

EXERCISE 2.3

Question # 1

Let $y = x^4 + 2x^3 + x^2$

Differentiating w.r.t. x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^4 + 2x^3 + x^2) \\ &= \frac{d}{dx}x^4 + 2\frac{d}{dx}x^3 + \frac{d}{dx}x^2 = 4x^{4-1} + 2(3x^{3-1}) + 2x^{2-1} \\ &= 4x^3 + 6x^2 + 2x\end{aligned}$$

Question # 2

Let $y = x^{-3} + 2x^{-\frac{3}{2}} + 3$

Diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^{-3} + 2x^{-\frac{3}{2}} + 3\right) \\ &= \frac{d}{dx}x^{-3} + 2\frac{d}{dx}x^{-\frac{3}{2}} + \frac{d}{dx}(3) = -3x^{-3-1} + 2\left(-\frac{3}{2}x^{-\frac{3}{2}-1}\right) + 0 \\ \Rightarrow \frac{dy}{dx} &= -3x^{-4} - 3x^{-\frac{5}{2}} \quad \text{or} \quad \frac{dy}{dx} = -3\left(\frac{1}{x^4} + \frac{1}{x^{5/2}}\right)\end{aligned}$$

Question # 3

Let $y = \frac{a+x}{a-x}$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{a+x}{a-x}\right) = \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2} \\ &= \frac{(a-x)(0+1) - (a+x)(0-1)}{(a-x)^2} = \frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2} \\ &= \frac{a-x+a+x}{(a-x)^2} = \frac{2a}{(a-x)^2} \quad \text{Answer}\end{aligned}$$

Question # 4

Let $y = \frac{2x-3}{2x+1}$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{2x-3}{2x+1}\right) \\ &= \frac{(2x+1)\frac{d}{dx}(2x-3) - (2x-3)\frac{d}{dx}(2x+1)}{(2x+1)^2} \\ &= \frac{(2x+1)(2-0) - (2x-3)(2+0)}{(2x+1)^2} = \frac{(2x+1)(2) - (2x-3)(2)}{(2x+1)^2} \\ &= \frac{2(2x+1-2x+3)}{(2x+1)^2} = \frac{2(4)}{(2x+1)^2} = \frac{8}{(2x+1)^2} \quad \text{Answer}\end{aligned}$$

Question # 5

$$\begin{aligned}\text{Let } y &= (x-5)(3-x) \\ &= 3x - x^2 - 15 + 5x = -x^2 + 8x - 15\end{aligned}$$

Now

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dx}(-x^2 + 8x - 15) \\ &= \frac{dy}{dx}(-x^2) + 8\frac{d}{dx}(x) - \frac{d}{dx}(15) = -2x^{2-1} + 8(1) - 0 = -2x + 8 \quad \text{Answer}\end{aligned}$$

Question # 6

$$\begin{aligned}\text{Let } y &= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 \\ &= (\sqrt{x})^2 + \left(\frac{1}{\sqrt{x}}\right)^2 - 2(\sqrt{x})\left(\frac{1}{\sqrt{x}}\right) = x + \frac{1}{x} - 2 = x + x^{-1} - 2\end{aligned}$$

Now diff. w.r.t x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x + x^{-1} - 2) = \frac{d}{dx}(x) + \frac{d}{dx}(x^{-1}) - \frac{d}{dx}(2) \\ &= 1 + (-1 \cdot x^{-1-1}) - 0 = 1 - x^{-2} \\ &= 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} \quad \text{Answer}\end{aligned}$$

Question # 7

$$\begin{aligned}\text{Consider } y &= \frac{(1 + \sqrt{x})(x - x^{3/2})}{\sqrt{x}} \\ &= \frac{(1 + \sqrt{x})}{\sqrt{x}} \cdot x(1 - x^{\frac{1}{2}}) = \frac{x(1 + \sqrt{x})(1 - \sqrt{x})}{\sqrt{x}} \quad \text{Since } x^{\frac{3}{2}} = x^{1+\frac{1}{2}} \\ &= \frac{(\sqrt{x})^2(1 - (\sqrt{x})^2)}{\sqrt{x}} = \sqrt{x}(1 - x) = x^{\frac{1}{2}}(1 - x) = x^{\frac{1}{2}} - x^{\frac{3}{2}}\end{aligned}$$

Now

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) \\ &= \frac{1}{2}x^{\frac{1}{2}-1} - \frac{3}{2}x^{\frac{3}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}} = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 3\sqrt{x}\right) \quad \text{Answer}\end{aligned}$$

Question # 8

$$\text{Let } y = \frac{(x^2 + 1)^2}{x^2 - 1}$$

Differentiating w.r.t. x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(\frac{(x^2 + 1)^2}{x^2 - 1}\right) \\ &= \frac{(x^2 - 1) \frac{d}{dx}(x^2 + 1)^2 - (x^2 + 1)^2 \frac{d}{dx}(x^2 - 1)}{(x^2 - 1)^2}\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{dy}{dx} &= \frac{(x^2-1) 2(x^2+1)^{2-1} \frac{d}{dx}(x^2+1) - (x^2+1)^2 (2x)}{(x^2-1)^2} \\
&= \frac{(x^2-1) 2(x^2+1)(2x) - (x^2+1)^2 (2x)}{(x^2-1)^2} \\
&= \frac{2x(x^2+1)[2(x^2-1) - (x^2+1)]}{(x^2-1)^2} = \frac{2x(x^2+1)[2x^2-2-x^2-1]}{(x^2-1)^2} \\
&= \frac{2x(x^2+1)(x^2-3)}{(x^2-1)^2} \quad \text{Answer}
\end{aligned}$$

Question # 9

Let $y = \frac{x^2+1}{x^2-3}$

Differentiating w.r.t. x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2+1}{x^2-3} \right) = \frac{(x^2-3) \frac{d}{dx}(x^2+1) - (x^2+1) \frac{d}{dx}(x^2-3)}{(x^2-3)^2} \\
&= \frac{(x^2-3)(2x) - (x^2+1)(2x)}{(x^2-3)^2} = \frac{2x(x^2-3-x^2-1)}{(x^2-3)^2} \\
&= \frac{2x(-4)}{(x^2-3)^2} = \frac{-8x}{(x^2-3)^2} \quad \text{Answer}
\end{aligned}$$

Question # 10

Let $y = \frac{\sqrt{1+x}}{\sqrt{1-x}} = \left(\frac{1+x}{1-x} \right)^{1/2}$

$$\begin{aligned}
\text{Now } \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{1+x}{1-x} \right)^{1/2} \\
&= \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{1+x}{1-x} \right) \\
&= \frac{1}{2} \left(\frac{1+x}{1-x} \right)^{-\frac{1}{2}} \left(\frac{(1-x) \frac{d}{dx}(1+x) - (1+x) \frac{d}{dx}(1-x)}{(1-x)^2} \right) \\
&= \frac{1}{2} \left(\frac{1-x}{1+x} \right)^{\frac{1}{2}} \left(\frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2} \right) \\
&= \frac{1}{2} \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}} \left(\frac{1-x+1+x}{(1-x)^2} \right) = \frac{(1-x)^{\frac{1}{2}}}{2(1+x)^{\frac{1}{2}}} \left(\frac{2}{(1-x)^2} \right) \\
&= \frac{1}{(1+x)^{\frac{1}{2}} (1-x)^{2-\frac{1}{2}}} = \frac{1}{\sqrt{1+x} (1-x)^{\frac{3}{2}}} \quad \text{Answer}
\end{aligned}$$

Question # 11

$$\text{Let } y = \frac{2x-1}{\sqrt{x^2+1}}$$

Differentiating w.r.t. x

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left(\frac{2x-1}{(x^2+1)^{1/2}} \right) = \frac{(x^2+1)^{1/2} \frac{d}{dx}(2x-1) - (2x-1) \frac{d}{dx}(x^2+1)^{1/2}}{\left((x^2+1)^{1/2}\right)^2} \\ &= \frac{(x^2+1)^{1/2} (2) - (2x-1) \frac{1}{2}(x^2+1)^{-1/2} \frac{d}{dx}(x^2+1)}{(x^2+1)} \\ &= \frac{2(x^2+1)^{1/2} - (2x-1) \frac{1}{2(x^2+1)^{1/2}} (2x)}{(x^2+1)} = \frac{1}{(x^2+1)} \left(2(x^2+1)^{1/2} - \frac{2x^2-x}{(x^2+1)^{1/2}} \right) \\ &= \frac{1}{(x^2+1)} \left(\frac{2x^2+2-2x^2+x}{(x^2+1)^{1/2}} \right) = \frac{x+2}{(x^2+1)\sqrt{x^2+1}} \text{ or } \frac{x+2}{(x^2+1)^{3/2}} \text{ Answer} \end{aligned}$$

Question # 12

Do yourself as Question # 10

Question # 13

$$\text{Let } y = \frac{\sqrt{x^2+1}}{\sqrt{x^2-1}} = \left(\frac{x^2+1}{x^2-1} \right)^{\frac{1}{2}}$$

Now do yourself as Question # 11

Question # 14

$$\begin{aligned} \text{Assume } y &= \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \\ &= \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \cdot \frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} - \sqrt{1-x}} \quad \text{Rationalizing} \\ &= \frac{(\sqrt{1+x} - \sqrt{1-x})^2}{(\sqrt{1+x})^2 - (\sqrt{1-x})^2} = \frac{(\sqrt{1+x})^2 + (\sqrt{1-x})^2 - 2(\sqrt{1+x})(\sqrt{1-x})}{1+x-1-x} \\ &= \frac{1+x+1-x-2\sqrt{(1+x)(1-x)}}{2x} = \frac{2-2\sqrt{1-x^2}}{2x} \\ &= \frac{2\left(1-(1-x^2)^{\frac{1}{2}}\right)}{2x} = \frac{1-(1-x^2)^{\frac{1}{2}}}{x} \end{aligned}$$

Now differentiation w.r.t x

$$\frac{dy}{dx} = \frac{d}{dx} \left(\frac{1-(1-x^2)^{\frac{1}{2}}}{x} \right)$$

$$\begin{aligned}
&= \frac{x \frac{d}{dx} \left(1 - (1-x^2)^{\frac{1}{2}} \right) - \left(1 - (1-x^2)^{\frac{1}{2}} \right) \frac{d}{dx} x}{x^2} \\
&= \frac{1}{x^2} \cdot \left[x \left(0 - \frac{1}{2} (1-x^2)^{\frac{1}{2}-1} \frac{d}{dx} (1-x^2) \right) - \left(1 - (1-x^2)^{\frac{1}{2}} \right) (1) \right] \\
&= \frac{1}{x^2} \cdot \left[x \left(-\frac{1}{2} (1-x^2)^{-\frac{1}{2}} (-2x) \right) - 1 + (1-x^2)^{\frac{1}{2}} \right] \\
&= \frac{1}{x^2} \cdot \left[\frac{x^2}{(1-x^2)^{\frac{1}{2}}} - 1 + (1-x^2)^{\frac{1}{2}} \right] = \frac{1}{x^2} \cdot \left[\frac{x^2 - (1-x^2)^{\frac{1}{2}} + 1 - x^2}{(1-x^2)^{\frac{1}{2}}} \right] \\
&= \frac{1}{x^2} \cdot \left[\frac{1 - (1-x^2)^{\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} \right] = \frac{1 - \sqrt{1-x^2}}{x^2 \sqrt{1-x^2}} \quad \text{Answer}
\end{aligned}$$

Question # 15

Let $y = \frac{x\sqrt{a+x}}{\sqrt{a-x}} = x \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}}$

Diff. w.r.t. x

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} x \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \\
&= x \frac{d}{dx} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} + \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \frac{d}{dx} x \quad \dots\dots\dots (i)
\end{aligned}$$

$$\begin{aligned}
\text{Now } \frac{d}{dx} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} &= \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}-1} \frac{d}{dx} \left(\frac{a+x}{a-x} \right) \\
&= \frac{1}{2} \left(\frac{a+x}{a-x} \right)^{-\frac{1}{2}} \left(\frac{(a-x) \frac{d}{dx} (a+x) - (a+x) \frac{d}{dx} (a-x)}{(a-x)^2} \right) \\
&= \frac{1}{2} \left(\frac{a-x}{a+x} \right)^{\frac{1}{2}} \left(\frac{(a-x)(1) - (a+x)(-1)}{(a-x)^2} \right) \\
&= \frac{1}{2} \frac{(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}} \left(\frac{a-x+a+x}{(a-x)^2} \right) = \frac{1}{2} \frac{1}{(a+x)^{\frac{1}{2}} (a-x)^{-\frac{1}{2}}} \cdot \left(\frac{2a}{(a-x)^2} \right) \\
&= \frac{a}{(a+x)^{\frac{1}{2}} (a-x)^{2-\frac{1}{2}}} = \frac{a}{(a+x)^{\frac{1}{2}} (a-x)^{\frac{3}{2}}}
\end{aligned}$$

Using in eq. (i)

$$\begin{aligned}
\frac{dy}{dx} &= x \cdot \frac{a}{(a+x)^{\frac{1}{2}} (a-x)^{\frac{3}{2}}} + \left(\frac{a+x}{a-x} \right)^{\frac{1}{2}} \quad (1) \\
&= \frac{ax}{(a+x)^{\frac{1}{2}} (a-x)^{\frac{3}{2}}} + \frac{(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}} \\
&= \frac{ax + (a+x)(a-x)}{(a+x)^{\frac{1}{2}} (a-x)^{\frac{3}{2}}} = \frac{ax + a^2 - x^2}{\sqrt{a+x} (a-x)^{\frac{3}{2}}} \quad \text{Answer}
\end{aligned}$$

Question # 16

$$\begin{aligned}\text{Since } y &= \sqrt{x} - \frac{1}{\sqrt{x}} \\ &= x^{\frac{1}{2}} - x^{-\frac{1}{2}}\end{aligned}$$

Diff. w.r.t. x

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}} \right) \\ &= \frac{1}{2} x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}\end{aligned}$$

Multiplying by $2x$

$$2x \frac{dy}{dx} = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

Adding y on both sides

$$\begin{aligned}2x \frac{dy}{dx} + y &= x^{\frac{1}{2}} + x^{-\frac{1}{2}} + y \\ \Rightarrow 2x \frac{dy}{dx} + y &= x^{\frac{1}{2}} + x^{-\frac{1}{2}} + x^{\frac{1}{2}} - x^{-\frac{1}{2}} \quad \because y = x^{\frac{1}{2}} - x^{-\frac{1}{2}} \\ \Rightarrow 2x \frac{dy}{dx} + y &= 2x^{\frac{1}{2}} \quad \Rightarrow 2x \frac{dy}{dx} + y = 2\sqrt{x} \quad \text{Proved}\end{aligned}$$

Question # 17

$$\text{Since } y = x^4 + 2x^2 + 2$$

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= \frac{d}{dx} (x^4 + 2x^2 + 2) \\ \Rightarrow \frac{dy}{dx} &= 4x^{4-1} + 2(2x^{2-1}) + 0 \\ &= 4x^3 + 4x \\ \Rightarrow \frac{dy}{dx} &= 4x(x^2 + 1) \dots\dots\dots (i)\end{aligned}$$

$$\text{Now } y = x^4 + 2x^2 + 2$$

$$\begin{aligned}\Rightarrow y - 1 &= x^4 + 2x^2 + 2 - 1 \\ &= x^4 + 2x^2 + 1 = (x^2 + 1)^2 \\ \Rightarrow \sqrt{y - 1} &= (x^2 + 1) \quad \text{i.e. } (x^2 + 1) = \sqrt{y - 1}\end{aligned}$$

Using it in eq. (i), we have

$$\Rightarrow \frac{dy}{dx} = 4x\sqrt{y - 1} \quad \text{as required.}$$
