Exercise 3.3

1. Evaluate the Determinant

i) $\begin{vmatrix} 5 & -2 & -4 \\ 3 & -1 & -3 \\ -2 & 1 & 2 \end{vmatrix}$

$= 5\begin{vmatrix} -1 & -3 \\ 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ -2 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5(-2+3) + 2(6-6) - 4(3-2)$

$= 5(1) + 2(0) - 4(1)$

$= 5 - 4 = 1$

ii) $\begin{vmatrix} 5 & 2 & -3 \\ 3 & -1 & 1 \\ -2 & 1 & -2 \end{vmatrix}$

$= 5\begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ -2 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & -1 \\ -2 & 1 \end{vmatrix}$

$= 5(2-1) - 2(-6+2) - 3(3-2)$

$= 5(1) - 2(-4) - 3(1)$

$= 5 + 8 - 3 = 10$

iii) $\begin{vmatrix} 1 & 2 & -3 \\ -1 & 3 & 4 \\ -2 & 5 & 6 \end{vmatrix}$

$= 1\begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ -2 & 6 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 3 \\ -2 & 5 \end{vmatrix}$

$= 1(18-20) - 2(-6+8) - 3(-5+6)$

$= 1(-2) - 2(2) - 3(1)$

$= -2 - 4 - 3 = -9$

iv) $\begin{vmatrix} a+l & a-l & a \\ a & a+l & a-l \\ a-l & a & a+l \end{vmatrix}$

$= \begin{vmatrix} 3a & a-l & a \\ 3a & a+l & a-l \\ 3a & a & a+l \end{vmatrix} \quad C_1 + (C_2 + C_3)$

$= 3a\begin{vmatrix} 1 & a-l & a \\ 1 & a+l & a-l \\ 1 & a & a+l \end{vmatrix}$

$= 3a\begin{vmatrix} 1 & a-l & a \\ 0 & 2l & -l \\ 0 & l & l \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$

Expand by C_1

$= 3a[1(2l^2 + l^2) - 0 + 0]$

$= 3a(3l^2) = 9al^2$

vi) $\begin{vmatrix} 2a & a & c \\ b & 2b & b \\ c & c & 2c \end{vmatrix}$

Sol. $abc\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix}$

$= abc\begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -3 & -1 & 2 \end{vmatrix}$

Expand by R_1

$= abc\left[0 - 0 + 1\begin{vmatrix} -1 & 1 \\ -3 & -1 \end{vmatrix}\right]$

$= abc[1(1+3)]$

$= 4abc$

2. Without expansion show that

i) $\begin{vmatrix} 6 & 7 & 8 \\ 3 & 4 & 5 \\ 2 & 3 & 4 \end{vmatrix} = 0$

Sol

L.H.S $\begin{vmatrix} 6 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - C_2 \end{matrix}$

$= 0$ ($\because$ C_2, C_3 are same)

$=$ R.H.S

ii) $\begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix} = 0$

Sol

L.H.S $\begin{vmatrix} -1 & 3 & -1 \\ 0 & 1 & 0 \\ 5 & -3 & 5 \end{vmatrix} \quad C_1 - C_2$

$= 0$ ($\because$ $C_1 = C_3$)

$=$ R.H.S

iii) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

Sol.

L.H.S $\begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix}$ $R_2 - R_1$, $R_3 - R_2$

$= 0 \quad (\because R_2 = R_3)$

$= R.H.S$

3. i) Show that

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}+\alpha_{13} \\ a_{21} & a_{22} & a_{23}+\alpha_{23} \\ a_{31} & a_{32} & a_{33}+\alpha_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{11} & a_{32} & \alpha_{33} \end{vmatrix}$$

Sol L.H.S Expand by C_3

$$=(a_{13}+\alpha_{13})\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - (\alpha_{23}+\alpha_{23})\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + (a_{33}+\alpha_{33})\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - a_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + a_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$+\alpha_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - \alpha_{23}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + \alpha_{33}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix}$$

ii) $\begin{vmatrix} 2 & 3 & 0 \\ 3 & 9 & 6 \\ 2 & 15 & 1 \end{vmatrix} = 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix}$

L.H.S Taking 3 common from R_2

$$= 3\begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & 2 \\ 2 & 15 & 1 \end{vmatrix}$$

Taking 3 common from C_2

$$= 9\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ 2 & 5 & 1 \end{vmatrix} = R.H.S$$

iii) $\begin{vmatrix} a+l & a & a \\ a & a+l & a \\ a & a & a+l \end{vmatrix} = l^2(3a+l)$

L.H.S

$$= \begin{vmatrix} 3a+l & 3a+l & 3a+l \\ a & a+l & a \\ a & a & a+l \end{vmatrix} \quad R_1+(R_2+R_3)$$

$$=(3a+l)\begin{vmatrix} 1 & 1 & 1 \\ a & a+l & a \\ a & a & a+l \end{vmatrix}$$

$$=(3a+l)\begin{vmatrix} 1 & 0 & 0 \\ a & l & 0 \\ a & 0 & l \end{vmatrix} \quad C_2 - C_1,\ C_3 - C_1$$

$$=(3a+l)\left[1\begin{vmatrix} l & 0 \\ 0 & l \end{vmatrix} - 0 + 0\right]$$

$$=(3a+l)\left[1(l^2-0)-0\right]$$

$$=(3a+l)l^2 = R.H.S$$

iv) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

L.H.S Multiplying C_1 by x, C_2 by y, C_3 by z and dividing Det by xyz

$$= \frac{1}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Taking xyz common from R_3

$$= \frac{xyz}{xyz}\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

Interchanging R_2 and R_3

$$= (-1)\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Interchanging R_1 and R_2

$$=(-1)(-1)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = R.H.S$$

v) $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$

Sol.

$$L.H.S = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \quad R_1-(R_2+R_3)$$

$$= -2\begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= -2\begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix} \quad R_2 - R_1,\ R_3 - R_1$$

$$= -2\left\{0 - c\begin{vmatrix} b & 0 \\ c & a \end{vmatrix} + b\begin{vmatrix} b & a \\ c & 0 \end{vmatrix}\right\}$$

$$= -2\{-c(ab-0) + b(0-ac)\}$$

$$= -2\{-abc - abc\}$$

$$= -2\{-2abc\} = 4abc = R.H.S$$

vi) $$\begin{vmatrix} b & -1 & a \\ a & b & 0 \\ 1 & a & b \end{vmatrix} = a^3 + b^3$$

Sol. Expand by R_1

$$= b\begin{vmatrix} b & 0 \\ a & b \end{vmatrix} - (-1)\begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} + a\begin{vmatrix} a & b \\ 1 & a \end{vmatrix}$$

$$= b(b^2 - 0) + 1(ab - 0) + a(a^2 - b)$$

$$= b^3 + ab + a^3 - ab$$

$$= a^3 + b^3 = R.H.S$$

vii) $$\begin{vmatrix} r\cos\phi & 1 & -\sin\phi \\ 0 & r & 0 \\ r\sin\phi & 0 & \cos\phi \end{vmatrix} = r^2$$

Sol. L.H.S

$$= r\cos\phi\begin{vmatrix} r & 0 \\ 0 & \cos\phi \end{vmatrix} - 1\begin{vmatrix} 0 & 0 \\ r\sin\phi & \cos\phi \end{vmatrix} - \sin\phi\begin{vmatrix} 0 & r \\ r\sin\phi & 0 \end{vmatrix}$$

$$r\cos\phi(r\cos\phi) - 0 - \sin\phi(0 - r^2\sin\phi)$$

$$= r^2\cos^2\phi + r^2\sin^2\phi$$

$$= r^2(1) = r^2 = R.H.S$$

viii) L.H.S $$\begin{vmatrix} a & b+c & a+b \\ b & c+a & b+c \\ c & a+b & c+a \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b+c & a+b \\ a+b+c & c+a & b+c \\ a+b+c & a+b & c+a \end{vmatrix} \quad C_1 + C_2$$

Taking $(a+b+c)$ common from C_1

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 1 & c+a & b+c \\ 1 & a+b & c+a \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b+c & a+b \\ 0 & a-b & c-a \\ 0 & a-c & c-b \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expand by C_1

$$= (a+b+c)\left(1\begin{vmatrix} a-b & c-a \\ a-c & c-b \end{vmatrix} - 0 + 0\right)$$

$$= (a+b+c)[(a-b)(c-b) - (a-c)(c-a)]$$

$$= (a+b+c)\{ac - ab - bc + b^2 - (ac - a^2 - c^2 + ac)\}$$

$$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ac)$$

$$= a^3 + b^3 + c^3 - 3abc = R.H.S$$

ix) $$\begin{vmatrix} a+\lambda & b & c \\ a & b+\lambda & c \\ a & b & c+\lambda \end{vmatrix} = \lambda^2(a+b+c+\lambda)$$

Sol. L.H.S

$$\begin{vmatrix} a+b+c+\lambda & b & c \\ a+b+c+\lambda & b+\lambda & c \\ a+b+c+\lambda & b & c+\lambda \end{vmatrix} \quad C_1 + (C_2 + C_3)$$

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 1 & b+\lambda & c \\ 1 & b & c+\lambda \end{vmatrix}$$

$$= (a+b+c+\lambda)\begin{vmatrix} 1 & b & c \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

Expand from C_1

$$= (a+b+c+\lambda)\left(1\begin{vmatrix} \lambda & 0 \\ 0 & \lambda \end{vmatrix} - 0 + 0\right)$$

$$= (a+b+c+\lambda)(1(\lambda^2 - 0))$$

$$= (a+b+c+\lambda)\lambda^2 = R.H.S$$

x) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Sol. L.H.S

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \quad \begin{matrix} C_1 - C_2 \\ C_2 - C_3 \end{matrix}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ (a-b)(a+b) & (b+c)(b-c) & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\left\{0 - 0 + 1\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}\right\}$$

$$= (a-b)(b-c)\{1(b+c-a-b)\}$$

$$= (a-b)(b-c)(c-a)$$

$$= R.H.S$$

xi)
L.H.S $\begin{vmatrix} b+c & a & a^2 \\ c+a & b & b^2 \\ a+b & c & c^2 \end{vmatrix}$

$= \begin{vmatrix} a+b+c & a & a^2 \\ a+b+c & b & b^2 \\ a+b+c & c & c^2 \end{vmatrix}$ C_1+C_2

$= (a+b+c) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

by $R_1-R_2,\ R_2-R_3$

$= (a+b+c) \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$

Expand by C_1

$= (a+b+c)\left[0-0+1\begin{vmatrix} a-b & (a-b)(a+b) \\ b-c & (b-c)(b+c) \end{vmatrix}\right]$

$= (a+b+c)(a-b)(b-c)\begin{vmatrix} 1 & a+b \\ 1 & b+c \end{vmatrix}$

$= (a+b+c)(a-b)(b-c)(b+c-a-b)$

$= (a+b+c)(a-b)(b-c)(c-a)$

$=$ R.H.S

4. If $A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 1 \end{pmatrix}$ $B = \begin{pmatrix} 5 & -2 & 5 \\ 3 & -1 & 4 \\ -2 & 1 & -2 \end{pmatrix}$

i) $A_{12},\ A_{22},\ A_{32},\ |A|$

Sol

$A_{ij} = (-1)^{i+j} M_{ij}$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 0 \\ -2 & 1 \end{vmatrix} = (-1)^3(0-0) = 0$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & -3 \\ -2 & 1 \end{vmatrix} = (-1)^4(1-6) = 1(-5) = -5$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix} = (-1)^5(0+0) = 0$

$|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$= 2(0) + (-2)(-5) + (-2)(0)$

$= 0 + 10 - 0 = 10$

ii) $B_{21},\ B_{22},\ B_{23},\ |B|$

$B_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix} = (-1)^3(4-5) = -1(-1) = 1$

$B_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 5 \\ -2 & -2 \end{vmatrix} = (-1)^4(-10+10) = 0$

$B_{23} = (-1)^{2+3}\begin{vmatrix} 5 & -2 \\ -2 & 1 \end{vmatrix} = (-1)^5(5-4) = -1\times1 = -1$

$|B|\quad b_{21}B_{21} + b_{22}B_{22} + b_{23}B_{23}$

$= 3(1) + (-1)(0) + 4(-1)$

$= 3 - 0 - 4 = -1$

5. Without expansion verify that

(i) $\begin{vmatrix} \alpha & \beta+\gamma & 1 \\ \beta & \alpha+\gamma & 1 \\ \gamma & \alpha+\beta & 1 \end{vmatrix}$

Sol.
L.H.S $= \begin{vmatrix} \alpha+\beta+\gamma & \beta+\gamma & 1 \\ \alpha+\beta+\gamma & \alpha+\gamma & 1 \\ \alpha+\beta+\gamma & \alpha+\beta & 1 \end{vmatrix}$ C_1+C_2

$= (\alpha+\beta+\gamma)\begin{vmatrix} 1 & \beta+\gamma & 1 \\ 1 & \alpha+\gamma & 1 \\ 1 & \alpha+\beta & 1 \end{vmatrix}$

$= (\alpha+\beta+\gamma)(0)\quad (\because C_1 = C_3)$

$= 0 =$ R.H.S

ii) $\begin{vmatrix} 1 & 2 & 3x \\ 2 & 3 & 6x \\ 3 & 6 & 9x \end{vmatrix} = 0$

Sol.
L.H.S $(3x)$ common from C_3.

$= 3x\begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 6 & 3 \end{vmatrix}$

$= 3x(0)\quad (\because C_1 = C_3)$

$= 0 =$ R.H.S

iii) $\begin{vmatrix} 1 & a^2 & \frac{a}{bc} \\ 1 & b^2 & \frac{b}{ca} \\ 1 & c^2 & \frac{c}{ab} \end{vmatrix} = 0$

Sol.
L.H.S Multiplying C_3 by abc and dividing Det by abc.

$= \frac{1}{abc}\begin{vmatrix} 1 & a^2 & abc\frac{a}{bc} \\ 1 & b^2 & abc\frac{b}{ac} \\ 1 & c^2 & abc\frac{c}{ab} \end{vmatrix}$

$= \frac{1}{abc}\begin{vmatrix} 1 & a^2 & a^2 \\ 1 & b^2 & b^2 \\ 1 & c^2 & c^2 \end{vmatrix}$

$= \frac{1}{abc}(0)\quad (\because C_2 = C_3)$

$= 0 =$ R.H.S

iv) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$

Sol. L.H.S $C_1 + (C_2 + C_3)$

$$= \begin{vmatrix} a-b+b-c+c-a & b-c & c-a \\ b-c+c-a+a-b & c-a & a-b \\ c-a+a-b+b-c & a-b & b-c \end{vmatrix}$$

$$= \begin{vmatrix} 0 & b-c & c-a \\ 0 & c-a & a-b \\ 0 & a-b & b-c \end{vmatrix} \quad \because \text{all elements of } C_1 \text{ are zero}$$

$= 0 = R.H.S$

v) $\begin{vmatrix} bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ a & b & c \end{vmatrix}$

Sol. L.H.S Multiplying R_2 by abc and dividing Det by abc

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ abc(\frac{1}{a}) & abc(\frac{1}{b}) & abc(\frac{1}{c}) \\ a & b & c \end{vmatrix}$$

$$= \frac{1}{abc} \begin{vmatrix} bc & ca & ab \\ bc & ac & ab \\ a & b & c \end{vmatrix}$$

$= \frac{1}{abc}(0) \quad (\because R_1 = R_2)$

$= 0 = R.H.S$

vi) $\begin{vmatrix} mn & l & l^2 \\ nl & m & m^2 \\ lm & n & n^2 \end{vmatrix} = \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$

Sol. L.H.S Multiplying R_1 by l, R_2 by m, R_3 by n and dividing det by lmn

$$= \frac{1}{lmn} \begin{vmatrix} lmn & l^2 & l^3 \\ lmn & m^2 & m^3 \\ lmn & n^2 & n^3 \end{vmatrix}$$

$$= \frac{1}{lmn} lmn \begin{vmatrix} 1 & l^2 & l^3 \\ 1 & m^2 & m^3 \\ 1 & n^2 & n^3 \end{vmatrix}$$

$= R.H.S$

vii) $\begin{vmatrix} 2a & 2b & 2c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix} = 0$

Sol L.H.S

$$= 2 \begin{vmatrix} a & b & c \\ a+b & 2b & b+c \\ a+c & b+c & 2c \end{vmatrix}$$

$$= 2 \begin{vmatrix} a & b & c \\ b & b & b \\ c & c & c \end{vmatrix} \quad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$= 2bc \begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$= 2bc(0) = 0$ R.H.S $(\because R_2 = R_3)$

viii) $\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 7 & 2 & 7 \\ 6 & 3 & 5 \\ -3 & 5 & -3 \end{vmatrix} + \begin{vmatrix} 7 & 2 & -1 \\ 6 & 3 & -3 \\ -3 & 5 & 4 \end{vmatrix}$

Sol. R.H.S By adding C_3

$$\begin{vmatrix} 7 & 2 & 7+(-1) \\ 6 & 3 & 5+(-3) \\ -3 & 5 & -3+4 \end{vmatrix}$$

$$\begin{vmatrix} 7 & 2 & 6 \\ 6 & 3 & 2 \\ -3 & 5 & 1 \end{vmatrix} = L.H.S$$

ix) $\begin{vmatrix} -a & 0 & c \\ 0 & a & -b \\ b & -c & 0 \end{vmatrix} = 0$

Sol. Multiplying C_1 by c, C_2 by b, C_3 by a and dividing Det abc

$$= \frac{1}{abc} \begin{vmatrix} -ac & 0 & ac \\ 0 & ab & -ab \\ bc & -bc & 0 \end{vmatrix}$$

$C_1 + (C_2 + C_3)$

$$= \frac{1}{abc} \begin{vmatrix} 0 & 0 & ac \\ 0 & ab & -ab \\ 0 & -bc & 0 \end{vmatrix}$$

$= \frac{1}{abc}(0) \quad (\because \text{All entries of } C_1 \text{ are zero})$

$= R.H.S$

6. Find x if :

Sol

(i) $\begin{vmatrix} 3 & 1 & x \\ -1 & 3 & 4 \\ x & 1 & 0 \end{vmatrix} = -30$

$3\begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ x & 0 \end{vmatrix} + x\begin{vmatrix} -1 & 3 \\ x & 1 \end{vmatrix} = -30$

$3(0-4) - 1(0-4x) + x(-1-3x) = -30$

$-12 + 4x - x - 3x^2 + 30 = 0$

$-3x^2 + 3x + 18 = 0$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow x^2 - 3x + 2x - 6 = 0$

$\Rightarrow x(x-3) + 2(x-3) = 0$

$\Rightarrow (x-3)(x+2) = 0$

$\Rightarrow x - 3 = 0 \qquad x + 2 = 0$

$\Rightarrow x = 3 \qquad x = -2$

ii) $\begin{vmatrix} 1 & x-1 & 3 \\ -1 & x+1 & 2 \\ 2 & -2 & x \end{vmatrix} = 0$

$1\begin{vmatrix} x+1 & 2 \\ -2 & x \end{vmatrix} - (x-1)\begin{vmatrix} -1 & 2 \\ 2 & x \end{vmatrix} + 3\begin{vmatrix} -1 & x+1 \\ 2 & -2 \end{vmatrix} = 0$

$x^2 + x + 4 - (x-1)(-x-4) + 3(2-2x-2) = 0$

$x^2 + x + 4 + (x-1)(x+4) - 6x = 0$

$x^2 + x + 4 + x^2 - x + 4x - 4 - 6x = 0$

$2x^2 - 2x = 0 \Rightarrow 2x(x-1) = 0$

$2x = 0 \Rightarrow x = 0$

$x - 1 = 0 \Rightarrow x = 1$

iii) $\begin{vmatrix} 1 & 2 & 1 \\ 2 & x & 2 \\ 3 & 6 & x \end{vmatrix} = 0$

Sol. $1\begin{vmatrix} x & 2 \\ 6 & x \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & x \end{vmatrix} + 1\begin{vmatrix} 2 & x \\ 3 & 6 \end{vmatrix} = 0$

$x^2 - 12 - 2(2x-6) + 12 - 3x = 0$

$\Rightarrow x^2 - 12 - 4x + 12 + 12 - 3x = 0$

$\Rightarrow x^2 - 7x + 12 = 0$

$\Rightarrow (x-4)(x-3) = 0$

$\Rightarrow x - 4 = 0 \qquad x - 3 = 0$

$x = 4 \qquad x = 3$

7. Evaluate

(i) $\begin{vmatrix} 3 & 4 & 2 & 7 \\ 2 & 5 & 0 & 3 \\ 1 & 2 & -3 & 5 \\ 4 & 1 & -2 & 6 \end{vmatrix}$

Sol.

$\begin{vmatrix} 0 & -2 & 11 & -8 \\ 0 & 1 & 6 & -7 \\ 1 & 2 & -3 & 5 \\ 0 & -7 & 10 & 14 \end{vmatrix} \begin{matrix} R_1 - 3R_3 \\ R_2 - 2R_3 \\ \\ R_4 - 4R_3 \end{matrix}$

Expand by C_1

$= 1(-1)^{1+3}\begin{vmatrix} -2 & 11 & -8 \\ 1 & 6 & -7 \\ -7 & 10 & -14 \end{vmatrix}$

$= \begin{vmatrix} 0 & 23 & -22 \\ 1 & 6 & -7 \\ 0 & 52 & -63 \end{vmatrix} \begin{matrix} R_1 + 2R_2 \\ \\ R_3 + 7R_2 \end{matrix}$

$= -\begin{vmatrix} 23 & -22 \\ 52 & -63 \end{vmatrix}$

$= -(-23 \times 63 + 22 \times 52)$

$= -(-1449 + 1144) = 305$

ii) $\begin{vmatrix} 2 & 3 & 1 & -1 \\ 4 & 0 & 2 & 1 \\ 5 & 2 & -1 & 6 \\ 3 & -7 & 2 & -2 \end{vmatrix}$

$\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & -6 & 2 & 3 \\ 7 & 5 & -1 & 5 \\ -1 & -13 & 2 & 0 \end{vmatrix} \begin{matrix} C_1 - 2C_3 \\ C_2 - 3C_3 \\ C_4 + C_3 \end{matrix}$

Expand by R_1

$= 1(-1)^{1+3}\begin{vmatrix} 0 & -6 & 3 \\ 7 & 5 & 5 \\ -1 & -13 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 3 \\ 7 & 15 & 5 \\ -1 & -13 & 0 \end{vmatrix} \quad C_2 + 2C_3$

$= 3\begin{vmatrix} 7 & 15 \\ -1 & -13 \end{vmatrix} = 3(-91 + 15)$

$= 3(-76) = -228$

iii) $\begin{vmatrix} -3 & 9 & 1 & 1 \\ 0 & 3 & -1 & 2 \\ 9 & 7 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{vmatrix}$

$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 6 & -15 & -3 & 2 \\ 12 & -2 & -2 & 1 \\ -5 & +9 & 9 & -1 \end{vmatrix} \begin{matrix} C_1 + 3C_4 \\ C_2 - 9C_4 \\ C_3 - C_4 \end{matrix}$

$$= 1(-1)^{1+4}\begin{vmatrix} 6 & -15 & -3 \\ 12 & -2 & -2 \\ -5 & 9 & 2 \end{vmatrix}$$

$$= (-1)\, 3\times 2 \begin{vmatrix} 2 & -5 & -1 \\ 6 & -1 & -1 \\ -5 & 9 & 2 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 0 & 0 & -1 \\ 4 & 4 & -1 \\ -1 & -1 & 0 \end{vmatrix} \quad \begin{matrix} C_1+2C_3 \\ C_2-5C_3 \end{matrix}$$

$$= -6(-1)^{1+2}\begin{vmatrix} 4 & 4 \\ -1 & -1 \end{vmatrix}$$

$$= 6(-4+4) = 0$$

8. Show that

$$\begin{vmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix} = (x+3)(x-1)^3$$

Sol. $R_1 + (R_2+R_3+R_4)$

$$= \begin{vmatrix} x+3 & x+3 & x+3 & x+3 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3)\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{vmatrix}$$

$$= (x+3)\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x-1 & 0 & 0 \\ 1 & 0 & x-1 & 0 \\ 1 & 0 & 0 & x-1 \end{vmatrix} \quad \begin{matrix} C_2-C_1 \\ C_3-C_1 \\ C_4-C_1 \end{matrix}$$

Expand by R_1

$$= (x+3)\begin{vmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$$= (x+3)(x-1)\begin{vmatrix} x-1 & 0 \\ 0 & x-1 \end{vmatrix}$$

$$= (x+3)(x-1)\left[(x-1)^2 - 0\right]$$

$$= (x+3)(x-1)^3 = R.H.S$$

9. Find $|AA^t|$, $|A^tA|$

i) if $A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$ $A^t = \begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$

$$AA^t = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4+1 & 6+2-3 \\ 6+2-3 & 4+1+9 \end{pmatrix} = \begin{pmatrix} 14 & 5 \\ 5 & 14 \end{pmatrix}$$

$$|AA^t| = \begin{vmatrix} 14 & 5 \\ 5 & 14 \end{vmatrix} = 196 - 25 = 171$$

$$A^tA = \begin{pmatrix} 3 & 2 \\ 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4 & 6+2 & -3+6 \\ 6+2 & 4+1 & -2+3 \\ -3+6 & -2+3 & 1+9 \end{pmatrix}$$

$$|A^tA| = \begin{vmatrix} 13 & 8 & 3 \\ 8 & 5 & 1 \\ 3 & 1 & 10 \end{vmatrix}$$

$$= 13\begin{vmatrix} 5 & 1 \\ 1 & 10 \end{vmatrix} - 8\begin{vmatrix} 8 & 1 \\ 3 & 10 \end{vmatrix} + 3\begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$$

$$= 13(50-1) - 8(80-3) + 3(8-15)$$

$$= 637 - 616 - 21 = 0$$

ii) $$AA^t = \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+16 & 6+4 & 3+4 & 6+12 \\ 6+4 & 4+1 & 2+1 & 4+3 \\ 3+4 & 2+1 & 1+1 & 2+3 \\ 6+12 & 4+3 & 2+3 & 4+9 \end{pmatrix}$$

$$|AA^t| = \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 8 & 7 & 5 & 13 \end{vmatrix}$$

$$= \begin{vmatrix} 25 & 10 & 7 & 18 \\ 10 & 5 & 3 & 7 \\ 7 & 3 & 2 & 5 \\ 1 & -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 35 & 7 & -7 \\ 0 & 15 & 3 & -3 \\ 0 & 10 & 2 & -2 \\ 1 & -1 & 0 & 1 \end{vmatrix} \quad \begin{matrix} R_1-25R_4 \\ R_2-10R_4 \\ R_3-7R_4 \end{matrix}$$

Expand by C_1

$$= -1 \begin{vmatrix} 35 & 7 & -7 \\ 5 & 3 & -3 \\ 0 & 2 & -2 \end{vmatrix}$$

$$\because \begin{vmatrix} 35 & 7 & 7 \\ 5 & 3 & 3 \\ 0 & 2 & 2 \end{vmatrix} = 0 = R.H.S \quad (\because C_2 = C_3)$$

$$A^t A = \begin{pmatrix} 3 & 2 & 1 & 2 \\ 4 & 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 2 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 9+4+1+4 & 12+2+1+6 \\ 12+2+1+6 & 16+1+1+9 \end{pmatrix}$$

$$|A^t A| = \begin{vmatrix} 18 & 21 \\ 21 & 27 \end{vmatrix} = 18 \times 27 - 21 \times 21$$

$$= 486 - 441 = 45.$$

10. If A is a square matrix of order 3. Then show that

$$|KA| = K^3 |A|$$

Sol. let
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$KA = K \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & Ka_{13} \\ Ka_{21} & Ka_{22} & Ka_{23} \\ Ka_{31} & Ka_{32} & Ka_{33} \end{vmatrix}$$

$$= K \cdot K \cdot K \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|KA| = K^3 |A|$$

11. Find the values of λ if A and B are singular

(i)
$$|A| = \begin{vmatrix} 4 & \lambda & 3 \\ 7 & 3 & 6 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= 4(3-18) - \lambda(7-12) + 3(21-6)$$
$$= -60 + 5\lambda + 45 = 5\lambda - 15$$

For singular $|A| = 0$

$$5\lambda - 15 = 0 \Rightarrow \lambda = 3$$

ii)
$$|B| = \begin{vmatrix} 5 & 1 & 2 & 0 \\ 8 & 2 & 5 & 1 \\ 3 & 2 & 0 & 1 \\ 2 & \lambda & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 1 & 0 & 0 \\ -2 & 2 & 1 & 1 \\ -7 & 2 & 4 & 1 \\ 2-5\lambda & \lambda & -1-2\lambda & 3 \end{vmatrix} \quad \begin{matrix} C_1 - 5C_2 \\ C_3 - 2C_2 \end{matrix}$$

$$= 1(-1)^{1+2} \begin{vmatrix} -2 & 1 & 1 \\ -7 & -4 & 1 \\ 2-5\lambda & -1-2\lambda & 3 \end{vmatrix}$$

$$= -1 \begin{vmatrix} 0 & 0 & 1 \\ -5 & -5 & 1 \\ 8-5\lambda & -4-2\lambda & 3 \end{vmatrix} \quad \begin{matrix} C_1 + 2C_3 \\ C_2 - C_3 \end{matrix}$$

$$= -1 \cdot 1 \begin{vmatrix} -5 & -5 \\ 8-5\lambda & -4-2\lambda \end{vmatrix}$$

$$= -\left[20 + 10\lambda + 40 - 25\lambda\right]$$
$$= 15\lambda - 60$$

For singular $|B| = 0$

$$15\lambda - 60 = 0 \Rightarrow \lambda = 4$$

12.

(i)
$$|A| = \begin{vmatrix} 1 & 0 & 3 \\ 3 & 1 & -1 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= 1(4+2) - 0 + 3(6-0)$$
$$= 6 + 18 = 24 \neq 0$$

A is non singular

ii)
$$|B| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 0 \\ 2 & -3 & 5 \end{vmatrix}$$

$$= 2(5+0) - 3(5-0) - 1(-3-2)$$
$$= 10 - 15 + 5 = 0$$

B is singular

iii)
$$|C| = \begin{vmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & -1 & -3 \\ 2 & 3 & 1 & 2 \\ 3 & -1 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & -3 & -2 \\ 2 & 1 & -3 & 4 \\ 3 & -4 & -3 & 7 \end{vmatrix} \quad \begin{matrix} C_2 - C_1 \\ C_3 - 2C_1 \\ C_4 + C_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & -3 & -2 \\ 1 & -3 & 4 \\ -4 & -3 & 7 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 6 \\ -4 & -15 & -1 \end{vmatrix} \quad \begin{matrix} C_2+3C_1 \\ C_3+2C_1 \end{matrix}$$

$= 1(0+90) = 90 \neq 0$

C is non Singular

13. $A = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$

Show that $A^{-1}A = I_3$

Sol. $|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$

Sol

Cofactors of $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$A_{ij} = (-1)^{i+j} M_{ij}$

$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ -4 & 1 \end{vmatrix} = (-1)^2(-1+12) = 11$

$A_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = (-1)^3(1-6) = -(-5) = 5$

$A_{13} = (-1)^{1+3}\begin{vmatrix} 1 & -1 \\ 2 & -4 \end{vmatrix} = (-1)^4(-4+2) = 1(-2) = -2$

$A_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 0 \\ -4 & 1 \end{vmatrix} = (-1)^3(1+0) = -1$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 0 \\ 2 & 1 \end{vmatrix} = (-1)^4(2-0) = 2$

$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 2 & -4 \end{vmatrix} = (-1)^5(-8-2) = 10$

$A_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = (-1)^4(3+0) = 3$

$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} = (-1)^5(6-0) = -6$

$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = (-1)^6(-2-1) = -3$

Cofactors of $A = \begin{pmatrix} 11 & 5 & -2 \\ -1 & 2 & 10 \\ 3 & -6 & -3 \end{pmatrix}$

$\text{Adj} A = (\text{Cofactors of } A)^t = \begin{pmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{pmatrix}$

$|A| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{vmatrix}$

$= 2(-1+12) - 1(1-6) + 0$

$= 22 + 5 = 27$

$A^{-1} = \frac{1}{|A|} \text{Adj} A$

$= \frac{1}{27}\begin{pmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{pmatrix}$

$A^{-1}A = \frac{1}{27}\begin{pmatrix} 11 & -1 & 3 \\ 5 & 2 & -6 \\ -2 & 10 & -3 \end{pmatrix}\begin{pmatrix} 2 & 1 & 0 \\ 1 & -1 & 3 \\ 2 & -4 & 1 \end{pmatrix}$

$= \frac{1}{27}\begin{pmatrix} 22-1+6 & 11+1-12 & 0-3+3 \\ 10+2-12 & 5-2+24 & 0+6-6 \\ -4+10-6 & -2-10+12 & 0+30-3 \end{pmatrix}$

$= \frac{1}{27}\begin{pmatrix} 27 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

R.H.S

14. Verify That $(AB)^{-1} = B^{-1}A^{-1}$

(i) $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$

Sol $AB = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}$

$= \begin{pmatrix} -3+8 & 1-2 \\ 3+0 & -1+0 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 3 & -1 \end{pmatrix}$

$|AB| = \begin{vmatrix} 5 & -1 \\ 3 & -1 \end{vmatrix} = -5+3 = -2$

$(AB)^{-1} = \frac{1}{|AB|} \text{Adj} AB = \frac{1}{-2}\begin{pmatrix} -1 & +1 \\ -3 & 5 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix} \ldots (i)$

R.H.S

$|A| = \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = 0+2 = 2$

$A^{-1} = \frac{1}{|A|} \text{Adj} A = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$

$|B| = \begin{vmatrix} -3 & 1 \\ 4 & -1 \end{vmatrix} = 3-4 = -1$

$B^{-1} = \frac{1}{|B|} \text{Adj} B = \frac{1}{-1}\begin{pmatrix} -1 & -1 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}$

$B^{-1}A^{-1} = \frac{1}{+2}\begin{pmatrix} 1 & 1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 0+1 & -2+1 \\ 0+3 & -8+3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 3 & -5 \end{pmatrix}$ ...(ii)

(i) & (ii) $(AB)^{-1} = B^{-1}A^{-1}$

ii) $A=\begin{pmatrix}5 & 1\\ 2 & 2\end{pmatrix}$ $B=\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}$

$AB=\begin{pmatrix}5 & 1\\ 2 & 2\end{pmatrix}\begin{pmatrix}4 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}20+2 & 15+1\\ 8+4 & 6+2\end{pmatrix}$

$AB=\begin{pmatrix}22 & 16\\ 10 & 8\end{pmatrix}$

$|AB|=\begin{vmatrix}22 & 16\\ 12 & 8\end{vmatrix}=176-192=-16$

$(AB)^{-1}=\frac{1}{|AB|}\,Adj\,AB=\frac{1}{-16}\begin{pmatrix}8 & -16\\ -12 & 22\end{pmatrix}$

For R.H.S $B^{-1}A^{-1}$

$|A|=\begin{vmatrix}5 & 1\\ 2 & 2\end{vmatrix}=10-2=8$

$A^{-1}=\frac{1}{|A|}\,Adj\,A=\frac{1}{8}\begin{pmatrix}2 & -1\\ -2 & 5\end{pmatrix}$

$|B|=\begin{vmatrix}4 & 3\\ 2 & 1\end{vmatrix}=4-6=-2$

$B^{-1}=\frac{1}{|B|}\,Adj\,B=\frac{1}{-2}\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}$

$B^{-1}A^{-1}=\frac{1}{-16}\begin{pmatrix}1 & -3\\ -2 & 4\end{pmatrix}\begin{pmatrix}2 & -1\\ -2 & 5\end{pmatrix}$

$=\frac{1}{-16}\begin{pmatrix}2+6 & -1-15\\ -4-8 & 2+20\end{pmatrix}=\frac{1}{-16}\begin{pmatrix}8 & -16\\ -12 & 22\end{pmatrix}$

L.H.S = R.H.S

15. Verify That
$(AB)^t=B^tA^t$ if

Sol. $A=\begin{pmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{pmatrix}$ $B=\begin{pmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{pmatrix}$

$AB=\begin{pmatrix}1 & -1 & 2\\ 0 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 3 & 2\\ 0 & -1\end{pmatrix}$

$=\begin{pmatrix}1-3+0 & 1-2-2\\ 0+9+0 & 0+6-1\end{pmatrix}=\begin{pmatrix}-2 & -3\\ 9 & 5\end{pmatrix}$

$(AB)^t=\begin{pmatrix}-2 & 9\\ -3 & 5\end{pmatrix}$

R.H.S

$B^tA^t=\begin{pmatrix}1 & 3 & 0\\ 1 & 2 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 3\\ 2 & 1\end{pmatrix}$

$=\begin{pmatrix}1-3+0 & 0+9+0\\ 1-2-2 & 0+6-1\end{pmatrix}=\begin{pmatrix}-2 & 9\\ -3 & 5\end{pmatrix}$

L.H.S = R.H.S

16. If $A=\begin{pmatrix}2 & -1\\ 3 & 1\end{pmatrix}$
verify That $(A^{-1})^t=(A^t)^{-1}$

Sol. L.H.S

$|A|=\begin{vmatrix}2 & -1\\ 3 & 1\end{vmatrix}=2+3=5$

$A^{-1}=\frac{1}{|A|}\,Adj\,A=\frac{1}{5}\begin{pmatrix}1 & 1\\ -3 & 2\end{pmatrix}$

$(A^{-1})^t=\frac{1}{5}\begin{pmatrix}1 & -3\\ 1 & 2\end{pmatrix}$ ---- (i)

For R.H.S

$A^t=\begin{pmatrix}2 & 3\\ -1 & 1\end{pmatrix}$ $|A^t|=2+3=5$

$(A^t)^{-1}=\frac{1}{|A^t|}\,Adj\,A^t=\frac{1}{5}\begin{pmatrix}1 & -3\\ 1 & 2\end{pmatrix}$ .. (ii)

L.H.S = R.H.S

17. If A and B are non Singular

(i) $(AB)^{-1}=B^{-1}A^{-1}$

Sol. Let $(AB)(B^{-1}A^{-1})$

$=A(BB^{-1})A^{-1}=AIA^{-1}=AA^{-1}=I$

Now $(B^{-1}A^{-1})(AB)$

$=B^{-1}(A^{-1}A)B=B^{-1}IB=B^{-1}B=I$

Thus $(AB)(B^{-1}A^{-1})=(B^{-1}A^{-1})(AB)=I$

It shows That AB is The inverse of $B^{-1}A^{-1}$

$\Rightarrow (AB)^{-1}=B^{-1}A^{-1}$

ii) $(A^{-1})^{-1}=A$

Sol. Since $A^{-1}A=I$
and $AA^{-1}=I$

It shows That A^{-1} be the inverse of A

$(A^{-1})^{-1}=A$
