

EXERCISE 1.3

Important Limits

- I. $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$, where n is integer and $a > 0$.
- II. $\lim_{x \rightarrow 0} \frac{\sqrt{x+a} - \sqrt{a}}{x} = \frac{1}{2\sqrt{a}}$.
- III. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.
- IV. $\lim_{x \rightarrow \infty} (1 + x)^{\frac{1}{x}} = e$.
- V. $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \ln a$, where $a > 0$.
- VI. $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \ln e = 1$.
- VII. If θ is measured in radian, then $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$.

Question # 1

- (i) $\lim_{x \rightarrow 3} (2x + 4) = \lim_{x \rightarrow 3} (2x) + \lim_{x \rightarrow 3} (4) = 2\lim_{x \rightarrow 3} (x) + 4 = 2(3) + 4 = 10$.
- (ii) $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = 3(1)^2 - 2(1) + 4 = 3 - 2 + 4 = 5$.
- (iii) $\lim_{x \rightarrow 3} \sqrt{x^2 + x + 4} = \sqrt{(3)^2 + (3) + 4} = \sqrt{9 + 3 + 4} = \sqrt{16} = 4$.
- (iv) $\lim_{x \rightarrow 2} x\sqrt{x^2 - 4} = 2\sqrt{2^2 - 4} = 0$.
- (v) $\begin{aligned} \lim_{x \rightarrow 2} (\sqrt{x^3 + 1} - \sqrt{x^2 + 5}) &= \lim_{x \rightarrow 2} (\sqrt{x^3 + 1}) - \lim_{x \rightarrow 2} (\sqrt{x^2 + 5}) \\ &= (\sqrt{(2)^3 + 1}) - (\sqrt{(2)^2 + 5}) \\ &= \sqrt{8 + 1} - \sqrt{4 + 5} = \sqrt{9} - \sqrt{9} = 0. \end{aligned}$
- (vi) $\lim_{x \rightarrow -2} \frac{2x^3 + 5x}{3x - 2} = \frac{2(-2)^3 + 5(-2)}{3(-2) - 2} = \frac{-16 - 10}{-6 - 2} = \frac{-26}{-8} = \frac{13}{4}$.
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Question # 2

- (i) $\begin{aligned} \lim_{x \rightarrow -1} \frac{x^3 - x}{x + 1} &= \lim_{x \rightarrow -1} \frac{x(x^2 - 1)}{x + 1} = \lim_{x \rightarrow -1} \frac{x(x + 1)(x - 1)}{x + 1} \\ &= \lim_{x \rightarrow -1} x(x - 1) = (-1)(-1 - 1) = 2 \end{aligned}$
- (ii) $\lim_{x \rightarrow 0} \left(\frac{3x^3 + 4x}{x^2 + x} \right) = \lim_{x \rightarrow 0} \frac{x(3x^2 + 4)}{x(x + 1)} = \lim_{x \rightarrow 0} \frac{3x^2 + 4}{x + 1} = \frac{3(0) + 4}{0 + 1} = 4$.

$$\begin{aligned}
\text{(iii)} \quad & \lim_{x \rightarrow 2} \frac{x^3 - 8}{x^2 + x - 6} \\
&= \lim_{x \rightarrow 2} \frac{x^3 - (2)^3}{x^2 + 3x - 2x - 6} = \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{x(x+3) - 2(x+3)} \\
&= \lim_{x \rightarrow 2} \frac{(x-2)(x^2 + 2x + 4)}{(x+3)(x-2)} = \lim_{x \rightarrow 2} \frac{(x^2 + 2x + 4)}{(x+3)} \\
&= \frac{(2)^2 + 2(2) + 4}{(2+3)} = \frac{12}{5}
\end{aligned}$$

$$\begin{aligned}
\text{(iv)} \quad & \lim_{x \rightarrow 1} \frac{x^3 - 3x^2 + 3x - 1}{x^3 - x} \\
&= \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x^2 - 1)} = \lim_{x \rightarrow 1} \frac{(x-1)^3}{x(x-1)(x+1)} \\
&= \lim_{x \rightarrow 1} \frac{(x-1)^2}{x(x+1)} = \lim_{x \rightarrow 1} \frac{(1-1)^2}{(1)(1+1)} = 0
\end{aligned}$$

$$\begin{aligned}
\text{(v)} \quad & \lim_{x \rightarrow -1} \left(\frac{x^3 + x^2}{x^2 - 1} \right) = \lim_{x \rightarrow -1} \frac{x^2(x+1)}{(x+1)(x-1)} = \lim_{x \rightarrow -1} \frac{x^2}{(x-1)} \\
&= \frac{(-1)^2}{(-1-1)} = -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\text{(vi)} \quad & \lim_{x \rightarrow 4} \frac{2x^2 - 32}{x^3 - 4x^2} = \lim_{x \rightarrow 4} \frac{2(x^2 - 16)}{x^2(x-4)} \\
&= \lim_{x \rightarrow 4} \frac{2(x+4)(x-4)}{x^2(x-4)} = \lim_{x \rightarrow 4} \frac{2(x+4)}{x^2} \\
&= \frac{2(4+4)}{4^2} = \frac{16}{16} = 1.
\end{aligned}$$

$$\begin{aligned}
\text{(vii)} \quad & \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} = \lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2} \left(\frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} \right) \\
&= \lim_{x \rightarrow 2} \frac{(\sqrt{x})^2 - (\sqrt{2})^2}{(x-2)(\sqrt{x} + \sqrt{2})} = \lim_{x \rightarrow 2} \frac{x - 2}{(x-2)(\sqrt{x} + \sqrt{2})} \\
&= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\text{(viii)} \quad & \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \\
&= \lim_{h \rightarrow 0} \frac{(\sqrt{x+h})^2 - (\sqrt{x})^2}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{\sqrt{x+0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}$$

$$\begin{aligned}
\text{(ix)} \quad & \lim_{x \rightarrow a} \frac{x^n - a^n}{x^m - a^m} \\
&= \lim_{x \rightarrow a} \frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x-a)(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
&= \lim_{x \rightarrow a} \frac{(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})}{(x^{m-1} + x^{m-2}a + x^{m-3}a^2 + \dots + a^{m-1})} \\
&= \frac{a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1}}{a^{m-1} + a^{m-2}a + a^{m-3}a^2 + \dots + a^{m-1}} \\
&= \frac{a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} \text{ (n terms)}}{a^{m-1} + a^{m-1} + a^{m-1} + \dots + a^{m-1} \text{ (m terms)}} \\
&= \frac{na^{n-1}}{ma^{m-1}} = \frac{n}{m} a^{n-1-m+1} = \frac{n}{m} a^{n-m}
\end{aligned}$$

Law of Sine

If θ is measured in radian, then $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

See proof on book at page 25

Question # 3

$$\text{(i)} \quad \lim_{x \rightarrow 0} \frac{\sin 7x}{x}$$

$$\text{Put } t = 7x \Rightarrow \frac{t}{7} = x$$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\begin{aligned}
\lim_{x \rightarrow 0} \frac{\sin 7x}{x} &= \lim_{t \rightarrow 0} \frac{\sin t}{\frac{t}{7}} \\
&= 7 \lim_{t \rightarrow 0} \frac{\sin t}{t} = 7(1) = 7
\end{aligned}$$

By law of sine.

$$\text{(ii)} \quad \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x}$$

$$\text{Since } 180^\circ = \pi \text{ rad} \Rightarrow 1^\circ = \frac{\pi}{180} \text{ rad} \Rightarrow x^\circ = \frac{x\pi}{180} \text{ rad}$$

$$\text{So } \lim_{x \rightarrow 0} \frac{\sin x^\circ}{x} = \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x}$$

Now put $\frac{\pi x}{180} = t$ i.e. $x = \frac{180t}{\pi}$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin \frac{\pi x}{180}}{x} &= \lim_{x \rightarrow 0} \frac{\sin t}{\frac{180t}{\pi}} \\ &= \frac{\pi}{180} \lim_{x \rightarrow 0} \frac{\sin t}{t} = \frac{\pi}{180} (1) = \frac{\pi}{180} \end{aligned} \quad \text{by law of sine}$$

$$\begin{aligned} \text{(iii)} \quad \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} &= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\ &= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin \theta (1 + \cos \theta)} \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{(1 + \cos \theta)} = \frac{\sin(0)}{1 + \cos(0)} = \frac{0}{1 + 1} = 0 \end{aligned}$$

$$\text{(iv)} \quad \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x}$$

Put $t = \pi - x \Rightarrow x = \pi - t$

When $x \rightarrow \pi$ then $t \rightarrow 0$, so

$$\begin{aligned} \lim_{x \rightarrow \pi} \frac{\sin x}{\pi - x} &= \lim_{t \rightarrow 0} \frac{\sin(\pi - t)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\sin t}{t} \quad \because \sin(\pi - t) = \sin\left(2 \cdot \frac{\pi}{2} - t\right) = \sin t \\ &= 1 \end{aligned} \quad \text{By law of sine.}$$

$$\begin{aligned} \text{(v)} \quad \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \sin ax \cdot \frac{1}{\sin bx} \\ &= \lim_{x \rightarrow 0} \sin ax \cdot \left(\frac{ax}{ax}\right) \frac{1}{\sin bx \cdot \left(\frac{bx}{bx}\right)} = \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot ax \frac{1}{\frac{\sin bx}{bx} \cdot bx} \\ &= \frac{a}{b} \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \cdot \frac{1}{\lim_{x \rightarrow 0} \frac{\sin bx}{bx}} = \frac{a}{b} \cdot (1) \cdot \frac{1}{(1)} = \frac{a}{b} \quad \text{by law of sine} \end{aligned}$$

$$\begin{aligned} \text{(vi)} \quad \lim_{x \rightarrow 0} \frac{x}{\tan x} &= \lim_{x \rightarrow 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot \cos x \\ &= \lim_{x \rightarrow 0} \frac{1}{\frac{\sin x}{x}} \cdot \cos x = \frac{1}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \cdot \lim_{x \rightarrow 0} \cos x = \frac{1}{1} \cdot 1 = 1 \end{aligned}$$

$$\begin{aligned} \text{(vii)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \quad \because \sin^2 x = \frac{1 - \cos 2x}{2} \\ &\quad \therefore 2 \sin^2 x = 1 - \cos 2x \end{aligned}$$

$$= 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 = 2 \left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2 = 2(1)^2 = 2$$

(vii) *Do yourself by rationalizing*

$$\begin{aligned} \text{(viii)} \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta} &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \sin \theta \\ &= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \cdot \lim_{\theta \rightarrow 0} \sin \theta = (1) \cdot (0) = 0 \end{aligned}$$

$$\begin{aligned} \text{(x)} \quad \lim_{x \rightarrow 0} \frac{\sec x - \cos x}{x} &= \lim_{x \rightarrow 0} \frac{1}{\cos x} - \cos x = \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{\cos x} = (1) \frac{\sin(0)}{\cos(0)} = (1) \cdot \frac{0}{1} = 0 \end{aligned}$$

$$\begin{aligned} \text{(xi)} \quad \lim_{x \rightarrow 0} \frac{1 - \cos p\theta}{1 - \cos q\theta} &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{p\theta}{2}}{2 \sin^2 \frac{q\theta}{2}} \quad \because \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2} \\ &= \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2}} = \lim_{x \rightarrow 0} \sin^2 \frac{p\theta}{2} \cdot \frac{\left(\frac{p\theta}{2}\right)^2}{\left(\frac{p\theta}{2}\right)^2} \cdot \frac{1}{\sin^2 \frac{q\theta}{2} \cdot \frac{\left(\frac{q\theta}{2}\right)^2}{\left(\frac{q\theta}{2}\right)^2}} \\ &= \lim_{x \rightarrow 0} \frac{\sin^2 \frac{p\theta}{2}}{\left(\frac{p\theta}{2}\right)^2} \cdot \left(\frac{p\theta}{2}\right)^2 \cdot \frac{1}{\frac{\sin^2 \frac{q\theta}{2}}{\left(\frac{q\theta}{2}\right)^2} \cdot \left(\frac{q\theta}{2}\right)^2} = \lim_{x \rightarrow 0} \left(\frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left(\frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} \cdot \frac{p^2 \cancel{\theta^2} / 4}{q^2 \cancel{\theta^2} / 4} \\ &= \frac{p^2}{q^2} \left(\lim_{x \rightarrow 0} \frac{\sin \frac{p\theta}{2}}{\frac{p\theta}{2}} \right)^2 \cdot \frac{1}{\left(\lim_{x \rightarrow 0} \frac{\sin \frac{q\theta}{2}}{\frac{q\theta}{2}} \right)^2} = \frac{p^2}{q^2} (1)^2 \cdot \frac{1}{(1)^2} = \frac{p^2}{q^2} \end{aligned}$$

$$\begin{aligned}
\text{(xii)} \quad & \lim_{\theta \rightarrow 0} \frac{\tan \theta - \sin \theta}{\sin^3 \theta} \\
&= \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta}{\cos \theta} - \sin \theta}{\sin^3 \theta} = \lim_{\theta \rightarrow 0} \frac{\frac{\sin \theta - \sin \theta \cos \theta}{\cos \theta}}{\sin^3 \theta} \\
&= \lim_{\theta \rightarrow 0} \frac{\sin \theta - \sin \theta \cos \theta}{\sin^3 \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{\sin \theta (1 - \cos \theta)}{\sin^3 \theta \cos \theta} \\
&= \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} = \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin^2 \theta \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} \\
&= \lim_{\theta \rightarrow 0} \frac{1 - \cos^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} = \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\sin^2 \theta \cos \theta (1 + \cos \theta)} \\
&= \lim_{\theta \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} = \lim_{x \rightarrow 0} \frac{1}{\cos \theta (1 + \cos \theta)} \\
&= \frac{1}{\cos(1)(1 + \cos(1))} = \frac{1}{1 \cdot (1 + 1)} = \frac{1}{2}
\end{aligned}$$

Note:

$$\text{a)} \quad \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\text{b)} \quad \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e \quad \text{where} \quad e = 2.718281\dots$$

See proof of (a) and (b) on book at page 23

$$\text{c)} \quad \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a \quad \text{or} \quad \ln a$$

Proof:

$$\text{Put } y = a^x - 1 \quad \dots\dots\dots (i)$$

$$\text{When } x \rightarrow 0 \quad \text{then } y \rightarrow 0$$

$$\text{Also from (i)} \quad 1 + y = a^x$$

Taking log on both sides

$$\ln(1 + y) = \ln a^x \Rightarrow \ln(1 + y) = x \ln a$$

$$\because \ln x^m = m \ln x$$

$$\Rightarrow x = \frac{\ln(1 + y)}{\ln a}$$

$$\begin{aligned}
\text{Now } \lim_{x \rightarrow 0} \frac{a^x - 1}{x} &= \lim_{y \rightarrow 0} \frac{y}{\frac{\ln(1 + y)}{\ln a}} \\
&= \lim_{y \rightarrow 0} \frac{y \ln a}{\ln(1 + y)} = \lim_{y \rightarrow 0} \frac{\ln a}{\frac{1}{y} \ln(1 + y)}
\end{aligned}$$

$$\begin{aligned}
&= \lim_{y \rightarrow 0} \frac{\ln a}{\ln(1+y)^{\frac{1}{y}}} = \frac{\ln a}{\lim_{y \rightarrow 0} \ln(1+y)^{\frac{1}{y}}} & \because \ln x^m = m \ln x \\
&= \frac{\ln a}{\ln\left(\lim_{y \rightarrow 0} (1+y)^{\frac{1}{y}}\right)} = \frac{\ln a}{\ln(e)} & \because \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e \\
&= \frac{\ln a}{1} = \ln a & \because \ln e = 1
\end{aligned}$$

Question # 4

- (i) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{2n} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \right]^2 = e^2$
- (ii) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^{\frac{n}{2}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n}\right)^n \right]^{\frac{1}{2}} = e^{\frac{1}{2}} = \sqrt{e}$
- (iii) $\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^n = \left[\lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n}\right)^{-n} \right]^{-1} = e^{-1} = \frac{1}{e}$
- (iv) $\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^n$
 $= \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^{\frac{3n}{3}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{1}{3n}\right)^{3n} \right]^{\frac{1}{3}} = e^{\frac{1}{3}}$
- (v) $\lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^n = \lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^{\frac{4n}{4}} = \left[\lim_{n \rightarrow +\infty} \left(1 + \frac{4}{n}\right)^{\frac{n}{4}} \right]^4 = e^4.$
- (vi) $\lim_{x \rightarrow 0} (1+3x)^{\frac{2}{x}}$
 $= \lim_{x \rightarrow 0} (1+3x)^{\frac{6}{3x}} = \left[\lim_{x \rightarrow 0} (1+3x)^{\frac{1}{3x}} \right]^6 = e^6$
- (vii) $\lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} (1+2x^2)^{\frac{2}{2x^2}} = \left[\lim_{x \rightarrow 0} (1+2x^2)^{\frac{1}{2x^2}} \right]^2 = e^2$
- (viii) $\lim_{h \rightarrow 0} (1-2h)^{\frac{1}{h}} = \lim_{h \rightarrow 0} (1-2h)^{\frac{-2}{-2h}} = \left[\lim_{h \rightarrow 0} (1-2h)^{\frac{1}{-2h}} \right]^{-2} = e^{-2} = \frac{1}{e^2}$
- (ix) $\lim_{x \rightarrow \infty} \left(\frac{x}{1+x}\right)^x$
 $= \lim_{x \rightarrow \infty} \left(\frac{1+x}{x}\right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + \frac{x}{x}\right)^{-x} = \lim_{x \rightarrow \infty} \left(\frac{1}{x} + 1\right)^{-x}$

$$= \left[\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e^{-1} = \frac{1}{e} \right]$$

$$(x) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} \quad ; \quad x < 0$$

Put $x = -t$ where $t > 0$

When $x \rightarrow 0$ then $t \rightarrow 0$, so

$$\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} = \lim_{t \rightarrow 0} \frac{e^{-1/t} - 1}{e^{-1/t} + 1} = \frac{e^{-1/0} - 1}{e^{-1/0} + 1}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0 -}{0 +}$$

$$= -1$$

$$\because e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$$

$$(xi) \quad \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} \quad ; \quad x > 0$$

$$= \lim_{x \rightarrow 0} \frac{e^{1/x} \left(1 - \frac{1}{e^{1/x}} \right)}{e^{1/x} \left(1 + \frac{1}{e^{1/x}} \right)} = \lim_{x \rightarrow 0} \frac{\left(1 - \frac{1}{e^{1/x}} \right)}{\left(1 + \frac{1}{e^{1/x}} \right)}$$

$$= \frac{1 - \frac{1}{e^{1/0}}}{1 + \frac{1}{e^{1/0}}} = \frac{1 - \frac{1}{e^{\infty}}}{1 + \frac{1}{e^{\infty}}} = \frac{1 - \frac{1}{\infty}}{1 + \frac{1}{\infty}} = \frac{1 - 0}{1 + 0} = 1$$
