Exercise 5.3

Question # 1

$$\frac{9x-7}{(x^2+1)(x+3)}$$

Solution

$$\frac{9x-7}{(x^2+1)(x+3)}$$

Resolving it into partial fraction.

$$\frac{9x-7}{(x^2+1)(x+3)}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x+3)}$$

Multiplying both sides by $(x^2+1)(x+3)$.

$9x-7=(Ax+B)(x+3)+C(x^2+1)$ ...(i)

Put $x+3=0 \Rightarrow x=-3$ in equation (i).

$$9(-3)-7=\big(A(-3)+B\big)(0)+C\left((-3)^2+1\right) \quad \Rightarrow -27-7=0+C(9+1)$$

$$\Rightarrow -34=10C \qquad \Rightarrow C=-\frac{34}{10} \qquad \Rightarrow \boxed{C=-\frac{17}{5}}$$

Now equation (i) can be written as

$$9x-7=A(x^2+3x)+B(x+3)+C(x^2+1)$$

Comparing the coefficients of x^2, x and x^0.

$$0=A+C \quad \ldots \text{(ii)}$$

$$9=3A+B \quad \ldots \text{(iii)}$$

$$-7=+3B+C \quad \ldots \text{(iv)}$$

Putting value of C in equation (ii)

$$0=A-\frac{17}{5} \qquad \Rightarrow \boxed{A=\frac{17}{5}}$$

Now putting value of A in equation (iii)

$$9=3\left(\frac{17}{5}\right)+B \qquad \Rightarrow 9=\frac{51}{5}+B$$

$$\Rightarrow 9-\frac{51}{5}=B \qquad \Rightarrow \boxed{B=-\frac{6}{5}}$$

Hence

$$\frac{9x-7}{(x^2+1)(x+3)}=\frac{\frac{17}{5}x-\frac{6}{5}}{x^2+1}+\frac{-\frac{17}{5}}{(x+3)}$$

$$=\frac{\frac{17x-6}{5}}{x^2+1}-\frac{\frac{17}{5}}{(x+3)} \qquad =\frac{17x-6}{5(x^2+1)}-\frac{17}{5(x+3)} \quad \textit{Answer}$$

Question # 2 $\dfrac{1}{(x^2+1)(x+1)}$

Solution $\dfrac{1}{(x^2+1)(x+1)}$

Now Consider

$$\frac{1}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$$

Multiplying both sides by $(x^2+1)(x+1)$.

$$1=(Ax+B)(x+1)+C(x^2+1)\text{.........(i)}$$

Put $x+1=0 \;\Rightarrow\; x=-1$ in equation (i)

$$1=0+C\left((-1)^2+1\right) \qquad \Rightarrow 1=2C \qquad \Rightarrow \boxed{C=\frac{1}{2}}$$

Now eq. (i) can be written as

$$1=A(x^2+x)+B(x+1)+C(x^2+1)$$

Comparing the coefficients of x^2, x and x^0.

$$0=A+C\ \ldots\text{(ii)}$$
$$0=A+B\ \ldots\text{(iii)}$$
$$1=A+C\ \ldots\text{(iv)}$$

Putting value of C in equation (ii)

$$0=A+\frac{1}{2} \quad\Rightarrow \boxed{A=-\frac{1}{2}}$$

Putting value of A in equation (iii)

$$0=-\frac{1}{2}+B \Rightarrow \boxed{B=\frac{1}{2}}$$

Hence $$\frac{1}{(x^2+1)(x+1)}=\frac{-\frac{1}{2}x+\frac{1}{2}}{x^2+1}+\frac{\frac{1}{2}}{x+1}=\frac{\frac{-x+1}{2}}{x^2+1}+\frac{\frac{1}{2}}{x+1}$$

$$=\frac{-x+1}{2(x^2+1)}+\frac{1}{2(x+1)}=\frac{1-x}{2(x^2+1)}+\frac{1}{2(x+1)}$$ *Answer*

Question # 3 $\dfrac{3x+7}{(x^2+4)(x+3)}$

Solution $\dfrac{3x+7}{(x^2+4)(x+3)}$

Resolving it into partial fraction.

$$\frac{3x+7}{(x^2+4)(x+3)}=\frac{Ax+B}{x^2+4}+\frac{C}{x+3}$$

$$\left[\begin{array}{c} \textit{Now do yourself, you will get} \\ A = \frac{2}{13},\ B = \frac{33}{13} \textit{ and } C = -\frac{2}{13} \end{array}\right]$$

Question # 4 $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$

Solution $\dfrac{x^2+15}{(x^2+2x+5)(x-1)}$

Resolving it into partial fraction.

$$\frac{x^2+15}{(x^2+2x+5)(x-1)} = \frac{Ax+B}{x^2+2x+5} + \frac{C}{x-1}$$

$$\Rightarrow\ x^2+15 = (Ax+B)(x-1) + C(x^2+2x+5)\ldots\text{(i)}$$

Put $x-1=0 \ \Rightarrow x=1$ in equation (i)

$$(1)^2+15 = \big(A(1)+B\big)(0) + C\big((1)^2+2(1)+5\big)$$

$$\Rightarrow\ 1+15 = 0 + C(1+2+5)$$

$$\Rightarrow 16 = 8C \quad \Rightarrow \frac{16}{8} = C \quad \Rightarrow \boxed{C = 2}$$

Now equation (i) can be written as

$$x^2+15 = A(x^2-x) + B(x-1) + C(x^2+2x+5)$$

Comparing the coefficients of x^2, x and x^0.

$$1 = A + C \ \ldots \text{(ii)}$$

$$0 = -A + B + 2C \ \ldots \text{(iii)}$$

$$15 = -B + 5C \ \ldots \text{(iv)}$$

Putting value of C in equation (ii).

$$1 = A+2 \quad \Rightarrow 1-2 = A$$

$$\Rightarrow \boxed{A = -1}$$

Putting value of A and C in equation (iii)

$$0 = -(-1) + B + 2(2) \quad \Rightarrow 0 = 1 + B + 4$$

$$\Rightarrow 0 = B+5 \quad \Rightarrow \boxed{B = -5}$$

Hence $$\frac{x^2+15}{(x^2+2x+5)(x-1)} = \frac{(-1)x-5}{x^2+2x+5} + \frac{2}{x-1}$$

$$= \frac{-x-5}{x^2+2x+5} + \frac{2}{x-1} \qquad \textit{Answer}$$

Question # 5 $\dfrac{x^2}{(x^2+4)(x+2)}$

Solution $\dfrac{x^2}{(x^2+4)(x+2)}$

Resolving it into partial fraction.

$$\frac{x^2}{(x^2+4)(x+2)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+2}$$

$$\left[\begin{array}{c} \textit{Now do yourself, you will get} \\ A = \tfrac{1}{2},\ B = -1 \ \textit{and}\ C = -\tfrac{1}{2} \end{array}\right]$$

Question # 6 $\quad \dfrac{x^2+1}{x^3+1}$

Solution $\quad \dfrac{x^2+1}{x^3+1} = \dfrac{x^2+1}{(x+1)(x^2-x+1)} \qquad \because\ x^3+1 = (x+1)(x^2-x+1)$

Now consider

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1} \qquad \ldots \text{(A)}$$

$$x^2+1 = A\left(x^2-x+1\right) + \left(Bx+C\right)\left(x+1\right)$$

$$x^2+1 = A\left(x^2-x+1\right) + \left(Bx^2+Bx+Cx+C\right)$$

$$x^2+1 = \left(A+B\right)x^2 + \left(B+C-A\right)x + \left(A+C\right)$$

By comparing the coefficients of x^2 , x and x^0

$A+B=1 \quad \ldots (1)$

$B+C-A=0 \quad \ldots (2)$

$A+C=1 \quad \ldots (3)$

From (3), We have

$C=1-A$

Put the value of C in (2).

$-2A+B=-1 \ \ldots (4)$

Subtract (1) from (4), We have

$3A=2 \Rightarrow A=\dfrac{2}{3}$

Put the value of A in (1) and (3), we have

$B=\dfrac{1}{3}, \quad C=\dfrac{1}{3}$

Put the values of $A, B\ and\ C$ in (A), We have

$$\frac{x^2+1}{(x+1)(x^2-x+1)} = \frac{2}{3(x+1)} + \frac{(x+1)}{3\left(x^2-x+1\right)}$$

Question # 7 $\quad \dfrac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

Solution $\quad \dfrac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}$

Consider

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{Ax+B}{x^2+3}+\frac{C}{x+1}+\frac{D}{x-1}$$

$\Rightarrow\; x^2+2x+2=(Ax+B)(x+1)(x-1)+C(x^2+3)(x-1)+D(x^2+3)(x+1)\ldots\text{(i)}$

Put $x+1=0 \;\Rightarrow\; x=-1$ in equation (i)

$$(-1)^2+2(-1)+2=0+C\big((-1)^2+3\big)\big((-1)-1\big)+0 \qquad \Rightarrow 1-2+2=C(4)(-2)$$

$$\Rightarrow 1=-8C \qquad \Rightarrow \boxed{C=-\frac{1}{8}}$$

Now put $x-1=0 \;\Rightarrow\; x=1$ in equation (i)

$$\Rightarrow\; (1)^2+2(1)+2=0+0+D\big((1)^2+3\big)\big((1)+1\big) \qquad \Rightarrow 1+2+2=D(4)(2)$$

$$\Rightarrow 5=8D \qquad \Rightarrow \boxed{D=\frac{5}{8}}$$

Equation (i) can be written as

$$x^2+2x+2=(Ax+B)(x^2-1)+C(x^3-x^2+3x-3)+D(x^3+x^2+3x+3)$$

$$\Rightarrow\; x^2+2x+2=A(x^3-x)+B(x^2-1)+C(x^3-x^2+3x-3)+D(x^3+x^2+3x+3)$$

Comparing the coefficients of x^3, x^2, x and x^0.

$0=A+C+D\ldots$ (ii)

$1=B-C+D\;\ldots$ (iii)

$2=-A+3C+3D\;\ldots$ (iv)

$2=-B-3C+3D\;\ldots$ (v)

Putting values of C and D in (ii)

$$0=A-\frac{1}{8}+\frac{5}{8} \qquad \Rightarrow 0=A+\frac{1}{2} \qquad \Rightarrow \boxed{A=-\frac{1}{2}}$$

Putting values of C and D in (iii)

$$1=B-\left(-\frac{1}{8}\right)+\frac{5}{8} \qquad \Rightarrow 1=B+\frac{1}{8}+\frac{5}{8} \qquad \Rightarrow 1=B+\frac{3}{4}$$

$$\Rightarrow 1-\frac{3}{4}=B \qquad \Rightarrow \boxed{B=\frac{1}{4}}$$

Hence

$$\frac{x^2+2x+2}{(x^2+3)(x+1)(x-1)}=\frac{-\frac{1}{2}x+\frac{1}{4}}{x^2+3}+\frac{-\frac{1}{8}}{x+1}+\frac{\frac{5}{8}}{x-1}$$

$$=\frac{\frac{-2x+1}{4}}{x^2+3}+\frac{-\frac{1}{8}}{x+1}+\frac{\frac{5}{8}}{x-1}$$

$$=\frac{-2x+1}{4(x^2+3)}+\frac{-1}{8(x+1)}+\frac{5}{8(x-1)}$$

$$=\frac{1-2x}{4(x^2+3)}-\frac{1}{8(x+1)}+\frac{5}{8(x-1)} \qquad \textit{Answer}$$

Question # 8 $\qquad \dfrac{1}{(x-1)^2(x^2+2)}$

Solution $\qquad \dfrac{1}{(x-1)^2(x^2+2)}$

Resolving it into partial fraction.

$$\frac{1}{(x-1)^2(x^2+2)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{Cx+D}{x^2+2}$$

$$\Rightarrow\ 1=A(x-1)(x^2+2)+B(x^2+2)+(Cx+D)(x-1)^2 \ldots\text{(i)}$$

Put $x-1=0 \Rightarrow x=1$ in equation (i)

$$1=0+B\left((1)^2+2\right)+0$$

$$\Rightarrow\ 1=3B \qquad \Rightarrow \boxed{B=\frac{1}{3}}$$

Now equation (i) can be written as

$$1=A(x^3-x^2+2x-2)+B(x^2+2)+(Cx+D)(x^2-2x+1)$$

$$\Rightarrow\ 1=A(x^3-x^2+2x-2)+B(x^2+2)+C(x^3-2x^2+x)+D(x^2-2x+1)$$

Comparing the coefficients of x^3, x^2, x and x^0.

$0=A+C \ \ldots$ (ii)

$0=-A+B-2C+D \ \ldots$ (iii)

$0=2A+C-2D \ \ldots$ (iv)

$1=-2A+2B+D \ \ldots$ (v)

Multiplying eq. (iii) by 2 and adding in (iv)

$$\begin{array}{l} 0=-2A+2B-4C+2D \\ 0=\ \ 2A \qquad\quad +\ C-2D \\ \hline 0= \qquad\quad 2B-3C \end{array}$$

Putting value of B in above

$$0=2\left(\frac{1}{3}\right)-3C \quad \Rightarrow 0=\frac{2}{3}-3C \quad \Rightarrow 3C=\frac{2}{3} \quad \Rightarrow \boxed{C=\frac{2}{9}}$$

Putting value of C in eq. (ii)

$$0=A+\frac{2}{9} \quad \Rightarrow \boxed{A=-\frac{2}{9}}$$

Putting value of A and B in eq. (v)

$$1=-2\left(-\frac{2}{9}\right)+2\left(\frac{1}{3}\right)+D \ \Rightarrow 1=\frac{4}{9}+\frac{2}{3}+D$$

$$\Rightarrow\ 1-\frac{4}{9}-\frac{2}{3}=D \quad \Rightarrow \boxed{D=-\frac{1}{9}}$$

Hence

$$\frac{1}{(x-1)^2(x^2+2)}=\frac{-\frac{2}{9}}{x-1}+\frac{\frac{1}{3}}{(x-1)^2}+\frac{\left(\frac{2}{9}\right)x+\left(-\frac{1}{9}\right)}{x^2+2}$$

$$=\frac{-\frac{2}{9}}{x-1}+\frac{\frac{1}{3}}{(x-1)^2}+\frac{\frac{2x-1}{9}}{x^2+2}$$

$$=\frac{-2}{9(x-1)}+\frac{1}{3(x-1)^2}+\frac{2x-1}{9(x^2+2)}$$

Question # 9 $\dfrac{x^4}{1-x^4}$

Solution $\dfrac{x^4}{1-x^4}$

$$\begin{array}{r|l} & -1 \\ \hline 1-x^4 & x^4 \\ & \underset{-}{x^4}\ \underset{+}{-1} \\ \hline & 1 \end{array}$$

$$=-1+\frac{1}{1-x^4}=-1+\frac{1}{(1-x^2)(1+x^2)}$$

$$=-1+\frac{1}{(1-x)(1+x)(1+x^2)}$$

Now consider

$$\frac{1}{(1-x)(1+x)(1+x^2)}=\frac{A}{1-x}+\frac{B}{1+x}+\frac{Cx+D}{1+x^2}$$

$$\left[\begin{array}{c}\textit{Now find values of } A, B, C \textit{ and } D \textit{ yourself.} \\ \textit{You will get } A=\frac{1}{4},\ B=\frac{1}{4},\ C=0 \textit{ and } D=\frac{1}{2}\end{array}\right]$$

So

$$\frac{1}{(1-x)(1+x)(1+x^2)}=\frac{\frac{1}{4}}{1-x}+\frac{\frac{1}{4}}{1+x}+\frac{(0)x+\frac{1}{2}}{1+x^2}$$

$$=\frac{1}{4(1-x)}+\frac{1}{4(1+x)}+\frac{1}{2(1+x^2)}$$

Hence

$$\frac{x^4}{1-x^4}=-1+\frac{1}{4(1-x)}+\frac{1}{4(1+x)}+\frac{1}{2(1+x^2)} \quad \textit{Answer}$$

Question # 10 $\dfrac{x^2-2x+3}{x^4+x^2+1}$

Solution $\dfrac{x^2-2x+3}{x^4+x^2+1}$

$$\begin{aligned}\because\ x^4+x^2+1&=x^4+2x^2+1-x^2\\&=(x^2+1)^2-x^2\\&=(x^2+1+x)(x^2+1-x)\\&=(x^2+x+1)(x^2-x+1)\end{aligned}$$

$$=\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}$$

Now Consider

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)}=\frac{Ax+B}{x^2+x+1}+\frac{Cx+D}{x^2-x+1}$$

$$\Rightarrow x^2-2x+3=(Ax+B)(x^2-x+1)+(Cx+D)(x^2+x+1)\ldots\text{(i)}$$

$$\Rightarrow x^2-2x+3=A(x^3-x^2+x)+B(x^2-x+1)+C(x^3+x^2+x)+D(x^2+x+1)$$

Comparing the coefficients of x^3, x^2, x and x^0.

$0=A+C \;\ldots$ (ii)

$1=-A+B+C+D \ldots$ (iii)

$-2=A-B+C+D \;\ldots$ (iv)

$3=B+D \;\ldots$ (v)

Subtracting (ii) and (iv)

$$\begin{array}{rcccc} 0= & A & & +C & \\ -2= & A & -B & +C & +D \\ \underline{\scriptstyle +} & \underline{\scriptstyle -} & \underline{\scriptstyle +} & \underline{\scriptstyle -} & \underline{\scriptstyle -} \\ 2= & & B & & -D \end{array}$$

$\Rightarrow 2=B-D \;\ldots$ (vi)

Adding (v) and (vi)

$$\begin{array}{l} 3=B+D \\ \underline{2=B-D} \\ 5=2B \end{array}$$

$$\Rightarrow \boxed{B=\frac{5}{2}}$$

Putting value of B in (v)

$$3=\frac{5}{2}+D$$

$$\Rightarrow 3-\frac{5}{2}=D \qquad \Rightarrow \boxed{D=\frac{1}{2}}$$

Putting value of B and D in (iii)

$$1=-A+\frac{5}{2}+C+\frac{1}{2}$$

$$\Rightarrow 1-\frac{5}{2}-\frac{1}{2}=-A+C$$

$$\Rightarrow -2=-A+C \quad \ldots \text{(vii)}$$

Adding (ii) and (vii)

$$\begin{array}{rl} 0= & A+C \\ \underline{-2=} & \underline{-A+C} \\ -2= & 2C \end{array}$$

$$\Rightarrow \boxed{C=-1}$$

Putting value of C in equation (ii)

$$0=A-1$$

$$\Rightarrow \boxed{A=1}$$

Hence

$$\frac{x^2-2x+3}{(x^2+x+1)(x^2-x+1)} = \frac{(1)\,x+\frac{5}{2}}{x^2+x+1} + \frac{(-1)x+\frac{1}{2}}{x^2-x+1}$$

$$= \frac{\frac{2x+5}{2}}{x^2+x+1} + \frac{\frac{-2x+1}{2}}{x^2-x+1}$$

$$= \frac{2x+5}{2(x^2+x+1)} + \frac{-2x+1}{2(x^2-x+1)}$$

$$= \frac{2x+5}{2(x^2+x+1)} + \frac{1-2x}{2(x^2-x+1)} \qquad \textit{Answer}$$