EXERCISE 3.7

Example 4

Find the area bounded by the curve

$$f(x) = x^3 - 2x^2 + 1$$

and the x-axis in the first quadrant.

Solution

Put $f(x) = 0$

$\Rightarrow \; x^3 - 2x + 1 = 0$

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -2 & 0 & 1 \\ & \downarrow & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & \boxed{0} \end{array}$$

$\Rightarrow \; (x-1)(x^2 - x - 1) = 0$

$\Rightarrow \; x - 1 = 0 \quad \text{or} \quad x^2 - x - 1 = 0$

$\Rightarrow \; x = 1 \quad \text{or} \quad x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$

$$= \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Thus the curve cuts the x-axis at $x = 1, \; \dfrac{1 \pm \sqrt{5}}{2}$

Since we are taking area in the first quad. only

$\therefore \; x = 1, \; \dfrac{1+\sqrt{5}}{2}$ ignoring $\dfrac{1-\sqrt{5}}{2}$ as it is $-$ive.

Intervals in 1st quad. are $[0,1]$ & $\left[1, \dfrac{1+\sqrt{5}}{2}\right]$

Since $f(x) \geq 0$ whenever $x \in [0,1]$

and $f(x) \leq 0$ whenever $x \in \left[1, \dfrac{1+\sqrt{5}}{2}\right]$

$\therefore$ Area in 1st quad. $= \displaystyle\int_0^1 \left(x^3 - 2x^2 + 1\right) dx$

$$= \left| \frac{x^4}{4} - 2\frac{x^3}{3} + x \right|_0^1$$

$$= \left(\frac{1}{2} - \frac{2}{3} + 1\right) - 0$$

$$= \frac{7}{12} \text{ sq. unit}$$

Question # 1

$y = x^2 + 1 \quad ; \quad x = 1 \text{ to } x = 2$

$\therefore \; y \geq 0$ whenever $x \in [1,2]$

$\therefore$ Area $= \displaystyle\int_1^2 (x^2 + 1) dx$

$$= \int_1^2 x^2 dx + \int_1^2 dx$$

$$= \left| \frac{x^3}{3} \right|_1^2 + |\,x\,|_1^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(1)^3}{3}\right) + (2-1)$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right) + 1$$

$$= \frac{7}{3} + 1 = \frac{10}{3} \text{ sq. unit.}$$

Question # 2

$y = 5 - x^2 \quad ; \quad x = -1 \text{ to } x = 2$

$\therefore \; y > 0$ whenever $x \in (-1, 2)$

$\therefore$ Area $= \displaystyle\int_{-1}^2 (5 - x^2) dx$

$$= \left| 5x - \frac{x^3}{3} \right|_{-1}^2$$

$$= \left(5(2) - \frac{(2)^3}{3}\right) - \left(5(-1) - \frac{(-1)^3}{3}\right)$$

$$= \left(10 - \frac{8}{3}\right) - \left(-5 + \frac{1}{3}\right)$$

$$= \frac{22}{3} - \left(-\frac{14}{3}\right) = \frac{22}{3} + \frac{14}{3}$$

$$= \frac{36}{3} = 12 \text{ sq. unit}$$

Question # 3

$y = 3\sqrt{x} \quad ; \quad x = 1 \text{ to } x = 4$

Since $y \geq 0$ when $x \in [1,4]$

$\therefore$ Area $= \displaystyle\int_1^4 3\sqrt{x}\, dx$

$$= \int_1^4 3x^{\frac{1}{2}}\, dx = 3\int_1^4 x^{\frac{1}{2}}\, dx$$

$$= 3\left| \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} \right|_1^4 = 3\left| \frac{x^{\frac{3}{2}}}{3/2} \right|_1^4$$

$$= 3 \times \frac{2}{3} \left| x^{\frac{3}{2}} \right|_1^4 = 2\left((4)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right)$$

$$= \frac{3}{4}\left((4)^{\frac{4}{3}} - (1)^{\frac{4}{3}}\right) = 2\left((2^2)^{\frac{3}{2}} - 1\right)$$

$$= 2(8-1) = 14 \text{ sq. unit}$$

Question # 4

$y = \cos x \quad ; \quad x = -\dfrac{\pi}{2} \text{ to } x = \dfrac{\pi}{2}$

$\because \; y > 0$ whenever $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

$$\therefore \text{ Area } = \int_{-\pi/2}^{\pi/2} \cos x \, dx$$

$$= \left| \sin x \right|_{-\pi/2}^{\pi/2}$$

$$= \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)$$

$$= 1 + 1 = 2 \text{ sq. unit}$$

Question # 5

$y = 4x - x^2$

Putting $y = 0$, we have

$4x - x^2 = 0$

$\Rightarrow x(4-x) = 0$

$\Rightarrow x = 0$ or $x = 4$

Now $y > 0$ when $x \in (0,4)$

$$\therefore \text{ Area } = \int_0^4 \left(4x - x^2\right) dx$$

$$= \left| \frac{4x^2}{2} - \frac{x^3}{3} \right|_0^4 = \left| 2x^2 - \frac{x^3}{3} \right|_0^4$$

$$= \left(2(4)^2 - \frac{(4)^3}{3}\right) - \left(2(0)^2 - \frac{(0)^3}{3}\right)$$

$$= \left(32 - \frac{64}{3}\right) - (0 - 0)$$

$$= \frac{32}{2} \text{ sq. unit.}$$

Question # 6

$y = x^2 + 2x - 3$

Putting $y = 0$, we have

$x^2 + 2x - 3 = 0$

$\Rightarrow x^2 + 3x - x - 2 = 0$

$\Rightarrow x(x+3) - 1(x+3) = 0$

$\Rightarrow (x+3)(x-1) = 0$

$\Rightarrow x = -3$ or $x = 1$

Now $y \le 0$ whenever $x \in [-3,1]$

$$\therefore \text{ Area } = -\int_{-3}^{1} \left(x^2 + 2x - 3\right) dx$$

$$= -\left| \frac{x^3}{3} + \frac{2x^2}{2} - 3x \right|_{-3}^{1}$$

$$= -\left| \frac{x^3}{3} + x^2 - 3x \right|_{-3}^{1}$$

$$= -\left(\frac{(1)^3}{3} + (1)^2 - 3(1) \right) + \left(\frac{(-3)^3}{3} + (-3)^2 - 3(-3) \right)$$

$$= -\left(\frac{1}{3} + 1 - 3\right) + \left(\frac{-27}{3} + 9 + 9\right)$$

$$= -\left(-\frac{5}{3}\right) + (-9 + 18)$$

$$= \frac{5}{3} + 9 = \frac{32}{3} \text{ sq. unit}$$

Question # 7

$y = x^3 + 1$

Putting $y = 0$, we have

$x^3 + 1 = 0$

$\Rightarrow (x+1)(x^2 - x + 1) = 0$

$\Rightarrow x + 1 = 0$ or $x^2 - x + 1 = 0$

$\Rightarrow x = -1$ or $x = \dfrac{1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{1-4}}{2}$

$\Rightarrow x = \dfrac{1 \pm \sqrt{-3}}{2}$

Which is not possible.

Now $y \ge 0$ when $x \in [-1, 2]$

$$\therefore \text{ Area } = \int_{-1}^{2} \left(x^3 + 1\right) dx$$

$$= \left| \frac{x^4}{4} + x \right|_{-1}^{2}$$

$$= \left(\frac{(2)^4}{4} + 2 \right) - \left(\frac{(-1)^4}{4} - 1 \right)$$

$$= \left(\frac{16}{4} + 2\right) - \left(\frac{1}{4} - 1\right)$$

$$= 6 - \frac{3}{4} = \frac{27}{4} \text{ sq. unit}$$

Question # 8

$y = x^3 - 2x + 4 \quad ; \quad x = 1$

Putting $y = 0$, we have

$x^3 - 2x + 4 = 0$

By synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & \downarrow & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\Rightarrow (x+2)(x^2 - 2x + 2) = 0$

$\Rightarrow x + 2 = 0$ or $x^2 - 2x + 2 = 0$

$\Rightarrow x = -2$ or $x = \dfrac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2}$

$= \dfrac{2 \pm \sqrt{4-8}}{2}$

$= \dfrac{2 \pm \sqrt{-4}}{2}$

This is imaginary.

Now $y \ge 0$ when $x \in [-2, 1]$

$$\therefore \text{ Area } = \int_{-2}^{1} \left(x^3 - 2x + 4\right) dx$$

$$= \int_{-2}^{1} x^3 dx - 2\int_{-2}^{1} x \, dx + 4\int_{-2}^{1} dx$$

$$= \left|\frac{x^4}{4}\right|_{-2}^{1} - 2\left|\frac{x^2}{2}\right|_{-2}^{1} + 4\left|x\right|_{-2}^{1}$$

$$= \left(\frac{(1)^4}{4} - \frac{(-2)^4}{4}\right) - 2\left(\frac{(1)^2}{2} - \frac{(-2)^2}{2}\right) + 4\left(1-(-2)\right)$$

$$= \left(\frac{1}{4} - \frac{16}{4}\right) - 2\left(\frac{1}{2} - \frac{4}{2}\right) + 4(1+2)$$

$$= \left(\frac{1}{4} - 4\right) - 2\left(\frac{1}{2} - 2\right) + 4(3)$$

$$= \left(-\frac{15}{4}\right) - 2\left(-\frac{3}{2}\right) + 12$$

$$= -\frac{15}{4} + 3 + 12 \quad = \frac{45}{4} \text{ sq. unit}$$

Question # 9

$$y = x^3 - 4x$$

Putting $y = 0$, we have

$$x^3 - 4x = 0$$

$$\Rightarrow \quad x\left(x^2 - 4\right) = 0$$

$$\Rightarrow \quad x(x+2)(x-2) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -2 \quad \text{or} \quad x = 2$$

Now $y \geq 0$ whenever $x \in [-2, 0]$

And $y \leq 0$ whenever $x \in [0, 2]$

$$\therefore \text{ Area } = \int_{-2}^{0} y\,dx - \int_{0}^{2} y\,dx$$

$$= \int_{-2}^{0} \left(x^3 - 4x\right)dx - \int_{0}^{2} \left(x^3 - 4x\right)dx$$

$$= \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{-2}^{0} - \left|\frac{x^4}{4} - 4\frac{x^2}{2}\right|_{0}^{2}$$

$$= \left|\frac{x^4}{4} - 2x^2\right|_{-2}^{0} - \left|\frac{x^4}{4} - 2x^2\right|_{0}^{2}$$

$$= \left(\frac{(0)^4}{4} - 2(0)^2\right) - \left(\frac{(-2)^4}{4} - 2(-2)^2\right) - \left(\frac{(2)^4}{4} - 2(2)^2\right) + \left(\frac{(0)^4}{4} - 2(0)^2\right)$$

$$= (0-0) - \left(\frac{16}{4} - 8\right) - \left(\frac{16}{4} - 8\right) + (0-0)$$

$$= -(4-8) - (4-8) \quad = -(-4) - (-4)$$

$$= 4 + 4 = 8 \text{ sq. unit.}$$

Question # 10

$$y = x(x-1)(x+1)$$

Putting $y = 0$, we have

$$x(x-1)(x+1) = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = 1 \quad \text{or} \quad x = -1$$

Now $y \geq 0$ whenever $x \in [-1, 0]$

And $y \leq 0$ whenever $x \in [0, 1]$

$$\therefore \text{ Area } = \int_{-1}^{0} y\,dx - \int_{0}^{1} y\,dx$$

$$= \int_{-1}^{0} x(x-1)(x+1)\,dx - \int_{0}^{1} x(x-1)(x+1)\,dx$$

$$= \int_{-1}^{0} \left(x^3 - x\right)dx - \int_{0}^{1} \left(x^3 - x\right)dx$$

$$= \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{-1}^{0} - \left|\frac{x^4}{4} - \frac{x^2}{2}\right|_{0}^{1}$$

$$= \left(\frac{(0)^4}{4} - \frac{(0)^2}{2}\right) - \left(\frac{(-1)^4}{4} - \frac{(-1)^2}{2}\right) - \left(\frac{(1)^4}{4} - \frac{(1)^2}{2}\right) + \left(\frac{(0)^4}{4} - \frac{(0)^2}{2}\right)$$

$$= (0-0) - \left(\frac{1}{4} - \frac{1}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) + (0-0)$$

$$= 0 - \left(-\frac{1}{4}\right) - \left(-\frac{1}{4}\right) + 0$$

$$= \frac{1}{4} + \frac{1}{4} \quad = \frac{1}{2} \text{ sq. unit}$$

Question # 11

$$y^2 = 3 - x \quad ; \quad x = -1 \text{ to } x = 2$$

$$\Rightarrow \quad y = \pm\sqrt{3-x}$$

The branch of curve above the x-axis is

$$y = \sqrt{3-x}$$

$$\because \quad y \geq 0 \quad \text{when} \quad x \in [-1, 2]$$

$$\therefore \text{ Area } = \int_{-1}^{2} \sqrt{3-x}\,dx$$

$$= \int_{-1}^{2} (3-x)^{\frac{1}{2}}\,dx$$

$$= \left|\frac{(3-x)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)(-1)}\right|_{-1}^{2}$$

$$= \left|\frac{(3-x)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(-1)}\right|_{-1}^{2} \quad = -\left|\frac{2(3-x)^{\frac{3}{2}}}{3}\right|_{-1}^{2}$$

$$= -\frac{2(3-2)^{\frac{3}{2}}}{3} + \frac{2(3-(-1))^{\frac{3}{2}}}{3}$$

$$= -\frac{2(1)^{\frac{3}{2}}}{3} + \frac{2(4)^{\frac{3}{2}}}{3}$$

$$= -\frac{2}{3} + \frac{2(2)^3}{3}$$

$$= -\frac{2}{3}+\frac{16}{3} = \frac{14}{3} \text{ sq. unit}$$

Question # 12

$$g(x) = \cos\frac{1}{2}x \quad ; \quad x=-\pi \text{ to } x=\pi$$

$\because\ g(x) \geq 0$ when $x\in[-\pi,\pi]$

$$\therefore \text{Area} = \int_{-\pi}^{\pi}\cos\frac{1}{2}x\,dx$$

$$= \left|\frac{\sin\frac{x}{2}}{1/2}\right|_{-\pi}^{\pi} = 2\left|\sin\frac{x}{2}\right|_{-\pi}^{\pi}$$

$$= 2\left(\sin\left(\frac{\pi}{2}\right)-\sin\left(\frac{-\pi}{2}\right)\right)$$

$$= 2\big(1-(-1)\big) = 2(1+1)$$

$$= 2(2) = 4 \text{ sq. unit.}$$

Question # 13

$$y = \sin 2x \quad ; \quad x=0 \text{ to } x=\frac{\pi}{3}$$

$\because\ y\geq 0$ when $x\in\left[0,\frac{\pi}{3}\right]$

$$\therefore \text{Area} = \int_{0}^{\pi/3}\sin 2x\,dx$$

$$= \left|-\frac{\cos 2x}{2}\right|_{0}^{\pi/3} = -\left(\cos\frac{2\pi}{3}-\cos(0)\right)$$

$$= -\left(-\frac{1}{2}-1\right) = -\left(-\frac{3}{2}\right) = \frac{3}{2} \text{ sq. unit.}$$

Question # 14

$$y = \sqrt{2ax-x^2}$$

Putting $y=0$, we have

$$\sqrt{2ax-x^2} = 0$$

On squaring

$$2ax-x^2 = 0$$

$$\Rightarrow\ x(2a-x) = 0$$

$$\Rightarrow\ x=0 \quad\text{or}\quad 2a-x = 0 \Rightarrow\ x=2a$$

$\because\ y\geq 0$ when $x\in[0,2a]$

$$\therefore\ \text{Area} = \int_{0}^{2a}\sqrt{2ax-x^2}\,dx$$

$$= \int_{0}^{2a}\sqrt{a^2-a^2+2ax-x^2}\,dx$$

$$= \int_{0}^{2a}\sqrt{a^2-\left(a^2-2ax+x^2\right)}\,dx$$

$$= \int_{0}^{2a}\sqrt{a^2-(a-x)^2}\,dx$$

Put $a-x = a\sin\theta$

$\Rightarrow\ -dx = a\cos\theta\,d\theta$

$\Rightarrow\ dx = -a\cos\theta\,d\theta$

When $x=0$

$$a-0 = a\sin\theta \Rightarrow a\sin\theta = a$$

$$\Rightarrow\ \sin\theta = 1 \Rightarrow\ \theta = \frac{\pi}{2}$$

When $x=2a$

$$a-2a = a\sin\theta \Rightarrow -a = a\sin\theta$$

$$\Rightarrow\ -1 = \sin\theta \Rightarrow\ \theta = -\frac{\pi}{2}$$

$$\text{So area} = \int_{\pi/2}^{-\pi/2}\sqrt{a^2-a^2\sin^2\theta}\,(-a\cos\theta d\theta)$$

$$= -a\int_{\pi/2}^{-\pi/2}\sqrt{a^2\left(1-\sin^2\theta\right)}\cos\theta\,d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2}\sqrt{a^2\cos^2\theta}\cos\theta\,d\theta$$

$$= -a\int_{\pi/2}^{-\pi/2}a\cos\theta\cdot\cos\theta\,d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2}\cos^2\theta\,d\theta$$

$$= -a^2\int_{\pi/2}^{-\pi/2}\left(\frac{1+\cos\theta}{2}\right)d\theta$$

$$= -\frac{a^2}{2}\int_{\pi/2}^{-\pi/2}(1+\cos\theta)\,d\theta$$

$$= -\frac{a^2}{2}\left|\theta+\sin\theta\right|_{\pi/2}^{-\pi/2}$$

$$= -\frac{a^2}{2}\left(-\frac{\pi}{2}+\sin\left(-\frac{\pi}{2}\right)-\frac{\pi}{2}+\sin\left(\frac{\pi}{2}\right)\right)$$

$$= -\frac{a^2}{2}(-\pi+-1+1)$$

$$= -\frac{a^2}{2}(-\pi) = \frac{a^2\pi}{2} \text{ sq. unit}$$