Quadratic Formula:

Derivation of quadratic formula by using completing square method.

The quadratic equation in standard form is

$ax^2 + bx + c = 0, \ a \neq 0$

Dividing each term of the equation by a, we get

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Shifting constant term $\frac{c}{a}$ to the right, we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Adding $\left(\frac{b}{a}\right)^2$ on both sides, we obtain

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

or $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Taking square root of both sides, we get

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

or $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Thus, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is known as "quadratic formula".

SOLVED EXERCISE 1.2

Q1. Solve the following equations using quadratic formula:

(i) $2 - x^2 = 7x$

Solution:

$$2 - x^2 = 7x$$
$$-x^2 - 7x + 2 = 0$$
$$-(x^2 + 7x - 2) = 0$$
$$\Rightarrow \quad x^2 + 7x - 2 = 0$$

Compare it with, we have

$$ax^2 + bx + c = 0$$

Here $a = 1, b = 7, c = -2$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{-7 \pm \sqrt{49 + 8}}{2}$$

$$x = \frac{-7 \pm \sqrt{57}}{2}$$

Thus, solution set $= \left\{\dfrac{-7 \pm \sqrt{57}}{2}\right\}$

(ii) $8x + 1 = 0$

Solution:

$$5x^2 + 8x + 1 = 0$$

Compare it with, we have

$$ax^2 + bx + c = 0$$

Here $a = 5, b = 8, c = 1$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(5)(1)}}{2(5)}$$

$$x = \frac{-8 \pm \sqrt{64 - 20}}{10}$$

$$x = \frac{-8 \pm \sqrt{44}}{10}$$

$$x = \frac{-8 \pm 2\sqrt{11}}{10}$$

$$x = \frac{2\left(-4 \pm \sqrt{11}\right)}{10}$$

$$x = \frac{-4 \pm \sqrt{11}}{5}$$

Thus, solution set = $= \left\{\dfrac{-4 \pm \sqrt{11}}{5}\right\}$

(iii) $\sqrt{3}\,x^2 + x = 4\sqrt{3}$

Solution:

$$\sqrt{3}\,x^2 + x = 4\sqrt{3}$$

$$\sqrt{3}\,x^2 + x - 4\sqrt{3} = 0$$

Compare it with, we have

$$ax^2 + bx + c = 0$$

Here $a = \sqrt{3}, b = 1, c = -4\sqrt{3}$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4\left(\sqrt{3}\right)\left(-4\sqrt{3}\right)}}{2\left(\sqrt{3}\right)}$$

$$x = \frac{-1 \pm \sqrt{1 - 48}}{2\sqrt{3}}$$

$$x = \frac{-1 \pm \sqrt{49}}{2\sqrt{3}}$$

$$x = \frac{-1 \pm 7}{2\sqrt{3}}$$

$$x = \frac{-1 \pm 7}{2\sqrt{3}} \quad \text{or} \quad x = \frac{-1 - 7}{2\sqrt{3}}$$

$$x = \frac{6}{2\sqrt{3}} \qquad x = \frac{-8}{2\sqrt{3}}$$

$$x = \frac{3}{\sqrt{3}} \qquad x = -\frac{4}{\sqrt{3}}$$

$$x = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$x = \frac{3\sqrt{3}}{\left(\sqrt{3}\right)^2}$$

$$x = \frac{3\sqrt{3}}{3}$$

$$x = \sqrt{3}$$

Thus, solution set = $\left\{\sqrt{3}, -\dfrac{4}{\sqrt{3}}\right\}$

(iv) $4x^2 - 14 = 3x$

Solution:

$$4x^2 - 14 = 3x$$

$$4x^2 - 3x - 14 = 0$$

Compare it with, we have

$$ax^2 + bx + c = 0$$

$$ax^2 + bx + c = 0$$

Here $a = 4, b = -3, c = -14$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-14)}}{2(4)}$$

$$x = \frac{3 \pm \sqrt{9 + 224}}{8}$$

$$x = \frac{3 \pm \sqrt{233}}{8}$$

Thus, solution set $= \left\{\dfrac{3 \pm \sqrt{233}}{8}\right\}$

(v) $6x^2 - 3 - 1x = 0$

Solution:

$$6x^2 - 3 - 1x = 0$$

$$6x^2 - 7x - 3 = 0$$

Compare it with, we have

$$ax^2 + bx + c = 0$$

Here $a = 6, b = -7, c = -3$

Now $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(6)(-3)}}{2(6)}$$

$$x = \frac{7 \pm \sqrt{49 + 72}}{12}$$

$$x = \frac{7 \pm \sqrt{121}}{12}$$

$$x = \frac{7 \pm 11}{12}$$

$$x = \frac{7 \pm 11}{12} \quad , \quad x = \frac{7-11}{12}$$

$$= \frac{18}{12} \qquad\qquad = -\frac{4}{12}$$

$$= \frac{3}{2} \qquad\qquad = -\frac{1}{3}$$

Thus, solution set = $\left\{-\frac{1}{3}, \frac{3}{2}\right\}$

(vi) $\quad 3x^2 + 8x + 2 = 0$

Solution:

$$3x^2 + 8x + 2 = 0$$

Compare it with

$$ax^2 + bx + c = 0$$

Here $\quad a = 3, b = 8, c = 2$

Now $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-8 \pm \sqrt{(8)^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-8 \pm \sqrt{64 - 24}}{6}$$

$$x = \frac{-8 \pm \sqrt{40}}{6}$$

$$x = \frac{-8 \pm \sqrt{10}}{6}$$

$$x = \frac{2\left(-4 \pm \sqrt{10}\right)}{6}$$

$$x = \frac{-4 \pm \sqrt{10}}{3}$$

Thus, solution set = $\left\{\frac{-4 \pm \sqrt{10}}{3}\right\}$

(vii) $\quad \frac{3}{x-6} - \frac{4}{x-5} = 1$

Solution:

$$\frac{3}{x-6} - \frac{4}{x-5} = 1$$

$$\frac{3(x-5)-4(x-6)}{(x-6)(x-5)}=1$$

$$3x-15-4x+24=(x-6)(x-5)$$

$$-x+9=x^2-11x+30$$

$$x^2-11x+x+30-9=0$$

$$x^2-10x+21=0$$

Compare it with, we have

$$ax^2+bx+c=0$$

Here $a=1, b=-10, c=21$

Now $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(21)}}{2(1)}$$

$$x=\frac{10\pm\sqrt{100-84}}{2}$$

$$x=\frac{10\pm\sqrt{16}}{2}$$

$$x=\frac{-8\pm 4}{2}$$

$$x=\frac{10+4}{2}, \qquad x=\frac{10-4}{2}$$

$$x=\frac{14}{2} \qquad x=\frac{6}{2}$$

$$x=7 \qquad x=3$$

Thus, solution set = $\{3, 7\}$

(viii) $\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$

Solution:

$$\frac{x+2}{x-1}-\frac{4-x}{2x}=2\frac{1}{3}$$

$$\frac{2x(x+2)-(x-1)(4-x)}{2x(x-1)}=\frac{7}{3}$$

$$\frac{(2x^2+4x)-(4x-x^2-4+x)}{2x^2-2x}=\frac{7}{3}$$

$$\frac{2x^2+4x+x^2-5x+4}{2x^2-2x}=\frac{7}{3}$$

$$7(2x^2-2x)=3(3x^2-x+4)$$

$$14x^2-14x=9x^2-3x+12$$

$$14x^2-9x^2-14x+3x-12=0$$

$$5x^2-11x-12=0$$

Compare it with, we have

$$ax^2+bx+c=0$$

Here $a=5, b=-11, c=-12$

Now $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-(-11)\pm\sqrt{(-11)^2-4(5)(-12)}}{2(5)}$$

$$x=\frac{11\pm\sqrt{121-240}}{10}$$

$$x=\frac{11\pm\sqrt{361}}{10}$$

$$x=\frac{11\pm19}{10}$$

$$x=\frac{11+19}{10}, \qquad x=\frac{11-10}{10}$$

$$x=\frac{30}{10} \qquad x=\frac{-8}{10}$$

$$x=3 \qquad x=-\frac{4}{5}$$

Thus, solution set = $\left\{3,-\dfrac{4}{5}\right\}$

(ix) $\dfrac{a}{x-b}+\dfrac{b}{x-a}=2$

Solution:

$$\frac{a}{x-b}+\frac{b}{x-a}=2$$

$$\frac{a(x-a)+b(x-b)}{(x-a)(x-b)}=2$$

$$ax-a^2+bx-b^2=2(x-a)(x-b)$$

$$ax+bx-a^2-b^2=2(x^2-ax-bx+ab)$$

$$ax+bx-a^2-b^2=2x^2-2ax-2bx+2ab$$

$$2x^2-3ax-3bx+2ab+a^2+b^2=0$$

$$2x^2-3(a+b)x+(2ab+a^2+b^2)=0$$

$$2x^2-3(a+b)x+(a+b)^2=0$$

Compare it with, we have

$$ax^2+bx+c=0$$

Here $a = 2, b = -3(a + b); c = (a + b)^2$

Now $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-[-3(a+b)]\pm\sqrt{[-3(a+b)]^2-4(2)(a+b)^2}}{2(2)}$$

$$x=\frac{3(a+b)\pm\sqrt{9(a+b)^2-8(a+b)^2}}{4}$$

$$x=\frac{3(a+b)\pm\sqrt{(a+b)^2}}{4}$$

$$x=\frac{3(a+b)\pm(a+b)}{4}$$

$$x=\frac{3(a+b)+(a+b)}{4},\quad x=\frac{3(a+b)-(a+b)}{4}$$

$$x=\frac{3a+3b+a+b}{4}\qquad x=\frac{3a+3b-a-b}{4}$$

$$x=\frac{4a+4b}{4}\qquad x=\frac{2a+2b}{4}$$

$$x=\frac{4(a+b)}{4}\qquad x=\frac{2(a+b)}{4}$$

$$x=a+b,\qquad x=\frac{1}{2}(a+b)$$

Thus, solution set = $\left\{(a+b),\frac{1}{2}(a+b)\right\}$

(x) $-(\ell+m)-\ell x^2+(2\ell+m)x=0$

Solution:

$$-(\ell+m)-\ell x^2+(2\ell+m)x=0, \ell=0$$

$$-\ell x^2+(2\ell+m)x-(\ell+m)=0$$

$$-\left[\ell x^2-(2\ell+m)x+(\ell+m)\right]=0$$

$$\Rightarrow \ell x^2-(2\ell+m)x+(\ell+m)=0$$

Compare it with, we have

$$ax^2+bx+c=0$$

Here $a=\ell$, $b=-(2\ell+m)$, $c=(\ell+m)$

Now $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$x=\frac{-\left[-(2\ell+m)\pm\sqrt{\left[-(2\ell+m)\right]^2-4(\ell)(\ell+m)}\right]}{2\ell}$$

$$x=\frac{(2\ell+m)\pm\sqrt{(2\ell+m)^2-4\ell(\ell+m)}}{2\ell}$$

$$x=\frac{(2\ell+m)\pm\sqrt{4\ell^2+4\ell m+m^2-4\ell^2-4\ell m}}{2\ell}$$

$$x=\frac{(2\ell+m)\pm\sqrt{m^2}}{2\ell}$$

$$x=\frac{(2\ell+m)\pm m}{2\ell}$$

$$x=\frac{2\ell+2m}{.2\ell}, \qquad x=\frac{2\ell+2m-m}{2\ell}$$

$$x=\frac{2\ell+2m}{2l} \qquad =\frac{2l}{2l}$$

$$=\frac{2(l+m)}{2\ell} \qquad =\ell$$

$$=\frac{\ell+m}{\ell}$$

Thus, solution set = $\left\{\ell,\dfrac{\ell+m}{\ell}\right\}$

SOLVED EXERCISE 1.3

Q1. Solve the following equations.

(1) $2x^4-11x^2-5=0$