

## EXERCISE 2.9

### Increasing and Decreasing Function (P-104)

Let  $f$  be defined on an interval  $(a, b)$  and let  $x_1, x_2 \in (a, b)$ . Then

1.  $f$  is increasing on the interval  $(a, b)$  if  $f(x_2) > f(x_1)$  whenever  $x_2 > x_1$
2.  $f$  is decreasing on the interval  $(a, b)$  if  $f(x_2) < f(x_1)$  whenever  $x_2 > x_1$

### Theorem ((P-105)

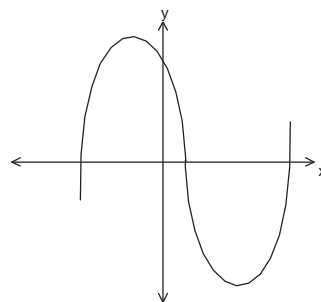
Let  $f$  be differentiable on the open interval  $(a, b)$

- 1-  $f$  is increasing on  $(a, b)$  if  $f'(x) > 0$  for each  $x \in (a, b)$ .
- 2-  $f$  is decreasing on  $(a, b)$  if  $f'(x) < 0$  for each  $x \in (a, b)$ .

### First Derivative Test (P-109)

Let  $f$  be differentiable in neighbourhood of  $c$ , where  $f'(c) = 0$ .

1. The function has relative maxima at  $x = c$  if  $f'(x) > 0$  before  $x = c$  and  $f'(x) < 0$  after  $x = c$ .
2. The function has relative minima at  $x = c$  if  $f'(x) < 0$  before  $x = c$  and  $f'(x) > 0$  after  $x = c$ .



### Second Derivative Test (P-111)

Let  $f$  be differential function in a neighbourhood of  $c$ , where  $f'(c) = 0$ . Then

- 1-  $f$  has relative maxima at  $c$  if  $f''(c) < 0$ .
- 2-  $f$  has relative minima at  $c$  if  $f''(c) > 0$ .

### Question # 1(i)

$$f(x) = \sin x \quad ; \quad x \in [-\pi, \pi]$$

$$\Rightarrow f'(x) = \cos x$$

$$\text{Put } f'(x) = 0 \Rightarrow \cos x = 0$$

$$\Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$$

So we have sub-intervals  $\left(-\pi, -\frac{\pi}{2}\right)$ ,  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ ,  $\left(\frac{\pi}{2}, \pi\right)$

$$f'(x) = \cos x < 0 \text{ whenever } x \in \left(-\pi, -\frac{\pi}{2}\right)$$

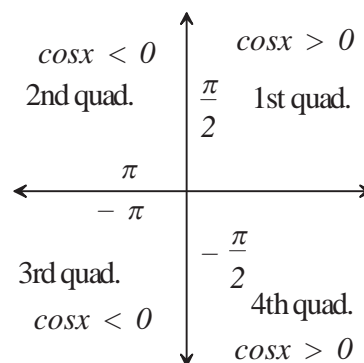
So  $f$  is decreasing on the interval  $\left(-\pi, -\frac{\pi}{2}\right)$ .

$$f'(x) = \cos x > 0 \text{ whenever } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

So  $f$  is increasing on the interval  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ .

$$f'(x) = \cos x < 0 \text{ whenever } x \in \left(\frac{\pi}{2}, \pi\right)$$

So  $f$  is decreasing on the interval  $\left(\frac{\pi}{2}, \pi\right)$ .



**Question # 1(ii)**

$$f(x) = \cos x \quad ; \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\Rightarrow f'(x) = -\sin x$$

$$\text{Put } f'(x) = 0 \Rightarrow -\sin x = 0 \Rightarrow \sin x = 0 \Rightarrow x = 0$$

So we have sub-intervals  $\left(-\frac{\pi}{2}, 0\right)$  and  $\left(0, \frac{\pi}{2}\right)$ .

Now  $f'(x) = -\sin x > 0$  whenever  $x \in \left(-\frac{\pi}{2}, 0\right)$

So  $f$  is increasing on  $\left(-\frac{\pi}{2}, 0\right)$

$$f'(x) = -\sin x < 0 \text{ whenever } x \in \left(0, \frac{\pi}{2}\right)$$

So  $f$  is decreasing on  $\left(0, \frac{\pi}{2}\right)$ .

---

**Question # 1(iii)**

$$f(x) = 4 - x^2 \quad ; \quad x \in [-2, 2]$$

$$\Rightarrow f'(x) = -2x$$

$$\text{Put } f'(x) = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$$

So we have subintervals  $(-2, 0)$  and  $(0, 2)$

$$\because f'(x) = -2x > 0 \text{ whenever } x \in (-2, 0)$$

$\therefore f$  is increasing on the interval  $(-2, 0)$

$$\text{Also } f'(x) = -2x < 0 \text{ whenever } x \in (0, 2)$$

$\therefore f$  is decreasing on  $(0, 2)$

---

**Question # 1(iv)**

$$f(x) = x^2 + 3x + 2 \quad ; \quad x \in [-4, 1]$$

$$\Rightarrow f'(x) = 2x + 3$$

$$\text{Put } f'(x) = 0 \Rightarrow 2x + 3 = 0 \Rightarrow x = -\frac{3}{2}$$

So we have sub-intervals  $\left(-4, -\frac{3}{2}\right)$  and  $\left(-\frac{3}{2}, 1\right)$

Now  $f'(x) = 2x + 3 < 0$  whenever  $x \in \left(-4, -\frac{3}{2}\right)$

So  $f$  is decreasing on  $\left(-4, -\frac{3}{2}\right)$

$$\text{Also } f'(x) > 0 \text{ whenever } x \in \left(-\frac{3}{2}, 1\right)$$

Therefore  $f$  is increasing on  $\left(-\frac{3}{2}, 1\right)$ .

---

**Question # 2(i)**

$$f(x) = 1 - x^3$$

Diff. w.r.t  $x$

$$f'(x) = -3x^2 \dots\dots (i)$$

$$\text{For stationary points, put } f'(x) = 0 \Rightarrow -3x^2 = 0 \Rightarrow x = 0$$

Diff (i) w.r.t  $x$

$$f''(x) = -6x \dots\dots\dots (ii)$$

Now put  $x = 0$  in (ii)

$$f''(0) = -6(0) = 0$$

So second derivative test fails to determinate the extreme points.

Put  $x = 0 - \varepsilon = -\varepsilon$  in (i)

$$f'(x) = -3(-\varepsilon)^2 = -3\varepsilon^2 < 0$$

Put  $x = 0 + \varepsilon = \varepsilon$  in (i)

$$f'(x) = -3(\varepsilon)^2 = -3\varepsilon^2 < 0$$

As  $f'(x)$  does not change its sign before and after  $x = 0$ .

Since at  $x = 0$ ,  $f(x) = 1$  therefore  $(0,1)$  is the point of inflexion.

### Question # 2(ii)

$$f(x) = x^2 - x - 2$$

Diff. w.r.t.  $x$

$$f'(x) = 2x - 1 \dots\dots\dots (i)$$

For stationary points, put  $f'(x) = 0$

$$\Rightarrow 2x - 1 = 0 \quad \Rightarrow 2x = 1 \quad \Rightarrow x = \frac{1}{2}$$

Diff (i) w.r.t  $x$

$$f''(x) = \frac{d}{dx}(2x - 1) = 2$$

$$\text{As } f''\left(\frac{1}{2}\right) = 2 > 0$$

Thus  $f(x)$  is minimum at  $x = \frac{1}{2}$

$$\text{And } f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 2 = -\frac{9}{4}$$

### Question # 2(iii)

$$f(x) = 5x^2 - 6x + 2$$

Diff. w.r.t.  $x$

$$f'(x) = 10x - 6 \dots\dots\dots (i)$$

For stationary points, put  $f'(x) = 0$

$$\Rightarrow 10x - 6 = 0 \quad \Rightarrow 10x = 6 \quad \Rightarrow x = \frac{6}{10} \quad \Rightarrow x = \frac{3}{5}$$

Diff (i) w.r.t  $x$

$$f''(x) = \frac{d}{dx}(10x - 6) = 10$$

$$\text{As } f''\left(\frac{3}{5}\right) = 10 > 0$$

Thus  $f(x)$  is minimum at  $x = \frac{3}{5}$

$$\text{And } f\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right)^2 - 6\left(\frac{3}{5}\right) + 2 = \frac{9}{5} - \frac{18}{5} + 2 = \frac{1}{5}$$

### Question # 2(iv)

$$f(x) = 3x^2$$

Diff. w.r.t  $x$

$$f'(x) = 6x \dots\dots\dots (i)$$

For stationary points, put  $f'(x) = 0$

$$\Rightarrow 6x = 0 \Rightarrow x = 0$$

Diff. (i) w.r.t  $x$

$$f''(x) = 6$$

At  $x = 0$

$$f''(0) = 6 > 0$$

$\Rightarrow f$  has minimum value at  $x = 0$

$$\text{And } f(0) = 3(0)^2 = 0$$

### Question # 2(vi)

$$f(x) = 2x^3 - 2x^2 - 36x + 3$$

Diff. w.r.t  $x$

$$f'(x) = \frac{d}{dx}(2x^3 - 2x^2 - 36x + 3) = 6x^2 - 4x - 36 \dots\dots\dots(i)$$

For stationary points, put  $f'(x) = 0$

$$\Rightarrow 6x^2 - 4x - 36 = 0$$

$$\Rightarrow 3x^2 - 2x - 12 = 0 \quad \div \text{ing by } 2$$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 - 4(3)(-18)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{4 + 216}}{6} = \frac{2 \pm \sqrt{220}}{6} = \frac{2 \pm 2\sqrt{55}}{6} = \frac{1 \pm \sqrt{55}}{3}$$

Diff. (i) w.r.t  $x$

$$f''(x) = \frac{d}{dx}(6x^2 - 4x - 36) = 12x - 4$$

$$\text{Now } f''\left(\frac{1 + \sqrt{55}}{3}\right) = 12\left(\frac{1 + \sqrt{55}}{3}\right) - 4$$

$$= 4(1 + \sqrt{55}) - 4 = 4 + 4\sqrt{55} - 4 = 4\sqrt{55} > 0$$

$$\Rightarrow f(x) \text{ has relative minima at } x = \frac{1 + \sqrt{55}}{3}.$$

$$\begin{aligned} \text{And } f\left(\frac{1 + \sqrt{55}}{3}\right) &= 2\left(\frac{1 + \sqrt{55}}{3}\right)^3 - 2\left(\frac{1 + \sqrt{55}}{3}\right)^2 - 36\left(\frac{1 + \sqrt{55}}{3}\right) + 3 \\ &= \frac{2}{27}(1 + \sqrt{55})^3 - \frac{2}{9}(1 + \sqrt{55})^2 - 12(1 + \sqrt{55}) + 3 \\ &= \frac{2}{27}(1 + 3\sqrt{55} + 3 \cdot 55 + 55\sqrt{55}) - \frac{2}{9}(1 + 2\sqrt{55} + 55) - 12(1 + \sqrt{55}) + 3 \\ &= \frac{2}{27}(166 + 58\sqrt{55}) - \frac{2}{9}(56 + 2\sqrt{55}) - 12(1 + \sqrt{55}) + 3 \\ &= \frac{332}{27} + \frac{116}{27}\sqrt{55} - \frac{112}{9} - \frac{4}{9}\sqrt{55} - 12 - 12\sqrt{55} + 3 \\ &= -\frac{247}{27} - \frac{220}{27}\sqrt{55} = -\frac{1}{27}(247 + 220\sqrt{55}) \end{aligned}$$

$$\text{Also } f''\left(\frac{1 - \sqrt{55}}{3}\right) = 12\left(\frac{1 - \sqrt{55}}{3}\right) - 4$$

$$= 4(1 - \sqrt{55}) - 4 = 4 - 4\sqrt{55} - 4 = -4\sqrt{55} < 0$$

$$\Rightarrow f(x) \text{ has relative maxima at } x = \frac{1 - \sqrt{55}}{3}.$$

$$\text{And Since } f\left(\frac{1 - \sqrt{55}}{3}\right) = -\frac{1}{27}(247 + 220\sqrt{55})$$

Therefore by replacing  $\sqrt{55}$  by  $-\sqrt{55}$ , we have

$$f\left(\frac{1-\sqrt{55}}{3}\right) = -\frac{1}{27}(247-220\sqrt{55})$$

---

### Question # 2(vii)

$$f(x) = x^4 - 4x^2$$

*\*Correction*

Diff. w.r.t.  $x$

$$f'(x) = 4x^3 - 8x \dots\dots\dots (i)$$

For critical points put  $f'(x) = 0$

$$\Rightarrow 4x^3 - 8x = 0 \Rightarrow 4x(x^2 - 2) = 0$$

$$\Rightarrow 4x = 0 \text{ or } x^2 - 2 = 0$$

$$\Rightarrow x = 0 \text{ or } x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$

Now diff. (i) w.r.t  $x$

$$f''(x) = 12x^2 - 8$$

For  $x = -\sqrt{2}$

$$f''(-\sqrt{2}) = 12(-\sqrt{2})^2 - 8 = 24 - 8 = 16 > 0$$

$\Rightarrow f$  has relative minima at  $x = -\sqrt{2}$

$$\text{And } f(-\sqrt{2}) = (-\sqrt{2})^4 - 4(-\sqrt{2})^2 = 4 - 8 = -4$$

For  $x = 0$

$$f''(0) = 12(0) - 8 = -8 < 0$$

$\Rightarrow f$  has relative maxima at  $x = 0$

$$\text{And } f(0) = (0)^4 - 4(0)^2 = 0$$

For  $x = \sqrt{2}$

$$f''(\sqrt{2}) = 12(\sqrt{2})^2 - 8 = 24 - 8 = 16 > 0$$

$\Rightarrow f$  has relative minima at  $x = \sqrt{2}$

$$\text{And } f(\sqrt{2}) = (\sqrt{2})^4 - 4(\sqrt{2})^2 = 4 - 8 = -4$$

---

### Question # 2(viii)

$$f(x) = (x-2)^2(x-1)$$

$$= (x^2 - 4x + 4)(x-1) = x^3 - 4x^2 + 4x - x^2 + 4x - 4$$

$$= x^3 - 5x^2 + 8x - 4$$

Diff. w.r.t.  $x$

$$f'(x) = 3x^2 - 10x + 8$$

For critical (stationary) points, put  $f'(x) = 0$

$$\Rightarrow 3x^2 - 10x + 8 = 0 \Rightarrow 3x^2 - 6x - 4x + 8 = 0$$

$$\Rightarrow 3x(x-2) - 4(x-2) = 0 \Rightarrow (x-2)(3x-4) = 0$$

$$\Rightarrow (x-2) = 0 \text{ or } (3x-4) = 0$$

$$\Rightarrow x = 2 \text{ or } x = \frac{4}{3}$$

Now diff. (i) w.r.t  $x$

$$f''(x) = 6x - 10$$

For  $x = 2$

$$f''(2) = 6(2) - 10 = 2 > 0$$

$\Rightarrow f$  has relative minima at  $x=2$

$$\text{And } f(2) = (2-2)^2(2-1) = 0$$

$$\text{For } x = \frac{4}{3}$$

$$f''\left(\frac{4}{3}\right) = 6\left(\frac{4}{3}\right) - 10 = 8 - 10 = -2 < 0$$

$\Rightarrow f$  has relative maxima at  $x = \frac{4}{3}$

$$\text{And } f\left(\frac{4}{3}\right) = \left(\frac{4}{3} - 2\right)^2\left(\frac{4}{3} - 1\right) = \left(-\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \left(\frac{4}{9}\right)\left(\frac{1}{3}\right) = \frac{4}{27}$$

---

### Question # 2(ix)

$$f(x) = 5 + 3x - x^3$$

Diff. w.r.t  $x$

$$f'(x) = 3 - 3x^2 \dots\dots\dots (i)$$

For stationary points, put  $f'(x) = 0$

$$\Rightarrow 3 - 3x^2 = 0 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Diff. (i) w.r.t  $x$

$$f''(x) = -6x$$

For  $x = 1$

$$f''(1) = -6(1) = -6 < 0$$

$\Rightarrow f$  has relative maxima at  $x = 1$

$$\text{And } f(1) = 5 + 3(1) - (1)^3 = 5 + 3 - 1 = 7$$

For  $x = -1$

$$f''(-1) = -6(-1) = 6 > 0$$

$\Rightarrow f$  has relative minima at  $x = -1$ , and

$$f(-1) = 5 + 3(-1) - (-1)^3 = 5 - 3 + 1 = 3$$

---

### Question # 3

$$f(x) = \sin x + \cos x \quad \text{where } x \in [0, 2\pi]$$

Diff. w.r.t  $x$

$$f'(x) = \cos x - \sin x \dots\dots\dots (i)$$

For stationary points, put  $f'(x) = 0$

$$\cos x - \sin x = 0$$

$$\Rightarrow -\sin x = -\cos x \Rightarrow \frac{\sin x}{\cos x} = 1 \Rightarrow \tan x = 1$$

$$\Rightarrow x = \tan^{-1}(1) \Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4} \quad \text{when } x \in [0, 2\pi]$$

Now diff. (i) w.r.t  $x$

$$f''(x) = -\sin x - \cos x$$

$$\text{For } x = \frac{\pi}{4}$$

$$f''\left(\frac{\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) < 0$$

$\Rightarrow f$  has relative maxima at  $x = \frac{\pi}{4}$

$$\text{And } f\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) = (\sqrt{2})^2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$$

For  $x = \frac{5\pi}{4}$

$$\begin{aligned} f''\left(\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) \\ &= -\left(-\frac{1}{\sqrt{2}}\right) - \left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\left(\frac{1}{\sqrt{2}}\right) > 0 \end{aligned}$$

$\Rightarrow f$  has relative minima at  $x = \frac{5\pi}{4}$

$$\text{And } f\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}\right) + \cos\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -2\left(\frac{1}{\sqrt{2}}\right) = -\sqrt{2}$$

---

#### Question # 4

$$y = \frac{\ln x}{x}$$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\ln x}{x}\right) = \frac{x \cdot \frac{1}{x} - \ln x \cdot (1)}{x^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{1 - \ln x}{x^2} \dots\dots\dots (i) \end{aligned}$$

For critical points, put  $\frac{dy}{dx} = 0$

$$\Rightarrow \frac{1 - \ln x}{x^2} = 0 \Rightarrow 1 - \ln x = 0 \Rightarrow \ln x = 1$$

$$\Rightarrow \ln x = \ln e \Rightarrow x = e \quad \because \ln e = 1$$

Diff. (i) w.r.t  $x$

$$\begin{aligned} \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}\left(\frac{1 - \ln x}{x^2}\right) \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{x^2 \cdot \left(-\frac{1}{x}\right) - (1 - \ln x) \cdot (2x)}{(x^2)^2} = \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-3x + 2x \ln x}{x^4} \end{aligned}$$

At  $x = e$

$$\begin{aligned} \left.\frac{d^2y}{dx^2}\right|_{x=e} &= \frac{-3e + 2e \cdot \ln e}{e^4} \\ &= \frac{-3e + 2e \cdot (1)}{e^4} = \frac{-e}{e^4} = -\frac{1}{e^3} < 0 \end{aligned}$$

$\Rightarrow y$  has a maximum value at  $x = e$ .

---

#### Question # 5

$$y = x^x$$

Taking log on both sides

$$\ln y = \ln x^x \Rightarrow \ln y = x \ln x$$

Diff. w.r.t  $x$

$$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx} x \ln x \\ \Rightarrow \frac{1}{y} \frac{dy}{dx} &= x \cdot \frac{d}{dx} \ln x + \ln x \cdot \frac{dx}{dx} \\ &= x \cdot \frac{1}{x} + \ln x \cdot (1) \\ \Rightarrow \frac{dy}{dx} &= y(1 + \ln x) \Rightarrow \frac{dy}{dx} = x^x(1 + \ln x) \dots\dots\dots (i) \end{aligned}$$

For critical point, put  $\frac{dy}{dx} = 0$

$$\Rightarrow x^x(1 + \ln x) = 0 \Rightarrow 1 + \ln x = 0 \text{ as } x^x \neq 0$$

$$\Rightarrow \ln x = -1 \Rightarrow \ln x = -\ln e \quad \because \ln e = 1$$

$$\Rightarrow \ln x = \ln e^{-1} \Rightarrow x = e^{-1} \Rightarrow x = \frac{1}{e}$$

Diff. (i) w.r.t  $x$

$$\frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dx} x^x (1 + \ln x)$$

$$\begin{aligned} \Rightarrow \frac{d^2 y}{dx^2} &= x^x \frac{d}{dx} (1 + \ln x) + (1 + \ln x) \frac{d}{dx} x^x \\ &= x^x \cdot \frac{1}{x} + (1 + \ln x) \cdot x^x (1 + \ln x) \quad \text{from (i)} \\ &= x^x \left( \frac{1}{x} + (1 + \ln x)^2 \right) \end{aligned}$$

At  $x = \frac{1}{e}$

$$\begin{aligned} \left. \frac{d^2 y}{dx^2} \right|_{x=1/e} &= \left( \frac{1}{e} \right)^e \left( \frac{1}{1/e} + \left( 1 + \ln \frac{1}{e} \right)^2 \right) \\ &= \left( \frac{1}{e} \right)^e \left( e + (1 + \ln e^{-1})^2 \right) = \left( \frac{1}{e} \right)^e \left( e + (1 - \ln e)^2 \right) \\ &= \left( \frac{1}{e} \right)^e \left( e + (1 - 1)^2 \right) = \left( \frac{1}{e} \right)^e \cdot e > 0 \end{aligned}$$

$\Rightarrow y$  has a minimum value at  $x = \frac{1}{e}$

---