Exercise 8.1

Principle of Mathematical Induction

A given statement $S(n)$ is true for each positive integer n if two below conditions hold

Condition I: $S(1)$ is true i.e. $S(n)$ is true for $n = 1$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers.

Use mathematical induction to prove the following formulae for every positive integer n

Question # 1

$$1+5+9+\ldots+(4n-3)=n(2n-1)$$

Solution Suppose $S(n):\ 1+5+9+\ldots\ldots\ldots\ldots+(4n-3)=n(2n-1)$

Put $n = 1$

$$S(1):\ 1=1(2(1)-1) \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+5+9+\ldots\ldots\ldots\ldots+(4k-3)=k(2k-1)\ \ \ldots\ldots\ldots\ldots\ (\text{i})$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+5+9+\ldots\ldots\ldots\ldots+(4(k+1)-3)=(k+1)(2(k+1)-1)$$

$$\Rightarrow\ 1+5+9+\ldots\ldots\ldots\ldots+(4k+1)=(k+1)(2k+2-1)$$

$$=(k+1)(2k+1)$$

$$=2k^2+2k+k+1$$

$$=2k^2+3k+1$$

Adding $4k+1$ on both sides of equation (i)

$$1+5+9+\ldots\ldots\ldots\ldots+(4k-3)+(4k+1)=k(2k-1)+4k+1$$

$$\Rightarrow\ 1+5+9+\ldots\ldots\ldots\ldots+(4k+1)=2k^2-k+4k+1$$

$$=2k^2+3k+1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 2

$$1+3+5+\ldots+(2n-1)=n^2$$

Solution Suppose

$$S(n):\ 1+3+5+\ldots\ldots\ldots\ldots+(2n-1)=n^2$$

Put $n = 1$

$$S(1):\ 1=(1)^2 \quad \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+3+5+\ldots\ldots\ldots\ldots+(2k-1)=k^2\ \ \ldots\ldots\ldots\ldots\ (\text{i})$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+3+5+\dots\dots\dots\dots+(2(k+1)-1)=(k+1)^2$$

$$\Rightarrow\ 1+3+5+\dots\dots\dots\dots+(2k+1)=(k+1)^2$$

Adding $2k+1$ on both sides of equation (i)

$$1+3+5+\dots\dots\dots\dots+(2k-1)+(2k+1)=k^2+2k+1$$

$$\Rightarrow\ 1+3+5+\dots\dots\dots\dots+(2k+1)=(k+1)^2$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 3

$$1+4+7+\dots+(3n-2)=\frac{n(3n-1)}{2}$$

Solution Suppose

$$S(n):\ 1+4+7+\dots\dots\dots\dots+(3n-2)=\frac{n(3n-1)}{2}$$

Put $n = 1$

$$S(1):\ 1=\frac{1(3(1)-1)}{2} \quad \Rightarrow\ 1=\frac{2}{2} \quad \Rightarrow 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+4+7+\dots\dots\dots\dots+(3k-2)=\frac{k(3k-1)}{2} \quad \dots\dots\dots\dots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+4+7+\dots\dots\dots\dots+(3(k+1)-2)=\frac{(k+1)(3(k+1)-1)}{2}$$

$$\Rightarrow\ 1+4+7+\dots\dots\dots\dots+(3k+1)=\frac{(k+1)(3k+3-1)}{2}$$

$$=\frac{(k+1)(3k+2)}{2}$$

Adding $3k+1$ on both sides of equation (i)

$$1+4+7+\dots\dots\dots\dots+(3k-2)+(3k+1)=\frac{k(3k-1)}{2}+3k+1$$

$$\Rightarrow\ 1+4+7+\dots\dots\dots\dots+(3k+1)=\frac{k(3k-1)+2(3k+1)}{2}$$

$$=\frac{3k^2-k+6k+2}{2}$$

$$=\frac{3k^2+5k+2}{2}$$

$$=\frac{3k^2+3k+2k+2}{2}$$

$$=\frac{3k(k+1)+2(k+1)}{2}$$

$$=\frac{(k+1)(3k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 4

$$1+2+4+\ldots+2^{n}=2^{n}-1$$

Solution Suppose

$$S(n):\ 1+2+4+\ldots\ldots\ldots\ldots+2^{n-1}=2^{n}-1$$

Put $n = 1$

$$S(1):\ 1=2^{1}-1 \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+2+4+\ldots\ldots\ldots\ldots+2^{k-1}=2^{k}-1 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+2+4+\ldots\ldots\ldots\ldots+2^{k+1-1}=2^{k+1}-1$$

$$\Rightarrow\ 1+2+4+\ldots\ldots\ldots\ldots+2^{k}=2^{k+1}-1$$

adding 2^{k} on both sides of equation (i)

$$1+2+4+\ldots\ldots\ldots\ldots+2^{k-1}+2^{k}=2^{k}-1+2^{k}$$

$$\Rightarrow\ 1+2+4+\ldots\ldots\ldots\ldots+2^{k}=2(2^{k})-1 \qquad \because\ 2^{k}+2^{k}=2(2^{k})$$

$$=2^{k+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 5

$$1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^{n-1}}=2\left[1-\frac{1}{2^{n}}\right]$$

Solution Suppose

$$S(n):\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{n-1}}=2\left(1-\frac{1}{2^{n}}\right)$$

Put $n = 1$

$$S(1):\ 1=2\left(1-\frac{1}{2^{1}}\right) \ \Rightarrow\ 1=2\left(\frac{1}{2}\right) \ \Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{k-1}}=2\left(1-\frac{1}{2^{k}}\right) \quad \ldots\ldots\ldots\ldots \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{k+1-1}}=2\left(1-\frac{1}{2^{k+1}}\right)$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{2^{k}}=2-\frac{2}{2^{k+1}}$$

$$= 2 - \frac{2}{2^k \cdot 2}$$

$$= 2 - \frac{1}{2^k}$$

Adding $\frac{1}{2^k}$ on both sides of equation (i)

$$1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots\ldots + \frac{1}{2^{k-1}} + \frac{1}{2^{k+1-1}} = 2\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^k}$$

$$\Rightarrow \quad 1 + \frac{1}{2} + \frac{1}{4} + \ldots\ldots\ldots\ldots + \frac{1}{2^{k+1-1}} = 2 - \frac{2}{2^k} + \frac{1}{2^k}$$

$$= 2 - \frac{1}{2^k}(2-1)$$

$$= 2 - \frac{1}{2^k}(1) \quad = 2 - \frac{1}{2^k}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 6 *Do yourself as Question # 1*

Question # 7

$$2 + 6 + 18 + \ldots + 2 \times 3^{n-1} = 3^n - 1$$

Solution Suppose

$$S(n):\ 2 + 6 + 18 + \ldots\ldots\ldots\ldots + 2 \times 3^{n-1} = 3^n - 1$$

Put $n = 1$

$$S(1):\ 2 = 3^1 - 1 \quad \Rightarrow \quad 2 = 2$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 2 + 6 + 18 + \ldots\ldots\ldots\ldots + 2 \times 3^{k-1} = 3^k - 1 \quad \ldots\ldots\ldots\ldots \text{(i)}$$

The statement for $n = k+1$ becomes

$$S(k+1):\ 2 + 6 + 18 + \ldots\ldots\ldots\ldots + 2 \times 3^{k+1-1} = 3^{k+1} - 1$$

Adding 2×3^k on both sides of equation (i)

$$2 + 6 + 18 + \ldots\ldots\ldots\ldots + 2 \times 3^{k-1} + 2 \times 3^k = 3^k - 1 + 2 \times 3^k$$

$$\Rightarrow \quad 2 + 6 + 18 + \ldots\ldots\ldots\ldots + 2 \times 3^{k+1-1} = 3(3^k) - 1 \qquad \because\ 3^k + 2 \times 3^k = 3(3^k)$$

$$= 3^{k+1} - 1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 8

$$1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}$$

Solution Suppose

$$S(n):\ 1 \times 3 + 2 \times 5 + 3 \times 7 + \ldots\ldots\ldots\ldots + n \times (2n+1) = \frac{n(n+1)(4n+5)}{6}$$

Put $n = 1$

$$S(1):\ 1\times 3=\frac{1(1+1)(4(1)+5)}{6} \quad \Rightarrow\ 3=\frac{(2)(9)}{6} \quad \Rightarrow\ 3=3$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1\times 3+2\times 5+3\times 7+.............+k\times(2k+1)=\frac{k(k+1)(4k+5)}{6} \quad .... \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1): 1\times 3+2\times 5+3\times 7+.............+(k+1)\times(2(k+1)+1)=\frac{(k+1)(k+1+1)(4(k+1)+5)}{6}$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+.............+(k+1)\times(2k+3)=\frac{(k+1)(k+2)(4k+9)}{6}$$

Adding $(k+1)\times(2k+3)$ on both sides of equation (i)

$$1\times 3+2\times 5+3\times 7+.............+k\times(2k+1)+(k+1)\times(2k+3)=\frac{k(k+1)(4k+5)}{6}+(k+1)\times(2k+3)$$

$$\Rightarrow\ 1\times 3+2\times 5+3\times 7+.............+(k+1)\times(2k+3)=(k+1)\left(\frac{k(4k+5)}{6}+(2k+3)\right)$$

$$=(k+1)\left(\frac{k(4k+5)+6(2k+3)}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+5k+12k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+17k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k^2+8k+9k+18}{6}\right)$$

$$=(k+1)\left(\frac{4k(k+2)+9(k+2)}{6}\right)$$

$$=(k+1)\left(\frac{(k+2)(4k+9)}{6}\right)$$

$$=\frac{(k+1)(k+2)(4k+9)}{6}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 9

$$1\times 2+2\times 3+3\times 4+...+n\times(n+1)=\frac{n(n+1)(n+2)}{3}$$

Solution *Do yourself as Question # 8*

Question # 10

$$1\times2+3\times4+5\times6+\ldots+(2n-1)\times2n=\frac{n(n+1)(4n-1)}{3}$$

Solution *Do yourself as Question # 8*

Question # 11

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Solution Suppose

$$S(n):\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots\ldots\ldots\ldots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$$

Put $n = 1$

$$S(1):\ \frac{1}{1\times2}=1-\frac{1}{1+1}\quad\Rightarrow\ \frac{1}{2}=1-\frac{1}{2}\quad\Rightarrow\ \frac{1}{2}=\frac{1}{2}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots\ldots\ldots\ldots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}\quad\ldots\ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{(k+1)(k+1+1)}=1-\frac{1}{k+1+1}$$

$$\Rightarrow\ 1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots\ldots\ldots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}$$

Adding $\dfrac{1}{(k+1)(k+2)}$ on both sides of equation (i)

$$\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots\ldots\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$\Rightarrow\ \frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\ldots\ldots\ldots\ldots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+1}\left(1-\frac{1}{(k+2)}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+2-1}{k+2}\right)$$

$$=1-\frac{1}{k+1}\left(\frac{k+1}{k+2}\right)$$

$$=1-\frac{1}{k+2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 12

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\ldots\ldots\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Solution Suppose

$$S(n):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Put $n = 1$

$$S(1):\ \frac{1}{1\times 3}=\frac{1}{2(1)+1}\quad\Rightarrow\ \frac{1}{3}=\frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}\ \ldots\ldots\ldots\ (\text{i})$$

The statement for $n=k+1$ becomes

$$S(k+1):\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{\big(2(k+1)-1\big)\big(2(k+1)+1\big)}=\frac{k+1}{2(k+1)+1}$$

$$\Rightarrow\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Adding $\frac{1}{(2k+1)(2k+3)}$ on both sides of equation (i)

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$\Rightarrow\ \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots\ldots\ldots\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{1}{2k+1}\left(k+\frac{1}{(2k+3)}\right)$$

$$=\frac{1}{2k+1}\left(\frac{k(2k+3)+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k^2+3k+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k^2+2k+k+1}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{2k(k+1)+1(k+1)}{2k+3}\right)$$

$$=\frac{1}{2k+1}\left(\frac{(2k+1)(k+1)}{2k+3}\right)$$

$$=\left(\frac{k+1}{2k+3}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 13

$$\frac{1}{2\times 5}+\frac{1}{5\times 8}+\frac{1}{8\times 11}+\ldots\ldots\ldots\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{2(3n+2)}$$

Solution *Do yourself as Question # 12*

Question # 14

$$r+r^2+r^3+\dots\dots\dots\dots+r^n=\frac{r(1-r^n)}{1-r} \qquad (r\neq 1)$$

Solution Suppose $S(n):\ r+r^2+r^3+\dots\dots\dots\dots+r^n=\dfrac{r(1-r^n)}{1-r}$

Put $n=1$

$$S(1):\ r=\frac{r\left(1-r^1\right)}{1-r} \Rightarrow r=r$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ r+r^2+r^3+\dots\dots\dots\dots+r^k=\frac{r(1-r^k)}{1-r} \quad \dots\dots\dots\dots \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ r+r^2+r^3+\dots\dots\dots\dots+r^{k+1}=\frac{r(1-r^{k+1})}{1-r}$$

Adding r^{k+1} on both sides of equation (i)

$$r+r^2+r^3+\dots\dots\dots\dots+r^k+r^{k+1}=\frac{r(1-r^k)}{1-r}+r^{k+1}$$

$$\Rightarrow r+r^2+r^3+\dots\dots\dots\dots+r^{k+1}=\frac{r(1-r^k)+r^{k+1}(1-r)}{1-r}$$

$$=\frac{r-r^{k+1}+r^{k+1}-r^{k+2}}{1-r}$$

$$=\frac{r-r^{k+2}}{1-r}$$

$$=\frac{r(1-r^{k+1})}{1-r}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 15

$$a+(a+d)+(a+2d)+\dots\dots\dots\dots\dots+\left[a+(n-1)d\right]=\frac{n}{2}\left[2a+(n-1)d\right]$$

Solution Suppose

$$S(n):\ a+(a+d)+(a+2d)+\dots\dots\dots\dots\dots+\left[a+(n-1)d\right]=\frac{n}{2}\left[2a+(n-1)d\right]$$

Put $n=1$

$$S(1):\ a=\frac{1}{2}\left[2a+(1-1)d\right] \Rightarrow a=\frac{1}{2}\left[2a+(0)d\right] \Rightarrow a=\frac{1}{2}\left[2a\right]=a$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ a+(a+d)+(a+2d)+\ldots\ldots\ldots\ldots+\left[a+(k-1)d\right]=\frac{k}{2}\left[2a+(k-1)d\right]\ \ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ a+(a+d)+(a+2d)+\ldots\ldots\ldots\ldots+\left[a+(k+1-1)d\right]=\frac{k+1}{2}\left[2a+(k+1-1)d\right]$$

$$\Rightarrow\ a+(a+d)+(a+2d)+\ldots\ldots\ldots\ldots+\left[a+kd\right]=\frac{k+1}{2}\left[2a+kd\right]$$

Adding $a+kd$ on both sides of equation (i)

$$a+(a+d)+(a+2d)+\ldots\ldots\ldots\ldots+\left[a+(k-1)d\right]+\left[a+kd\right]=\frac{k}{2}\left[2a+(k-1)d\right]+\left[a+kd\right]$$

$$\begin{aligned}
\Rightarrow\ a+(a+d)+(a+2d)+\ldots\ldots\ldots\ldots+\left[a+kd\right]&=\frac{k}{2}\left[2a+kd-d\right]+\left[a+kd\right]\\
&=\frac{k\left[2a+kd-d\right]+2\left[a+kd\right]}{2}\\
&=\frac{2ak+k^2d-kd+2a+2kd}{2}\\
&=\frac{2ak+k^2d+kd+2a}{2}\\
&=\frac{2ak+2a+k^2d+kd}{2}\\
&=\frac{2a(k+1)+kd(k+1)}{2}\\
&=\frac{(k+1)(2a+kd)}{2}\\
&=\frac{k+1}{2}\left[2a+kd\right]
\end{aligned}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 16

$$1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+n\cdot\underline{|n}=\underline{|n+1}-1$$

Solution

Suppose $S(n):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+n\cdot\underline{|n}=\underline{|n+1}-1$

Put $n=1$

$S(1):\ 1\cdot\underline{|1}=\underline{|1+1}-1\ \Rightarrow\ 1=\underline{|2}-1\ \Rightarrow\ 1=2-1\ \Rightarrow\ 1=1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+k\cdot\underline{|k}=\underline{|k+1}-1\ \ldots\ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+(k+1)\cdot\underline{|k+1}=\underline{|k+1+1}-1$$

$$\Rightarrow\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+(k+1)\cdot\underline{|k+1}=\underline{|k+2}-1$$

Adding $(k+1)\cdot\underline{|k+1}$ on both sides of equation (i)

$$1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+k\cdot\underline{|k}+(k+1)\cdot\underline{|k+1}=\underline{|k+1}-1+(k+1)\cdot\underline{|k+1}$$

$$\Rightarrow\ 1\cdot\underline{|1}+2\cdot\underline{|2}+3\cdot\underline{|3}+\ldots\ldots\ldots\ldots+(k+1)\cdot\underline{|k+1}=\underline{|k+1}+\underline{|k+1}(k+1)-1$$
$$=\underline{|k+1}\left(1+k+1\right)-1$$
$$=\underline{|k+1}\left(k+2\right)-1$$
$$=\left(k+2\right)\underline{|k+1}-1$$
$$=\underline{|k+2}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 17

$a_n = a_1 + (n-1)d$ When, $a_1, a_1+d, a_1+2d, \ldots$ from an A.P.

Solution

Suppose $S(n):\ a_n = a_1 + (n-1)d$

Put $n=1$

$S(1):\ a_1 = a_1 + (1-1)d \ \Rightarrow\ a_1 = a_1 + 0d = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ a_k = a_1 + (k-1)d$ .... (i)

The statement for $n=k+1$ becomes

$S(k+1):\ a_{k+1} = a_1 + (k+1-1)d$
$= a_1 + (k)d$

Adding d on both sides of equation (i)

$a_k + d = a_1 + (k-1)d + d$
$\Rightarrow\ a_{k+1} = a_1 + (k-1+1)d$
$\Rightarrow\ a_{k+1} = a_1 + (k)d$

$\because\ a_2 = a_1 + d$
$a_3 = a_2 + d$
$\therefore\ a_{k+1} = a_k + d$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 18

$a_n = a_1 r^{n-1}$ When, $a_1, a_1 r, a_1 r^2, \ldots$ from an G.P.

Solution Suppose $S(n):\ a_n = a_1 r^{n-1}$

Put $n=1$

$S(1):\ a_1 = a_1 r^{1-1} \ \Rightarrow\ a_1 = a_1 r^0 = a_1$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ a_k = a_1 r^{k-1}$ .... (i)

The statement for $n=k+1$ becomes

$S(k+1):\ a_{k+1} = a_1 r^{k+1-1}$
$= a_1 r^k$

Multiplying r on both sides of equation (i)

$a_k \cdot r = a_1 r^{k-1} \cdot r^1$
$\Rightarrow\ a_{k+1} = a_1 r^{k-1+1}$

$\because\ a_2 = a_1 r$
$a_3 = a_2 r$
$\therefore\ a_{k+1} = a_k r$

$$\Rightarrow \quad a_{k+1} = a_1 r^k$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer *n*.

Question # 19

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

Solution

Suppose $\quad S(n):\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$

Put $n = 1$

$$S(1):\ 1^2 = \frac{1(4(1)^2-1)}{3} \ \Rightarrow\ 1 = \frac{1(4-1)}{3} \ \Rightarrow\ 1 = \frac{3}{3} = 1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$$S(k):\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2k-1)^2 = \frac{k(4k^2-1)}{3} \ \ldots\ldots\ldots \text{(i)}$$

The statement for $n = k+1$ becomes

$$S(k+1):\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2(k+1)-1)^2 = \frac{(k+1)(4(k+1)^2-1)}{3}$$

$$\Rightarrow\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2k+1)^2 = \frac{(k+1)(4(k^2+2k+1)-1)}{3}$$

$$= \frac{(k+1)(4k^2+8k+4-1)}{3}$$

$$= \frac{(k+1)(4k^2+8k+3)}{3}$$

$$= \frac{4k^3+8k^2+3k+4k^2+8k+3}{3}$$

$$= \frac{4k^3+12k^2+11k+3}{3}$$

Adding $(2k+1)^2$ on both sides of equation (i)

$$1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2k-1)^2 + (2k+1)^2 = \frac{k(4k^2-1)}{3} + (2k+1)^2$$

$$\Rightarrow\ 1^2 + 3^2 + 5^2 + \ldots\ldots\ldots\ldots\ldots + (2k+1)^2 = \frac{k(4k^2-1)+3(2k+1)^2}{3}$$

$$= \frac{k(4k^2-1)+3(4k^2+4k+1)}{3}$$

$$= \frac{4k^3-k+12k^2+12k+3}{3}$$

$$=\frac{4k^3+12k^2+11k+3}{3}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 20

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{n+2}{3}=\binom{n+3}{4}$$

Solution

Suppose $\quad S(n):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{n+2}{3}=\binom{n+3}{4}$

Put $n=1$

L.H.S $=\binom{3}{3}=1$

R.H.S $=\binom{1+3}{4}=\binom{4}{4}=1$

L.H.S = R.H.S

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+2}{3}=\binom{k+3}{4}\quad\ldots\ldots\ldots\text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+1+2}{3}=\binom{k+1+3}{4}$$

$$\Rightarrow\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+4}{4}$$

Adding $\binom{k+3}{3}$ on both sides of equation (i)

$$\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+2}{3}+\binom{k+3}{3}=\binom{k+3}{4}+\binom{k+3}{3}$$

$$\Rightarrow\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+3+1}{4}\qquad\because\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$$

$$\Rightarrow\binom{3}{3}+\binom{4}{3}+\binom{5}{3}+\ldots\ldots\ldots\ldots+\binom{k+3}{3}=\binom{k+4}{4}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 21

Prove by mathematical induction that for all positive integral values of n.

(i) n^2+n is divisible by 2 (ii) 5^n-2^n is divisible by 3
(iii) 5^n-1 is divisible by 4 (iv) $8\times10^n-2$ is divisible by 6
(v) n^3-n is divisible by 6

Solution

(i) Suppose $S(n):\ n^2+n$

Put $n=1$

$S(1):\ 1^2+1=2$

$S(1)$ is clearly divisible by 2, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ k^2+k$

Then there exists quotient Q such that

$$k^2+k=2Q$$

$$\begin{array}{r|l} & Q \\ 2 & k^2+k \\ & \underline{-k^2-k} \\ & 0 \end{array}$$

The statement for $n=k+1$

$$S(k+1):\ (k+1)^2+k+1$$
$$=k^2+2k+1+k+1$$
$$=k^2+k+2k+2$$
$$=2Q+2k+2 \qquad \because\ k^2+k=2Q$$
$$=2(Q+k+1)$$

Clearly $S(k+1)$ is divisible by 2.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

(ii) Suppose $S(n):\ 5^n-2^n$

Put $n=1$

$S(1):\ 5^1-2^1=3$

$S(1)$ is clearly divisible by 3, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ 5^k-2^k$

Then there exists quotient Q such that

$$5^k-2^k=3Q$$

The statement for $n=k+1$

$$S(k+1):\ 5^{k+1}-2^{k+1}$$
$$=\ 5\cdot5^k-2\cdot2^k$$
$$=\ 5\cdot5^k-5\cdot2^k+5\cdot2^k-2\cdot2^k$$
$$=\ 5\left(5^k-2^k\right)+2^k(5-2)$$
$$=\ 5(3Q)+2^k\cdot3 \qquad \because\ 5^k-2^k=3Q$$
$$=3(5Q+2^k)$$

Clearly $S(k+1)$ is divisible by 3.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^{+}$.

(iii) *Same as Question # 21 (ii)*

Hint: $S(k+1):\ 5^{k+1}-1$

$= 5\cdot 5^{k}-1 \quad = 5\cdot 5^{k}-5+5-1$

$= 5\left(5^{k}-1\right)+4 \quad = 5(4Q)-4 \qquad \because\ 5^{k}-1=4Q$

(iv) Suppose $S(n):\ 8\times 10^{n}-2$

Put $n=1$

$S(1):\ 8\times 10^{1}-2=80-2=78=6\times 13$

$S(1)$ is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ 8\times 10^{k}-2$

Then there exists quotient Q such that

$8\times 10^{k}-2 = 6Q$

The statement for $n=k+1$

$S(k+1):\ 8\times 10^{k+1}-2$

$=8\times 10\cdot 10^{k}-2$

$= 8\times 10\cdot 10^{k}-2\cdot 10+2\cdot 10-2 \qquad$ –ing & +ing $2\cdot 10$

$= 10\left(8\times 10^{k}-2\right)+20-2$

$= 10\left(6Q\right)+18 \qquad \because\ 8\times 10^{k}-2=6Q$

$= 6\left(10Q+3\right)$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^{+}$.

(v) Suppose $S(n):\ n^{3}-n$

Put $n=1$

$S(1):\ 1^{3}-1=0$

$S(1)$ i.e. 0 is clearly divisible by 6, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ k^{3}-k$

Then there exists quotient Q such that

$k^{3}-k=6Q$

The statement for $n=k+1$

$S(k+1):\ (k+1)^{3}-(k+1)$

$=k^{3}+3k^{2}+3k+1-k-1$

$=k^{3}+3k^{2}+3k-k$

$=(k^{3}-k)+3(k^{2}+k)$

$=6Q+3(2Q')$

$=6Q+6Q'$

Since $n^{2}+n$ is divisible by 2

Therefore $n^{2}+n=2Q'$

Or $k^{2}+k=2Q'$

Clearly $S(k+1)$ is divisible by 6.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^{+}$.

Question # 22

$$\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^n}=\frac{1}{2}\left(1-\frac{1}{3^n}\right)$$

Solution Suppose $S(n):\ \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^n}=\frac{1}{2}\left(1-\frac{1}{3^n}\right)$

Put $n=1$

$$S(1):\ \frac{1}{3}=\frac{1}{2}\left(1-\frac{1}{3^1}\right)\ \Rightarrow\ \frac{1}{3}=\frac{1}{2}\left(\frac{2}{3}\right)\ \Rightarrow\ \frac{1}{3}=\frac{1}{3}$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^k}=\frac{1}{2}\left(1-\frac{1}{3^k}\right)\ \ldots\ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^{k+1}}=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$$

Adding $\frac{1}{3^k}$ on both sides of equation (i)

$$\frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^k}+\frac{1}{3^{k+1}}=\frac{1}{2}\left(1-\frac{1}{3^k}\right)+\frac{1}{3^{k+1}}$$

$$\Rightarrow\ \frac{1}{3}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots\ldots+\frac{1}{3^{k+1}}=\frac{1}{2}-\frac{1}{2\cdot 3^k}+\frac{1}{3\cdot 3^k}$$

$$=\frac{1}{2}-\frac{1}{3^k}\left(\frac{1}{2}-\frac{1}{3}\right)$$

$$=\frac{1}{2}-\frac{1}{3^k}\left(\frac{3-2}{6}\right)\ =\frac{1}{2}-\frac{1}{3^k}\left(\frac{1}{6}\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{3^k}\left(\frac{1}{3}\right)\right)$$

$$=\frac{1}{2}\left(1-\frac{1}{3^{k+1}}\right)$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 23

$$1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots\ldots+(-1)^{n-1}\cdot n^2=\frac{(-1)^{n-1}\cdot n(n+1)}{2}$$

Solution Suppose

$$S(n):\ 1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{n-1}\cdot n^2=\frac{(-1)^{n-1}\cdot n(n+1)}{2}$$

Put $n = 1$

$$S(1):\ 1^2=\frac{(-1)^{1-1}\cdot 1(1+1)}{2} \quad\Rightarrow\ 1=\frac{(-1)^0\cdot 2}{2} \quad\Rightarrow\ 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k-1}\cdot k^2=\frac{(-1)^{k-1}\cdot k(k+1)}{2} \quad\ldots\ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k+1-1}\cdot(k+1)^2=\frac{(-1)^{k+1-1}\cdot(k+1)(k+1+1)}{2}$$

$$\Rightarrow\ 1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k}\cdot(k+1)^2=\frac{(-1)^{k}\cdot(k+1)(k+2)}{2}$$

Adding $(-1)^k\cdot(k+1)^2$ on both sides of equation (i)

$$1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k-1}\cdot k^2+(-1)^k\cdot(k+1)^2=\frac{(-1)^{k-1}\cdot k(k+1)}{2}+(-1)^k\cdot(k+1)^2$$

$$\Rightarrow\ 1^2-2^2+3^2-4^2+\ldots\ldots\ldots\ldots+(-1)^{k}\cdot(k+1)^2=\frac{(-1)^{k-1}\cdot k(k+1)+2(-1)^k\cdot(k+1)^2}{2}$$

$$=\frac{(-1)^k(k+1)\left[(-1)^{-1}k+2(k+1)\right]}{2}$$

$$=\frac{(-1)^k(k+1)\left[-k+2k+2\right]}{2}$$

$$=\frac{(-1)^k(k+1)(k+2)}{2}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 24

$$1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2n-1)^3=n^2(2n^2-1)$$

Solution Suppose $S(n):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2n-1)^3=n^2(2n^2-1)$

Put $n = 1$

$$S(1):\ 1^3=1^2\left(2(1)^2-1\right) \quad\Rightarrow\ 1=1(2-1) \quad\Rightarrow 1=1$$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$$S(k):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k-1)^3=k^2(2k^2-1) \quad\ldots\ldots\ldots\ldots\ \text{(i)}$$

The statement for $n=k+1$ becomes

$$S(k+1):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2(k+1)-1)^3=(k+1)^2(2(k+1)^2-1)$$
$$\Rightarrow\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k+1)^3=(k^2+2k+1)(2(k^2+2k+1)-1)$$
$$=(k^2+2k+1)(2k^2+4k+2-1)$$
$$=(k^2+2k+1)(2k^2+4k+1)$$

$$=2k^4+4k^3+2k^2+4k^3+8k^2+4k+k^2+2k+1$$
$$=2k^4+8k^3+11k^2+6k+1$$

Adding $(2k+1)^3$ on both sides of equation (i)

$$S(k):\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k-1)^3+(2k+1)^3=k^2(2k^2-1)+(2k+1)^3$$
$$\Rightarrow\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k+1)^3=k^2(2k^2-1)+(2k)^3+3(2k)^2(1)+3(2k)(1)^2+(1$$
$$\Rightarrow\ 1^3+3^3+5^3+\ldots\ldots\ldots\ldots+(2k+1)^3=2k^4-k^2+8k^3+12k^2+6k+1$$
$$=2k^4+8k^3+11k^2+6k+1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integer n.

Question # 25

$x+1$ is a factor of $x^{2n}-1;\left(x\neq -1\right)$

Solution Suppose $S(n):\ x^{2n}-1$

Put $n=1$

$S(1):\ x^{2(1)}-1=x^2-1=(x-1)(x+1)$

$x+1$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^{2k}-1$

Then there exists quotient Q such that

$$x^{2k}-1=(x+1)Q$$

The statement for $n=k+1$

$S(k+1):\ x^{2(k+1)}-1$

$$=x^{2k+2}-1$$
$$=x^{2k+2}-x^{2k}+x^{2k}-1 \qquad \text{+ing and –ing } x^{2k}$$
$$=x^{2k}(x^2-1)+(x^{2k}-1)$$
$$=x^{2k}(x-1)(x+1)+(x+1)Q \qquad \because\ x^{2k}-1=(x+1)Q$$
$$=(x+1)\left(x^{2k}(x-1)+Q\right)$$

Clearly $x+1$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 26

$x-y$ is a factor of $x^n-y^n;\left(x\neq y\right)$

Solution Suppose $S(n):\ x^n-y^n$

Put $n=1$

$S(1):\ x^1 - y^1 = x - y$

$x-y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^k - y^k$

Then there exists quotient Q such that

$$x^k - y^k = (x-y)Q$$

The statement for $n=k+1$

$S(k+1):\ x^{k+1} - y^{k+1}$

$$= x\cdot x^k - y\cdot y^k$$

$$= x\cdot x^k - x\cdot y^k + x\cdot y^k - y\cdot y^k \qquad \text{–ing \& +ing } x\,y^k$$

$$= x(x^k - y^k) + y^k(x-y)$$

$$= x(x-y)Q + y^k(x-y) \qquad \because x^k - y^k = (x-y)Q$$

Clearly $x-y$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Question # 27

$x+y$ is a factor of $x^{2n-1} + y^{2n-1}; (x\neq y)$

Solution Suppose $S(n):\ x^{2n-1} + y^{2n-1}$

Put $n=1$

$S(1):\ x^{2(1)-1} + y^{2(1)-1} = x^1 + y^1 = x + y$

$x+y$ is clearly factor of $S(1)$, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$

$S(k):\ x^{2k-1} + y^{2k-1}$

Then there exists quotient Q such that

$$x^{2k-1} + y^{2k-1} = (x+y)Q$$

The statement for $n=k+1$

$S(k+1):\ x^{2(k+1)-1} + y^{2(k+1)-1}$

$$= x^{2k+2-1} + y^{2k+2-1}$$

$$= x^{2k+2-1} - x^{2k-1}y^2 + x^{2k-1}y^2 + y^{2k+2-1} \qquad \text{+ing and –ing } x^{2k-1}y^2$$

$$= x^{2k-1}(x^2 - y^2) + y^2(x^{2k-1} + y^{2k-1})$$

$$= x^{2k-1}(x-y)(x+y) + y^2(x+y)Q \qquad \because x^{2k-1} + y^{2k-1} = (x+y)Q$$

$$= (x+y)\left(x^{2k-1}(x-y) + y^2\,Q\right)$$

Clearly $x+y$ is a factor of $S(k+1)$.

Since the truth for $n=k$ implies the truth for $n=k+1$ therefore the given statement is true for $\forall\ n\in\mathbb{Z}^+$.

Principle of Extended Mathematical Induction

A given statement $S(n)$ is true for $n\geq i$ if the following two conditions hold

Condition I: $S(i)$ is true i.e. $S(n)$ is true for $n = i$ and

Condition II: $S(k+1)$ is true whenever $S(k)$ is true for any positive integer k,

Then $S(n)$ is true for all positive integers

Question # 28

Use mathematical induction to show that

$1+2+2^2+.............+2^n = 2^{n+1}-1$ for all non-negative integers n.

Solution Suppose $S(n):\ 1+2+2^2+.............+2^n = 2^{n+1}-1$

Put $n = 0$

$S(1):\ 1 = 2^{0+1}-1 = \ \Rightarrow\ 1 = 2-1 \ \Rightarrow\ 1 = 1$

> ***Note***: Non- negative number are $0,1,2,3,..........$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n = k$

$S(k):\ 1+2+2^2+.............+2^k = 2^{k+1}-1$ .... (i)

The statement for $n = k+1$ becomes

$$S(k+1):\ 1+2+2^2+.............+2^{k+1} = 2^{k+1+1}-1$$
$$= 2^{k+2}-1$$

Adding 2^{k+1} on both sides of equation (i)

$$1+2+2^2+.............+2^k+2^{k+1} = 2^{k+1}-1+2^{k+1}$$
$$\Rightarrow\ 1+2+4+.............+2^{k+1} = 2(2^{k+1})-1 \qquad \because\ 2^{k+1}+2^{k+1} = 2(2^{k+1})$$
$$= 2^{k+1+1}-1$$
$$= 2^{k+1+1}-1$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all non-negative integers n.

Question # 29

If A and B are square matrices and $\mathrm{AB} = \mathrm{BA}$, then show by mathematical induction that $\mathrm{AB}^n = \mathrm{B}^n\mathrm{A}$ for any positive integer n.

Solution Suppose $S(n):\ AB^n = B^nA$

Put $n = 1$

$S(1):\ \mathrm{AB}^1 = B^1A \ \Rightarrow\ \mathrm{AB} = BA$

$S(1)$ is true as we have given $\mathrm{AB} = BA$, Thus condition I is satisfied

Now suppose that given statement is true for $n = k$

$S(k):\ AB^k = B^kA$ .... (i)

The statement for $n = k+1$

$S(k+1):\ AB^{k+1} = B^{k+1}A$

Post-multiplying equation (i) by B.

$$(AB^k)B = (B^kA)B$$
$$\Rightarrow\ A(B^kB) = B^k(AB) \qquad \text{by associative law}$$
$$\Rightarrow\ AB^{k+1} = B^k(BA) \qquad \because\ \mathrm{AB} = BA \ \ (\textit{given})$$
$$= (B^kB)A = B^{k+1}A$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all positive integers n.

Question # 30

Prove by Principle of mathematical induction that n^2-1 is divisible by 8 when n is and odd positive integer.

Solution Suppose $S(n):\ n^2-1$

Put $n=1$

$$S(1):\ (1)^2-1=0$$

$S(1)$ is clearly divisible by 8, Thus condition I is satisfied

Now suppose that given statement is true for $n=k$ where k is odd.

$$S(k):\ k^2-1$$

Then there exists quotient Q such that

$$k^2-1=8\,Q$$

As $k+2$ is the next odd integer after k The statement for $n=k+1$

$$S(k+2):\ (k+2)^2-1$$
$$=k^2+4k+4-1$$
$$=k^2-1+4k+4$$
$$=8Q+4(k+1) \qquad \because\ k^2+k=2Q$$

Since k is odd therefore $k+1$ is even so their exists integer t such that $k+1=2t$

$$\Rightarrow\ S(k+2):=8Q+4(2t)$$
$$=8Q+8t$$

Clearly $S(k+2)$ is divisible by 8 so condition II is satisfied.

Therefor the given statement is true for odd positive integers.

Question # 31

Use the principle of mathematical induction to prove that $\ln x^n = n\ln x$ for any integral $n\geq 0$ if x is a positive number.

Solution Suppose $S(n):\ \ln x^n=n\ln x$

Put $n=1$

$$S(1):\ \ln x^1=(1)\ln x \qquad \Rightarrow\ \ln x=\ln x$$

$S(1)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$$S(k):\ \ln x^k=k\ln x\ \ldots\ldots\ldots\ldots\ldots\ (\text{i})$$

The statement for $n=k+1$

$$S(k+1):\ \ln x^{k+1}=(k+1)\ln x$$

Now adding $\ln x$ on both sides of equation (i)

$$\ln x^k+\ln x=k\ln x+\ln x$$
$$\Rightarrow\ \ln x^k\cdot x=(k+1)\ln x \qquad \because\ \ln x+\ln y=\ln x\,y$$
$$\Rightarrow\ \ln x^{k+1}=(k+1)\ln x$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all $n\in\mathbb{Z}^+$.

Question # 32

$n!>2^n-1$ for integral values of $n\geq 4$.

Solution Suppose $S(n):\ n! > 2^n - 1$; $n \geq 4$

Put $n = 4$

$S(4):\ 4! > 2^4 - 1 \quad \Rightarrow \quad 24 > 16 - 1 \quad \Rightarrow \quad 24 > 15$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ \ k! > 2^k - 1 \ldots\ldots\ldots\ldots\ldots$ (i)

The statement for $n = k+1$

$S(k+1):\ (k+1)! > 2^{k+1} - 1$

Multiplying both sides of equation (i) by $k+1$

$$(k+1)k! > (k+1)(2^k - 1)$$

$$\Rightarrow\ (k+1)! > (k+1+2-2)(2^k - 1) \qquad \because\ (k+1)k! = (k+1)!$$

$$\Rightarrow\ (k+1)! > (k-1+2)(2^k - 1)$$

$$\Rightarrow\ (k+1)! > k\cdot 2^k - k - 2^k + 1 + 2\cdot 2^k - 2$$

$$\Rightarrow\ (k+1)! > (k\cdot 2^k - 2^k - k) + 2^{k+1} - 1$$

$$\Rightarrow\ (k+1)! > 2^{k+1} - 1 \qquad \because\ k\cdot 2^k - 2^k - k \geq 0 \ \ \forall\ \ k \geq 4$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 33

$n^2 > n+3$ for integral values of $n \geq 3$.

Solution Suppose $S(n):\ n^2 > n+3$; $n \geq 3$

Put $n = 3$

$S(3):\ 3^2 > 3+3 \quad \Rightarrow \quad 9 > 6$

$S(3)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n = k$

$S(k):\ \ k^2 > k+3 \ldots\ldots\ldots\ldots\ldots$ (i)

The statement for $n = k+1$

$S(k+1):\ (k+1)^2 > k+1+3 \quad \Rightarrow \quad (k+1)^2 > k+4$

Adding $2k+1$ on both sides of equation (i)

$$k^2 + 2k + 1 > k + 3 + 2k + 1$$

$$\Rightarrow\ (k+1)^2 > k + 4 + 2k$$

$$\Rightarrow\ (k+1)^2 > k + 4 \qquad \text{ignoring } 2k \text{ as } 2k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 34

$4^n > 3^n + 2^{n-1}$ for integral values of $n \geq 2$.

Solution Suppose $S(n):\ 4^n > 3^n + 2^{n-1}$; $n \geq 2$

Put $n = 2$

$S(2):\ 4^2 > 3^2 + 2^{2-1} \quad \Rightarrow \quad 16 > 9 + 2 \quad \Rightarrow \quad 16 > 11$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$$S(k):\ 4^k > 3^k + 2^{k-1} \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The statement for $n=k+1$

$$S(k+1):\ 4^{k+1} > 3^{k+1} + 2^{k+1-1}$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k$$

Multiplying both sides of equation (i) by 4.

$$4(4^k) > 4(3^k + 2^{k-1})$$

$$\Rightarrow\ 4^{k+1} > 4\cdot 3^k + 4\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > (3+1)\cdot 3^k + (2+2)\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3\cdot 3^k + 3^k + 2\cdot 2^{k-1} + 2\cdot 2^{k-1}$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k + (3^k + 2^k)$$

$$\Rightarrow\ 4^{k+1} > 3^{k+1} + 2^k \qquad \text{ignoring } 3^k + 2^k \text{ as } 3^k + 2^k > 0$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 3$.

Question # 35

$3^n < n!$ for integral values of $n \geq 6$.

Solution Suppose $S(n):\ 3^n < n!$; $n > 6$

Put $n=7$

$$S(7):\ 3^7 < 7! \quad \Rightarrow\ 2187 < 5040$$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$$S(k):\ 3^k < k! \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

The statement for $n=k+1$

$$S(k+1):\ 3^{k+1} < (k+1)!$$

Multiplying both sides of equation (i) by $k+1$.

$$(k+1)3^k < (k+1)k!$$

$$\Rightarrow\ \big((k-2)+3\big)3^k < (k+1)!$$

$$\Rightarrow\ (k-2)3^k + 3^{k+1} < (k+1)!$$

$$\Rightarrow\ 3^{k+1} < (k+1)! \qquad \because\ (k-2)3^k > 0 \ \ \forall\ \ k > 6$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n > 6$.

Question # 36

$n! > n^2$ for integral values of $n \geq 4$.

Solution Suppose $S(n):\ n! > n^2$; $n \geq 4$

Put $n=4$

$$S(4):\ 4! > 4^2 \quad \Rightarrow\ 24 > 16$$

$S(4)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$S(k):\ \ k! > k^2$ .... (i)

The statement for $n=k+1$

$S(k+1):\ (k+1)! > (k+1)^2$

Multiplying both sides of equation (i) by $k+1$.

$(k+1)k! > (k+1)k^2$

$\Rightarrow\ (k+1)! > (k+1)(k+1)$ $\qquad \because\ k+1 < k^2\ \ \forall\ \ k \geq 4$

$\Rightarrow\ (k+1)! > (k+1)^2$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq 4$.

Question # 37

$3+5+7+..............+(2n+5)=(n+2)(n+4)$ for integral values of $n \geq -1$.

Solution Suppose $S(n):\ 3+5+7+..............+(2n+5)=(n+2)(n+4)$; $n \geq -1$

Put $n=-1$

$S(-1):\ 3=(-1+2)(-1+4)\ \ \Rightarrow\ 3=(1)(3)\ \ \Rightarrow\ 3=3$

Thus condition I is satisfied

Now suppose that $S(n)$ is true for $n=k$

$S(k):\ 3+5+7+..............+(2k+5)=(k+2)(k+4)$ .... (i)

The statement for $n=k+1$ becomes

$S(k+1):\ 3+5+7+..............+(2(k+1)+5)=\big((k+1)+2\big)\big((k+1)+4\big)$

$\Rightarrow\ 3+5+7+..............+(2k+7)=\big(k+3\big)\big(k+5\big)$

Adding $(2k+7)$ on both sides of equation (i)

$S(k):\ 3+5+7+..............+(2k+5)+(2k+7)=(k+2)(k+4)+(2k+7)$

$\Rightarrow\ 3+5+7+..............+(2k+7)=k^2+2k+4k+8+2k+7$

$=k^2+8k+15$

$=k^2+5k+3k+15$

$=k(k+5)+3(k+5)$

$=(k+5)(k+3)$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \geq -1$.

Question # 38

$1+nx \leq (1+x)^n$ for integral values of $n \geq -1$.

Solution Suppose $S(n):\ 1+nx \leq (1+x)^n$; $n \geq 2$

Put $n=2$

$S(2):\ 1+2x \leq (1+x)^2\ \ \Rightarrow\ 1+2x \leq 1+2x+x^2$

$S(2)$ is true so condition I is satisfied.

Now suppose that given statement is true for $n=k$

$S(k):\ \ 1+kx \leq (1+x)^k$ .... (i)

The statement for $n=k+1$

$S(k+1):\ 1+(k+1)x \leq (1+x)^{k+1}$

Multiplying both sides of equation (i) by $1+x$.

$$(1+kx)(1+x) \le (1+x)^k(1+x)$$

$$\Rightarrow \quad 1+kx+x+kx^2 \le (1+x)^{k+1}$$

$$\Rightarrow \quad 1+kx+x \le (1+x)^{k+1} \qquad \because \; kx^2 > 0$$

$$\Rightarrow \quad 1+(k+1)x \le (1+x)^{k+1}$$

Thus $S(k+1)$ is true if $S(k)$ is true, so condition II is satisfied and $S(n)$ is true for all integers $n \ge 2$