Symmetric functions of the roots of a quadratic equation:
Define symmetric functions of the roots of a quadratic equation:
Definition:
Symmetric functions are those functions in which the roots involved are such that the value of the expressions involving them remain unaltered, when roots are interchanged.
For example, if

$$f(\alpha, \beta) = \alpha^2 + \beta^2 \text{, then}$$
$$f(\beta, \alpha) = \beta^2 + \alpha^2 = \alpha^2 + \beta^2 \qquad (\because \beta^2 + \alpha^2 = \alpha^2 + \beta^2)$$
$$= f(\alpha, \beta)$$

SOLVED EXERCISE 2.4

1. If α, β are the roots of the equation $x^2 + px + q = 0$, then evaluate

(i) $\alpha^2 + \beta^2$

Solution:

$\alpha^2 + \beta^2$
$x^2 + px + q = 0$

Here $a = 1, b = p, c = q$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$= -\frac{p}{1}$ $\qquad = \frac{q}{1}$

$= -p$ $\qquad = q$

Now $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$

(ii) $\alpha^3\beta + \alpha\beta^3$

Solution:

$\alpha^3\beta + \alpha\beta^3$
$x^2 + px + q = 0$

Here $a = 1, b = p, c = q$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$= -\frac{p}{1}$ $\qquad = \frac{q}{1}$

$= -p$ $\qquad = q$

Now $\alpha^3 + \beta^3 = \alpha\beta\,(\alpha^2 + \beta^2) - 2\alpha\beta$

$$=\alpha\beta\left[(\alpha+\beta)^2-2\alpha\beta\right]$$
$$=q\left[(-p)^2-2q\right]$$
$$=q\left(p^2-2q\right)$$

(iii) $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

Solution:

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$$
$$x^2+px+q=0$$

Here $a=1, b=p, c=q$

As α, β be the roots of given equation

Then $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

$$=-\frac{p}{1} \qquad =\frac{q}{1}$$
$$=-p \qquad =q$$

Now $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}=\dfrac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$
$$=q\frac{(-p)^2-2q}{q}=\frac{1}{q}(p^2-2q)$$

2. If α, β are the roots of the equation $4x^2-5x+6=0$, then find the values of

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

Solution:

$$\frac{1}{\alpha}+\frac{1}{\beta}$$
$$4x^2-5x+6=0$$

Here $a=4, b=-5, c=6$

As α,β be the roots of given equation

Then $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$

$$= -\frac{(-5)}{4} \qquad = \frac{6}{4}$$

$$= \frac{5}{4} \qquad = \frac{3}{2}$$

Now $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha \beta} = \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{4} \times \frac{2}{3} = \frac{5}{6}$

(ii) $\alpha^2\beta^2$

Solution:

$$4x^2 - 5x + 6 = 0$$

Here $a = 4, b = -5, c = 6$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$$= -\frac{(-5)}{4} \qquad = \frac{6}{4}$$

$$= \frac{5}{4} \qquad = \frac{3}{2}$$

Now $\alpha^2 \beta^2 = (\alpha \beta)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

Solution:

$$4x^2 - 5x + 6 = 0$$

Here $a = 4, b = -5, c = 6$

As α, β be the roots of given equation

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$$= -\frac{(-5)}{4} \qquad = \frac{6}{4}$$

$$= \frac{5}{4} \qquad = \frac{3}{2}$$

Now $\frac{1}{\alpha^2 \beta} + \frac{1}{\alpha \beta^2} = \frac{\alpha + \beta}{\alpha^2 \beta^2} = \frac{\alpha + \beta}{(\alpha \beta)^2}$

$$=\frac{\frac{5}{4}}{\left(\frac{3}{2}\right)^2}=\frac{\frac{5}{4}}{\frac{9}{4}}=\frac{5}{4}\times\frac{4}{9}=\frac{5}{9}$$

(iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$

Solution:

$$4x^2-5x+6=0$$

Here $a=4,\ b=-5,\ c=6$

As $\propto,\beta$ be the roots of given equation

Then $\propto+\beta=-\frac{b}{a}$ and $\propto\beta=\frac{c}{a}$

$$=-\frac{(-5)}{4} \qquad =\frac{6}{4}$$

$$=\frac{5}{4} \qquad =\frac{3}{2}$$

Now $\frac{\propto^2}{\beta}+\frac{\beta^2}{\propto}=\frac{\propto^3+\beta^3}{\propto\beta}=\frac{(\propto+\beta)^3-3\propto\beta(\propto+\beta)}{\propto\beta}$

$$=\frac{\left(\frac{5}{4}\right)^3-3\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)}{\frac{3}{2}}=\frac{\frac{125}{54}-\frac{45}{8}}{\frac{3}{2}}$$

$$=\frac{125-360}{64}\times\frac{2}{3}=-\frac{235}{64}\times\frac{2}{3}$$

$$=-\frac{235}{96}$$

3. If α, β are the roots of the equation $lx^2+mx+=0$ $(l\neq 0)$, then find the values of:

(i) $\alpha^3\beta^2+\alpha^2\beta^3$

Solution:

$$lx^2+mx+n=0$$

Here $a=l,\ b=m,\ c=n$

Then $\propto+\beta=-\frac{b}{a}$ and $\propto\beta=\frac{c}{a}$

$$=-\frac{m}{l} \qquad =\frac{n}{l}$$

Now $\alpha^3\beta^2 + \alpha^2\beta^3 = \alpha^2\beta^2(\alpha + \beta) = (\alpha\beta)^2(\alpha + \beta)$

$$= \left(\frac{n}{l}\right)^2\left(-\frac{m}{l}\right) = -\frac{mn^2}{l^2}$$

(ii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$\frac{1}{\alpha^2} + \frac{1}{\beta^2}$

Solution:

$lx^2 + mx + n = 0$

Here $a = l, b = m, c = n$

As

Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

$= -\frac{m}{l}$ $= \frac{n}{l}$

Now $\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$$= \frac{\left(-\frac{m}{l}\right)^2 - 2\left(\frac{n}{l}\right)}{\left(\frac{n}{l}\right)^2} = \frac{\frac{m^2}{l^2} - \frac{2n}{l}}{\frac{n^2}{l^2}}$$

$$= \frac{m^2 - 2nl}{l^2} \times \frac{l^2}{n^2} = \frac{m^2 - 2nl}{n^2}$$

Formation of a quadratic equation:

If α and β are the roots of the required quadratic equation.

Let $x = \alpha$ and $x = \beta$

i.e., $x - \alpha = 0$, $x - \beta = 0$

and $(x - \alpha)(x - \beta) = 0$

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

which is the required quadratic equation in the standard form.

Find a quadratic equation from given roots and establish the formula

x^2 (sum of the roots) x+ product of the roots = 0.

Let α, β be the roots of the quadratic equation

$ax^2 + bx + c = 0$, $(a \neq 0)$ (i)

Then $\alpha + \beta -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Rewrite eq. (i) as $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

or $$x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

or x^2 – (sum of roots) x + product of roots = 0 , that is,

$x^2 - Sx + P = 0$ where $S = \alpha + \beta$ and $P = \alpha\beta$

SOLVED EXERCISE 2.5

1. Write the quadratic equations having following roots.

(a) 1, 5

Solution:

Since 1 and 5 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 1 + 5 = 6

P = Product of roots = (1) (5) = 5

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 6x + 5 = 0$

(b) 4, 9

Solution:

Since 4 and 9 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 4 + 9 = 13

P = Product of roots = (4) (9) = 36

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - 13x + 36 = 0$

(c) - 2, 3

Solution:

Since –2 and 3 are the roots of the required quadratic equation, therefore,

S = Sum of roots = – 2 + 3 = 1

P = Product of roots = (–2) (3) = – 6

Thus, the required quadratic equation is

$x^2 - Sx + P = 0$

$x^2 - (1)x + (-6) = 0$

$x^2 - x - 6 = 0$

(d) 0, – 3

Solution:

Since 0 and –3 are the roots of the required quadratic equation, therefore,

S = Sum of roots = 0 + (–3) = – 3