Exercise 13.1

1. Evaluate without using tables / calculator:

i) $\sin^{-1}(1)$ ii) $\sin^{-1}(-1)$ iii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

iv) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ v) $\cos^{-1}\left(\frac{1}{2}\right)$ vi) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

vii) $\cot^{-1}(-1)$ viii) $\csc^{-1}\left(\frac{-2}{\sqrt{3}}\right)$ ix) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

2. Without using table/ Calculator show that:

i) $\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13}$ ii) $2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$

iii) $\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$

3) Find the value of each expression:

i) $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$ ii) $\sec\left(\cos^{-1}\frac{1}{2}\right)$ iii) $\tan\left(\cos^{-1}\frac{\sqrt{3}}{2}\right)$

iv) $\csc(\tan^{-1}(-1))$ v) $\sec\left(\sin^{-1}(-\frac{1}{2})\right)$ vi) $\tan(\tan^{-1}(-1))$

vii) $\sin\left(\sin^{-1}(\frac{1}{2})\right)$ viii) $\tan\left(\sin^{-1}(-\frac{1}{2})\right)$ ix) $\sin(\tan^{-1}(-1))$

Solution Are Given Below

Solution # 1

(i) Suppose $y = \sin^{-1}(1)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = 1$

$\Rightarrow y = \frac{\pi}{2}$ $\because \sin\left(\frac{\pi}{2}\right) = 1$ *Answer*

(ii) Suppose $y = \sin^{-1}(-1)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \sin y = -1$

$$\Rightarrow \quad y=-\frac{\pi}{2} \qquad \because \ \sin\left(-\frac{\pi}{2}\right)=-1 \ \textit{Answer}$$

(iii) Suppose $y=\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ where $y\in[0,\pi]$

$$\Rightarrow \cos y=\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad y=\frac{\pi}{6} \qquad \because \ \cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2} \ \textit{Answer}$$

(iv) Suppose $y=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ where $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$$\Rightarrow \tan y=-\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad y=-\frac{\pi}{6} \qquad \because \ \tan\left(-\frac{\pi}{6}\right)=-\frac{1}{\sqrt{3}} \ \textit{Answer}$$

(v) Suppose $y=\cos^{-1}\left(\frac{1}{2}\right)$ where $y\in[0,\pi]$

$$\Rightarrow \cos y=\frac{1}{2}$$

$$\Rightarrow \quad y=\frac{\pi}{3} \qquad \because \ \cos\left(\frac{\pi}{3}\right)=\frac{1}{2}. \ \textit{Answer}$$

(vi) Suppose $y=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ where $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$

$$\Rightarrow \tan y=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad y=\frac{\pi}{6} \qquad \because \ \tan\left(\frac{\pi}{6}\right)=\frac{1}{\sqrt{3}}. \ \textit{Answer}$$

(vii) Suppose $y=\cot^{-1}(-1)$ where $y\in\,]0,\pi[$

$$\Rightarrow \cot y=-1$$

$$\Rightarrow \frac{1}{\cot y}=\frac{1}{-1}$$

$$\Rightarrow \tan y=-1$$

$$\Rightarrow \quad y=\frac{3\pi}{4} \qquad \because \ \tan\left(\frac{3\pi}{4}\right)=-1 \ \textit{Answer}$$

(viii) Suppose $y=\text{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ where $y\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $y\neq 0$

$$\Rightarrow \text{cosec}\, y=-\frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{1}{\operatorname{cosec} y} = \frac{1}{-2/\sqrt{3}}$$

$$\Rightarrow \sin y = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \quad y = -\frac{\pi}{3} \qquad \because \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \textit{ Answer}$$

(ix) Suppose $y = \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \sin y = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow y = -\frac{\pi}{4} \qquad \because \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Solution # 2

(i) Suppose $\alpha = \sin^{-1}\frac{5}{13}$ ………….. (i) where $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \sin\alpha = \frac{5}{13}$$

Now $\cos\alpha = \pm\sqrt{1-\sin^2\alpha}$

Since $\alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ therefore *cos* is +ive.

$$\cos\alpha = +\sqrt{1-\sin^2\alpha}$$

$$= \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}}$$

$$= \sqrt{\frac{144}{169}} = \frac{12}{13}$$

Now $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{5/13}{12/13} = \frac{5}{12}$

$$\Rightarrow \quad \alpha = \tan^{-1}\frac{5}{12} \text{ ………….. (ii)}$$

From (i) and (ii)

$$\tan^{-1}\frac{5}{12} = \sin^{-1}\frac{5}{13} \qquad \textit{Proved}$$

(ii) Suppose $\alpha = 2\cos^{-1}\frac{4}{5}$ ………….. (i) where $\frac{\alpha}{2} \in [0, \pi]$

$$\Rightarrow \frac{\alpha}{2} = \cos^{-1}\frac{4}{5} \quad \Rightarrow \cos\frac{\alpha}{2} = \frac{4}{5}$$

Now $\quad \sin\frac{\alpha}{2} = \pm\sqrt{1-\cos^2\frac{\alpha}{2}}$

Since $\frac{\alpha}{2} \in [0,\pi]$ therefore *sin* is +ive.

$$\sin\frac{\alpha}{2} = +\sqrt{1-\cos^2\frac{\alpha}{2}}$$

$$= \sqrt{1-\frac{16}{25}} \quad = \sqrt{1-\frac{16}{25}}$$

$$= \sqrt{\frac{9}{25}} \quad = \frac{3}{5}$$

Now $\quad \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \quad \Rightarrow \quad \sin\alpha = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)$

$\Rightarrow \sin\alpha = \frac{24}{25} \quad \Rightarrow \quad \alpha = \sin^{-1}\frac{24}{25}$ ………….. (ii)

From (i) and (ii)

$2\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{24}{25}$ *Prove*

(iii) Suppose $\alpha = \cos^{-1}\frac{4}{5}$ ………….. (i) where $\alpha \in [0,\pi]$

$\Rightarrow \cos\alpha = \frac{4}{5}$

Now $\quad \sin\alpha = \pm\sqrt{1-\cos^2\alpha}$

Since $\alpha \in [0,\pi]$ therefore *sin* is +ive.

$$\sin\alpha = +\sqrt{1-\cos^2\alpha}$$

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}.$$

Now $\quad \cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{4/5}{3/5} = \frac{4}{3}$

$\Rightarrow \quad \alpha = \cot^{-1}\frac{4}{3}$ ………….. (ii)

From (i) and (ii)

$\cos^{-1}\frac{4}{5} = \cot^{-1}\frac{4}{3}$ *Proved*

Solution # 3

(i) Suppose $y = \sin^{-1}\frac{1}{\sqrt{2}}$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow \quad \sin y = \frac{1}{\sqrt{2}}$

$$\Rightarrow \quad y=\frac{\pi}{4} \qquad \because \sin\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Now $\cos\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)=\cos y=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ *Answer*

(ii) Suppose $y=\cos^{-1}\frac{1}{2}$ where $y\in[0,\pi]$

$$\Rightarrow \quad \cos y=\frac{1}{2}$$

$$\Rightarrow \quad y=\frac{\pi}{3} \qquad \because \cos\frac{\pi}{3}=\frac{1}{2}$$

Now $\sec\left(\cos^{-1}\frac{1}{2}\right)=\sec y=\frac{1}{\cos y}=\frac{1}{\cos \pi/3}=\frac{1}{1/2}=2$ *Answer*

(iii) *Do yourself* □

(iv) Suppose $y=\tan^{-1}(-1)$ where $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$\Rightarrow \tan y=-1$$

$$\Rightarrow \quad y=-\frac{\pi}{4} \qquad \because \tan\left(-\frac{\pi}{4}\right)=-1$$

Now $\operatorname{cosec}\left(\tan^{-1}(-1)\right)=\operatorname{cosec} y=\frac{1}{\sin y}=\frac{1}{\sin\left(-\pi/4\right)}=\frac{1}{-1/\sqrt{2}}=-\sqrt{2}$ *Answer*

(v) *Do yourself*

(vi) Suppose $y=\tan^{-1}(-1)$ where $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$\Rightarrow \tan y=-1$$

$$\Rightarrow \quad y=-\frac{\pi}{4} \qquad \because \tan\left(-\frac{\pi}{4}\right)=-1$$

Now $\tan\left(\tan^{-1}(-1)\right)=\tan y=\tan\left(-\frac{\pi}{4}\right)=-1$ *Answer*

(vii) Suppose $y=\sin^{-1}\frac{1}{2}$ where $y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$$\Rightarrow \quad \sin y=\frac{1}{2}$$

$$\Rightarrow \quad y=\frac{\pi}{6} \qquad \because \sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

Now $\sin\left(\sin^{-1}\frac{1}{2}\right)=\sin y=\sin\frac{\pi}{6}=\frac{1}{2}$ *Answer* [

(viii) Suppose $y = \sin^{-1}\left(-\frac{1}{2}\right)$ where $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \quad \sin y = -\frac{1}{2}$$

$$\Rightarrow \quad y = -\frac{\pi}{6} \qquad \because \ \sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$$

Now $\tan\left(\sin^{-1}\left(-\frac{1}{2}\right)\right) = \tan y = \tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$ *Answer* □

(ix) *Do yourself*