Exercise 10.3

Question # 1
Find the values of $\sin 2\alpha, \cos 2\alpha$ and $\tan 2\alpha$ when:

(i) $\sin\alpha = \dfrac{12}{13}$ (ii) $\cos\alpha = \dfrac{3}{5}$ where $o < \alpha < \dfrac{\pi}{2}$

Solution

(i) $\sin\alpha = \dfrac{12}{13}$; $0 < \alpha < \dfrac{\pi}{2}$

Since $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

As α is in the first quadrant so value of cos is +ive

$$\cos\alpha = \sqrt{1-\sin^2\alpha}$$

$$= \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} \qquad \Rightarrow \cos\alpha = \frac{5}{13}$$

and $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{12/13}{5/13} \qquad \Rightarrow \tan\alpha = \dfrac{12}{5}$

Now $\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$= 2\left(\frac{12}{13}\right)\left(\frac{5}{13}\right) \qquad \Rightarrow \boxed{\sin 2\alpha = \frac{120}{169}}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} \qquad \Rightarrow \boxed{\cos 2\alpha = \frac{119}{169}}$$

$$\tan 2\alpha = \frac{2\alpha\tan}{1-\tan^2\alpha}$$

$$= \frac{2\left(12/5\right)}{1-\left(12/5\right)^2} = \frac{24/5}{1-144/25} = \frac{24/5}{-119/25} = -\frac{24}{5}\cdot\frac{25}{119}$$

$$\Rightarrow \boxed{\tan 2\alpha = \frac{120}{119}}$$

(ii) $\cos\alpha = \dfrac{3}{5}$; $0 < \alpha < \dfrac{\pi}{2}$

Prove the following identities (**Question 2 – 13**)

Question # 2

$$\cot\alpha - \tan\alpha = 2\cot 2\alpha$$

Solution L.H.S $= \cot\alpha - \tan\alpha = \dfrac{\cos\alpha}{\sin\alpha} - \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\cos^2\alpha - \sin^2\alpha}{\sin\alpha\cos\alpha}$

$$= \frac{2(\cos^2\alpha - \sin^2\alpha)}{2\sin\alpha\cos\alpha} = \frac{2\cos 2\alpha}{\sin 2\alpha} = 2\cot 2\alpha = \text{R.H.S}$$

Question # 3

$$\frac{\sin 2\alpha}{1+\cos 2\alpha} = \tan\alpha$$

Solution

$$\text{L.H.S} = \frac{\sin 2\alpha}{1+\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{2\cos^2\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha} = \tan\alpha = \text{R.H.S}$$

$\because \cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$

$\therefore 2\cos^2\alpha = 1+\cos 2\alpha$

Question # 4

$$\frac{1-\cos\alpha}{\sin\alpha} = \tan\frac{\alpha}{2}$$

Solution

$$\text{L.H.S} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

$$= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \tan\frac{\alpha}{2} = \text{R.H.S}$$

$\because \sin^2\dfrac{\alpha}{2} = \dfrac{1-\cos\alpha}{2}$

$\because 2\sin^2\dfrac{\alpha}{2} = 1-\cos\alpha$

$\sin\alpha = 2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}$

Question # 5

$$\frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} = \sec 2\alpha - \tan 2\alpha$$

Solution

$$\text{L.H.S} = \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha}$$

$$= \frac{\cos\alpha - \sin\alpha}{\cos\alpha + \sin\alpha} \times \frac{\cos\alpha - \sin\alpha}{\cos\alpha - \sin\alpha}$$

$$= \frac{(\cos\alpha - \sin\alpha)^2}{\cos^2\alpha - \sin^2\alpha} = \frac{\cos^2\alpha + \sin^2\alpha - 2\sin\alpha\cos\alpha}{\cos 2\alpha}$$

$$= \frac{1-\sin 2\alpha}{\cos 2\alpha} = \frac{1}{\cos 2\alpha} - \frac{\sin 2\alpha}{\cos 2\alpha} = \sec 2\alpha - \tan 2\alpha = \text{R.H.S}$$

Question 6

$$\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}=\frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}$$

Solution L.H.S $=\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}}$

$\because\ \sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}=1$

$\sin\alpha=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$

$$=\sqrt{\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}}$$

$$=\sqrt{\frac{\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)^2}{\left(\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}\right)^2}}=\frac{\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}}=\text{R.H.S}$$

Question # 7

$$\frac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta}=\cot\frac{\theta}{2}$$

Solution L.H.S $=\dfrac{\operatorname{cosec}\theta+2\operatorname{cosec}2\theta}{\sec\theta}$

$$=\frac{\frac{1}{\sin\theta}+\frac{2}{\sin2\theta}}{\frac{1}{\cos\theta}}=\cos\theta\left(\frac{1}{\sin\theta}+\frac{2}{2\sin\theta\cos\theta}\right)$$

$$=\cos\theta\left(\frac{1}{\sin\theta}+\frac{1}{\sin\theta\cos\theta}\right)=\cos\theta\left(\frac{\cos\theta+1}{\sin\theta\cos\theta}\right)$$

$$=\frac{\cos\theta+1}{\sin\theta}=\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}=\cot\frac{\theta}{2}=\text{R.H.S}$$

Question # 8

$$1+\tan\alpha\tan2\alpha=\sec2\alpha$$

Solution L.H.S $=1+\tan\alpha\tan2\alpha=1+\left(\dfrac{\sin\alpha}{\cos\alpha}\right)\left(\dfrac{\sin2\alpha}{\cos2\alpha}\right)$

$$=\frac{\cos\alpha\cos2\alpha+\sin\alpha\sin2\alpha}{\cos\alpha\cos2\alpha}=\frac{\cos(2\alpha-\alpha)}{\cos\alpha\cos2\alpha}$$

$$=\frac{\cos\alpha}{\cos\alpha\cos2\alpha}=\frac{1}{\cos2\alpha}=\sec2\alpha=\text{R.H.S}$$

Question # 9

$$\frac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta} == \tan 2\theta\tan\theta$$

$\because\ \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\because\ \cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$\therefore\ 2\cos^2\theta - 1 = \cos 2\theta$

***S*olution** L.H.S $= \dfrac{2\sin\theta\sin 2\theta}{\cos\theta+\cos 3\theta}$

$$= \frac{2\sin\theta\sin 2\theta}{\cos\theta + 4\cos^3\theta - 3\cos\theta}$$

$$= \frac{2\sin\theta\sin 2\theta}{4\cos^3\theta - 2\cos\theta} = \frac{2\sin\theta\sin 2\theta}{2\cos\theta(2\cos^2\theta - 1)}$$

$$= \frac{\sin\theta\sin 2\theta}{\cos\theta\cos 2\theta} = \tan\theta\tan 2\theta = \tan 2\theta\tan\theta \ = \text{R.H.S}$$

Question # 10

$$\frac{\sin 3\theta}{\sin\theta} - \frac{\cos 3\theta}{\cos\theta} = 2$$

Solution L.H.S $= \dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = \dfrac{\sin 3\theta\cos\theta - \cos 3\theta\sin\theta}{\sin\theta\cos\theta}$

$$= \frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta} = \frac{\sin 2\theta}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S}$$

Question # 11

$$\frac{\cos 3\theta}{\cos\theta} + \frac{\sin 3\theta}{\sin\theta} = 4\cos 2\theta$$

Solution L.H.S $= \dfrac{\cos 3\theta}{\cos\theta} + \dfrac{\sin 3\theta}{\sin\theta} = \dfrac{\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{\sin\theta\cos\theta}$

$$= \frac{\sin(\theta+3\theta)}{\sin\theta\cos\theta} = \frac{\sin 4\theta}{\sin\theta\cos\theta} = \frac{2\sin 2\theta\cos 2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2(2\sin\theta\cos\theta)\cos 2\theta}{\sin\theta\cos\theta} = 4\cos 2\theta \ = \text{R.H.S}$$

Question # 12

$$\frac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \sec\theta$$

Solution L.H.S $= \dfrac{\tan\frac{\theta}{2} + \cot\frac{\theta}{2}}{\cot\frac{\theta}{2} - \tan\frac{\theta}{2}} = \dfrac{\dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} + \dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}}{\dfrac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \dfrac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}} = \dfrac{\dfrac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}{\dfrac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}}$

$$=\frac{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}=\frac{1}{\cos\theta}=\sec\theta=\text{R.H.S}$$

Question # 13

$$\frac{\sin 3\theta}{\cos\theta}+\frac{\cos 3\theta}{\sin\theta}=2\cot 2\theta$$

Solution L.H.S $=\dfrac{\sin 3\theta}{\cos\theta}+\dfrac{\cos 3\theta}{\sin\theta}=\dfrac{\sin 3\theta\sin\theta+\cos 3\theta\cos\theta}{\sin\theta\cos\theta}$

$$=\frac{\cos(3\theta-\theta)}{\sin\theta\cos\theta}=\frac{\cos 2\theta}{\sin\theta\cos\theta}=\frac{2\cos 2\theta}{2\sin\theta\cos\theta}$$

$$=\frac{2\cos 2\theta}{\sin 2\theta}=2\cot 2\theta=\text{R.H.S}$$

Question # 14

Reduce $\sin^4\theta$ to an expression involving only functions of multiples of θ raised to the first power.

Solution $\sin^4\theta=\left(\sin^2\theta\right)^2=\left(\dfrac{1-\cos 2\theta}{2}\right)^2$

$$=\frac{1-2\cos 2\theta+\cos^2 2\theta}{4}=\frac{1}{4}\left(1-2\cos 2\theta+\cos^2 2\theta\right)$$

$$=\frac{1}{4}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)=\frac{1}{4}\left(\frac{2-4\cos 2\theta+1+\cos 4\theta}{2}\right)$$

$$=\frac{1}{8}\left(3-4\cos 2\theta+\cos 4\theta\right)$$

Question # 15

Find the values of $\sin\theta$ and $\cos\theta$, without using table or calculator, when θ

(i) 18° (ii) 36° (iii) 54° (iv) 72°

Solution

(i) Let $\theta=18^\circ \Rightarrow 5\theta=90^\circ \Rightarrow 3\theta+2\theta=90^\circ \Rightarrow 2\theta=90^\circ-3\theta$

$\sin 2\theta=\sin(90-3\theta)$

$\Rightarrow \sin 2\theta=\cos 3\theta$

$\Rightarrow 2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

$\Rightarrow 2\sin\theta=4\cos^2\theta-3$ $\quad\div$ing by $\cos\theta$

$\because \cos 3\theta=4\cos^3\theta-3\cos\theta$

$\therefore \sin 2\theta=2\sin\theta\cos\theta$

$\Rightarrow 2\sin\theta=4(1-\sin^2\theta)-3$

$\Rightarrow 2\sin\theta=4-4\sin^2\theta-3 \quad \Rightarrow 2\sin\theta=1-4\sin^2\theta$

$\Rightarrow 4\sin^2\theta+2\sin\theta-1=0$

This is quadratic in $\sin\theta$ with $a=4$, $b=1$ and $c=-1$

$$\sin\theta = \frac{-2\pm\sqrt{(2)^2-4(4)(-1)}}{2(4)} = \frac{-2\pm\sqrt{4+16}}{8} = \frac{-2\pm\sqrt{20}}{8}$$

$$= \frac{-2\pm 2\sqrt{5}}{8} = \frac{-1\pm\sqrt{5}}{8}$$

Since $\theta = 18^\circ$ lies in the first quadrant so value of sin can not be negative therefore

$$\sin\theta = \frac{-1+\sqrt{5}}{4} \quad \Rightarrow \quad \boxed{\sin 18^\circ = \frac{\sqrt{5}-1}{4}} \qquad \because \theta = 18^\circ$$

Now

$$\cos^2 18^\circ = 1-\sin^2 18^\circ \qquad \Rightarrow \cos^2 18^\circ = 1-\left(\frac{\sqrt{5}-1}{4}\right)^2$$

$$\Rightarrow \cos^2 18^\circ = 1-\frac{\left(\sqrt{5}\right)^2-2\sqrt{5}+1}{16} = 1-\frac{5-2\sqrt{5}+1}{16}$$

$$= 1-\frac{6-2\sqrt{5}}{16} = \frac{16-6+\sqrt{5}}{16} = \frac{10+2\sqrt{5}}{16}$$

$$\Rightarrow \cos 18^\circ = \sqrt{\frac{10+2\sqrt{5}}{16}} \qquad \Rightarrow \boxed{\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

(ii) Since $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$

$$\Rightarrow \cos 2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \cos 2(18) = 2\cos^2(18) - 1$$

$$\Rightarrow \cos 36 = 2\left(\frac{\sqrt{10+2\sqrt{5}}}{4}\right)^2 - 1$$

$$= 2\left(\frac{10+2\sqrt{5}}{16}\right) - 1 = \frac{10+2\sqrt{5}}{8} - 1$$

$$= \frac{10+2\sqrt{5}-8}{8} = \frac{2+2\sqrt{5}}{8} \qquad \Rightarrow \boxed{\cos 36^\circ = \frac{1+\sqrt{5}}{4}}$$

Now $\sin^2 36 = 1-\cos^2 36$

$$= 1-\left(\frac{1+\sqrt{5}}{2}\right)^2 = 1-\frac{1+2\sqrt{5}+\left(\sqrt{5}\right)^2}{16}$$

$$= 1-\frac{1+2\sqrt{5}+5}{16} = 1-\frac{6+2\sqrt{5}}{16}$$

$$= \frac{16-6-2\sqrt{5}}{16} = \frac{10-2\sqrt{5}}{16}$$

$$\Rightarrow \sin 36^\circ = \sqrt{\frac{10-2\sqrt{5}}{16}} \qquad \Rightarrow \boxed{\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

(iii) Now $\sin(90-36) = \cos 36^\circ$ $\qquad \because \sin(90-\theta) = \cos\theta$

$$\Rightarrow \sin 54^\circ = \cos 36^\circ \qquad \Rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}}$$

And $\cos(90-36) = \sin 36^\circ$

$$\Rightarrow \cos 54^\circ = \sin 36^\circ \qquad \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}}$$

(iv) Now $\sin(90-18) = \cos 18^\circ$ $\qquad \because \sin(90-\theta) = \cos\theta$

$$\Rightarrow \sin 72^\circ = \cos 18^\circ \qquad \Rightarrow \boxed{\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}}$$

and $\cos(90-18) = \sin 18^\circ$

$$\Rightarrow \cos 72^\circ = \sin 18^\circ \qquad \Rightarrow \boxed{\cos 72^\circ = \frac{\sqrt{5}-1}{4}}$$

Alternative Method for Q # 15 (iii)

Let $\theta = 54^\circ \Rightarrow 5\theta = 270^\circ \Rightarrow 3\theta + 2\theta = 270^\circ \Rightarrow 2\theta = 270^\circ - 3\theta$

$$\sin 2\theta = \sin(270 - 3\theta)$$

$$\Rightarrow \sin 2\theta = \sin(3(90) - 3\theta)$$

$$\Rightarrow \sin 2\theta = -\cos 3\theta$$

$\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\therefore \sin 2\theta = 2\sin\theta\cos\theta$

$$\Rightarrow 2\sin\theta\cos\theta = -(4\cos^3\theta - 3\cos\theta)$$

$$\Rightarrow 2\sin\theta\cos\theta = -4\cos^3\theta + 3\cos\theta)$$

$$\Rightarrow 2\sin\theta = -4\cos^2\theta + 3 \qquad \div\text{ing by } \cos\theta$$

$$\Rightarrow 2\sin\theta = -4(1-\sin^2\theta) + 3$$

$$\Rightarrow 2\sin\theta = -4 + 4\sin^2\theta + 3 \qquad \Rightarrow 2\sin\theta = 4\sin^2\theta - 1$$

$$\Rightarrow 4\sin^2\theta - 2\sin\theta - 1 = 0$$

This is quadratic in $\sin\theta$ with $a = 4$, $b = 1$ and $c = -1$

$$\sin\theta = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(4)(-1)}}{2(4)} = \frac{2 \pm \sqrt{4+16}}{8} = \frac{2 \pm \sqrt{20}}{8}$$

$$= \frac{2 \pm 2\sqrt{5}}{8} = \frac{1 \pm \sqrt{5}}{4}$$

Since $\theta = 54^\circ$ lies in the first quadrant so value of sin can not be negative therefore

$$\sin\theta = \frac{1+\sqrt{5}}{4} \qquad \Rightarrow \boxed{\sin 54^\circ = \frac{1+\sqrt{5}}{4}} \qquad \because\ \theta = 54^\circ$$

Now

$$\cos^2 54^\circ = 1 - \sin^2 54^\circ \qquad \Rightarrow \cos^2 54^\circ = 1 - \left(\frac{1+\sqrt{5}}{4}\right)^2$$

$$\Rightarrow \cos^2 54^\circ = 1 - \frac{\left(\sqrt{5}\right)^2 + 2\sqrt{5} + 1}{16} = 1 - \frac{5 + 2\sqrt{5} + 1}{16}$$

$$= 1 - \frac{6 + 2\sqrt{5}}{16} = \frac{16 - 6 - \sqrt{5}}{16} = \frac{10 - 2\sqrt{5}}{16}$$

$$\Rightarrow \cos 54^\circ = \sqrt{\frac{10 - 2\sqrt{5}}{16}} \qquad \Rightarrow \boxed{\cos 54^\circ = \frac{\sqrt{10 - 2\sqrt{5}}}{4}}$$