EXERCISE 3.2

Theorem on Anti-Derivatives

i) $\int cf(x)dx = c\int f(x)dx$ where c is constant.

ii) $\int[f(x)\pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

Important Integral

Since $\dfrac{d}{dx}x^{n+1} = (n+1)x^n$

Taking integral w.r.t x

$$\int \frac{d}{dx}x^{n+1}dx = \int (n+1)x^n dx$$

$$\Rightarrow\ x^{n+1} = (n+1)\int x^n dx$$

$$\Rightarrow\ \boxed{\int x^n\ dx = \frac{x^{n+1}}{n+1}} \quad \text{where } n \neq -1$$

If $n=-1$ then

$$\int x^{-1}dx = \int \frac{1}{x}dx$$

Since $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$

Therefore $\boxed{\int \frac{1}{x}dx = \ln|x| + c}$

Note: Since log of negative numbers does not exist therefore in above formula mod assure that we are taking a log of +ive quantity.

Question # 1(i)

$$\int (3x^2-2x+1)dx = 3\int x^2dx - 2\int xdx + \int dx$$
$$= 3\cdot\frac{x^{2+1}}{2+1} - 2\cdot\frac{x^{1+1}}{1+1} + x + c$$
$$= 3\cdot\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + x + c$$
$$= x^3 - x^2 + x + c$$

Question # 1(ii)

$$\int\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx = \int\left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx$$
$$= \int x^{\frac{1}{2}}dx + \int x^{-\frac{1}{2}}dx$$
$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + c \quad \textit{Ans.}$$

Question # 1(iii)

$$\int x(\sqrt{x}+1)dx = \int x\left(x^{\frac{1}{2}}+1\right)dx$$
$$= \int\left(x^{\frac{3}{2}} + x\right)dx$$
$$= \int x^{\frac{3}{2}}dx + \int x\,dx$$
$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{x^{1+1}}{1+1} + c \quad = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{2} + c$$
$$= \frac{2}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$$

Important Integral

Since $\dfrac{d}{dx}(ax+b)^{n+1} = (n+1)(ax+b)^n\cdot a$

Taking integral

$$\int\frac{d}{dx}(ax+b)^{n+1}\,dx = \int (n+1)(ax+b)^n\cdot a\ dx$$

$$\Rightarrow\ (ax+b)^{n+1} = (n+1)\cdot a\int (ax+b)^n\ dx$$

$$\Rightarrow\ \boxed{\int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{(n+1)\cdot a}}$$

Question # 1(iv)

$$\int (2x+3)^{\frac{1}{2}}\,dx = \frac{(2x+3)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 2} + c$$
$$= \frac{(2x+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)\cdot 2} + c$$
$$= \frac{1}{3}(2x+3)^{\frac{3}{2}} + c$$

Question # 1(v)

$$\int\left(\sqrt{x}+1\right)^2 dx = \int\left((\sqrt{x})^2 + 2\sqrt{x} + 1\right)dx$$
$$= \int\left(x + 2(x)^{\frac{1}{2}} + 1\right)dx$$
$$= \int x\,dx + 2\int (x)^{\frac{1}{2}}\,dx + \int dx$$
$$= \frac{x^{1+1}}{1+1} + 2\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + x + c$$
$$= \frac{x^2}{2} + 2\frac{x^{\frac{3}{2}}}{\cancel{3}/2} + x + c$$
$$= \frac{x^2}{2} + \frac{4x^{\frac{3}{2}}}{3} + x + c$$

Question # 1(vi)

$$\int\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 dx = \int\left(x + \frac{1}{x} - 2\right)dx$$
$$= \int xdx + \int\frac{1}{x}dx - 2\int dx$$
$$= \frac{x^2}{2} + \ln|x| - 2x + c$$

Question # 1(vii)

$$\int \frac{3x+2}{\sqrt{x}}dx = \int \frac{3x+2}{x^{1/2}}dx$$
$$= \int \frac{3x}{x^{1/2}} + \frac{2}{x^{1/2}}dx$$
$$= \int \left(3x^{1/2} + 2x^{-1/2}\right)dx$$
$$= 3\int x^{1/2}dx + 2\int x^{-1/2}dx$$

Now do yourself.

Question # 1(viii)

$$\int \frac{\sqrt{y}\,(y+1)}{y}dy$$
$$= \int \frac{\sqrt{y}\,(y+1)}{\left(\sqrt{y}\right)^2}dy = \int \frac{(y+1)}{\sqrt{y}}dy$$
$$= \int\left(\frac{y}{\sqrt{y}} + \frac{1}{\sqrt{y}}\right)dy = \int\left(y^{\frac{1}{2}} + y^{-\frac{1}{2}}\right)dy$$
$$= \int y^{\frac{1}{2}}dy + \int y^{-\frac{1}{2}}dy$$
$$= \frac{y^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + \frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \frac{2}{3}y^{\frac{3}{2}} + 2y^{\frac{1}{2}} + c$$

Question # 1(ix)

$$\int \frac{\left(\sqrt{\theta}-1\right)^2}{\sqrt{\theta}}\,d\theta = \int \frac{\left(\theta - 2\sqrt{\theta}+1\right)}{\sqrt{\theta}}\,d\theta$$
$$= \int\left(\frac{\theta}{\sqrt{\theta}} - \frac{2\sqrt{\theta}}{\sqrt{\theta}} + \frac{1}{\sqrt{\theta}}\right)d\theta$$
$$= \int\left(\theta^{\frac{1}{2}} - 2 + \theta^{\frac{1}{2}}\right)d\theta$$

No do yourself

Question # 1(x)

Do yourself as above

Important Integral

We know $\dfrac{d}{dx}e^{ax} = a\cdot e^{ax}$

Taking integral

$$\int \frac{d}{dx}e^{ax}\,dx = \int a\cdot e^{ax}dx$$
$$\Rightarrow\ e^{ax} = a\int e^{ax}dx$$
$$\Rightarrow\ \boxed{\int e^{ax}dx = \frac{e^{ax}}{a}}$$

Also note that $\displaystyle\int e^{(ax+b)}dx = \frac{e^{(ax+b)}}{a}$

Question # 1(xi)

$$\int \frac{e^{2x}+e^x}{e^x}\,dx = \int\left(\frac{e^{2x}}{e^x} + \frac{e^x}{e^x}\right)dx$$
$$= \int\left(e^x+1\right)\,dx$$
$$= \int e^x dx + \int dx$$
$$= e^x + x + c \quad \underline{Ans}$$

Question # 2(i)

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$$
$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}\cdot\frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}$$
$$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{x+a-x-b}\,dx$$
$$= \int \frac{(x+a)^{\frac{1}{2}}-(x+b)^{\frac{1}{2}}}{a-b}\,dx$$
$$= \frac{1}{a-b}\left[\int (x+a)^{\frac{1}{2}}\,dx - \int (x+b)^{\frac{1}{2}}dx\right]$$
$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1} - \frac{(x+b)^{\frac{1}{2}}}{\frac{1}{2}+1}\right]+c$$
$$= \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$
$$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right]+c \quad Ans.$$

Important Integral

Since $\dfrac{d}{dx}Tan^{-1}x = \dfrac{1}{1+x^2}$

Also $\dfrac{d}{dx}\left(-Cot^{-1}x\right) = \dfrac{1}{1+x^2}$

Therefore $\displaystyle\int \frac{1}{1+x^2}dx = Tan^{-1}x$ or $-Cot^{-1}x$

Similarly $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx = Sin^{-1}x$ or $-Cos^{-1}x$

$\displaystyle\int \frac{1}{x\sqrt{x^2-1}}dx = Sec^{-1}x$ or $-Csc^{-1}x$

Question # 2(ii)

$$\int \frac{1-x^2}{1+x^2}dx$$
$$= \int\left(-1+\frac{2}{1+x^2}\right)dx$$
$$= -\int dx + 2\int \frac{1}{1+x^2}dx$$
$$= -x + 2Tan^{-1}x + c$$

$$\begin{array}{r|l} & \ \ -1 \\ 1+x^2 & \overline{\ 1-x^2} \\ & \underline{\begin{array}{l}-1-x^2\\ {\scriptstyle +\ \ +}\end{array}} \\ & \ \ \ 2 \end{array}$$

Question # 2(iii)

$$\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$$
$$= \int \frac{dx}{\sqrt{x+a}+\sqrt{x}}\cdot\frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}}$$
$$= \int \frac{\sqrt{x+a}-\sqrt{x}}{x+a-x}\,dx$$
$$= \int \frac{(x+a)^{\frac{1}{2}}-(x)^{\frac{1}{2}}}{a}\,dx$$

$$=\frac{1}{a}\left[\int(x+a)^{\frac{1}{2}}\,dx-\int(x)^{\frac{1}{2}}dx\right]$$
$$=\frac{1}{a}\left[\frac{(x+a)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(x)^{\frac{1}{2}}}{\frac{1}{2}+1}\right]+c$$
$$=\frac{1}{a}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x)^{\frac{3}{2}}}{\frac{3}{2}}\right]+c$$
$$=\frac{2}{3a}\left[(x+a)^{\frac{3}{2}}-x^{\frac{3}{2}}\right]+c \quad Ans.$$

Question # 2(iv)

$$\int(a-2x)^{\frac{3}{2}}\,dx=\frac{(a-2x)^{\frac{3}{2}+1}}{\left(\frac{3}{2}+1\right)\cdot(-2)}+c$$
$$=\frac{(a-2x)^{\frac{5}{2}}}{\left(\frac{5}{2}\right)\cdot(-2)}+c$$
$$=-\frac{(a-2x)^{\frac{5}{2}}}{5}+c$$

Question # 2(v)

$$\int\frac{(1+e^x)^3}{e^x}dx=\int\frac{(1+3e^x+3e^{2x}+e^{3x})}{e^x}dx$$
$$=\int\left(\frac{1}{e^x}+\frac{3e^x}{e^x}+\frac{3e^{2x}}{e^x}+\frac{e^{3x}}{e^x}\right)dx$$
$$=\int\left(e^{-x}+3+3e^x+e^{2x}\right)dx$$
$$=\frac{e^{-x}}{-1}+3x+3e^x+\frac{e^{2x}}{2}+c$$
$$=-e^{-x}+3x+3e^x+\frac{1}{2}e^{2x}+c$$

Important Integrals

We know $\frac{d}{dx}\cos ax=-a\sin ax$

Taking integral

$$\int\frac{d}{dx}\cos ax\,dx=-\int a\sin ax\,dx$$
$$\Rightarrow\quad \cos ax=-a\int\sin ax\,dx$$
$$\Rightarrow\quad \boxed{\int\sin ax\,dx=-\frac{\cos ax}{a}}$$

Also $\frac{d}{dx}\sin ax=a\cdot\cos ax$

$$\therefore\quad \boxed{\int\cos ax\,dx=\frac{\sin ax}{a}}$$

Similarly

$$\boxed{\int\sec^2 ax\,dx=\frac{\tan ax}{a}}$$
$$\boxed{\int\text{cosec}^2 ax\,dx=-\frac{\cot ax}{a}}$$
$$\boxed{\int\sec ax\tan ax\,dx=\frac{\sec ax}{a}}$$
$$\boxed{\int\csc ax\cot ax\,dx=-\frac{\csc ax}{a}}$$

Also note that

$$\int\sin(ax+b)\,dx=-\frac{\cos(ax+b)}{a}$$
$$\int\cos(ax+b)\,dx=\frac{\sin(ax+b)}{a}\quad\text{and so on.}$$

Question # 2(vi)

$$\int\sin(a+b)x\,dx=-\frac{\cos(a+b)x}{a+b}$$

Question # 2(vii)

$$\int\sqrt{1-\cos 2x}\,dx$$
$$=\int\sqrt{2\sin^2 x}\,dx\qquad\because\ \sin^2 x=\frac{1-\cos 2x}{2}$$
$$=\sqrt{2}\int\sin x\,dx=\sqrt{2}(-\cos x)+c$$
$$=-\sqrt{2}\cos x+c$$

Important Formula

$$\because\ \frac{d}{dx}[f(x)]^{n+1}=(n+1)[f(x)]^n\frac{d}{dx}f(x)$$
$$\Rightarrow\ \frac{d}{dx}[f(x)]^{n+1}=(n+1)[f(x)]^n f'(x)$$

Taking integral

$$\int\frac{d}{dx}[f(x)]^{n+1}dx=\int(n+1)[f(x)]^n f'(x)dx$$
$$\Rightarrow\ [f(x)]^{n+1}=(n+1)\int[f(x)]^n f'(x)dx$$
$$\Rightarrow\ \boxed{\int[f(x)]^n f'(x)dx=\frac{[f(x)]^{n+1}}{(n+1)}}\quad;\ n\neq-1$$

Also $\frac{d}{dx}\ln|f(x)|=\frac{1}{f(x)}\cdot f'(x)$

Taking integral

$$\ln|f(x)|=\int\frac{f'(x)}{f(x)}dx$$

i.e. $\boxed{\int\frac{f'(x)}{f(x)}dx=\ln|f(x)|+c}$

Question # 2(viii)

Let $I=\int\ln x\times\frac{1}{x}\,dx$

Put $f(x)=\ln x\ \Rightarrow\ f'(x)=\frac{1}{x}$

So $I=\int[f(x)]f'(x)\,dx$

$$=\frac{[f(x)]^{1+1}}{1+1}+c=\frac{[f(x)]^2}{2}+c$$
$$=\frac{(\ln x)^2}{2}+c$$

Question # 2(ix)

$$\int \sin^2 x\, dx = \int\left(\frac{1-\cos 2x}{2}\right) dx$$
$$= \int\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right) dx$$
$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + c$$
$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Question # 3(x)

$$\int \frac{1}{1+\cos x} dx$$
$$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx \qquad \because \cos^2\frac{x}{2} = \frac{1+\cos x}{2}$$
$$= \frac{1}{2}\int \sec^2\frac{x}{2} dx \quad = \frac{1}{2}\frac{\tan\frac{x}{2}}{\cancel{1}/2} + c \; = \tan\frac{x}{2} + c$$

Alternative

$$\int \frac{1}{1+\cos x} dx = \int \frac{1}{1+\cos x}\times\frac{1-\cos x}{1-\cos x} dx$$
$$= \int \frac{1-\cos x}{1-\cos^2 x} dx$$
$$= \int \frac{1-\cos x}{\sin^2 x} dx$$
$$= \int\left(\frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right) dx$$
$$= \int\left(\operatorname{cosec}^2 x - \frac{\cos x}{\sin x\cdot\sin x}\right) dx$$
$$= \int \operatorname{cosec}^2 x dx - \int \operatorname{cosec} x \cot x\, dx$$
$$= -\cot x - \left(-\operatorname{cosec} x\right) + c$$
$$= \operatorname{cosec} x - \cot x + c$$

Question # 2(xi)

Let $I = \int \frac{ax+b}{ax^2+bx+c} dx$

Put $f(x) = ax^2 + 2bx + c$

$\Rightarrow f'(x) = 2ax + 2b$

$\Rightarrow f'(x) = 2\left(ax+b\right) \;\Rightarrow\; \frac{1}{2}f'(x) = ax+b$

So $I = \int \frac{\frac{1}{2}f'(x)}{f(x)} dx$

$$= \frac{1}{2}\int\frac{f'(x)}{f(x)}dx \quad = \frac{1}{2}\ln\left|f(x)\right| + c_1$$
$$= \frac{1}{2}\ln\left|ax^2+bx+c\right| + c_1$$

Review

- $2\sin\alpha\cos\beta = \sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)$
- $2\cos\alpha\sin\beta = \sin\left(\alpha+\beta\right) - \sin\left(\alpha-\beta\right)$
- $2\cos\alpha\cos\beta = \cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)$
- $-2\sin\alpha\sin\beta = \cos\left(\alpha+\beta\right) - \cos\left(\alpha-\beta\right)$

Question # 2(xii)

$$\int \cos 3x \sin 2x\, dx$$
$$= \frac{1}{2}\int 2\cos 3x \sin 2x\, dx$$
$$= \frac{1}{2}\int\left[\sin(3x+2x) - \sin(3x-2x)\right] dx$$
$$= \frac{1}{2}\int\left[\sin 5x - \sin x\right] dx$$
$$= \frac{1}{2}\left[-\frac{\cos 5x}{5} - \left(-\cos x\right)\right] + c$$
$$= -\frac{1}{2}\left[\frac{\cos 5x}{5} - \cos x\right] + c$$

Question # 2(xiii)

$$\int \frac{\cos 2x - 1}{1+\cos 2x} dx \qquad \because \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= -\int \frac{1-\cos 2x}{1+\cos 2x} dx \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$
$$= -\int \frac{2\sin^2 x}{2\cos^2 x} dx$$
$$= -\int \tan^2 x\, dx \quad = -\int\left(\sec^2 x - 1\right) dx$$
$$= -\int \sec^2 x\, dx + \int dx$$
$$= -\tan x + x + c$$

Question # 2(xiv)

$$\int \tan^2 x\, dx = \int\left(\sec^2 x - 1\right) dx$$
$$= \int \sec^2 x\, dx - \int dx$$
$$= \tan x - x + c$$

Important Integral

Since $\frac{d}{dx}\ln\left|ax+b\right| = \frac{1}{ax+b}\cdot\frac{d}{dx}\left(ax+b\right)$

$\Rightarrow \frac{d}{dx}\ln\left|ax+b\right| = \frac{1}{ax+b}\cdot a$

On Integrating

$\Rightarrow \ln\left|ax+b\right| = a\int\frac{1}{ax+b} dx$

$$\Rightarrow \boxed{\int\frac{1}{ax+b} dx = \frac{\ln\left|ax+b\right|}{a}}$$