EXERCISE 2.8

Taylor Series Expansion of Function

$$f(x+h)= f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\frac{h^3}{3!}f'''(x)+....$$

Maclaurin Series

$$f(x)= f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

Question # 1(i)

Let $f(x) = \ln(1+x) \quad \Rightarrow \quad f(0) = \ln(1+0) = 0$

$$f'(x) = \frac{d}{dx}\ln(1+x) = \frac{1}{1+x} \quad \Rightarrow \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(x) = \frac{d}{dx}(1+x)^{-1} = -(1+x)^{-2} \quad \Rightarrow \quad f''(0) = -(1+0)^{-2} = -1$$

$$f'''(x) = \frac{d}{dx}\left[-(1+x)^{-2}\right] = +2(1+x)^{-3} \quad \Rightarrow \quad f'''(0) = 2(1+0)^{-3} = 2$$

$$f^{(iv)}(x) = \frac{d}{dx}2(1+x)^{-3} = -6(1+x)^{-4} \quad \Rightarrow \quad f^{(iv)}(0) = -6(1+0)^{-4} = -6$$

By Maclaurin series

$$f(x)= f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \ln(1+x) = 0+x(1)+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2)+\frac{x^4}{4!}(-6)+......$$

$$= x-\frac{x^2}{2\cdot 1}+\frac{x^3}{3\cdot 2\cdot 1}(2)-\frac{x^4}{4\cdot 3\cdot 2\cdot 1}(6)+......$$

$$= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+......$$

Question # 1(ii)

Let $f(x) = \cos x \Rightarrow f(0) = \cos(0) = 1$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad f'(0) = -\sin(0) = 0$$

$$f''(x) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow \quad f''(0) = -\cos(0) = -1$$

$$f'''(x) = \frac{d}{dx}(-\cos x) = +\sin x \quad \Rightarrow \quad f'''(0) = \sin(0) = 0$$

$$f^{(iv)}(x) = \frac{d}{dx}\sin x = \cos x \quad \Rightarrow \quad f^{(iv)}(x) = \cos(0) = 1$$

$$f^{(v)}(x) = \frac{d}{dx}\cos x = -\sin x \quad \Rightarrow \quad f^{(v)}(x) = -\sin(0) = 0$$

$$f^{(vi)}(0) = \frac{d}{dx}(-\sin x) = -\cos x \quad \Rightarrow \quad f^{(vi)}(0) = -\cos(0) = -1$$

Now by Maclaurin series

$$f(x)= f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+......$$

$$\Rightarrow \cos x = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \frac{x^6}{6!}(-1) + \ldots\ldots$$
$$= 1 + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + \ldots\ldots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

Question # 1(iii)

Let $f(x) = \sqrt{1+x}$

$$= (1+x)^{\frac{1}{2}} \qquad \Rightarrow f(0) = (1+0)^{\frac{1}{2}} = 1$$
$$f'(x) = \frac{d}{dx}(1+x)^{\frac{1}{2}} = \frac{1}{2}(1+x)^{-\frac{1}{2}}(1) = \frac{1}{2}(1+x)^{-\frac{1}{2}}$$
$$\Rightarrow f'(0) = \frac{1}{2}(1+0)^{-\frac{1}{2}} = \frac{1}{2}$$
$$f''(x) = \frac{d}{dx}\left[\frac{1}{2}(1+x)^{-\frac{1}{2}}\right] = -\frac{1}{4}(1+x)^{-\frac{3}{2}}$$
$$\Rightarrow f''(0) = -\frac{1}{4}(1+0)^{-\frac{3}{2}} = -\frac{1}{4}$$
$$f'''(x) = -\frac{1}{4}\frac{d}{dx}\left[(1+x)^{-\frac{3}{2}}\right] = -\frac{1}{4}\left[-\frac{3}{2}(1+x)^{-\frac{5}{2}}\right] = \frac{3}{8}(1+x)^{-\frac{5}{2}}$$
$$\Rightarrow f'''(0) = \frac{3}{8}(1+0)^{-\frac{5}{2}} = \frac{3}{8}$$

Now by Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$
$$\Rightarrow \sqrt{1+x} = 1 + x\cdot\frac{1}{2} + \frac{x^2}{2!}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{3!}\cdot\frac{3}{8} + \ldots..$$
$$= 1 + x\cdot\frac{1}{2} + \frac{x^2}{2}\cdot\left(-\frac{1}{4}\right) + \frac{x^3}{6}\cdot\frac{3}{8} + \ldots..$$
$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \ldots..$$

Question # 1(iv)

Let $f(x) = e^x \Rightarrow f(0) = e^0 = 1$

$$f'(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'(0) = e^0 = 1$$
$$f''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f''(0) = e^0 = 1$$
$$f'''(x) = \frac{d}{dx}\left(e^x\right) = e^x \Rightarrow f'''(0) = e^0 = 1$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$
$$\Rightarrow e^x = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \ldots..$$
$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots..$$

Questoion # 1(v)

Let $f(x) = e^{2x} \quad \Rightarrow \quad f(0) = e^{2(0)} = e^{0} = 1$

$$f'(x) = \frac{d}{dx}\left(e^{2x}\right) = 2e^{2x} \qquad \Rightarrow \quad f'(0) = 2e^{2(0)} = 2(1) = 2$$

$$f''(x) = 2\frac{d}{dx}\left(e^{2x}\right) = 2\left(2e^{2x}\right) = 4e^{2x} \quad \Rightarrow \quad f''(0) = 4e^{2(0)} = 4(1) = 4$$

$$f'''(x) = 4\frac{d}{dx}\left(e^{2x}\right) = 4\left(2e^{2x}\right) = 8e^{2x} \quad \Rightarrow \quad f'''(0) = 8e^{2(0)} = 8$$

By Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \ldots\ldots$$

$$\Rightarrow \quad e^{2x} = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(8) + \ldots\ldots$$

$$= 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \ldots\ldots$$

Question # 2

Let $f(x) = \cos x$

$$f'(x) = \frac{d}{dx}\cos x = -\sin x$$

$$f''(x) = -\frac{d}{dx}\sin x = -\cos x$$

$$f'''(x) = -\frac{d}{dx}\cos x = -\left(-\sin x\right) = \sin x$$

By Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \ldots.$$

$$\Rightarrow \quad \cos\left(x+h\right) = \cos x + h\left(-\sin x\right) + \frac{h^2}{2!}\left(-\cos x\right) + \frac{h^3}{3!}\left(\sin x\right) + \ldots\ldots$$

$$\Rightarrow \quad \cos\left(x+h\right) = \cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \ldots\ldots$$

Put $x = 60^\circ$ and $h = 1^\circ = \dfrac{\pi}{180} = 0.01745$ rad

$$\cos\left(60+1\right) = \cos 60 - \left(0.01745\right)\sin 60 - \frac{\left(0.01745\right)^2}{\lfloor 2}\cos 60 + \frac{\left(0.01745\right)^3}{\lfloor 3}\sin 60 + \ldots\ldots$$

$$\Rightarrow \quad \cos 61^\circ = 0.5 - \left(0.01745\right)\left(0.866\right) - \frac{\left(0.000305\right)}{2}\left(0.5\right) + \frac{\left(0.00000531\right)}{6}\left(0.866\right) + \ldots\ldots$$

$$= 0.5 - 0.0151117 - 0.000076125 + 0.000000072 + \ldots\ldots$$

$$= 0.484812247 \quad \approx \quad 0.4848$$

Question # 3

Let $f(x) = 2^x$

$$f'(x) = \frac{d}{dx}2^x \qquad \because \quad \frac{d}{dx}a^x = a^x \ln a$$

$$= 2^x \ln 2$$

$$f''(x) = \ln 2 \cdot \frac{d}{dx}2^x = \ln 2\left(2^x \ln 2\right) = \left(\ln 2\right)^2 2^x$$

$$f'''(x) = \left(\ln 2\right)^2 \frac{d}{dx}2^x = \left(\ln 2\right)^2 2^x \cdot \ln 2 = \left(\ln 2\right)^3 2^x$$

Now by Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...$$

$$\Rightarrow \quad 2^{x+h} = 2^x + h \cdot 2^x \cdot \ln 2 + \frac{h^2}{2!} \cdot (\ln 2)^2 2^x + \frac{h^3}{3!} (\ln 2)^3 2^x + ...$$

$$= 2^x \left[1 + (\ln 2)h + (\ln 2)^2 \frac{h^2}{2!} + (\ln 2)^3 \frac{h^3}{3!} + ... \right]$$

Question # 2(Old Book)

Let $f(x) = \sin x$ and $h = y$

$$f'(x) = \frac{d}{dx}\sin x = \cos x$$

$$f''(x) = \frac{d}{dx}\cos x = -\sin x$$

$$f'''(x) = -\frac{d}{dx}\sin x = -\cos x$$

$$f^{(iv)}(x) = -\frac{d}{dx}\cos x = -(-\sin x) = \sin x$$

$$f^{(v)}(x) = \frac{d}{dx}\sin x = \cos x$$

Now by Taylor series

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...$$

$$\Rightarrow \sin(x+y) = \sin x + y(\cos x) + \frac{y^2}{2!}(-\sin x) + \frac{y^3}{3!}(-\cos x) + \frac{y^4}{4!}(\sin x) + \frac{y^5}{5!}(\cos x) + ...$$

$$= \sin x \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - .... \right) + \cos x \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - .... \right)$$

Since $\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - ....$ and $\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - ....$

Therefore

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
