EXERCISE 2.10

Question # 1

Let x and $30-x$ be two positive integers and P denotes product integers then

$$P = x(30-x)$$
$$= 30x - x^2$$

Diff. w.r.t. x

$$\frac{dP}{dx} = 30-2x \text{ .... } (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = -2 \text{ ...... } (ii)$$

For critical points, put $\frac{dP}{dx}=0$

$$\Rightarrow 30-2x=0 \quad \Rightarrow -2x=-30 \quad \Rightarrow x=15$$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=2} = -2 < 0$$

$\Rightarrow$ P is maximum at $x=15$

Other $+$tive integer $= 30-x = 30-15 = 15$

Hence 15 and 15 are the required positive numbers.

Question # 2

Let x be the part of 20 then other is $20-x$.

Let S denotes sum of squares then

$$S = x^2+(20-x)^2$$
$$= x^2+400-40x+x^2$$
$$= 2x^2-40x+400$$

Diff. w.r.t x

$$\frac{dS}{dx} = 4x-40 \text{ ...... } (i)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 4 \text{ .... } (ii)$$

For stationary points put $\frac{dS}{dx}=0$

$$\Rightarrow 4x-40=0 \quad \Rightarrow 4x=40 \quad \Rightarrow x=10$$

Putting value of x in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=10} = 4 > 0$$

$\Rightarrow$ S is minimum at $x=10$

Other integer $= 20-x = 20-10 = 10$

Hence 10, 10 are the two parts of 20.

Question # 3

Let x and $12-x$ be two $+$tive integers and P denotes product of one with square of the other then

$$P = x(12-x)^2$$
$$\Rightarrow P = x\left(144-24x+x^2\right)$$
$$= x^3-24x^2+144x$$

Diff. w.r.t x

$$\frac{dP}{dx} = 3x^2-48x+144 \text{ .... } (i)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 6x-48 \;\ldots\ldots.\; (ii)$$

For critical points put $\dfrac{dP}{dx}=0$

$$3x^2-48x+144 = 0$$

$$\Rightarrow\; x^2-16x+48 = 0 \qquad \Rightarrow\; x^2-4x-12x+48 = 0$$

$$\Rightarrow\; x(x-4)-12(x-4) = 0 \qquad \Rightarrow\; (x-4)(x-12) = 0$$

$$\Rightarrow\; x=4 \;\text{ or }\; x=12$$

We can not take $x=12$ as sum of integers is 12. So put $x=4$ in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=4} = 6(4)-48$$

$$= 24-48 = -24 < 0$$

$\Rightarrow$ P is maximum at $x=4$.

So the other integer $= 12-4 = 8$

Hence 4, 8 are the required integers.

Question # 4

Let the remaining sides of the triangles are x and y

Perimeter $= 16$

$$\Rightarrow\; 6+x+y = 16$$

$$\Rightarrow\; x+y = 16-10 \qquad \Rightarrow\; x+y = 6 \qquad \Rightarrow\; y=10-x \;\ldots\ldots..\; (i)$$

Now suppose A denotes the square of the area of triangle then

$$A = s(s-a)(s-b)(s-c)$$

Where $s=\dfrac{a+b+c}{2}=\dfrac{6+x+y}{2}$

$=\dfrac{6+x+10-x}{2}$ from (i)

$=\dfrac{16}{2} = 8$

So $A = 8(8-6)(8-x)(8-y)$

$$= 8(2)(8-x)(8-10+x) = 16(8-x)(-2+x)$$

$$= 16\left(-16+2x+8x-x^2\right)$$

$$\Rightarrow\; A = 16\left(-16+10x-x^2\right)$$

Diff. w.r.t x

$$\frac{dA}{dx} = 16(10-2x) \;\ldots\ldots\ldots\; (i)$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = 16(-2) = -32$$

For critical points put $\dfrac{dA}{dx}=0$

$$16(10-2x) = 0 \;\Rightarrow\; (10-2x) = 0 \;\Rightarrow\; -2x = -10 \;\Rightarrow\; x=5$$

Putting value of x in (ii)

$$\left.\frac{d^2A}{dx^2}\right|_{x=5} = -32 < 0$$

$\Rightarrow$ A is maximum at $x=5$

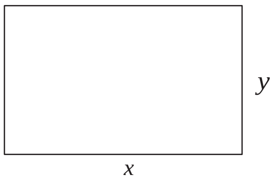
Putting value of x in (i)

$$y = 10-5 = 5$$

Hence length of remaining sides of triangles are $5cm$ and $5cm$.

Question # 5

Let x and y be the length and breadth of rectangle, then



Area $= A = xy \;\ldots\ldots..\; (i)$

Perimeter $= 60$

$$\Rightarrow\; x+x+y+y = 60 \qquad \Rightarrow\; 2x+2y = 60$$

$$\Rightarrow \quad x+y = 30 \quad \Rightarrow \quad y = 30-x \; \ldots\ldots. (ii)$$

Putting in (i)

$$A = x(30-x) \quad \Rightarrow \quad A = 30x-x^2$$

Diff. w.r.t x

$$\frac{dA}{dx} = 30-2x \; \ldots\ldots\ldots.. (iii)$$

Again diff. w.r.t x

$$\frac{d^2A}{dx^2} = -2 \; \ldots\ldots\ldots\ldots (iv)$$

For critical points put $\frac{dA}{dx}=0$

$$30-2x = 0 \quad \Rightarrow \quad -2x = -30 \quad \Rightarrow \quad x = 15$$

Putting value of x in (iv)

$$\left.\frac{d^2A}{dx^2}\right|_{x=15} = -2 < 0$$

$\Rightarrow$ A is maximum at $x=15$

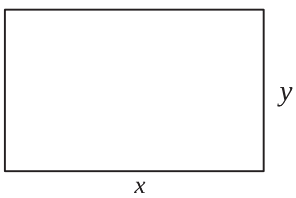
Putting value of x in (ii)

$$y = 30-15 = 15$$

Hence dimension of rectangle is $15cm$, $15cm$.

Question # 6

Let x and y be the length and breadth of the rectangle then



$$\text{Area} = xy$$

$$\Rightarrow \quad 36 = xy \quad \Rightarrow \quad y = {}^{36}\!/_{x} \; \ldots\ldots\ldots\ldots (i)$$

Now perimeter $= 2x+2y$

$$\Rightarrow \quad P = 2x+2\left({}^{36}\!/_{x}\right)$$

$$= 2\left(x+36x^{-1}\right)$$

Diff. P w.r.t x

$$\frac{dP}{dx} = 2\left(1-36x^{-2}\right) \; \ldots\ldots\ldots. (ii)$$

Again diff. w.r.t x

$$\frac{d^2P}{dx^2} = 2\left(0-36\left(-2x^{-3}\right)\right) = 2\left(72x^{-3}\right) = \frac{144}{x^3}$$

For critical points put $\frac{dP}{dx}=0$

$$2\left(1-36x^{-2}\right)=0 \quad \Rightarrow \quad 1-\frac{36}{x^2}=0 \quad \Rightarrow \quad 1=\frac{36}{x^2} \quad \Rightarrow \quad x^2 = 36 \quad \Rightarrow \quad x = \pm 6$$

Since length can not be negative therefore $x=6$

Putting value of x in (ii)

$$\left.\frac{d^2P}{dx^2}\right|_{x=6} = \frac{144}{(6)^3} > 0$$

Hence P is minimum at $x=6$.

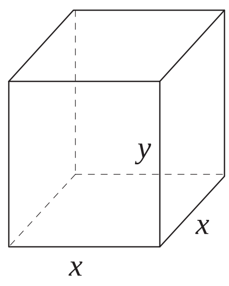
Putting in eq. (i)

$$y = \frac{36}{6} = 6$$

Hence $6cm$ and $6cm$ are the lengths of the sides of the rectangle.

Question # 7

Let x be the lengths of the sides of the base and y be the height of the box.



Then Volume $= x\cdot x\cdot y$

$$\Rightarrow \quad 4 = x^2y \quad \Rightarrow \quad y = \frac{4}{x^2} \; \ldots\ldots\ldots (i)$$

Suppose S denotes the surface area of the box, then

$$S = x^2+4xy$$

$$\Rightarrow \ S = x^2 + 4x\left(\frac{4}{x^2}\right) \qquad \Rightarrow \ S = x^2 + 16x^{-1}$$

Diff. S w.r.t x

$$\frac{dS}{dx} = 2x - 16x^{-2} \ \ldots\ldots\ldots.. (ii)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 16\left(-2x^{-3}\right) = 2 + \frac{32}{x^3} \ \ldots\ldots\ldots\ldots (iii)$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 16x^{-2} = 0 \quad \Rightarrow \ 2x - \frac{16}{x^2} = 0 \quad \Rightarrow \ \frac{2x^3 - 16}{x^2} = 0$$

$$\Rightarrow \ 2x^3 - 16 = 0 \quad \Rightarrow \ 2x^3 = 16 \quad \Rightarrow \ x^3 = 8 \quad \Rightarrow \ x = 2$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=2} = 2 + \frac{32}{(2)^3} > 0$$

$\Rightarrow$ S is min. when $x = 2$

Putting value of x in (i)

$$y = \frac{4}{(2)^2} = 1$$

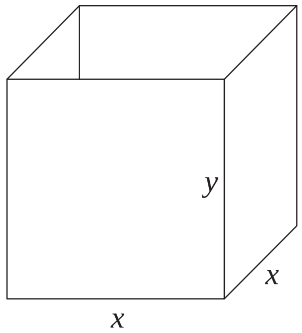
Hence $2dm$, $2dm$ and $1dm$ is the dimension of the box.

Question # 8 *Do yourself as question # 5.*

Question # 9

Let y be the height of the open tank.

Then Volume $= x \cdot x \cdot y$



$$\Rightarrow \ V = x^2 y \qquad \Rightarrow \ y = \frac{V}{x^2} \ \ldots\ldots\ldots.. (i)$$

If S denotes the surface area the open tank, then

$$S = x^2 + 4xy$$

$$= x^2 + 4x\left(\frac{V}{x^2}\right) \quad \Rightarrow \ S = x^2 + 4Vx^{-1}$$

Diff. w.r.t x

$$\frac{dS}{dx} = 2x - 4Vx^{-2} \ \ldots\ldots.. (ii)$$

Again diff. w.r.t x

$$\frac{d^2S}{dx^2} = 2 - 4V\left(-2x^{-3}\right) = 2 + \frac{8V}{x^3} \ \ldots\ldots\ldots\ldots (iii)$$

For critical points, put $\frac{dS}{dx} = 0$

$$2x - 4Vx^{-2} = 0 \quad \Rightarrow \ 2x - \frac{4V}{x^2} = 0 \quad \Rightarrow \ \frac{2x^3 - 4V}{x^2} = 0 \quad \Rightarrow \ 2x^3 - 4V = 0$$

$$\Rightarrow \ 2x^3 = 4V \quad \Rightarrow \ x^3 = 2V \quad \Rightarrow \ x = (2V)^{\frac{1}{3}}$$

Putting in (ii)

$$\left.\frac{d^2S}{dx^2}\right|_{x=(2V)^{\frac{1}{3}}} = 2 + \frac{8V}{\left((2V)^{\frac{1}{3}}\right)^3} = 2 + \frac{8V}{2V} = 2 + 4 = 6 > 0$$

$\Rightarrow$ S is minimum when $x = (2V)^{\frac{1}{3}}$ i.e. $x^3 = 2V \Rightarrow V = \frac{x^3}{2}$

Putting in (i)

$$y = \frac{x^3/2}{x^2} = \frac{x}{2}$$

Hence height of the open tank is $\frac{x}{2}$.

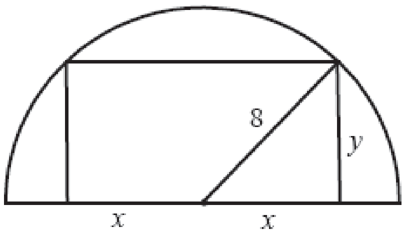
Question # 10

Let $2x$ & y be dimension of rectangle.

Then from figure, using Pythagoras theorem

$$x^2 + y^2 = 8^2 \quad \Rightarrow \quad y^2 = 64 - x^2 \quad \text{.......... } (i)$$

Now Area of the rectangle is given by



$$A = 2x \cdot y$$

Squaring both sides

$$A^2 = 4x^2y^2$$

$$= 4x^2\left(64 - x^2\right)$$

$$= 256x^2 - 4x^4$$

Now suppose $f = A^2 = 256x^2 - 4x^4$ .... (ii)

Diff. w.r.t x

$$\frac{df}{dx} = 512x - 16x^3 \quad \text{.......... } (iii)$$

Again diff. w.r.t x

$$\frac{d^2 f}{dx^2} = 512 - 48x^2 \quad \text{......... } (iv)$$

For critical points, put $\frac{df}{dx} = 0$

$$\Rightarrow 512x - 16x^3 = 0$$

$$\Rightarrow 16x\left(32 - x^2\right) = 0$$

$$\Rightarrow 16x = 0 \quad \text{or} \quad 32 - x^2 = 0$$

$$\Rightarrow x = 0 \quad \text{or} \quad x^2 = 32$$

$$\Rightarrow x = \pm 4\sqrt{2}$$

Since x can not be zero or $-$ive, therefore

$$x = 4\sqrt{2}$$

Putting in (iv)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=4\sqrt{2}} = 512 - 48\left(4\sqrt{2}\right)^2$$

$$= 512 - 48(32) = 512 - 1536 = -1024 < 0$$

$\Rightarrow$ Area is max. for $x = 4\sqrt{2}$

Hence length $= 2x = 2\left(4\sqrt{2}\right)$

Breadth $= y = \sqrt{64 - \left(4\sqrt{2}\right)^2} = \sqrt{64 - 32} = \sqrt{32} = 4\sqrt{2}$

Hence dimension is $8\sqrt{2}$ cm and $4\sqrt{2}$ cm.

Question # 11

Let $P(x, y)$ be point and let $A(3, -1)$

Then $d = \left|AP\right| = \sqrt{(x-3)^2 + (y+1)^2}$

$$\Rightarrow d^2 = (x-3)^2 + (y+1)^2$$

$$= (x-3)^2 + \left(x^2 - 1 + 1\right)^2 \qquad \because \; y = x^2 - 1 \text{ (given)}$$

$$\Rightarrow d^2 = (x-3)^2 + x^4$$

Let $f = d^2 = (x-3)^2 + x^4$

Diff. w.r.t x

$$\frac{df}{dx} = 2(x-3)+4x^3 \text{ .... } (i)$$

Again diff. w.r.t x

$$\frac{d^2 f}{dx^2} = 2+12x^2 \text{ .... } (ii)$$

For stationary points, put $\frac{df}{dx} = 0$

$$2(x-3)+4x^3 = 0$$

$$\Rightarrow 2x-6+4x^3 = 0$$

$$\Rightarrow 4x^3+2x-6 = 0$$

$$\Rightarrow 2x^3+x-3 = 0 \qquad \div \text{ing by 2}$$

By synthetic division

$$\begin{array}{c|cccc} 1 & 2 & 0 & 1 & -3 \\ & \downarrow & 2 & 2 & 3 \\ \hline & 2 & 2 & 3 & \underline{|0} \end{array}$$

$$\Rightarrow x = 1 \quad \text{or} \quad 2x^2+2x+3 = 0$$

$$\Rightarrow x = \frac{-2\pm\sqrt{4-4(2)(3)}}{4}$$

$$= \frac{-2\pm\sqrt{-20}}{4}$$

This is complex and not acceptable.

Now put $x=1$ in (ii)

$$\left.\frac{d^2 f}{dx^2}\right|_{x=1} = 2+12(1)^2 = 14 > 0$$

$\Rightarrow$ d is maximum at $x=1$.

$$y = 1^2-1 = 0$$

$\because$ $(1,0)$ is the required point.

Question # 12

Do yourself as Q # 11
