EXERCISE 7.5

① (i) Given that

$\underline{u} = 3\underline{i} + 2\underline{k} = 3\underline{i} + 0\underline{j} + 2\underline{k}$

$\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$

$\underline{w} = -\underline{j} + 4\underline{k} = 0\underline{i} - \underline{j} + 4\underline{k}$

Volume of parallelopiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix}$$

$= 3(8+1) - 0 + 2(-1-0)$

$= 3(9) + 2(-1) = 27 - 2 = 25$ (cubic units)

(ii) Given that $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$

$\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$ and

$\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Volume of parallelopiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$$

$= 1(-1-6) + 4(1+4) - 1(-3+2)$

$= 1(-7) + 4(5) - 1(-1)$

$= -7 + 20 + 1 = 14$ (cubic units)

(iii) Given that $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$

$\underline{w} = \underline{j} + \underline{k} = 0\underline{i} + \underline{j} + \underline{k}$

Volume of parallelopiped $= [\underline{u}\ \underline{v}\ \underline{w}]$

$$= \begin{vmatrix} 1 & -2 & 3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$= 1(-1+1) + 2(2-0) + 3(2-0)$

$= 1(0) + 2(2) + 3(2) = 0 + 4 + 6$

$= 10$ (cubic units)

② Given that $\underline{a} = 3\underline{i} - \underline{j} + 5\underline{k}$

$\underline{b} = 4\underline{i} + 3\underline{j} - 2\underline{k}$

$\underline{c} = 2\underline{i} + 5\underline{j} + \underline{k}$

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \begin{vmatrix} 3 & -1 & 5 \\ 4 & 3 & -2 \\ 2 & 5 & 1 \end{vmatrix}$$

$\underline{a}\cdot(\underline{b}\times\underline{c}) = 3(3+10) + 1(4+4) + 5(20-6)$

$= 3(13) + 1(8) + 5(14)$

$= 39 + 8 + 70 = 117$

$\Rightarrow \underline{a}\cdot(\underline{b}\times\underline{c}) = 117$ —— ①

$$\underline{b}\cdot(\underline{c}\times\underline{a}) = \begin{vmatrix} 4 & 3 & -2 \\ 2 & 5 & 1 \\ 3 & -1 & 5 \end{vmatrix}$$

$= 4(25+1) - 3(10-3) - 2(-2-15)$

$= 4(26) - 3(7) - 2(-17)$

$= 104 - 21 + 34 = 117$

$\Rightarrow \underline{b}\cdot(\underline{c}\times\underline{a}) = 117$ —— ②

$$\underline{c}\cdot(\underline{a}\times\underline{b}) = \begin{vmatrix} 2 & 5 & 1 \\ 3 & -1 & 5 \\ 4 & 3 & -2 \end{vmatrix}$$

$= 2(2-15) - 5(-6-20) + 1(9+4)$

$= 2(-13) - 5(-26) + 1(13)$

$= -26 + 130 + 13 = 117$

$\Rightarrow \underline{c}\cdot(\underline{a}\times\underline{b}) = 117$ —— ③

$\therefore$ from ①, ② and ③, we get

$\underline{a}\cdot(\underline{b}\times\underline{c}) = \underline{b}\cdot(\underline{c}\times\underline{a}) = \underline{c}\cdot(\underline{a}\times\underline{b})$

(proved)

③ Let $\underline{u} = \underline{i} - 2\underline{j} + 3\underline{k}$

$\underline{v} = -2\underline{i} + 3\underline{j} - 4\underline{k}$

$\underline{w} = \underline{i} - 3\underline{j} + 5\underline{k}$

$$[\underline{u}\ \underline{v}\ \underline{w}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$= 1(15-12) + 2(-10+4) + 3(6-3)$

$= 1(3) + 2(-6) + 3(3)$

$= 3 - 12 + 9 = 0$

$\therefore \underline{u}, \underline{v}$ and $\underline{w}$ are coplaner.

④ (i) Let $\underline{u} = \underline{i} - \underline{j} + \underline{k}$

$\underline{v} = \underline{i} - 2\underline{j} - 3\underline{k}$

$\underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$

$\because \underline{u}, \underline{v}$ and $\underline{w}$ are coplanes

$\therefore [\underline{u}\ \underline{v}\ \underline{w}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 0$$

$\Rightarrow 1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6) = 0$

$\Rightarrow -10 - 3\alpha + 14 - \alpha + 6 = 0$

$\Rightarrow -4\alpha + 10 = 0$

$\Rightarrow -4\alpha = -10$

$\Rightarrow \alpha = \frac{-10}{-4} \Rightarrow \boxed{\alpha = \frac{5}{2}}$ Ans.

(ii) Let $\underline{u} = \underline{i} - 2\alpha\underline{j} - \underline{k}$

$\underline{v} = \underline{i} - \underline{j} + 2\underline{k}$ and

$\underline{w} = \alpha\underline{i} - \underline{j} + \underline{k}$

$\because$ $\underline{u}$, $\underline{v}$ and $\underline{w}$ are coplaner

$\therefore [\underline{u}\ \underline{v}\ \underline{w}] = 0$

$$\Rightarrow \begin{vmatrix} 1 & -2\alpha & -1 \\ 1 & -1 & 2 \\ \alpha & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow 1(-1+2) + 2\alpha(1-2\alpha) - 1(-1+\alpha) = 0$

$\Rightarrow 1 + 2\alpha - 4\alpha^2 + 1 - \alpha = 0$

$\Rightarrow -4\alpha^2 + \alpha + 2 = 0$

$\Rightarrow -1(4\alpha^2 - \alpha - 2) = 0$

$\Rightarrow 4\alpha^2 - \alpha - 2 = 0$

$\Rightarrow \alpha = \dfrac{1 \pm \sqrt{1+32}}{8} = \dfrac{1 \pm \sqrt{33}}{8}$ Ans.

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(5)(a)(i) $2\underline{i} \times 2\underline{j} \cdot \underline{k} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= 2(2-0) - 0 + 0 = 2(2) = 4$ Ans

(ii) $3\underline{j} \cdot \underline{k} \times \underline{i} = \begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 0 - 3(0-1) + 0$

$= 3$ Ans.

(iii) $[\underline{k}\ \underline{i}\ \underline{j}] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 - 0 + 1(1-0) = 1$

Ans

(iv) $[\underline{i}\ \underline{i}\ \underline{k}] = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(0-0) - 0 + 0 = 0$

Ans.

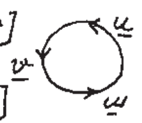
(b) Prove that

$\underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{v}\cdot(\underline{w}\times\underline{u}) + \underline{w}\cdot(\underline{u}\times\underline{v})$

$= 3\ \underline{u}\cdot(\underline{v}\times\underline{w})$

Sol:- L.H.S. $= \underline{u}\cdot(\underline{v}\times\underline{w}) + \underline{v}\cdot(\underline{w}\times\underline{u}) + \underline{w}\cdot(\underline{u}\times\underline{v})$

$= [\underline{u}\ \underline{v}\ \underline{w}] + [\underline{v}\ \underline{w}\ \underline{u}] + [\underline{w}\ \underline{u}\ \underline{v}]$



$= [\underline{u}\ \underline{v}\ \underline{w}] + [\underline{u}\ \underline{v}\ \underline{w}] + [\underline{u}\ \underline{v}\ \underline{w}]$

$\because [\underline{v}\ \underline{w}\ \underline{u}] = [\underline{u}\ \underline{v}\ \underline{w}]$

& $[\underline{w}\ \underline{u}\ \underline{v}] = [\underline{u}\ \underline{v}\ \underline{w}]$

$\therefore$ L.H.S $= 3\,[\underline{u}\ \underline{v}\ \underline{w}] = 3\,\underline{u}\cdot(\underline{v}\times\underline{w}) =$ R.H.S.

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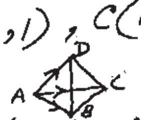
(6)(i) Let $A(0,1,2)$, $B(3,2,1)$, $C(1,2,1)$ and $D(5,5,6)$



Now $\overrightarrow{AB} = (3-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = 3\underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AC} = (1-0)\underline{i} + (2-1)\underline{j} + (1-2)\underline{k} = \underline{i} + \underline{j} - \underline{k}$

$\overrightarrow{AD} = (5-0)\underline{i} + (5-1)\underline{j} + (6-2)\underline{k} = 5\underline{i} + 4\underline{j} + 4\underline{k}$

Volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix}$$

$= \frac{1}{6}[3(4+4) - 1(4+5) - 1(4-5)]$

$= \frac{1}{6}[24 - 9 + 1] = \frac{1}{6}(16) = \frac{16}{6} = \frac{8}{3}$

(cubic units) Ans.

(ii) Let $A(2,1,8)$, $B(3,2,9)$, $C(2,1,4)$ and $D(3,3,10)$.

Now $\overrightarrow{AB} = (3-2)\underline{i} + (2-1)\underline{j} + (9-8)\underline{k} = \underline{i} + \underline{j} + \underline{k}$

$\overrightarrow{AC} = (2-2)\underline{i} + (1-1)\underline{j} + (4-8)\underline{k} = 0\underline{i} + 0\underline{j} - 4\underline{k}$

$\overrightarrow{AD} = (3-2)\underline{i} + (3-1)\underline{j} + (10-8)\underline{k} = \underline{i} + 2\underline{j} + 2\underline{k}$

Volume of tetrahedron $= \frac{1}{6}[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]$

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix}$$

$= \frac{1}{6}[1(0+8) - 1(0+4) + 1(0-0)]$

$= \frac{1}{6}[8 - 4 + 0] = \frac{1}{6}(4) = \frac{4}{6} = \frac{2}{3}$ Ans.

(cubic units)

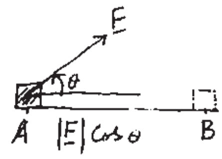
Work Done

:- If a constant force $\underline{F}$ acts on a body at any angle θ to the direction of motion, displaces it from A to B. Then

work done $= |\underline{F}|\cos\theta\,|\overrightarrow{AB}|$

$= \underline{F}\cdot\overrightarrow{AB}$

$= \underline{F}\cdot\underline{d}$



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(7) Given that $\underline{F} = 4\underline{i} + 3\underline{j} + 5\underline{k}$

$P_1(3,1,-2)$ and $P_2(2,4,6)$

$\underline{d} = \overrightarrow{P_1P_2} = (2-3)\underline{i} + (4-1)\underline{j} + (6+2)\underline{k}$

$\Rightarrow \underline{d} = -\underline{i} + 3\underline{j} + 8\underline{k}$

Work done $= \underline{F}\cdot\underline{d}$

$= (4\underline{i} + 3\underline{j} + 5\underline{k})\cdot(-\underline{i} + 3\underline{j} + 8\underline{k})$

$= (4)(-1) + (3)(3) + (5)(8)$

$= -4 + 9 + 40 = 45$ units.

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(8) Given that $\underline{F_1} = 4\underline{i} + \underline{j} - 3\underline{k}$

$\underline{F_2} = 3\underline{i} - \underline{j} - \underline{k}$

$\therefore$ net force $= \underline{F} = \underline{F_1} + \underline{F_2}$

$\Rightarrow F = 4i + j - 3k + 3i - j - k$

$\Rightarrow F = 7i + 0j - 4k$

$A(1,2,3)$, $B(5,4,1)$

$\therefore d = \overrightarrow{AB} = (5-1)i + (4-2)j + (1-3)k$

$\Rightarrow d = 4i + 2j - 2k$

$\therefore$ Work done $= F \cdot d$

$= (7i + 0j - 4k)\cdot(4i + 2j - 2k)$

$= (7)(4) + (0)(2) + (-4)(-2)$

$= 28 + 0 + 8 = 36$ units Ans

(9) Given that $A(5,-5,-7)$ and $B(6,2,-2)$

$d = \overrightarrow{AB} = (6-5)i + (2+5)j + (-2+7)k$

$\Rightarrow d = i + 7j + 5k$

$F_1 = 10i - j + 11k$

$F_2 = 4i + 5j + 9k$

$F_3 = -2i + j - 9k$

$\therefore$ net force $= F = F_1 + F_2 + F_3$

$\Rightarrow F = 12i + 5j + 11k$

$\therefore$ Work done $= F \cdot d$

$= (12i + 5j + 11k)\cdot(i + 7j + 5k)$

$= (12)(1) + (5)(7) + (11)(5)$

$= 12 + 35 + 55 = 102$ units.

Ans.

(10) Given that $|F| = 6$ units

$\because F = |F|\,\hat{F} \Rightarrow F = 6\hat{F}$ —— (1)

$\because F \parallel 2i - 2j + k$

$\therefore \hat{F} = \dfrac{2i - 2j + k}{\sqrt{4+4+1}} = \dfrac{2i - 2j + k}{3}$

$\therefore$ (1) becomes

$F = \cancel{6}^{\,2}\left(\dfrac{2i - 2j + k}{\cancel{3}}\right)$

$\Rightarrow F = 4i - 4j + 2k$

Let $A(1,2,3)$ and $B(5,3,7)$. Then

$d = \overrightarrow{AB} = (5-1)i + (3-2)j + (7-3)k$

$\Rightarrow d = 4i + j + 4k$

$\therefore$ Work done $= F \cdot d$

$= (4i - 4j + 2k)\cdot(4i + j + 4k)$

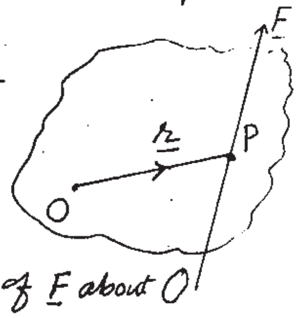
$= (4)(4) + (-4)(1) + (2)(4)$

$= 16 - 4 + 8 = 20$ units

Moment of Force

The turning effect of a force about a point is called moment of the force about that point.

If a force F acts a body at point P and rotates the body about point O.

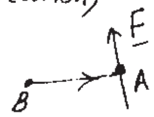


Then moment of F about O

$= \overrightarrow{M} = r \times F$

(11) Given that $F = 3i + 2j - 4k$

Let $A(1,-1,2)$ (point of application) and $B(2,-1,3)$



$r = \overrightarrow{BA}$

$r = (1-2)i + (-1+1)j + (2-3)k$

$r = -i + 0j - k$

$\therefore \overrightarrow{M} = r \times F = \begin{vmatrix} i & j & k \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$

$\Rightarrow \overrightarrow{M} = i(0+2) - j(4+3) + k(-2-0)$

$\Rightarrow \overrightarrow{M} = 2i - 7j - 2k$ Ans.

(12) Given that $F = 4i - 3k = 4i + 0j - 3k$

$A(2,-2,5)$ (point of application) and about $B(1,-3,1)$

$r = \overrightarrow{BA} = (2-1)i + (-2+3)j + (5-1)k$

$\Rightarrow r = i + j + 4k$

$\therefore \overrightarrow{M} = r \times F$

$= \begin{vmatrix} i & j & k \\ 1 & 1 & 4 \\ 4 & 0 & -3 \end{vmatrix}$

$= i(-3-0) - j(-3-16) + k(0-4)$

$= -3i + 19j - 4k$ Ans.

(13) Given that

$$\underline{F} = 2\underline{i} + \underline{j} - 3\underline{k}$$

$A(1,-2,1)$ and $B(2,0,-2)$

$\underline{r} = \overrightarrow{BA} = (1-2)\underline{i} + (-2-0)\underline{j} + (1+2)\underline{k}$

$\underline{r} = -\underline{i} - 2\underline{j} + 3\underline{k}$

$\therefore \underline{M} = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -2 & 3 \\ 2 & 1 & -3 \end{vmatrix}$$

$= \underline{i}(6-3) - \underline{j}(3-6) + \underline{k}(-1+4)$

$= 3\underline{i} + 3\underline{j} + 3\underline{k}$ Ans.

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(14) Given that $A(1,1,1)$ and $P(2,0,1)$

$\underline{r} = \overrightarrow{AP} = (2-1)\underline{i} + (0-1)\underline{j} + (1-1)\underline{k}$

$\underline{r} = \underline{i} - \underline{j} + 0\underline{k}$

$\underline{F}_1 = \underline{i} - 2\underline{j}$

$\underline{F}_2 = 3\underline{i} + 2\underline{j} - \underline{k}$

$\underline{F}_3 = 5\underline{j} + 2\underline{k}$

$\therefore$ net force $= \underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3$

$\Rightarrow \underline{F} = 4\underline{i} + 5\underline{j} + \underline{k}$

$\therefore \underline{M} = \underline{r} \times \underline{F}$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 4 & 5 & 1 \end{vmatrix}$$

$= \underline{i}(-1-0) - \underline{j}(1-0) + \underline{k}(5+4)$

$= -\underline{i} - \underline{j} + 9\underline{k}$ Ans.

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(15) Given that

$\underline{F} = 7\underline{i} + 4\underline{j} - 3\underline{k}$

$P(1,-2,3)$ and $Q(2,1,1)$. Then

$\underline{r} = \overrightarrow{QP} = (1-2)\underline{i} + (-2-1)\underline{j} + (3-1)\underline{k}$

$\underline{r} = -\underline{i} - 3\underline{j} + 2\underline{k}$

$$\therefore \underline{M} = \underline{r} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -3 & 2 \\ 7 & 4 & -3 \end{vmatrix}$$

$\underline{M} = \underline{i}(9-8) - \underline{j}(3-14) + \underline{k}(-4+21)$

$\underline{M} = \underline{i} + 11\underline{j} + 17\underline{k}$ Ans.