$$\frac{2}{3}+C=1 \qquad \because A=\frac{2}{3}$$

$$C=1-\frac{2}{3}$$

$$C=\frac{1}{3}$$

Thus required partial fractions are $\dfrac{2/3}{x+1}+\dfrac{1/3x+1/3}{x^2-x+1}$

Hence, $\dfrac{x^2+1}{x^3+1}=\dfrac{2}{3(x+1)}+\dfrac{x+1}{3(x^2-x+1)}$

Resolution of a fraction when D (x) has repeated irreducible quadratic factors.

Rule IV:

If a quadratic factor $(ax^1 + bx + c)$ with a $\neq 0$, occurs twice in the denominator, the corresponding partial fractions are

$$\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}$$

The constants A, B, C and D are found in the usual way.

SOLVED EXERCISE 4.4

Resolve into partial fractions.

(1) $\dfrac{x^3}{(x^2+4)^2}$

Solution:

Let $\dfrac{x^3}{(x^2+4)^2}=\dfrac{Ax+B}{x^2+4}+\dfrac{Cx+D}{(x^2+4)^2}$

Multiplying both sides by $(x^2+4)^2$, we get

$$x^3=(Ax+B)(x^2+4)+Cx+D \qquad \text{\_\_\_\_\_\_ (1)}$$

$$x^3=Ax^3+4Ax+Bx^2+4B+Cx+D$$

To find A, B, C and D, equating coefficient of x^3, x^2, x and constant on both sides of eq. (2),

We get.

Coefficient of x^3: $A=1$

Coefficient of x^2: $B=0$

Coefficient of x: $4A+C=0$ ______ (2)

Constant: $4B + D = 0$ ______ (3)

Put A = 1 in eq. (2), we get

$4(1) + C = 0$

$4 + C = 0$

$C = -4$

Put B = 0 in eq. (3), we get

$4(0) + D = 0$

$D = 0$

Thus required partial fractions are $\frac{(1)x+0}{x^2+4} + \frac{(-4)x+0}{(x^2+4)^2}$

Hence, $\frac{x^3}{(x^3+4)^2} = \frac{x}{x^2+4} - \frac{4x}{3(x^2+4)}$

(2) $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2}$

Solution:

Let $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

Multiplying both sides by $(x+1)(x^2+1)^2$, we get

$$x^4+3x^2+x+1 = A(x^2+1)^2 + (Bx+C)(x+1)(x^2+1) + (Dx+E)(x+1) \_\_\_ (1)$$

$$x^4+3x^2+x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x^2+x+1) + Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = Ax^4+2Ax^2+A+Bx^4+Bx^3+Bx^2+Bx+Cx^3+Cx^2$$
$$+Cx+C+Dx^2+Dx+Ex+E$$

$$x^4+3x^2+x+1 = Ax^4+Bx^4+Bx^3+Cx^3+2Ax^2+Bx^2+Cx^2$$
$$+Dx^2+Bx+Cx+Dx+Ex+A+C+E \_\_\_\_ (2)$$

To Find A, we put $x + 1 = 0 \Rightarrow x = -1$ in eq. (1), we get

$$(-1)^4+3(-1)^2(-1)+1 = A((-1)^2+1)^2$$

$$1+3-1+1 = A(1+1)^2$$

$$4 = A(2)^2$$

$$4 = 4A$$

Or $4A = 4$

$\Rightarrow$ $A = 1$

To find B, C, D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq. (2), we get.

Coefficient of x^4: $A + B = 1$ ______(3)
Coefficient of x^3: $B + C = 0$ ______(4)
Coefficient of x^2: $2A + B + C + D = 3$ ______(5)
Coefficient of x: $B + C + D + E = 1$ ______(6)

Put $A = 1$ in eq. (3), we get

$$1 + B = 1$$
$$B = 1 - 1$$
$$B = 0$$

Put $B = 0$ in eq. (4), we get

$$0 + C = 0$$
$$C = 0$$

Put $A = 1, B = 0, C = 0$ in eq. (5), we get

$$2(1) + 0 + 0 + D = 3$$
$$2 + D = 3$$
$$D = 3 - 2$$
$$D = 1$$

Put $B = 0, C = 0, D = 1$ in eq. (6) we get

$$0 + 0 + 1 + E = 1$$
$$1 + E = 1$$
$$E = 1 - 1$$
$$E = 0$$

Thus required partial fractions are $\frac{1}{x+1} + \frac{(0)x+0}{x^2+1} + \frac{(1)x+0}{(x^2+1)^2}$

Hence, $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{x+1} + \frac{x}{(x^2+1)^2}$

(3) $\frac{x^2}{(x+1)(x^2+1)^2}$

Solution:

$$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying both sides by $(x - 1)(x^2 + 1)^2$, we get

$$x^2 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-1) + (Dx+E)(x-1) \quad \_\_\_(1)$$

$$x^2 = A(x^4+2x^2+1) + (Bx+C)(x^3-x^2+x-1)$$
$$+Dx^2 - Dx + Ex - E$$

$$x^2 = Ax^4 + 2Ax^2 + A + Bx^4 - Bx^3 + Bx^2 - Bx + Cx^3$$
$$-Cx^2 + Cx - C + Dx^2 - Dx + Ex - E$$

$$x^2 = Ax^4 + Bx^4 - Bx^3 + Cx^3 + 2Ax^2 + Bx^2 - Cx^2 + Dx^2$$
$$-Bx + Cx - Dx + Ex + A - C - E \quad \_\_\_(2)$$

To find A, we put $x - 1 = 0 \Rightarrow x = 1$ in eq. (1), we get

$$(-1)^2 = A((-1)^2+1)^2$$
$$1 = A(1+1)^2$$
$$1 = A(2)^2$$
$$1 = 4A$$
$$\Rightarrow \quad A = \frac{1}{4}$$

To find B, C, D and E, equating coefficient of x^4, x^3, x^2 and x on both sides of eq. (2), we get.

Coefficient of x^4:	$A + B = 0$	______(3)
Coefficient of x^3:	$-B + C = 0$	______(4)
Coefficient of x^2:	$2A + B - C + D = 1$	______(5)
Coefficient of x:	$-B + C - D + E = 0$	______(6)

Put $A = \frac{1}{4}$ in eq. (4), we get

$$\frac{1}{4} + B = 0$$
$$B = -\frac{1}{4}$$

Put $B = \frac{1}{4}$, in eq. (4), we get

$$-\left(-\frac{1}{4}\right) + C = 0$$
$$\frac{1}{4} + C = 0$$
$$C = -\frac{1}{4}$$

Put $A = \frac{1}{4}, B = -\frac{1}{4}, C = -\frac{1}{4}$, in eq. (5), we get

$$2\left(\frac{1}{4}\right)+\left(-\frac{1}{4}\right)-\left(-\frac{1}{4}\right)+D=1$$

$$\frac{2}{4}-\frac{1}{4}+\frac{1}{4}+D=1$$

$$\frac{1}{2}+D=1$$

$$D=1-\frac{1}{2}$$

$$D=\frac{1}{2}.$$

Put $B=-\frac{1}{4}, C=-\frac{1}{4}, D=\frac{1}{2}$ in eq. (6) we get

$$-\left(-\frac{1}{4}\right)+\left(-\frac{1}{4}\right)-\frac{1}{2}+E=0$$

$$\frac{1}{4}-\frac{1}{4}-\frac{1}{2}+E=0$$

$$E=0\ \frac{1}{2}$$

Thus, required partial fractions are $\frac{1/4}{x-1}+\frac{(-1/4)x+(-1/4)}{x^2+1}+\frac{(1/2)x+(1/2)}{(x^2+1)^2}$

Hence, $\frac{x^2}{(x-1)(x^2+1)^2}=\frac{1}{4(x-1)}-\frac{x+1}{4(x^2+1)}+\frac{x+1}{2(x^2+1)^2}$

(4) $\frac{x^2}{(x-1)(x^2+1)^2}$

Solution:

(5) $\frac{x^4}{(x^2+2)^2}$

Solution:

$$\frac{x^4}{(x^2+2)^2}=\frac{x^4}{x^4+4x^2+4}$$

By long division, we have

$$\begin{array}{r|l} & \quad 1 \\ \hline x^4+4x^2+4 & x^4 \\ & \underline{\pm x^4 \pm 4x^2 \pm 4} \\ & \;-4x^2-4 \\ & -\left(4x^2+4\right) \end{array}$$

$$\frac{x^4}{\left(x^2+2\right)^2}=1-\frac{4x^2+4}{\left(x^2+2\right)^2}$$

Let $$\frac{4x^2+4}{\left(x^2+2\right)^2}=\frac{Ax+B}{\left(x^2+2\right)}+\frac{Cx+D}{\left(x^2+2\right)^2}$$

Multiply both sides by $\left(x^2+2\right)^2$, we get

$$4x^2+4=A\left(Ax+B\right)\left(x^2+2\right)+\left(Cx+D\right)$$

$$4x^2+4=Ax^3+2Ax+Bx^2+2B+Cx+D$$

$$4x^2+4=Ax^3+Bx^2+2Ax+Cx+2B+D \quad \_\_\_\_\_\_ (1)$$

To Find A, B, C, and D, equating coefficient of x^3, x^2, x and constant on both sides of eq. (1), we get.

Coefficient of x^3: $\quad A = 0$

Coefficient of x^2: $\quad B = 4$

Coefficient of x: $2A + C = 0$ _______ (2)

Constant: $2B + D = 0$ _______ (3)

Put $A = 0$ in eq. (2), we get.

$$2(0)+C=0$$
$$C=0$$

Put $B = 4$ in eq. (3), we get.

$$2(4)+D=4$$
$$8+D=4$$
$$D=4-8$$
$$D=-4$$

Thus, required partial fractions are $\dfrac{(0)x+4}{x^2+2}+\dfrac{(0)x+(-4)}{\left(x^2+2\right)^2}$

Hence, $$\frac{x^4}{\left(x^2+2\right)^2}=1-\left[\frac{4}{x^2+2}-\frac{4}{\left(x^2+2\right)^2}\right]$$

$$1-\frac{4}{x^2+2}+\frac{4}{\left(x^2+2\right)^2}$$

(6) $\dfrac{x^5}{(x^2+1)^2}$

Solution:

$$\frac{x^5}{(x^2+1)^2}=\frac{x^5}{x^4+2x^2+1}$$

By long division, we have

$$\begin{array}{r|l} & x \\ \hline x^4+2x^2+1 & x^5 \\ & \underline{\pm x^5 \pm 2x^3 \pm x} \\ & -2x^3-x \\ & -(4x^3+x) \end{array}$$

$$\frac{x^5}{(x^2+1)^2}=x-\frac{2x^3+x}{(x^2+1)^2}$$

Let $\dfrac{2x^3+x}{(x^2+1)^2}=\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$

Multiply both sides by $(x^2+1)^2$, we get

$$2x^3+x=(Ax+B)(x^2+1)+(Cx+D)$$

$$2x^3+x=Ax^3+Ax+Bx^2+B+Cx+D$$

$$2x^3+x=Ax^3+Bx^2+Ax+Cx+B+D \quad \text{\_\_\_\_\_\_}(1)$$

To Find A, B, C, and D, equating coefficient of x^3, x^2, x and constant on both sides of eq. (1), we get.

Coefficient of x^3: $A=2$

Coefficient of x^2: $B=0$

Coefficient of x: $A+C=1$ ______ (2)

Constant: $B+D=0$ ______ (3)

Put A = 2 in eq. (2), we get.

$$A+C=1$$
$$2+C=1$$
$$C=1-2$$
$$C=-1$$

Put B = 0 in eq. (3), we get.

$$0+D=0$$
$$D=0$$

Thus, required partial fractions are $\dfrac{2x+0}{x^2+1}+\dfrac{(-1)x+0}{(x^2+1)^2}$

$$\text{Hence, } \frac{x^5}{(x^2+1)^2} = x - \left[\frac{2x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right]$$

$$= x - \left[\frac{2x}{x^2+1} - \frac{x}{(x^2+1)^2}\right]$$

$$= x - \left[\frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}\right]$$

SOLVED MISCELLANEOUS EXERCISE - 4

Q1. Multiple Choice Questions

Four possible answers are given for the following questions. Tick (✓) the correct answer.

(i) The identity $(5x + 4)^2 = 25x^2 + 40x + 16$ is true for

(a) one value of x (b) two values of x

(c) all values of x (d) none of these

(ii) A function of the form $(x) = \frac{N(x)}{D(x)}$, with $D(x) \neq 0$, where N(x) and D(x) are polynomials in x is called

(a) an identity (b) an equation

(c) a fraction (c) none of these

(iii) A fraction in which the degree of the numerator is greater or equal the degree of denominator is called:

(a) a proper fraction (b) an improper fraction

(c) an equation (d) algebraic relation

(iv) A fraction in which the degree of numerator is less than the degree of the denominator is called

(a) an equation (b) an improper fraction

(c) an identity (d) a proper fraction

(v) $\frac{2x+1}{(x+1)(x-1)}$ is:

(a) an improper fraction (b) an equation

(c) a proper fraction (d) none of these

(vi) $(x + 3)^2 = x^2 + 6x + 9$ is:

(a) a linear equation (b) an equation

(c) an identity (d) none of these