$= (1)(1).......\text{n factors} \qquad \because \omega^3 = 1$

$= (1)^n$

$= 1$

= R.H.S.

Hence proved.

Roots and co-efficient of a quadratic equation:

We know that $\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $\dfrac{-b-\sqrt{b^2-4ac}}{2a}$ are roots of the equation $ax^2 + bx + c = 0$ where a, b are coefficients of x^2 and x respectively. While c is the constant term.

Relation between roots and co-efficient of a quadratic equation:

If $\alpha = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ $\beta = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$,

then we can find the sum and the product of the roots as follows.

Sum of the' roots = $\alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

Product of the roots = $\alpha\beta$

$$= \left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(\frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{(-b)^2-\left(\sqrt{b^2-4ac}\right)^2}{4a^2} = \frac{b^2-(b^2-4ac)}{4a^2}$$

$$= \frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

If we denote the sum of roots and product of roots by S and P respectively, then

$$S = -\frac{b}{a} = -\frac{\text{Co-efficient of } x}{\text{Co-efficient of } x^2} \quad \text{and} \quad P = -\frac{c}{a} = -\frac{\text{Constant term}}{\text{Co-efficient of } x^2}$$

SOLVED EXERCISE 2.3

1. **Without solving, find the sum and the product of the following quadratic equations.**

(i) $x^2 - 5x + 3 = 0$

Solution:

$$x^2 - 5x + 3 = 0$$

Here $a = 1, b = -5, c = 3$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} - \frac{-5}{1} = 5$

And product of roots $= \frac{c}{a} = \frac{3}{1} = 3$

(ii) $3x^2 + 7x - 11 = 0$

Solution:

$$3x^2 + 7x - 11 = 0$$

Here $a = 3, b = 7, c = -11$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{7}{3}$

And product of roots $= \propto\beta = \frac{c}{a} = -\frac{11}{3}$

(iii) $px^2 - qx + r = 0$

Solution:

$$px^2 - qx + r = 0$$

Here $a = p, b = -q, c = r$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{(-q)}{p} = \frac{q}{p}$

And product of roots $= \propto\beta = \frac{c}{a} = -\frac{r}{p}$

(iv) $(a + b)x^2 - ax + b = 0$

Solution:

$$(a + b)x^2 - ax + b = 0$$

Here $a = a + b = -a, c = b$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = - = -\frac{b}{a} = -\frac{(-a)}{a+b} = \frac{a}{a+b}$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{b}{a+b}$

(v) $(l + m)x^2 + (m + n)x + n - l = 0$

Solution:

$$(l + m)x^2 + (m + n)x + n - l = 0$$

Here $a = l + m, b = m + n, c = n - l$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{m+n}{l+m}$

And product of roots $= \propto\beta = \frac{c}{a} = -\frac{n-l}{l+m}$

(vi) $7x^2 - 5mx + 9n = 0$

Solution:

$7x^2 - 5mx + 9n = 0$

Here $a = 7, b = -5m, c = 9n$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{(-5m)}{7} = \frac{5m}{7}$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{9n}{7}$

2. Find the value of k, if

(i) Sum of the roots of the equation $2kx^2 - 3x + 4k = 0$ is twice the product of the roots.

Solution:

$2kx^2 - 3x + 4k = 0$

Here $a = 2k, b = -3, c = 4k$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{(-3)}{2k} = \frac{3}{2k}$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{4k}{2k} = 2$

As sum of the roots is twice the product of the roots, so

$\propto + \beta = 2\propto\beta$

$\frac{3}{2k} = 2(2)$

$\frac{3}{2k} = 4$

or $k = \frac{3}{8}$

(ii) Sum of the roots of the equation $x^2 + (3k - 7)x + 5k = 0$ is $\frac{3}{2}$ times the product of the roots.

Solution:

$x^2 + (3k - 7)x + 5k = 0$

Here $a = 1, b = 3k - 7, c = 5k$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{3k-7}{1} = -3k + 7$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{5k}{1} = 5k$

As sum of the roots is twice the product of the roots, so

$$\propto + \beta = \frac{3}{2}\propto\beta$$

$$-3k + 7 = \frac{3}{2}(5k)$$

$$-3k + 7 = \frac{15k}{2}$$

$$-3k - \frac{15k}{2} = -7$$

$$\frac{-6k - 15k}{2} = -7$$

$$\frac{-21k}{2} = -7$$

$$k = (-7)\left(-\frac{2}{21}\right)$$

$$k = \frac{2}{3}$$

3. Find k, if

(i) Sum of the squares of the roots of the equation $4kx^2 + 3kx - 8 = 0$ is 2.

Solution:

$4kx^2 + 3kx - 8 = 0$ is 2

Here $a = 4k, c = 3k, c = -8$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{3k}{4k} = -\frac{3}{4}$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{-8}{4k}$

As sum of the roots is twice the product of the roots is 2, so,

$\propto^2 + \beta^2 = 2$

$(\propto + \beta)^2 - 2\propto\beta = 2$ $\quad\because (\propto + \beta)^2 = \propto^2 + \beta^2 + 2\propto\beta$

$\left(-\frac{3}{4}\right)^2 - 2\left(\frac{-8}{4k}\right) = 2$ $\quad \propto^2 + \beta^2 = (\propto + \beta)^2 - 2\propto\beta$

$$\frac{9}{16}+\frac{16}{4k}=2$$

$$\frac{16}{4k}=2-\frac{9}{16} \qquad \Rightarrow \frac{16}{4k}=\frac{32-9}{16}$$

$$\frac{16}{4k}=\frac{23}{16} \qquad \Rightarrow 23\times 4k=16\times 16$$

$$k=\frac{16\times 16}{23\times 4} \qquad \Rightarrow k=\frac{64}{23}$$

(ii) Sum of the squares of the roots of the equation $x^2-2kx+(2k+1)=0$ is 6.

Solution:

$x^2-2kx+(2k+1)=0$ is 6

Here $a=1, b=-2k, c=2k+1$

Let $\propto$ and β be the roots of the given equation

Then sum of roots $=\propto+\beta=-\frac{b}{a}=-\frac{(-2k)}{1}=2k$

And product of roots $=\propto\beta=\frac{c}{a}=\frac{2k+1}{1}=2k+1$

As sum of the roots is twice the product of the roots is 2, so,

$\propto^2+\beta^2=6$

$(\propto+\beta)^2-2\propto\beta=6$

$(2k)^2-2(2k+1)=6$

$4k^2-4k-2=6$

$4k^2-4k-2-6=0$

$4k^2-4k-8=0$

$4(k^2-k-2)$

$\Rightarrow \quad k^2-k-2=0$

$k^2-2k+k-2=0$

$k(k-2)+1(k-2)=0$

$(k+1)(k-2)=0$

Either $k+1=0$ or $k-2=0$

$k=-1 \qquad k=2$

4. Find p, if

(i) The roots of the equation $x^2-x+p^2=0$ differ by unity.

Solution:

$x^2-x+p^2=0$

Here $a=1, b=-1, c=p^2$

Let $\propto$ and $\propto-1$ be the roots of given equation.

Then $\propto+\propto-1=-\frac{b}{a}$ and $\propto(\propto-1)=\frac{c}{a}$

$$2\alpha - 1 = -\frac{(-1)}{1} \qquad \alpha^2 - 1 = \frac{p^2}{1}$$

$$2\alpha - 1 = 1 \qquad \alpha^2 - 1 = p$$

$$2\alpha = 1 + 1 \qquad \text{put} \quad \alpha = 1 \text{ in above eq., we get}$$

$$2\alpha = 2 \qquad (1)^2 - 1 = p$$

$$\Rightarrow \quad \alpha = 1 \qquad 1 - 1 = p$$

$$\text{or} \quad p = 0$$

(ii) the roots of the equation $x^2 + 3x + p - 2 = 0$ differ by 2.

Solution:

$$x^2 + 3x + p - 2 = 0$$

Here $a = 1, b = 3, c = p - 2$

Let α and $\alpha - 2$ be the roots of given equation.

Then $\alpha + \alpha - 2 = -\frac{b}{a}$ and $\alpha(\alpha - 2) = \frac{c}{a}$

$$2\alpha - 2 = -\frac{3}{1} \qquad \alpha^2 - 2 = \frac{p-2}{1}$$

$$2\alpha - 2 = -3 \qquad \alpha^2 - 2 = p - 2$$

$$2\alpha = -3 + 2 \qquad \alpha^2 - 2\alpha = p - 2$$

$$\alpha = -\frac{1}{2} \qquad \text{put} \quad \alpha = -\frac{1}{2} \text{ in above eq., we get}$$

$$\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) = p - 2$$

$$\frac{1}{4} + 1 = p - 2$$

$$\frac{1+4}{4} = p - 2$$

$$\frac{5}{4} = p - 2$$

$$p = \frac{5}{4} + 2$$

$$\text{or} \quad p = \frac{13}{4}$$

5. Find m, if

(i) The roots of the equation $x^2 - 7x + 3m - 5 = 0$ satisfy the relation $3\alpha + 2\beta = 4$

Solution:

$$x^2 - 7x + 3m - 5 = 0$$

Here $a = 1, b = -7, c = 3m - 5$

Let $\propto$ and β be the roots of given equation.

Then sum of roots $= \propto + \beta = -\frac{b}{a} = -\frac{(-7)}{1} = 7$

And product of roots $= \propto\beta = \frac{c}{a} = \frac{3m}{1} = 3m - 5$

Now $\propto + \beta = 7$ and $\propto\beta = 3m - 5$ _______ (ii)

$\beta = 7 - \propto$ _______ (i)

Since $3\propto + 2\beta = 4$ _______ (iii)

Put β in eq (iii), we have

$3\propto + 2(7 - \propto) = 4$

$3\propto + 14 - 2\propto = 4$

$3\propto - 2\propto + 14 = 4$

$\propto = 4 - 14$

$\propto = -10$

Put $\propto = -10$ in eq. (i), we get

$\beta = 7 + 10$

$\beta = +17$

Put $\propto = 10, \beta = -4$ in eq. (ii), we get

$(-10)(17) = 3m - 5$

$5 - 170 = 3m$

or $m = -165$

$m = -55$

(ii) The roots of the equation $x^2 + 7x + 3m - 5 = 0$ satisfy the relation $3\alpha - 2\beta = 4$

Solution:

$x^2 + 7x + 3m - 5 = 0$

Here $a = 1, b = 7, c = 3m - 5$

Let $\propto$ and β be the roots of given equation.

Then sum of roots $= \propto + \beta = -\frac{b}{a}$ and $\propto\beta = \frac{c}{a}$

$\propto + \beta = -\frac{7}{1}$ $\qquad \propto\beta = \frac{3m - 5}{1}$

$\propto + \beta = -7$ $\qquad \propto\beta = 3m - 5$ _____ (ii)

$\beta = -7 - \propto$ _______ (i)

Since $3\propto + 2\beta = 4$ _______ (iii)

Put β in eq. (iii), we have

$3\propto + 2(-7 - \propto) = 4$

$3\propto + 14 + 2\propto = 4$

$3\propto + 2\propto = 4 - 14$

$5\propto = -10$

$\propto = -2$

Put $\propto = -2$ in eq. (i), we get

$\beta = -7 - (-2)$
$\beta = -7 + 2$
$\beta = -5$

Put $\propto = -2$ and $\beta = -5$ in eq. (iii), we get
$(-2)(-5) = 3m - 5$
$10 = 3m - 5$

or $3m = 10 + 5$

$\Rightarrow$ $m = 5$

(iii) The roots of the equation $3x^2 - 2x + 7m + 2 = 0$ satisfy the relation $7\alpha - 3\beta =$

Solution:

$3x^2 - 2x + 7m + 2 = 0$

Here $a = 3, b = 2, c = 7m + 2$

Let $\propto$ and β be the roots of given equation.

Then sum of roots $= \propto + \beta = -\frac{b}{a}$ and $\propto\beta = \frac{c}{a}$

$\propto + \beta = -\frac{(-2)}{3}$ $\qquad \propto\beta = \frac{3m+2}{3}$ _____(ii)

$\propto + \beta = \frac{2}{3}$

$\beta = \frac{2}{3} - \propto$ ______(i)

Since $7\propto - 3\beta = 18$ ______(iii)

Put $\beta = \frac{2}{3} - \propto$ in eq (iii), we get

$7\propto - 3\left(\frac{2}{3} - \propto\right) = 18$

$7\propto - 2 + 3\propto = 18$

$7\propto - 2 + 3\propto = 18 + 2$

$10\propto = 20$

$\propto = 2$

Put $\propto = 2$ in eq. (i), we get

$\beta = \frac{2}{3} - 2$

$\beta = \frac{2-6}{3}$

$\beta = -\frac{4}{3}$

Put $\propto = 2$ and $\beta = -\frac{4}{3}$ in eq. (ii), we get

$$(2)\left(-\frac{4}{3}\right)=\frac{7m+2}{3}$$

$$-\frac{8}{3}=\frac{7m+2}{3}$$

$$-\frac{8}{3}\times 3=7m+2$$

$$-8=7m+2$$

or $\quad 7m=-8-2$

$$7m=-10$$

$$m=-\frac{10}{7}$$

6. Find m, if sum and product of the roots of the following equations is equal to a given number λ.

Solution:

(i) $(2m+3)x^2+(7m-5)x+(3m-10)=0$

Solution:

$$(2m+3)x^2+(7m-5)x+(3m-10)=0$$

Here $\quad a=2m+3,\ b=7m-5,\ c=3m-10$

Let $\quad \alpha$ and β be the roots of given equation

Then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

$$\alpha+\beta=-\frac{7m-5}{2m+3} \qquad \alpha\beta=\frac{3m-10}{2m+3}$$

As $\quad \alpha+\beta=\alpha\beta=\lambda$

So $\quad \lambda=-\frac{7m-5}{2m+3}$ ——(i) and $\lambda=\frac{3m-10}{2m+3}$ ——(ii)

Comparing eq. (i) and eq (ii). we get

$$-\frac{7m-5}{2m+3}=\frac{3m-10}{2m+3}$$

$$-(7m-5)(2m+3)=(2m+3)(3m-10)$$
$$-(14m^2+21m-10-15)=6m^2-20m+9m-30$$
$$-(14m^2+11m-15)=6m^2-11m-30$$
$$-14m^2-11m+15=6m^2-11m-30$$
$$-14m^2-6m^2-11m+11m+15+30=0$$
$$-20m^2+45=0$$
$$-20m^2=-45$$
$$m^2=\frac{45}{20}$$
$$m^2=\frac{9}{4}$$
$$\Rightarrow \quad m=\frac{3}{2}$$

(ii) $4x^2-(3+5m)x-(9m-17)=0$

Solution:

$$4x^2-(3+5m)x-(9m-17)=0$$

Here $a=4,\quad b=-(3+5m),\ c=-(9m-17)$

Let $\propto$ and β be the roots of given equation

Then $\propto+\beta=-\frac{b}{a}$ and $\propto\beta=\frac{c}{a}$

$$\propto+\beta=-\frac{[-(3+5m)]}{4} \qquad \propto\beta=\frac{9m-17}{4}$$

$$\propto+\beta=\frac{3+5m}{4}$$

As $\propto+\beta=\propto\beta=\lambda$

So $\lambda=-\frac{3+5m}{4}$ ______(i) and $\lambda=\frac{9m-17}{4}$ ______(ii)

Comparing eq. (i) and eq (ii), we get

$$\frac{3+5m}{4}=\frac{9m-17}{4}$$
$$4(3+5m)=-4(9m-17)$$
$$3+5m=-(9m-17)$$
$$\Rightarrow \quad 3+5m=-(9m+17)$$
$$5m+9m=17-3$$
$$14m=14$$
$$\Rightarrow \quad m=1$$

Symmetric functions of the roots of a quadratic equation:
Define symmetric functions of the roots of a quadratic equation:
Definition:
Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.
For example, if

$$f(\alpha, \beta) = \alpha^2 + \beta^2 \text{, then}$$
$$f(\beta, \alpha) = \beta^2 + \alpha^2 = \alpha^2 + \beta^2 \qquad (\because \beta^2 + \alpha^2 = \alpha^2 + \beta^2)$$
$$= f(\alpha, \beta)$$

SOLVED EXERCISE 2.4

1. If α, β are the roots of the equation $x^2 + px + q = 0$, then evaluate

(i) $\alpha^2 + \beta^2$

Solution:

$\alpha^2 + \beta^2$

$x^2 + px + q = 0$

Here $a = 1, b = p, c = q$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$= -\frac{p}{1}$ $\qquad = \frac{q}{1}$

$= -p$ $\qquad = q$

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$

(ii) $\alpha^3\beta + \alpha\beta^3$

Solution:

$\alpha^3\beta + \alpha\beta^3$

$x^2 + px + q = 0$

Here $a = 1, b = p, c = q$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$= -\frac{p}{1}$ $\qquad = \frac{q}{1}$

$= -p$ $\qquad = q$

Now $\alpha^3 + \beta^3 = \alpha\beta\,(\alpha^2 + \beta^2) - 2\alpha\beta$