Rule I:

If linear factor (ax + b) occurs as a factor of D(x), then there is a partial fraction of the form $\frac{A}{ax+b}$, where A is a constant to be found.

In $\frac{N(x)}{D(x)}$, the polynomial D{x) may be written as,

$D(x) = (a_1x + b)(a_2x + 63).....(a_nx + b_n)$ with all factors distinct.

We have, $\frac{N(x)}{D(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3} + ...+ \frac{A_n}{a_nx+b_n}$,

where $A_1, A_2, ... A_n$ are constants to-be determined.

Note:

General method applicable to resolve all rational fractions of the form $\frac{N(x)}{D(x)}$ is as follows:

(i) The numerator N(x) must be of lower degree than the denominator D(x).
(ii) If degree of N(x) is greater than the degree of D(x), then division is used and the remainder fraction R(x) can be broken into partial fractions.
(iii) Make substitution of constants accordingly
(iv) Multiply both the sides by L.C.M.
(v) Arrange the terms on both sides in descending order.
(vi) Equate the coefficients of like powers of x on both sides, we get as many as equations as there are constants in assumption.
(vii) Solving these equations, we can find the values of constants.

SOLVED EXERCISE 4.1

Resolve into partial fractions.

(1) $\frac{7x-9}{(x+1)(x-3)}$

Solution:

$$\frac{7x-9}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Multiplying both sides by (x+1) (x-3), we get

$7x - 9 = A(x-3) + B(x+1)$ ______ (1)

To find A, we put $x = 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$7(1) - 9 = A(-1-3) + B(-1+1)$

$7 - 9 = a(-4) + B(0)$

$= 16 = -4A$

$-4A = 2 - 16$

Dividing both sides '–4', we get

$A = -4$

To find B, we put $x - 3 = 0 \Rightarrow x = 3$ in eq. (1), we get

$$7(3) - 9 = A(3 - 3) + b(3 + 1)$$
$$21 - 9 = A(0) + B(4)$$
$$12 = 4B$$

Or $\quad 4B = 12$

Dividing both sides by '4', we get

$$B = 3$$

Thus required partial fractions are $\dfrac{-4}{x+1} + \dfrac{3}{x-3}$

Hence, $\dfrac{7x-9}{(x+1)(x-3)} = -\dfrac{4}{x+1} + \dfrac{3}{x-3}$

(2) $\dfrac{x-11}{(x-4)(x+3)}$

Solution:

$$\frac{x-11}{(x-4)(x+3)} = \frac{A}{x-4} + \frac{B}{x+3}$$

Multiplying both sides by $(x - 1)(x + 3)$, we get

$$x - 11 = A(x + 3) + B(x - 1) \quad \underline{\qquad} (1)$$

To find A, we put $x - 4 = 0 \Rightarrow x = 4$ in eq. (1), we get

$$4 - 11 = A(4 + 3) + B(4 - 4)$$
$$-7 = A(7) + B(0)$$
$$-7 = 7A$$

or $\quad 7A = -7$

Dividing both sides by '7', we get

$$A = -1$$

To find B, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$-3 - 11 = A(-3 + 3) + B(-3 - 4)$$
$$-14 = A(0) + B(-7)$$
$$-14 = -7B$$

Or $\quad -7B = -14$

Dividing both sides by '−7', we get

$$B = 2$$

Thus required partial fractions are $\dfrac{-1}{x-4} + \dfrac{2}{x+3}$

Hence, $\dfrac{x-11}{(x-4)(x+3)} = -\dfrac{1}{x-1} + \dfrac{2}{x+3}$

(3) $\dfrac{3x-1}{x^2-1}$

Solution:

$$\frac{3x-1}{x^2-1}=\frac{3x-1}{(x-1)(x+1)}$$

Let $\frac{3x-1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$

Multiplying both sides by $(x-1)(x+1)$, we get

$$3x-1=A(x+1)+B(x-1) \quad \_\_\_\_\_\_ (1)$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$3(1)-1=A(1+1)+B(1-1)$$
$$3-1=A(2)+B(0)$$
$$2=2A$$

or $\quad 2A=2$

Dividing both sides by '2', we get

$$A=1$$

To find B, we put $x+1=0 \Rightarrow x=-1$ in eq. (1), we get

$$3(-1)-1=A(-1+1)+B(-1-1)$$
$$-3-1=A(0)+B(-2)$$
$$-4=-2B$$

Or $\quad -2B=-4$

Dividing both sides by '−2', we get

$$B=2$$

Thus required partial fractions are $\frac{-1}{x-1}+\frac{2}{x+1}$

Hence, $\frac{3x-1}{x^2-1}=\frac{-1}{x-1}+\frac{2}{x+1}$

(4) $\frac{x-5}{x^2+2x-3}$

Solution:

$$\frac{x-5}{x^2+2x-3}=\frac{x-5}{x^2+3x-x-3}$$
$$=\frac{x-5}{x(x+3)-1(x+3)}=\frac{x-5}{(x-1)(x+3)}$$

Let $\frac{x-5}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$

Multiplying both sides by $(x-1)(x+3)$, we get

$$x-5=A(x+3)+B(x-1) \quad \_\_\_\_\_\_ (1)$$

To find A, we put $x-1=0 \Rightarrow x=1$ in eq. (1), we get

$$1-5=A(1+3)+B(1-1)$$
$$-4=A(1+3)+B(0)$$
$$-4=4A$$

Or $\quad 4A=-4$

Dividing both sides by '4', we get

$$A = -1$$

To find B, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$-3 - 5 = A(-3 + 3) + B(-3 - 1)$$
$$-8 = A(0) + B(-4)$$
$$-8 = -4B$$

Or $\quad -4B = -8$

Dividing both sides by '–4', we get

$$B = 2$$

Thus required partial fractions are $\dfrac{-1}{x-1} + \dfrac{2}{x+3}$

Hence, $\dfrac{x-5}{x^2+2x-3} = -\dfrac{1}{x-1} + \dfrac{2}{x+3}$

(5) $\dfrac{3x+3}{(x-1)(x+2)}$

Solution:

Let $\dfrac{3x+3}{(x-1)(x+2)} = \dfrac{A}{x-1} + \dfrac{B}{x+2}$

Multiplying both sides by $(x - 1)(x + 2)$, we get

$$3x + 3 = A(x + 2) + B(x - 1) \_\_\_\_\_\_ (1)$$

To find A, we put $x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$3(1) + 3 = A(1 + 2) + B(1 - 1)$$
$$3 + 3 = A(3) + B(0)$$
$$6 = 3A$$

Or $\quad 3A = 6$

Dividing both sides by '3', we get

$$A = 2$$

To find B, we put $x + 2 = 0 \Rightarrow x = -2$ in eq. (1), we get

$$3(2) + 3 = A(-2 + 2) + B(-2 - 1)$$
$$-6 + 3 = A(0) + B(-3)$$
$$-3 = -3B$$

Or $\quad -3B = -3$

Dividing both sides by '–3', we get

$$B = 1$$

Thus required partial fractions are $\dfrac{2}{x-1} + \dfrac{1}{x+2}$

Hence, $\dfrac{3x+3}{(x-1)(x+2)} = \dfrac{2}{x-1} + \dfrac{1}{x+2}$

(6) $\dfrac{7x-25}{(x-4)(x-3)}$

Solution:

Let $\dfrac{7x-25}{(x-4)(x-3)} = \dfrac{A}{x-4} + \dfrac{B}{x-3}$

Multiplying both sides by $(x-4)(x-3)$, we get

$$7x-25 = A(x-3) + B(x-4) \quad \_\_\_\_\_\_ (1)$$

To find A, we put $x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$7(4) - 25 = A(4-3) + B(4-4)$$
$$28 - 25 = A(1) + B(0)$$
$$3 = A$$

Or $\quad A = 3$

To find B, we put $x - 3 = 0 \Rightarrow x = 3$ in eq. (1), we get

$$7(3) - 25 = A(3-3) + B(3-4)$$
$$21 - 25 = A(0) + B(-1)$$
$$-B = -4$$

Or $\quad B = 4$

$$B = 1$$

Thus required partial fractions are $\dfrac{3}{x-4} + \dfrac{4}{x-3}$

Hence, $\dfrac{7x-25}{(x-4)(x-3)} = \dfrac{3}{x-4} + \dfrac{4}{x-3}$

(7) $\dfrac{x^2+2x+1}{(x-2)(x+3)}$

Solution:

$$\frac{x^2+2x+1}{(x-2)(x+3)} = \frac{x^2+2x+1}{x^2+3x-2x-6}$$
$$= \frac{x^2+2x+1}{x^2+x-6}$$

By long division, we have

$$\begin{array}{r} 1 \\ x^2+x-6\overline{)\,x^2+2x+1} \\ \underline{\pm x^2 \pm x \mp 6} \\ x = 7 \end{array}$$

$$\frac{x^2+2x+1}{(x-2)(x+3)} = 1 + \frac{x+7}{x^2+x-6}$$

$$= 1 + \frac{x+7}{(x-2)(x+3)}$$

Let $\dfrac{x+7}{(x-2)(x+3)} = \dfrac{A}{x-2} + \dfrac{B}{x+3}$

Multiplying both sides by $(x - 2)(x + 3)$, we get

$$x + 7 = A(x + 3) + B(x - 2) \quad \_\_\_\_\_\_ (1)$$

To find A, we put $x - 2 = 0 \Rightarrow x = 2$ in eq. (1), we get

$$2 + 7 = A(2 + 3) + B(2 - 2)$$

$$9 = A(5) + B(0)$$

$$9 = 5A$$

Or $\quad 5A = 9$

Dividing both sides by '5', we get

$$A = \frac{9}{5}$$

To find B, we put $x + 3 = 0 \Rightarrow x = -3$ in eq. (1), we get

$$-3 + 7 = A(-3 + 3) + B(-3 - 2)$$

$$4 = A(0) + B(-5)$$

$$4 = -5B$$

Or $\quad -5B = 4$

Dividing both sides by '−5', we get

$$B = \frac{4}{5}$$

Thus required partial fractions are $\dfrac{9/5}{x-2} + \dfrac{-4/5}{x+3}$

Hence, $\dfrac{x^2+2x+1}{(x-2)(x+3)} = 1 + \dfrac{9}{5(x-2)} - \dfrac{4}{5(x+3)}$

(8) $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

Solution:

By long division, we have

$$\begin{array}{r l}
 & 2x + 3 \\ \hline
3x^2 - 2x - 1 \,\big) & 6x^3 + 5x^2 - 7 \\
 & \underline{\pm 6x^3 \mp 4x^2 \mp 2x} \\
 & \qquad 9x^2 + 2x - 7 \\
 & \qquad \underline{\pm 9x^2 \mp 6x \mp 3} \\
 & \qquad\qquad 8x - 4
\end{array}$$

$$\frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{8x-4}{3x^2-3x+x-1}$$
$$= 2x+3+\frac{8x-4}{3x(x-1)+1(x-1)}$$
$$= 2x+3+\frac{8x-4}{(3x+1)(x-1)}$$

Let $\frac{8x-4}{(3x+1)(x-1)} = \frac{A}{3x+1}+\frac{B}{x-1}$

Multiplying both sides by $(3x + 1)(x - 1)$, we get

$$8x-4 = A(x-1)+B(3x+1) \quad \text{\_\_\_\_\_\_} (1)$$

To find A, we put $3x + 1 = 0 \Rightarrow 3x = -1 \Rightarrow x = - -\frac{1}{3}$ in eq. (1), we get

$$8\left(-\frac{1}{3}\right)-4 = A\left(-\frac{1}{3}-1\right)+B\left[3\left(-\frac{1}{3}\right)+1\right]$$
$$-\frac{8}{3}-4 = A\left(-\frac{4}{3}\right)+B(0)$$
$$-\frac{20}{3} = -\frac{4}{3}A$$

Or $\quad -\frac{4}{3}A = -\frac{20}{3}$

$\Rightarrow \quad \frac{4}{3}A = \frac{20}{3}$

$$A = \frac{20}{3}\times\frac{3}{4}$$
$$A = 5$$

To find B, we put $x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$8(1)-4 = A(1-1)+B[3(1)+1]$$
$$8-4 = A(0)+B(4)$$
$$4 = 4B$$

Or $\quad 4B = 4$

Dividing both sides by '4', we get

$$B = 1$$

Thus required partial fractions are $\frac{5}{3x+1}+\frac{1}{x-1}$

Hence, $\frac{6x^3+5x^2-7}{3x^2-2x-1} = 2x+3+\frac{5}{3x+1}+\frac{1}{x-1}$

Resolution of a fraction when D(x) consists of repeated linear factors:

Rule II:

If a linear factor (ax + b) occurs n times as a factor of D(x), then there are n partial fractions of the form.

$$\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n}$$ where A_1 A_2, …, A_n, are constants and $n \geq 2$ is a positive integer.

$$\frac{N(x)}{D(x)}=\frac{A_1}{(ax+b)}+\frac{A_2}{(ax+b)^2}+\ldots+\frac{A_n}{(ax+b)^n}$$