EXERCISE 2.1

Question # 1 (i)

Let $y=2x^2+1$

$\Rightarrow\ y+\delta y=2\left(x+\delta x\right)^2+1 \qquad \Rightarrow\ \delta y=2\left(x+\delta x\right)^2+1-y$

$\Rightarrow\ \delta y=2\left(x^2+2x\delta x+\delta x^2\right)+1-2x^2-1 \qquad \because\ y=2x^2+1$

$\Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2 \quad \Rightarrow\ \delta y=2x^2+4x\delta x+2\delta x^2-2x^2$

$\Rightarrow\ \delta y=4x\delta x+2\delta x^2$

$=\delta x\left(4x+2\delta x\right)$

Dividing by δx

$$\frac{\delta y}{\delta x}=4x+2\delta x$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(4x+2\delta x\right)$$

$$\Rightarrow\ \frac{dy}{dx}=4x+2(0)$$

$$\Rightarrow\ \frac{dy}{dx}=4x \quad \text{i.e.} \quad \boxed{\frac{d}{dx}\left(2x^2+1\right)=4x}$$

Question # 1 (ii)

Let $y=2-\sqrt{x}$

$\Rightarrow\ y+\delta y=2-\sqrt{x+\delta x} \qquad \Rightarrow\ \delta y=2-\sqrt{x+\delta x}-y$

$\Rightarrow\ \delta y=2-\sqrt{x+\delta x}-2+\sqrt{x} \qquad \Rightarrow\ \delta y=x^{\frac{1}{2}}-\left(x+\delta x\right)^{\frac{1}{2}}$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{\delta x}{x}\right)^{\frac{1}{2}}$$

$$\Rightarrow\ \delta y=x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(1+\frac{1}{2}\cdot\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots\right)$$

$$=x^{\frac{1}{2}}-x^{\frac{1}{2}}-x^{\frac{1}{2}}\left(\frac{\delta x}{2x}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\frac{\delta x^2}{x^2}+\ldots.\right)$$

$$=-x^{\frac{1}{2}}\delta x\left(\frac{1}{2x}-\frac{1}{6}\frac{\delta x}{x^2}+\ldots.\right)$$

Dividing by δx, we have

$$\frac{\delta y}{\delta x}=-x^{\frac{1}{2}}\left(\frac{1}{2x}-\frac{1}{6}\frac{\delta x}{x^2}+\ldots.\right)$$

Taking limit as

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=-x^{\frac{1}{2}}\lim_{\delta x\to 0}\left(\frac{1}{2x}-\frac{1}{6}\frac{\delta x}{x^2}+\ldots.\right)$$

$$\Rightarrow\ \frac{dy}{dx}=x^{\frac{1}{2}}\left(\frac{1}{2x}-0+0-\ldots.\right)$$

$$=x^{\frac{1}{2}}\cdot\frac{1}{2x}=\frac{1}{2}x^{\frac{1}{2}-1} \qquad \Rightarrow\ \boxed{\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}}$$

Question # 1 (iii)

Let $y=\dfrac{1}{\sqrt{x}} \quad \Rightarrow \quad y=(x)^{-\frac{1}{2}}$

Now do yourself

Question # 1 (iv)

Let $y=\dfrac{1}{x^3} \quad \Rightarrow \quad y=x^{-3}$

$$\Rightarrow\ y+\delta y=(x+\delta x)^{-3}$$

$$\Rightarrow\ \delta y=(x+\delta x)^{-3}-x^{-3}$$

$$\Rightarrow\ \delta y=x^{-3}\left[\left(1+\frac{\delta x}{x}\right)^{-3}-1\right]$$

$$=x^{-3}\left[\left(1-\frac{3\delta x}{x}+\frac{-3(-3-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots.\right)-1\right]$$

$$=x^{-3}\left[1-\frac{3\delta x}{x}+\frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2+\ldots..-1\right]$$

$$=x^{-3}\left[-\frac{3\delta x}{x}+\frac{-3(-4)}{2}\left(\frac{\delta x}{x}\right)^2+\ldots..\right]$$

$$=x^{-3}\cdot\frac{\delta x}{x}\left[-3+6\left(\frac{\delta x}{x}\right)-\ldots..\right]$$

Dividing both sides by δx, we get

$$\frac{\delta y}{\delta x}=x^{-3-1}\left[-3+6\left(\frac{\delta x}{x}\right)-\ldots..\right]$$

Taking limit on both sided, we get

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}x^{-4}\left[-3+6\left(\frac{\delta x}{x}\right)-\ldots..\right]$$

$$\Rightarrow\ \frac{dy}{dx}=x^{-4}\left[-3+0-0+\ldots..\right]$$

$$\Rightarrow\ \frac{dy}{dx}=-3x^{-4} \quad \text{or} \quad \boxed{\frac{dy}{dx}=-\frac{3}{x^4}}$$

Question # 1 (vi)

Let $y=\dfrac{1}{x-a}$

$$\Rightarrow\ y=(x-a)^{-1}$$

$$\Rightarrow\ y+\delta y=(x+\delta x-a)^{-1}$$

$$\Rightarrow\ \delta y=(x-a+\delta x)^{-1}-y$$

$$\Rightarrow\ \delta y=(x-a+\delta x)^{-1}-(x-a)^{-1}$$

$$=(x-a)^{-1}\left[\left(1+\frac{\delta x}{x-a}\right)^{-1}-1\right]$$

$$=(x-a)^{-1}\left[\left(1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+\ldots.\right)-1\right]$$

$$\Rightarrow\ \delta y=(x-a)^{-1}\left[1-\frac{\delta x}{x-a}+\frac{-1(-1-1)}{2!}\left(\frac{\delta x}{x-a}\right)^2+\ldots.-1\right]$$
$$=(x-a)^{-1}\left[-\frac{\delta x}{x-a}+\frac{-1(-2)}{2}\left(\frac{\delta x}{x-a}\right)^2+\ldots.\right]$$
$$=(x-a)^{-1}\cdot\frac{\delta x}{x-a}\left[-1+\left(\frac{\delta x}{x-a}\right)-\ldots.\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}=(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-\ldots.\right]$$

Taking limit when $\delta x\to 0$, we have

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}(x-a)^{-1-1}\left[-1+\left(\frac{\delta x}{x-a}\right)-\ldots.\right]$$
$$\Rightarrow\ \frac{dy}{dx}=(x-a)^{-2}\left[-1+0-0+\ldots.\right]\qquad\Rightarrow\ \boxed{\frac{dy}{dx}=\frac{-1}{(x-a)^2}}$$

Question # 1 (vi)

Let $y\ =\ x(x-3)$

$=\ x^2-3x$

Do yourself

Question # 1 (vii)

Let $\quad y\ =\ \dfrac{2}{x^4}=\ 2x^{-4}$

$\Rightarrow\ y+\delta y\ =\ 2(x+\delta x)^{-4}$

Do yourself

Question # 1(viii)

Let $\ y\ =\ x^2+\dfrac{1}{x^2}\ =\ x^2+x^{-2}$

$$\Rightarrow\ y+\delta y\ =\ (x+\delta x)^2+(x+\delta x)^{-2}$$
$$\Rightarrow\ \delta y\ =\ (x+\delta x)^2+(x+\delta x)^{-2}-x^2-x^{-2}$$
$$=\ (x+\delta x)^2-x^2+(x+\delta x)^{-2}-x^{-2}$$
$$=\ x^2+2x\delta x+\delta x^2-x^2+x^{-2}\left[\left(1+\frac{\delta x}{x}\right)^{-2}-1\right]$$
$$=\ 2x\delta x+\delta x^2+x^{-2}\left[\left(1-\frac{2\delta x}{x}+\frac{-2(-2-1)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots.\right)-1\right]$$
$$=\ 2x\delta x+\delta x^2+x^{-2}\left[-\frac{2\delta x}{x}+\frac{-2(-3)}{2}\left(\frac{\delta x}{x}\right)^2+\ldots.\right]$$
$$=\ \delta x(2x+\delta x)+x^{-2}\cdot\frac{\delta x}{x}\left[-2+3\left(\frac{\delta x}{x}\right)+\ldots.\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}\ =\ (2x+\delta x)+x^{-3}\left[-2+3\left(\frac{\delta x}{x}\right)+\ldots.\right]$$

Taking limit when $\delta x \to 0$, we have

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (2x + \delta x) + \lim_{\delta x \to 0} x^{-3} \left[-2 + 3\left(\frac{\delta x}{x}\right) + ... \right]$$

$$\Rightarrow \frac{dy}{dx} = (2x + 0) + x^{-3} [-2 + 0 - 0....]$$

$$\Rightarrow \boxed{\frac{dy}{dx} = 2x - \frac{2}{x^3}}$$

Question # 1(ix)

Let $y = (x+4)^{\frac{1}{3}}$

$$\Rightarrow y + \delta y = (x + \delta x + 4)^{\frac{1}{3}}$$

$$\Rightarrow \delta y = (x + \delta x + 4)^{\frac{1}{3}} - y$$

$$= (x + 4 + \delta x)^{\frac{1}{3}} - (x+4)^{\frac{1}{3}}$$

$$= (x+4)^{\frac{1}{3}} \left[\left(1 + \frac{\delta x}{x+4}\right)^{\frac{1}{3}} - 1 \right]$$

$$= (x+4)^{\frac{1}{3}} \left[\left(1 + \frac{1}{3}\frac{\delta x}{x+4} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{\delta x}{x+4}\right)^2 + ...\right) - 1 \right]$$

$$= (x+4)^{\frac{1}{3}} \left[\frac{\delta x}{3(x+4)} + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}\left(\frac{\delta x}{x+4}\right)^2 + ... \right]$$

$$= (x+4)^{\frac{1}{3}} \cdot \frac{\delta x}{x+4} \left[\frac{1}{3} - \frac{1}{9}\left(\frac{\delta x}{x+4}\right) + ... \right]$$

Dividing by δx

$$\frac{\delta y}{\delta x} = (x+4)^{\frac{1}{3}-1} \left[\frac{1}{3} - \frac{1}{9}\left(\frac{\delta x}{x+4}\right) + ... \right]$$

Taking limit when $\delta x \to 0$

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} (x+4)^{-\frac{2}{3}} \left[\frac{1}{3} - \frac{1}{9}\left(\frac{\delta x}{x+4}\right) + ... \right]$$

$$\Rightarrow \frac{dy}{dx} = (x+4)^{-\frac{2}{3}} \left[\frac{1}{3} - 0 + 0 - .... \right] \qquad \Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{3}(x+4)^{-\frac{2}{3}}}$$

Question # 1 (x)

Let $y = x^{\frac{3}{2}}$

$$\Rightarrow y + \delta y = (x + \delta x)^{\frac{3}{2}}$$

$$\Rightarrow \delta y = (x + \delta x)^{\frac{3}{2}} - x^{\frac{3}{2}}$$

$$= x^{\frac{3}{2}} \left[\left(1 + \frac{\delta x}{x}\right)^{\frac{3}{2}} - 1 \right]$$

$$= x^{\frac{3}{2}} \left[\left(1 + \frac{3}{2}\frac{\delta x}{x} + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2 + ...\right) - 1 \right]$$

$$= x^{\frac{3}{2}}\left[\frac{3\delta x}{2x}+\frac{\frac{3}{2}\left(\frac{1}{2}\right)}{2}\left(\frac{\delta x}{x}\right)^2+....\right]$$

$$= x^{\frac{3}{2}}\cdot\frac{\delta x}{x}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}= x^{\frac{3}{2}-1}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}= \lim_{\delta x\to 0} x^{\frac{1}{2}}\left[\frac{3}{2}+\frac{3}{8}\left(\frac{\delta x}{x}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx}= x^{\frac{1}{2}}\left[\frac{3}{2}-0+0-....\right] \qquad \Rightarrow \boxed{\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}}$$

Question # 1 (xi)

Let $y = x^{5/2}$

Do yourself as above.

Question # 1 (xii)

Let $y = x^m$

$$\Rightarrow y+\delta y = (x+\delta x)^m$$

$$\Rightarrow \delta y = (x+\delta x)^m - x^m$$

$$= x^m\left[\left(1+\frac{\delta x}{x}\right)^m-1\right]$$

$$= x^m\left[\left(1+m\cdot\frac{\delta x}{x}+\frac{m(m-1)}{2!}\left(\frac{\delta x}{x}\right)^2+....\right)-1\right]$$

$$= x^m\left[\frac{m\delta x}{x}+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)^2+....\right]$$

$$= x^m\cdot\frac{\delta x}{x}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Dividing by δx

$$\frac{\delta y}{\delta x}= x^{m-1}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

Taking limit when $\delta x\to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}= \lim_{\delta x\to 0} x^{m-1}\left[m+\frac{m(m-1)}{2}\left(\frac{\delta x}{x}\right)+....\right]$$

$$\Rightarrow \frac{dy}{dx}= x^{m-1}[m+0+0....] \qquad \Rightarrow \boxed{\frac{dy}{dx}= m x^{m-1}}$$

Question # 1 (xii)

Let $y = \dfrac{1}{x^m} = x^{-m}$

Do yourself as above, just change the m by $-m$ in above question.

Question # 1 (xvi)

Let $y = x^{40}$

$$\Rightarrow \ y+\delta y = (x+\delta x)^{40}$$

$$\Rightarrow \ \delta y = (x+\delta x)^{40} - x^{40}$$

$$= \left[\binom{40}{0}x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40}\right] - x^{40}$$

$$= (1)x^{40} + \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40} - x^{40}$$

$$= \binom{40}{1}x^{39}\delta x + \binom{40}{2}x^{38}\delta x^2 + \dots + \binom{40}{40}\delta x^{40}$$

Dividing by δx

$$\frac{\delta y}{\delta x} = \binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + \dots + \binom{40}{40}\delta x^{39}$$

Taking limit as $\delta x \to 0$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\binom{40}{1}x^{39} + \binom{40}{2}x^{38}\delta x + \dots + \binom{40}{40}\delta x^{39}\right]$$

$$\frac{dy}{dx} = \left[\binom{40}{1}x^{39} + 0 + 0 + \dots + 0\right]$$

$$\Rightarrow \ \frac{dy}{dx} = \binom{40}{1}x^{39} \quad \text{or} \quad \boxed{\frac{dy}{dx} = 40x^{39}}$$

Question # 1 (xiii)

Let $y = x^{-100}$

Do yourself Question # 1(xii), Replace m by -100.

Question # 2 (i)

Let $y = \sqrt{x+2} = (x+2)^{\frac{1}{2}}$

Now do yourself as Question # 1(ix)

Question # 2 (ii)

Let $y = \dfrac{1}{\sqrt{x+a}} = (x+a)^{-\frac{1}{2}}$

Now do yourself as Question # 1 (ix)
