SOLVED EXERCISE 2.2

1. Find the cube roots of -1, 8, −27, 64.

Solution:

(i) The tree cube roots of − 1

Let $x^3 = -1$

$$(x)^3 + (1)^3 = 0$$

$$(x+1)(x^2 - x + 1) = 0$$

Either $x + 1 = 0$ or $x^2 - x + 1 = 0$

$x = -1$

Here $a = 1, b = -1, c = 1$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{(-1) \pm \sqrt{(1)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{1-4}}{2}$$

$$x = \frac{1 \pm \sqrt{-3}}{2}$$

$$x = \frac{1+\sqrt{-3}}{2} \quad \text{or} \quad x = \frac{1-\sqrt{-3}}{2}$$

$$= -\left(\frac{-1-\sqrt{-3}}{2}\right) \qquad = -\left(\frac{-1+\sqrt{-3}}{2}\right)$$

$$= -\omega^2 \qquad = -\omega$$

Three cube roots of -1 are $-1, -\omega, -\omega^2$

(ii) The three cube roots of 8

Let $x^3 = 8$

$$x^3 - 8 = 0$$

$$(x)^3 - (2)^3 = 0$$

$$(x-2)(x^2 + 2x + 4) = 0$$

Either $x - 2 = 0$ or $x^2 + 2x + 4 = 0$

$x = 2$

Here $a = 1, b = 2, c = 4$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$x = \frac{-2\sqrt{4-16}}{2}$$

$$x = \frac{-2 \pm \sqrt{-12}}{2}$$

$$x = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$x = \frac{2\left(-1 \pm \sqrt{-3}\right)}{2}$$

$$x = \frac{2\left(-1 \pm i\sqrt{3}\right)}{2} \because i = \sqrt{-1}$$

$$x = \frac{2\left(-1 + i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x = 2\left(-\frac{1 - i\sqrt{3}}{2}\right)$$

$$= 2\omega \qquad\qquad = 2\omega^2$$

Three cube roots of 8 are $2, 2\omega, 2\omega^2$

(iii) The three cube roots of – 27

Let $x^3 = -27$

$x^3 + 27 = 0$

$(x)^3 - (3)^3 = 0$

$(x+3)(x^2 - 3x + 9) = 0$

Either $x + 3 = 0$ or $x^2 - 3x + 9 = 0$

$x = -3$ Here $a = 1$; $b = -3$, $c = 9$

Using quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(-3)^2 - 4(1)(9)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9-36}}{2}$$

$$x=\frac{3\pm\sqrt{-27}}{2}$$

$$x=\frac{3\pm3\sqrt{-3}}{2}$$

$$x=\frac{3\left(-1\pm i\sqrt{3}\right)}{2} \qquad \because i=\sqrt{-1}$$

$$x=\frac{3\left(1+i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x=3\left(-\frac{1-i\sqrt{3}}{2}\right)$$

$$x=\frac{-3\left(-1-i\sqrt{3}\right)}{2} \qquad \text{or} \qquad =-3\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$=-3\omega^2 \qquad\qquad =-3\omega$$

$\therefore$ Three cube roots of -27 are $-3, -3\omega, -3\omega^2$

(iv) The three cube roots of 64

Let $x^3=64$

$$x^3-64=0$$

$$(x)^3-(4)^3=0$$

$$(x-4)\left(x^2+4x+16\right)=0$$

Either $x-3=0$ or $\qquad x^2+4x+16=0$

$x=4$ $\qquad$ Here $a=1, b=4, c=16$

Using quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x=\frac{-4\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$$

$$x=\frac{-4\pm\sqrt{16-64}}{2}$$

$$x=\frac{4\pm\sqrt{-48}}{2}$$

$$x=\frac{-4\pm4\sqrt{-3}}{2}$$

$$x=\frac{4\left(-1\pm i\sqrt{3}\right)}{2} \qquad i=\sqrt{-1}$$

$$x = \frac{3\left(-1+i\sqrt{3}\right)}{2} \qquad \text{or} \qquad x = 4\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$= 4\omega$ $\qquad\qquad\qquad\qquad$ $= 4\omega^2$

Three cube roots of 64 are 4, 4, ω, $4\omega^2$

2. Evaluate

(i) $(1 - \omega - \omega^2)^7$

Solution:

$$\begin{aligned}(1 - \omega - \omega^2)^7 &= \left[1-\left(\omega+\omega^2\right)\right]^7 \\ &= \left[-1-(-1)\right]^7 \quad \because \omega+\omega^2=-1 \\ &= (1+1)^7 \\ &= 2^7 = 128\end{aligned}$$

(ii) $(1 - 3\omega - 3\omega^2)^5$

Solution:

$$\begin{aligned}(1 - 3\omega - 3\omega^2)^5 &= \left[1-3\left(\omega+\omega^2\right)\right]^5 \\ &= \left[1-3(-1)\right]^5 \quad \because \omega+\omega^2=-1 \\ &= (1+3)^5 \\ &= 4^5 = 1024\end{aligned}$$

(iii) $(9 + 4\omega + 4\omega^2)^3$

Solution:

$$\begin{aligned}(9 + 4\omega + 4\omega^2)^3 &= \left[9+4\left(\omega+\omega^2\right)\right]^3 \\ &= \left[9+4(-1)\right]^3 \quad \because \omega+\omega^2=-1 \\ &= (9-4)^3 \\ &= 5^5 = 125\end{aligned}$$

(iv) $(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2)$

Solution:

$$\begin{aligned}&(2 + 2\omega - 2\omega^2)(3 - 3\omega + 3\omega^2) \\ &= \left[2(1+\omega)-2\omega^2\right]\left[3\left(1+\omega^2\right)-3\omega\right] \\ &= \left[2\left(-\omega^2\right)-2\omega^2\right]\left[3(-\omega)-3\omega\right] \quad \because 1+\omega=-\omega^2\end{aligned}$$

$$=\left(-2\omega^2-2\omega^2\right)\left(-3\omega-3\omega\right) \qquad \because 1+\omega^2=-\omega$$

$$=\left(-4\omega^2\right)\left(-6\omega\right)$$

$$=(-4)(-6)\left(\omega^2.\omega\right)$$

$$=24\omega^3$$

$$=24(1) \qquad \because \omega^3=1$$

$$=24$$

(v) $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$

Solution:

$$\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$$

$$=(2\omega)^6+\left(2\omega^2\right)^6 \qquad \because \omega=\frac{-1+\sqrt{-3}}{2} \quad \text{and} \quad \omega^2=\frac{-1-1\sqrt{-3}}{2}$$

$$=2^6\left(\omega^6\right)+2^6\left(\omega^{12}\right) \qquad 2\omega=-1+\sqrt{-3} \qquad 2\omega^2=-1-\sqrt{-3}$$

$$=2^6\left[\left(\omega^3\right)^2\right]+2^6\left[\left(\omega^3\right)^4\right]$$

$$=2^6\left[(1)^2+(1)^4\right] \qquad \because \omega^3=1$$

$$=2^6[1+1]$$

$$=2^6.2=2^{6+1}=2^7$$

$$=128$$

(vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$

Solution:

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$$

$$=\omega^9+\left(2\omega^2\right)^9 \qquad \because \omega=\frac{-1+\sqrt{-3}}{2} \text{ and } \omega^2=\frac{-1-1\sqrt{-3}}{2}$$

$$=\omega^9+\omega^{18}$$

$$=\left(\omega^3\right)^3+\left(\omega^3\right)^6 \quad =(1)^3+(1)^6 \qquad \because \omega^3=1$$

$$=1+1 \quad =2$$

(vii) $\omega^{37}+\omega^{38}-5$

Solution:

$\omega^{37} + \omega^{38} - 5$

$= \omega^{37} + \omega^{38} - 5$

$= \omega^{36}.\omega + \omega^{36}.\omega^2 - 5$

$= \left(\omega^3\right)^{12}.\omega + \left(\omega^3\right)^{12}.\omega^2 - 5$

$= (1)^{12}.\omega + (1)^{12}.\omega^2 - 5 \qquad \because \omega^3 = 1$

$= \omega + \omega^2 - 5$

$= -1 - 5 \qquad \because \omega + \omega^2 = -1$

$= -6$

(viii) $\omega^{-13} + \omega^{-17}$

Solution:

$\omega^{-13} + \omega^{-17}$

$= \omega^{-13} + \omega^{-17}$

$= \omega^{-12-1} + \omega^{-15-2}$

$= \omega^{12}.\omega^{-1} + \omega^{-15}.\omega^{-2}$

$= \left(\omega^3\right)^{-4}.\omega^{-1} + \left(\omega^3\right)^{-5}.\omega^{-2}$

$= (1)^{-4}.\omega^{-1} + (1)^{-5}.\omega^{-2} \qquad \because \omega^3 = 1$

$= \omega^{-1} + \omega^{-2}$

$= \dfrac{1}{\omega} + \dfrac{1}{\omega^2}$

$= \dfrac{\omega^2 + \omega}{\omega^3} \qquad \because \omega^2 + \omega = -1 \text{ and } \omega^3 = 1$

$= \dfrac{-1}{1}$

$= -1$

3. Prove that $x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$.

Solution:

$x^3 + y^3 = (x + y)(x + \omega y)(x + \omega^2 y)$

R.H.S. $= (x + y)(x + \omega y)(x + \omega^2 y)$

$= (x + y)\left[x\left(x + \omega^2 y\right) + \omega y\left(\omega + \omega^2 y\right)\right]$

$= (x + y)\left[x^2 + \omega^2 xy + \omega xy + \omega^3 y^2\right]$

$= (x + y)\left[x^2 + \left(\omega^2 + \omega\right)xy + (1)y^2\right] \qquad \because \omega^3 = 1$

$$= (x+y)\left[x^2 + (-1)xy + y^2\right] \qquad \because \omega^2 + \omega = -1$$

$$= (x+y)(x^2 - xy + y^2)$$

$= L.H.S.$

Hence Proved

4. Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$.

Solution:

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z).$$

R.H.S. $= (x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

$$= (x+y+z)\left[x(x+\omega^2 y+\omega z)+\omega y(x+\omega^2 y+\omega z)+\omega^2 z(x+\omega^2 y+\omega z)\right]$$

$$= (x+y+z)\left[x^2+\omega^2 xy+\omega xz+\omega xy+\omega^3 y^2+\omega^2 yz+\omega^2 xz+\omega^4 z+\omega^3 z^2\right]$$

$$= (x+y+z)\left[x^2+\omega^2 xy+\omega xy+\omega^2 yz+\omega yz+\omega^2 xz+\omega xz+(1)y^2+(1)z^2\right]$$

$$= (x+y+z)\left[x^2+(\omega^2+\omega)xy+(\omega^2+\omega)yz+(\omega^2+\omega)xz+y^2+z^2\right] \because \omega^3 = 1$$

$$= (x+y+z)\left[x^2+(-1)xy+(-1)yz+(-1)xz+y^2+z^2\right] \quad \because \omega^2+\omega=-1$$

$$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

$$= x^3+y^3+z^3-3xz$$

$= L.H.S.$

Hence proved.

5. Prove that $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8)$ ..... 2n factors = 1.

Solution:

L.H.S. $= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$ ..... 2n factors

$= (1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)$.........2n factors

$= (1+\omega)(1+\omega^2)(1+\omega^3.\omega)(1+\omega^2.\omega^6)$........2n factors

$= (1+\omega)(1+\omega^2)(1+\omega^3.\omega)\left(1+\omega^2.(\omega^3)^2\right)$......2n factors

$= (1+\omega)(1+\omega^2)(1+(1)\omega)\left(1+\omega^2(1)^2\right)$.......2n factors $\quad \because \omega^3 = 1$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)$......2n factors

$= (-\omega)^2(-\omega)(-\omega^2)(-\omega)$.....2n factors $\qquad \because 1+\omega=-\omega^2$

$= \left[(-\omega)^2(-\omega)\right]\left[(-\omega^2)(-\omega)\right]$......n factors $\qquad \because 1+\omega^2=-\omega$

$= \left[\omega^3\right]\left[\omega^3\right]$......n factors

$= (1)(1)\ldots\ldots n$ factors $\quad \because \omega^3 = 1$

$= (1)^n$

$= 1$

= R.H.S.

Hence proved.

Roots and co-efficient of a quadratic equation:

We know that $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2 + bx + c = 0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co-efficient of a quadratic equation:

If $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$,

then we can find the sum and the product of the roots as follows.

Sum of the' roots $= \alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

Product of the roots $= \alpha\beta$

$$= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S = -\frac{b}{a} = -\frac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2} \quad \text{and} \quad P = -\frac{c}{a} = -\frac{\text{Constant term}}{\text{Co-efficient of } x^2}$$

SOLVED EXERCISE 2.3

1. **Without solving, find the sum and the product of the following quadratic equations.**

(i) $x^2 - 5x + 3 = 0$

Solution: