Exercise 12.8

IMPORTANT FORMULAS

The Law of Cosine

- $a^2 = b^2 + c^2 - 2bc\cos\alpha$
- $b^2 = c^2 + a^2 - 2ca\cos\beta$
- $c^2 = a^2 + b^2 - 2ab\cos\gamma$
- $\cos\alpha = \dfrac{b^2 + c^2 - a^2}{2bc}$
- $\cos\beta = \dfrac{c^2 + a^2 - b^2}{2ca}$
- $\cos\gamma = \dfrac{a^2 + b^2 - c^2}{2ab}$

.The Law of Sine

- $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$

.The Law of Tangent

- $\dfrac{a-b}{a+b} = \dfrac{\tan\dfrac{\alpha-\beta}{2}}{\tan\dfrac{\alpha+\beta}{2}}$
- $\dfrac{b-c}{b+c} = \dfrac{\tan\dfrac{\beta-\gamma}{2}}{\tan\dfrac{\beta+\gamma}{2}}$
- $\dfrac{c-a}{c+a} = \dfrac{\tan\dfrac{\gamma-\alpha}{2}}{\tan\dfrac{\gamma+\alpha}{2}}$

.Half Angle Formulas

- $\sin\dfrac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$
- $\sin\dfrac{\beta}{2} = \sqrt{\dfrac{(s-c)(s-a)}{ca}}$
- $\sin\dfrac{\gamma}{2} = \sqrt{\dfrac{(s-a)(s-b)}{ab}}$
- $\cos\dfrac{\alpha}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$
- $\cos\dfrac{\beta}{2} = \sqrt{\dfrac{s(s-b)}{ac}}$
- $\cos\dfrac{\gamma}{2} = \sqrt{\dfrac{s(s-c)}{ab}}$
- $\tan\dfrac{\alpha}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$
- $\tan\dfrac{\beta}{2} = \sqrt{\dfrac{(s-c)(s-a)}{s(s-b)}}$
- $\tan\dfrac{\gamma}{2} = \sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}$

where $s = \dfrac{a+b+c}{2} \quad \Rightarrow 2s = a+b+c$

.Area of the Triangle ($=\Delta$)

- $\Delta = \dfrac{1}{2}bc\sin\alpha = \dfrac{1}{2}ca\sin\beta = \dfrac{1}{2}ab\sin\gamma$
- $\Delta = \dfrac{a^2\sin\beta\sin\gamma}{2\sin\alpha} = \dfrac{b^2\sin\gamma\sin\alpha}{2\sin\beta} = \dfrac{c^2\sin\alpha\sin\beta}{2\sin\gamma}$
- $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$ (*Hero's Formula*)

.Circum Radius ($=R$)

- $R = \dfrac{a}{2\sin\alpha} = \dfrac{b}{2\sin\beta} = \dfrac{c}{2\sin\gamma}$
- $R = \dfrac{abc}{4\Delta}$

In-Radius (= r)

- $r = \dfrac{\Delta}{s}$

Escribed Circle

- $r_1 = \dfrac{\Delta}{s-a}$
- $r_2 = \dfrac{\Delta}{s-b}$
- $r_3 = \dfrac{\Delta}{s-c}$

Question # 1

Show that

(i) $r = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$ (ii) $s = 4R\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}$

Solution

(i)

$$\text{R.H.S} = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

$$= 4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$= 4R\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{(bc)(ac)(ab)}}$$

$$= 4R\sqrt{\frac{(s-a)^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= 4R\frac{(s-a)(s-b)(s-c)}{abc}$$

$$= 4\left(\frac{abc}{4\Delta}\right)\frac{(s-a)(s-b)(s-c)}{abc} \qquad \because\ R = \frac{abc}{4\Delta}$$

$$= \frac{(s-a)(s-b)(s-c)}{\Delta} \qquad = \frac{s(s-a)(s-b)(s-c)}{s\Delta}$$

$$= \frac{\Delta^2}{s\Delta} \qquad \because\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta}{s} = r = \text{L.H.S}$$

(ii)

$$\text{R.H.S} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 4R\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= 4R\sqrt{\frac{s^2\cdot s(s-a)(s-b)(s-c)}{(bc)(ac)(ab)}} \qquad = 4R\sqrt{\frac{s^2\Delta^2}{a^2b^2c^2}}$$

$$= 4R\frac{s\Delta}{abc} \qquad = 4\left(\frac{abc}{4\Delta}\right)s\frac{\Delta}{abc} \qquad \because\ R = \frac{abc}{4\Delta}$$

$$= s \quad = \text{L.H.S}$$

Question # 2

Show that:

$$r = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2} = b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2} = c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

Solution

We take

$$a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}=a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\frac{1}{\cos\frac{\alpha}{2}}$$

$$=a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\frac{1}{\sqrt{\frac{s(s-a)}{bc}}}$$

$$=a\sqrt{\frac{(s-a)(s-c)}{ac}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{s(s-a)}}$$

$$=a\sqrt{\frac{(s-a)(s-c)(s-a)(s-b)(bc)}{(ac)(ab)s(s-a)}}$$

$$=a\sqrt{\frac{(s-a)(s-b)(s-c)}{a^2s}}\qquad=a\sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2s^2}}$$

$$=a\frac{\sqrt{s(s-a)(s-b)(s-c)}}{as}\qquad=\frac{\Delta}{s}\qquad=r$$

$$\Rightarrow\ a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}=r\ \ \ldots\ldots\ldots\ldots\ldots\ (\text{i})$$

Similarly prove yourself

$$b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}=r\ \ \ldots\ldots\ldots\ldots\ (\text{ii})$$

$$c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}=r\ \ \ldots\ldots\ldots\ldots\ (\text{iii})$$

From (i), (ii) and (iii)

$$r=a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\sec\frac{\alpha}{2}=b\sin\frac{\gamma}{2}\sin\frac{\alpha}{2}\sec\frac{\beta}{2}=c\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sec\frac{\gamma}{2}$$

Question # 3

Show that:

(i) $r_1=4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$ (ii) $r_2=4R\cos\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

(iii) $r_3=4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}$

Solution

$$\text{R.H.S}=4R\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$=4R\sqrt{\frac{(s-b)(s-c)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$=4R\sqrt{\frac{(s-b)(s-c)s(s-b)s(s-c)}{(bc)(ac)(ab)}}\qquad=4R\sqrt{\frac{s^2(s-b)^2(s-c)^2}{a^2b^2c^2}}$$

$$= 4R\frac{s(s-b)(s-c)}{abc} \quad = 4\frac{abc}{4\Delta}\frac{s(s-b)(s-c)}{abc}\cdot\frac{(s-a)}{(s-a)} \qquad \because\ R = \frac{abc}{4\Delta}$$

$$= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)}$$

$$= \frac{\Delta^2}{\Delta(s-a)} \quad = \frac{\Delta}{(s-a)} \quad = r_1 \quad = \text{R.H.S}$$

(ii) & (iii) *Do yourself*

Question # 4

Show that:

(i) $r_1 = s\tan\dfrac{\alpha}{2}$ (ii) $r_2 = s\tan\dfrac{\beta}{2}$ (iii) $r_3 = s\tan\dfrac{\gamma}{2}$

Solution

$$\text{R.H.S} = s\tan\frac{\alpha}{2}$$

$$= s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \quad = s\sqrt{\frac{(s-b)(s-c)}{s(s-a)}\cdot\frac{s(s-a)}{s(s-a)}}$$

$$= s\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}}$$

$$= s\sqrt{\frac{\Delta^2}{s^2(s-a)^2}} = s\,\frac{\Delta}{s(s-a)} \quad = \frac{\Delta}{(s-a)} \quad = r_1 \ = \text{L.H.S}$$

(ii) & (iii) *Do yourself*

Question # 5

Prove that:

(i) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (ii) $r\,r_1r_2r_3 = \Delta^2$

(iii) $r_1 + r_2 + r_3 - r = 4R$ (iv) $r_1r_2r_3 = r\,s^2$

Solution

(i) $\text{L.H.S} = r_1r_2 + r_2r_3 + r_3r_1$

$$= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) + \left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right)$$

$$= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)}$$

$$= \Delta^2\left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)}\right)$$

$$= \Delta^2\left(\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)}\right) \quad = \Delta^2\left(\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)}\right)$$

$$= \Delta^2\left(\frac{3s-2s}{(s-a)(s-b)(s-c)}\right) \qquad \because\ s = \frac{a+b+c}{2}$$

$$=\Delta^2\left(\frac{s}{(s-a)(s-b)(s-c)}\cdot\frac{s}{s}\right)$$

$$=\Delta^2\left(\frac{s^2}{s(s-a)(s-b)(s-c)}\right) \quad =\Delta^2\left(\frac{s^2}{\Delta^2}\right) \quad =s^2 \quad =\text{R.H.S}$$

(ii) L.H.S $= r\,r_1\,r_2\,r_3$

$$=\left(\frac{\Delta}{s}\right)\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$$

$$=\frac{\Delta^4}{s(s-a)(s-b)(s-c)} \quad =\frac{\Delta^4}{\Delta^2} \quad =\Delta^2 \quad =\text{R.H.S}$$

(iii) L.H.S $=r_1+r_2+r_3-r$

$$=\frac{\Delta}{s-a}+\frac{\Delta}{s-b}+\frac{\Delta}{s-c}-\frac{\Delta}{s} \quad =\Delta\left(\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}-\frac{1}{s}\right)$$

$$=\Delta\left(\frac{(s-b)+(s-a)}{(s-a)(s-b)}++\frac{s-(s-c)}{s(s-c)}\right) \quad =\Delta\left(\frac{2s-b-a}{(s-a)(s-b)}+\frac{s-s+c}{s(s-c)}\right)$$

$$=\Delta\left(\frac{a+b+c-b-a}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right) \qquad \because\ 2s=a+b+c$$

$$=\Delta\left(\frac{c}{(s-a)(s-b)}+\frac{c}{s(s-c)}\right) \quad =c\Delta\left(\frac{1}{(s-a)(s-b)}+\frac{1}{s(s-c)}\right)$$

$$=c\Delta\left(\frac{s(s-c)-(s-a)(s-b)}{s(s-a)(s-b)(s-c)}\right) \quad =c\Delta\left(\frac{s^2-sc+s^2-as-bs+ab}{\Delta^2}\right)$$

$$=c\left(\frac{2s^2-s(a+b+c)+ab}{\Delta}\right) \quad =c\left(\frac{2s^2-s(2s)+ab}{\Delta}\right)$$

$$=c\left(\frac{2s^2-2s^2+ab}{\Delta}\right) \quad =\frac{abc}{\Delta} \quad =4\cdot\frac{abc}{4\Delta} \quad =4R \quad =\text{R.H.S}$$

(iv) L.H.S $=\left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right)$

$$=\frac{\Delta^3}{(s-a)(s-b)(s-c)} \quad =\frac{s\Delta^3}{s(s-a)(s-b)(s-c)}$$

$$=\frac{s\Delta^3}{\Delta^2} \quad =s\Delta \quad =s^2\,\frac{\Delta}{s} \quad =s^2\,r \quad =r\,s^2 \quad =\text{R.H.S}$$

Question # 6

Find R, r, r_1, r_2 and r_3, if measures of the sides of triangle ABC are

(i) $a=13$, $b=14$, $c=15$ (ii) $a=34,$ $b=20,$, $c=42$

Solution

(i) $a=13$, $b=14$, $c=15$

$$s=\frac{a+b+c}{2}=\frac{13+14+15}{2}=21$$
$$s-a=21-13=8$$
$$s-b=21-14=7$$
$$s-c=21-15=6$$

So $\Delta=\sqrt{s(s-a)(s-b)(s-c)}$
$$=\sqrt{21(8)(7)(6)}=\sqrt{7056}=84$$

Now
$$R=\frac{abc}{4\Delta}=\frac{(13)(14)(15)}{4(84)}=8.125$$
$$r=\frac{\Delta}{s}=\frac{84}{21}=4$$
$$r_1=\frac{\Delta}{s-a}=\frac{84}{8}=10.5$$
$$r_2=\frac{\Delta}{s-b}=\frac{84}{7}=12$$
$$r_3=\frac{\Delta}{s-c}=\frac{84}{6}=14$$

(ii) *Do yourself*

Question # 7

Prove that in an equilateral triangle,

(i) $r:R:r_1=1:2:3$ (ii) $r:R:r_1:r_2:r_3=1:2:3:3:3$

Solution

(i) *Do yourself*

(ii) In equilateral triangle all the sides are equal so $a=b=c$
$$s=\frac{a+b+c}{2}=\frac{a+a+a}{2}=\frac{3a}{2}$$
$$s-a=\frac{3a}{2}-a=\left(\frac{3}{2}-1\right)a=\frac{1}{2}a$$

Now
$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$$
$$=\sqrt{s(s-a)(s-a)(s-a)}=\sqrt{s(s-a)^3}=\sqrt{\frac{3a}{2}\left(\frac{1}{2}a\right)^3}$$
$$=\sqrt{\frac{3a}{2}\left(\frac{a^3}{8}\right)}=\sqrt{\frac{3a^4}{16}}=\frac{\sqrt{3}\,a^2}{4}$$

Now $r=\dfrac{\Delta}{s}=\dfrac{\sqrt{3}\,a^2/4}{3a/2}=\dfrac{\sqrt{3}\,a^2}{4}\cdot\dfrac{2}{3a}=\dfrac{\sqrt{3}\,a}{6}$

$$R=\frac{abc}{4\Delta}=\frac{a\cdot a\cdot a}{4\sqrt{3}a^2/4}=\frac{a}{\sqrt{3}}=\frac{a}{\sqrt{3}}\cdot\frac{\sqrt{3}}{\sqrt{3}}=\frac{\sqrt{3}\,a}{3}$$

$$r_1=\frac{\Delta}{s-a}=\frac{\sqrt{3}a^2/4}{\frac{1}{2}a}=\frac{\sqrt{3}\,a^2}{4}\cdot\frac{2}{a}=\frac{\sqrt{3}\,a}{2}$$

$$r_2=\frac{\Delta}{s-b}=\frac{\Delta}{s-a}=\frac{\sqrt{3}\,a}{2}$$

$$r_3=\frac{\Delta}{s-c}=\frac{\Delta}{s-a}=\frac{\sqrt{3}\,a}{2}$$

Now

$$\begin{aligned}
r:R:r_1:r_2:r_3 &=\frac{\sqrt{3}\,a}{6}:\frac{\sqrt{3}\,a}{3}:\frac{\sqrt{3}\,a}{2}:\frac{\sqrt{3}\,a}{2}:\frac{\sqrt{3}\,a}{2}\\
&=\frac{1}{6}:\frac{1}{3}:\frac{1}{2}:\frac{1}{2}:\frac{1}{2} && \div\text{ing by } \sqrt{3}\,a\\
&=1:2:3:3:3 && \times\text{ing by } 6
\end{aligned}$$

Question # 8

Prove that:

(i) $\Delta = r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$ (ii) $r=s\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(iii) $\Delta = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$

Solution

(i) R.H.S $=r^2\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

$$=r^2\,\frac{1}{\tan\frac{\alpha}{2}}\cdot\frac{1}{\tan\frac{\beta}{2}}\cdot\frac{1}{\tan\frac{\gamma}{2}}$$

$$=r^2\,\frac{1}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}}\cdot\frac{1}{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}}\cdot\frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}$$

$$=r^2\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\cdot\sqrt{\frac{s(s-b)}{(s-a)(s-c)}}\cdot\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$=r^2\sqrt{\frac{s^3(s-a)(s-b)(s-c)}{(s-a)^2(s-b)^2(s-c)^2}}=r^2\sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}}$$

$$=r^2\sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}\cdot\frac{s}{s}}=r^2\sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}}$$

$$=r^2\sqrt{\frac{s^4}{\Delta^2}}=r^2\,\frac{s^2}{\Delta}=\left(\frac{\Delta}{s}\right)^2\frac{s^2}{\Delta}\qquad\because\ r=\frac{\Delta}{s}$$

$$=\frac{\Delta^2}{s^2}\frac{s^2}{\Delta} = \Delta = \text{L.H.S}$$

(ii) *Do yourself*

(iii) $$\text{R.H.S} = 4Rr\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$=4Rr\sqrt{\frac{s(s-a)}{bc}}\cdot\sqrt{\frac{s(s-b)}{ca}}\cdot\sqrt{\frac{s(s-c)}{ab}}$$

$$=4Rr\sqrt{\frac{s(s-a)\cdot s(s-b)\cdot s(s-c)}{(bc)(ac)(ab)}}$$

$$=4Rr\sqrt{\frac{s^2\cdot s(s-a)(s-b)(s-c)}{a^2b^2c^2}} = 4Rr\sqrt{\frac{s^2\cdot\Delta^2}{a^2b^2c^2}}$$

$$=4Rr\frac{s\Delta}{abc} = 4\left(\frac{abc}{4\Delta}\right)\left(\frac{\Delta}{s}\right)\frac{s\Delta}{abc} = \Delta = \text{L.H.S}$$

Question # 9

Show that

(i) $\frac{1}{2rR} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$ (ii) $\frac{1}{r} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$

Solution

(i) L.H.S $=\frac{1}{2rR}$

$$=\frac{1}{2\left(\frac{\Delta}{s}\right)\left(\frac{abc}{4\Delta}\right)} = \frac{4s\Delta}{2\Delta abc} = \frac{2s}{abc} = \frac{a+b+c}{abc} \qquad \because 2s = a+b+c$$

$$=\frac{a}{abc}+\frac{b}{abc}+\frac{c}{abc}$$

$$=\frac{1}{bc}+\frac{1}{ac}+\frac{1}{ab} = \frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = \text{R.H.S}$$

(ii) $$\text{R.H.S} = \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$$

$$=\frac{1}{\frac{\Delta}{s-a}}+\frac{1}{\frac{\Delta}{s-b}}+\frac{1}{\frac{\Delta}{s-c}} = \frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}$$

$$=\frac{s-a+s-b+s-c}{\Delta} = \frac{3s-(a+b+c)}{\Delta}$$

$$=\frac{3s-2s}{\Delta} \qquad \because 2s = a+b+c$$

$$=\frac{s}{\Delta} = \frac{1}{\Delta/s} = \frac{1}{r} = \text{L.H.S}$$

Question # 10

We take $\dfrac{a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}} = a\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\,\dfrac{1}{\cos\frac{\alpha}{2}}$

Solution *Now see Question # 2*

Question # 11

Prove that: $abc(\sin\alpha+\sin\beta+\sin\gamma)=4\Delta s$

Solution

L.H.S $=abc(\sin\alpha+\sin\beta+\sin\gamma)$

Since $\Delta=\frac{1}{2}ab\sin\gamma=\frac{1}{2}bc\sin\alpha=\frac{1}{2}ca\sin\beta$

$\therefore\ \sin\gamma=\dfrac{2\Delta}{ab}\quad,\quad \sin\alpha=\dfrac{2\Delta}{bc}\quad,\quad \sin\beta=\dfrac{2\Delta}{ca}$

Thus L.H.S $=abc\left(\dfrac{2\Delta}{bc}+\dfrac{2\Delta}{ac}+\dfrac{2\Delta}{ab}\right)$

$=abc\left(\dfrac{2\Delta a+2\Delta b+2\Delta c}{abc}\right) \quad =2\Delta a+2\Delta b+2\Delta c$

$=2\Delta(a+b+c) \quad =2\Delta(2s) \qquad \because\ 2s=a+b+c$

$=4\Delta s \quad = \quad$ R.H.S

Question # 12

(i) $(r_1+r_2)\tan\frac{\gamma}{2}=c$ (ii) $(r_3-r)\cot\frac{\gamma}{2}=c$

Solution

(i) L.H.S $=(r_1+r_2)\tan\frac{\gamma}{2}$

$$=\left(\frac{\Delta}{s-a}+\frac{\Delta}{s-b}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

$$=\left(\frac{\Delta(s-b)+\Delta(s-a)}{(s-a)(s-b)}\right)\sqrt{\frac{(s-a)(s-b)}{s(s-c)}\cdot\frac{s(s-c)}{s(s-c)}}$$

$$=\Delta\left(\frac{s-b+s-a}{(s-a)(s-b)}\right)\sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}}$$

$$=\Delta\left(\frac{2s-a-b}{(s-a)(s-b)}\right)\sqrt{\frac{\Delta^2}{s^2(s-c)^2}}$$

$$=\Delta\left(\frac{a+b+c-a-b}{(s-a)(s-b)}\right)\frac{\Delta}{s(s-c)} \qquad \because\ 2s=a+b+c$$

$$=\frac{\Delta^2 c}{s(s-a)(s-b)(s-c)} \quad =\frac{\Delta^2 c}{\Delta^2} \quad =c \quad = \text{R.H.S}$$

(ii) L.H.S $= (r_3 - r)\cot\dfrac{\gamma}{2}$

$$= \left(\frac{\Delta}{s-c} - \frac{\Delta}{s}\right)\frac{1}{\tan\frac{\gamma}{2}} \quad = \Delta\left(\frac{1}{s-c} - \frac{1}{s}\right)\frac{1}{\sqrt{\dfrac{(s-a)(s-b)}{s(s-c)}}}$$

$$= \Delta\left(\frac{s-(s-c)}{s(s-c)}\right)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \quad = \Delta\left(\frac{c}{s(s-c)}\right)\sqrt{\frac{s(s-c)}{(s-a)(s-b)}\cdot\frac{s(s-c)}{s(s-c)}}$$

$$= \Delta\left(\frac{c}{s(s-c)}\right)\sqrt{\frac{s^2(s-c)^2}{s(s-a)(s-b)(s-c)}} \quad = \Delta\left(\frac{c}{s(s-c)}\right)\sqrt{\frac{s^2(s-c)^2}{\Delta^2}}$$

$$= \Delta\left(\frac{c}{s(s-c)}\right)\frac{s(s-c)}{\Delta} \quad = c \quad = \text{R.H.S}$$