \frac{1+\sin\theta}{1-\sin\theta} - \frac{1-\sin\theta}{1+\sin\theta}$$

$$= \frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$

$$= \frac{(1+2\sin\theta+\sin^2\theta)-(1-2\sin\theta+\sin^2\theta)}{1-\sin^2\theta}$$

$$= \frac{1+2\sin\theta+\sin^2\theta-1+2\sin\theta-\sin^2\theta}{\cos^2\theta}$$

$$= \frac{4\sin\theta}{\cos^2\theta}$$

$$= 4\frac{\sin\theta}{\cos\theta}.\frac{1}{\cos\theta}$$

$$= 4\tan\theta\sec\theta$$

$$= \text{R.H.S.}$$

Hence proved.

21. $\sin^3\theta = \sin\theta - \sin\theta\cos^2\theta$

Solution

$$\text{L.H.S.} = \sin\theta - \sin\theta\cos^2\theta$$

$$= \sin\theta(1-\cos^2\theta)$$

$$= \sin\theta.\sin^2\theta$$

$$= \sin^3\theta$$

$$= \text{R.H.S.}$$

Hence proved.

22. $\cos^4\theta - \sin^4\theta = \left(\cos^2\theta - \sin^2\theta\right)$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= \cos^4\theta - \sin^4\theta \\
&= \left(\cos^2\theta\right)^2 - \left(\sin^2\theta\right)^2 \\
&= \left(\cos^2\theta - \sin^2\theta\right)\left(\cos^2\theta + \sin^2\theta\right) \\
&= \left(\cos^2\theta - \sin^2\theta\right)(1) \\
&= \cos^2\theta - \sin^2\theta \\
&= \text{R.H.S.}
\end{aligned}
$$

Hence proved.

23. $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \dfrac{\sin\theta}{1-\cos\theta}$

Solution

$$
\begin{aligned}
\text{L.H.S.} &= \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} \\
&= \frac{\sqrt{1+\cos\theta}}{\sqrt{1-\cos\theta}} \times \frac{\sqrt{1+\cos\theta}}{\sqrt{1+\cos\theta}} \\
&= \frac{\left(\sqrt{1-\cos\theta}\right)^2}{\sqrt{(1)^2 - (\cos\theta)^2}} \\
&= \frac{1+\cos\theta}{\sqrt{1-\cos^2\theta}} \\
&= \frac{1+\cos\theta}{\sqrt{\sin^2\theta}} \\
&= \frac{1+\cos\theta}{\sin\theta} \times \frac{1-\cos\theta}{1-\cos\theta} \\
&= \frac{(1)^2 - (\cos\theta)^2}{\sin\theta(1-\cos\theta)} \\
&= \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)} \\
&= \frac{1-\cos^2\theta}{\sin\theta(1-\cos\theta)}
\end{aligned}
$$

$$= \frac{\sin^2\theta}{\sin\theta(1-\cos\theta)}$$

$$= \frac{\sin\theta}{1-\cos\theta}$$

$=$ R.H.S.

Hence proved.

24. $\sqrt{\frac{\sec\theta+1}{\sec\theta-1}} = \frac{\sec\theta+1}{\tan\theta}$

Solution

$$\text{L.H.S.} = \sqrt{\frac{\sec\theta+1}{\sec\theta-1}}$$

$$= \frac{\sqrt{\sec\theta+1}}{\sqrt{\sec\theta-1}}$$

$$= \frac{\sqrt{\sec\theta+1}}{\sqrt{\sec\theta-1}} \times \frac{\sec\theta+1}{\sec\theta+1}$$

$$= \frac{\left(\sqrt{\sec\theta+1}\right)^2}{\sqrt{\sec^2\theta-1}}$$

$$= \frac{\sec\theta+1}{\sqrt{\tan^2\theta}}$$

$$= \frac{\sec\theta+1}{\tan\theta}$$

$=$ R.H.S.

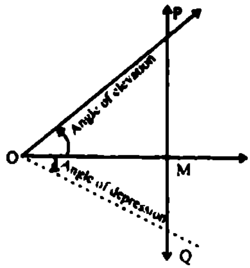
Hence proved.

Angle of Elevation and Angle of Depression:

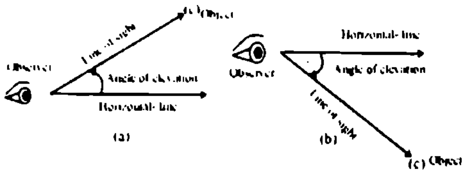
One of the objects of trigonometry is to find the distances between points or the heights of objects, without actually measuring these distances or heights.

Angle of elevation: Suppose O, P and Q are three points, P being at a higher level of O and Q being at lower level than O. Let a horizontal line drawn through O meet in M, the vertical line drawn through P and Q.

The angle MOP is called the angle of elevation of point P as seen from O. For looking at Q below the horizontal line we have to lower our eyes and $\angle MOQ$ is called the angle of depression.



We measure an angle of elevation from a horizontal line up to an object or an angle of depression from a horizontal line down to an object.

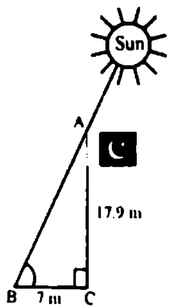


Find angle of elevation and angle of depression.

For finding distances height and angles by the use of trigonometric functions, consider the following examples:

Example 1

A flagpole 17.9 meter high casts a 7 meter shadow. Find the angle of elevation of the sun.



Solution

From the figure, we observe that is the angle of elevation.

Using the fact that

$$\tan\alpha = \frac{AC}{BC} = \frac{17.9}{7} = 255714$$

Solving for a gives us

$$\alpha = \tan^{-1}(2.55714)$$

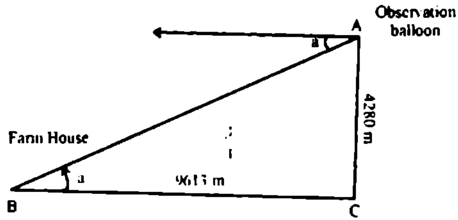
$$= (68.6666)^{\circ} = 68^{\circ}40'$$

$$\Rightarrow \qquad \alpha = 68^{\circ}40'$$

Example 2

An observation balloon is 4280 meter above the ground and 9613 meter away from a farmhouse. Find the angle of depressor of the farmhouse as observed from the observation balloon.

Solution



For problems of this type the angle of elevation of A from B is considered equal to the angle of depression of B from A, as shown in the diagram.

$$\tan\alpha = \frac{AC}{BC} = \frac{4280}{9613} = 0.44523$$

$$\alpha = \tan^{-1}(0.44523) = 24^{\circ}$$

So, angle of depression is 24°.

SOLVED EXERCISE 7.5

1. Find the angle of elevation of the sun if a 6 feet man casts a 3.5 feet shadow.

Solution

From the figure, we observe that $\propto$ is the angle of elevation.

Using the fact that

$$\tan\propto = \frac{\overline{BC}}{\overline{AC}}$$

$$\tan\propto = \frac{6}{3.5}$$