EXERCISE 6.9

1. By a rotation of axes, eliminate the xy-term in each of the following equations. Identify the conic and find its elements.

(i) $4x^2 - 4xy + y^2 - 6 = 0$

Solution. $4x^2 - 4xy + y^2 - 6 = 0$ ... (1)

Here $a = 4$, $b = 1$, $2h = -4$ the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-4}{4-1} = -\frac{4}{3} \Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = -\frac{4}{3}$$

$$6\tan\theta = 4\tan^2\theta - 4 \Rightarrow 4\tan^2\theta - 6\tan\theta - 4 = 0$$

$$2\tan^2\theta - 3\tan q - 2 = 0$$

$$\tan\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm \sqrt{25}}{4} = \frac{3 \pm 5}{4}$$

$$= 2, -\frac{1}{2} \implies \tan\theta = 2 \text{ (as } \theta \text{ is in the first quadrant)}$$

Now $\tan\theta = 2 = \frac{2}{1} \implies$ base $= 1$, $\perp = 2$, hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = \frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y \\ y &= X\sin\theta + Y\cos\theta = \frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y \end{aligned}\right\} \quad \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)^2 - 4\left(\frac{1}{\sqrt{5}}X - \frac{2}{\sqrt{5}}Y\right)\left(\frac{2}{\sqrt{5}}X + \frac{1}{\sqrt{5}}Y\right)$$
$$+\left(\frac{2}{\sqrt{5}}X + \frac{2}{\sqrt{5}}Y\right)^2 - 6 = 0$$

$$4\left(\frac{1}{5}X^2 - \frac{4}{5}XY + \frac{4}{5}Y^2\right) - 4\left(\frac{2}{5}X^2 - \frac{3}{5}XY + \frac{2}{5}Y^2\right)$$
$$+\left(\frac{4}{5}X^2 - \frac{4}{5}XY + \frac{1}{5}Y^2\right) - 6 = 0$$

$$\left(\frac{4}{5} - \frac{8}{5} + \frac{4}{5}\right)X^2 + \left(-\frac{16}{5} + \frac{12}{5} + \frac{4}{5}\right)XY + \left(\frac{16}{5} + \frac{8}{5} + \frac{1}{5}\right)Y^2 - 6 = 0$$

$$25Y^2 - 30 = 0 \implies Y^2 = \frac{6}{5} \implies Y = \pm\sqrt{\frac{6}{5}}$$

represents a pair of lines. To find their equations in xy - plane, we have

From (2), we have

$$X - 2Y = \sqrt{5}\,x \qquad (3)$$

$$2X + Y = \sqrt{5}\,y \qquad (4)$$

Multiplying (3) by 2, we get

$$2X - 4Y = 2\sqrt{5}\,x \qquad (5)$$

Subtracting equation (5) from (6), we get

$$5Y = \sqrt{5}\ y - 2\sqrt{5}\ x \implies Y = \frac{\sqrt{5}}{5}(y - 2x) = -\frac{1}{\sqrt{5}}(2x - y)$$

$$\pm\sqrt{\frac{6}{5}} = -\frac{1}{\sqrt{5}}(2x - y) \implies \pm\sqrt{6} = -(2x - y)$$

$$2x - y \pm \sqrt{6} = 0 \implies 2x - y + \sqrt{6} = 0\ ,\ 2x - y - \sqrt{6} = 0$$

(ii) Identify : $x^2 - 2xy + y^2 - 8x - 8y = 0$

Solution. $x^2 - 2xy + y^2 - 8x - 8y = 0$ ... (1)

Here $a = 1$, $b = 1$, $2h = -2$ the angle θ though which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-2}{1 - 1} = \infty \implies 2\theta = 90^\circ \implies \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{aligned}\right] \quad \text{... (2)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X - Y}{\sqrt{2}}\right)^2 - 2\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) + \left(\frac{X + Y}{\sqrt{2}}\right)^2 - 8\left(\frac{X - Y}{\sqrt{2}}\right) - 8\left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) - \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2X.Y + Y^2)$$
$$- \frac{8}{\sqrt{2}}(X - Y) - \frac{8}{\sqrt{2}}(X + Y) = 0$$

$$X^2 - 2XY + Y^2 - 2X^2 + 2Y^2 + X^2 + 2XY + Y^2 - 8\sqrt{2}\ X$$
$$+ 8\sqrt{2}\ Y - 8\sqrt{2}\ X - 8\sqrt{2}\ Y = 0$$

$$4Y^2 - 16\sqrt{2}\ X = 0 \implies Y^2 = 4\sqrt{2}\ X \quad (3)$$

which represents a parabola. In xy-plane, we have

From 2i), we have

$$X - Y = \sqrt{2}\ x \quad (4)$$

and $$X + Y = \sqrt{2}\ y \quad (5)$$

Adding (3) and (4), we get

$$2X = \sqrt{2}\ (x + y) \qquad \Rightarrow \quad X = \frac{1}{\sqrt{2}}\ (x + y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}\ (x + y) - Y = \sqrt{2}\ x \quad \Rightarrow \quad Y = -\sqrt{2}\ x + \frac{1}{\sqrt{2}}\ (x + y)$$

$$= \frac{-2x + x + y}{\sqrt{2}} = \frac{1}{\sqrt{2}}\ (y - x)$$

Thus $X = \frac{1}{\sqrt{2}}\ (x + y)$ and $Y = \frac{1}{\sqrt{2}}\ (y - x)$

Elements of the parabola are

Focus of (3) is $Y = 0,\ X = \sqrt{2}$

i.e., $\frac{1}{\sqrt{2}}\ (x + y) = \sqrt{2}$ and $\frac{1}{\sqrt{2}}\ (y - x) = 0$

$x + y = 2$ and $y - x = 0$

Adding: $x + y = 2$

$\underline{-x + y = 0}$

$2y = 2 \quad \Rightarrow \quad y = 1$

Put $y = 1$ in $x + y = 2$, we get

$x + 1 = 2 \quad \Rightarrow \quad x = 1$

$(1, 1)$ is the focus of (1)

Vertex of (3) is $X = 0,\ Y = 0$

i.e., $\frac{1}{\sqrt{2}}\ (x + y) = 0 \quad \Rightarrow \quad x + y = 0$

and $\frac{1}{\sqrt{2}}\ (y - x) = 0 \quad \Rightarrow \quad -x + y = 0$

Solving, we get $x = 0,\ y = 0$

Vertex: $(0, 0)$ is the vertex of (1).

Axis: $Y = 0$ i.e., $\frac{1}{\sqrt{2}}\ (y - x) = 0 \quad \Rightarrow \quad x - y = 0$

Equation of directrix of (3) is

$$X = -\sqrt{2} \quad \Rightarrow \quad \frac{x + y}{\sqrt{2}} = -\sqrt{2} \quad \Rightarrow \quad \frac{x + y}{\sqrt{2}} + \sqrt{2} = 0$$

$x + y + 2 = 0$ is the directrix in xy-coodinate system.

(iii) **Identify :** $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$

Solution. $x^2 + 2xy + y^2 + 2\sqrt{2}\,x - 2\sqrt{2}\,y + 2 = 0$ ... (1)

Here $a = 1,\ b = 1,\ 2h = 2$ the angle θ through which axes be rotated to given by

$$\tan 2q = \frac{2h}{a-b} = \frac{2}{1-1} = \frac{2}{0} \implies 2\theta = 90^\circ \implies \theta = 45^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\,.\,\frac{1}{\sqrt{2}} - Y\,.\,\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\,.\,\frac{1}{\sqrt{2}} + Y\,.\,\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right\} \quad ...(2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 2\sqrt{2}\left(\frac{X-Y}{\sqrt{2}}\right)$$

$$-2\sqrt{2}\left(\frac{X+Y}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{1}{2}(X^2 - 2XY + Y^2) + \frac{2}{2}(X^2 - Y^2) + \frac{1}{2}(X^2 + 2XY + Y^2)$$

$$+ 2(X - Y) - 2(X + Y) + 2 = 0$$

$$X^2 - 2XY + Y^2 + 2X^2 - 2Y^2 + X^2 + 2XY + Y^2 + 4X - 4Y - 4X$$

$$- 4Y + 4 = 0$$

$$4X^2 - 8Y + 4 = 0 \implies X^2 - 2Y + 1 = 0$$

$$X^2 = 2\left(Y - \frac{1}{2}\right) \quad ... (3)$$

Which represents a parabola.

From (ii), we have

$$X - Y = \sqrt{2}\,x \quad ... (4)$$

and $$X + Y = \sqrt{2}\,y \quad ... (5)$$

Adding (4) and (5), we get

$$2X = \sqrt{2}\,x + \sqrt{2}\,y \implies 2X = \sqrt{2}\,(x+y) \implies X = \frac{1}{\sqrt{2}}(x+y)$$

Put the value of X in (4), we get

$$\frac{1}{\sqrt{2}}(x+y)-Y=\sqrt{2}\,x \implies Y = \frac{1}{\sqrt{2}}(x+y)-\sqrt{2}\,x$$

$$= \frac{x+y-2x}{\sqrt{2}} = \frac{1}{\sqrt{2}}(y-x)$$

Thus $X = \frac{1}{\sqrt{2}}(x+y)$ and $Y = \frac{1}{\sqrt{2}}(y-x)$

Elements of parabola are

Focus of (3) is $X = 0,\ Y - \frac{1}{2} = \frac{1}{2} \implies Y = 1$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$ and $\frac{1}{\sqrt{2}}(y-x) = 1$

i.e., $x+y = 0$ and $y-x = \sqrt{2}$

Adding: $x+y = 0$

$$\underline{-x+y = \sqrt{2}}$$

$$2y = \sqrt{2} \implies y = \frac{1}{\sqrt{2}}$$

Put $y = \frac{1}{\sqrt{2}}$ in $x+y=0 \implies x = -y = -\frac{1}{\sqrt{2}}$

Focus: $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is the focus of (1)

Vertex of (3) is $X = 0,\ Y - \frac{1}{2} = 0 \implies Y = \frac{1}{2}$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0 \implies x+y = 0$

and $\frac{1}{\sqrt{2}}(y-x) = \frac{1}{2} \implies y-x = \frac{1}{\sqrt{2}}$

Solving, we get $x = -\frac{1}{2\sqrt{2}}$, $y = \frac{1}{2\sqrt{2}}$

Vertex $\left(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$ is the vertex of (1)

Axis $X = 0$ i.e., $\frac{1}{\sqrt{2}}(x+y) = 0$

$\implies x+y = 0$

Equation of directrix of (3) is

$$Y - \frac{1}{2} = -\frac{1}{2} \implies \frac{y-x}{\sqrt{2}} = 0 \implies y - x = 0 \implies x - y = 0$$

is the directrix in xy–coordinate system.

(iv) $\quad x^2 + xy + y^2 - 4 = 0$

Solution. $x^2 + xy + y^2 - 4 = 0 \qquad \ldots (1)$

Here $a = 1$, $b = 1$, $2h = 1$ the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a-b} = \frac{1}{1-1} = \frac{1}{0} = \infty \implies 2\theta = 90° \implies \theta = 45°$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 45° - Y\sin 45° = X\cdot\frac{1}{\sqrt{2}} - Y\cdot\frac{1}{\sqrt{2}} = \frac{X-Y}{\sqrt{2}} \\ y &= X\sin 45° + Y\cos 45° = X\cdot\frac{1}{\sqrt{2}} + Y\cdot\frac{1}{\sqrt{2}} = \frac{X+Y}{\sqrt{2}} \end{aligned}\right] \qquad \text{(ii)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-Y}{\sqrt{2}}\right)^2 + 2\left(\frac{X-Y}{\sqrt{2}}\right)\left(\frac{X+Y}{\sqrt{2}}\right) + \left(\frac{X+Y}{\sqrt{2}}\right)^2 + 4 = 0$$

$$\left(\frac{X^2 - 2XY + Y^2}{2}\right) + \left(\frac{X^2 - Y^2}{2}\right) + \left(\frac{X^2 + 2XY + Y^2}{2}\right) - 4 = 0$$

$$X^2 - 2XY + Y^2 + X^2 - Y^2 + X^2 + 2XY + Y^2 - 8 = 0$$

$$3X^2 + Y^2 = 8$$

$$\frac{X^2}{8/3} + \frac{Y^2}{8} = 1 \qquad \ldots (3)$$

Which represents an ellipse.

From (2), we have

$$X - Y = \sqrt{2}\, x \qquad \ldots (4)$$

$$X + Y = \sqrt{2}\, y \qquad \ldots (5)$$

Adding (4) and (5) $\quad 2X = \sqrt{2}\,x + \sqrt{2}\,y \implies X = \frac{1}{\sqrt{2}}(x+y)$

Subtracting (iv) and (v):

$$-2Y = \sqrt{2}\,x - \sqrt{2}\,y \implies Y = \frac{1}{\sqrt{2}}(y-x)$$

Elements of ellipse are

Centre of (3), is $X = 0$, $Y = 0$

$$\frac{1}{\sqrt{2}}(x+y) = 0 \quad \Rightarrow \quad x+y=0$$

and $\frac{1}{\sqrt{2}}(y-x) = 0 \quad \Rightarrow \quad -x+y=0 \quad \Rightarrow \quad x=0\,,\ y=0$

Hence $C\,(0\,,0)$ is the centre of (1)

Vertices of (3) are: $X = 0$, $Y = \pm 2\sqrt{2}$

$$X = 0 \Rightarrow \frac{1}{\sqrt{2}}(x+y) = 0 \quad \Rightarrow \quad x+y=0$$

and $\quad Y = \pm\sqrt{2} \Rightarrow \frac{1}{\sqrt{2}}(y-x) = \pm 2\sqrt{2} \quad \Rightarrow \quad -x+y = \pm 4$

and

$\Rightarrow \quad x+y=0$	$x+y=0$
$-x+y=4$	$-x+y=-4$
Adding: $2y = 4 \Rightarrow y = 2$	Adding: $2y = -4 \Rightarrow y = -2$
$\Rightarrow \quad x = -y = -2$	$x = -y = -(-2) = 2$
$(-2\,,2)$	$(2\,,-2)$

$(-2\,,2)\,,\,(2\,,-2)$, as vertices of (1)

Equation of major axis: $X = 0 \quad \Rightarrow \quad x+y=0$

Equation of minor axis: $Y = 0 \quad \Rightarrow \quad x-y=0$

Eccentricity: $e = \dfrac{\sqrt{a^2-b^2}}{a} = \dfrac{\sqrt{8-\frac{8}{3}}}{2\sqrt{2}} = \dfrac{4}{2\sqrt{6}} = \dfrac{2}{\sqrt{6}}$

Foci of (3) are $:X = 0$, $Y = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$

i.e., $\frac{1}{\sqrt{2}}(x+y) = 0\,,\ -\frac{1}{\sqrt{2}}(x-y) = \pm\sqrt{8}\left(\frac{2}{\sqrt{6}}\right)$

$\Rightarrow \quad x+y=0,$	$x+y=0$
$-x+y = \dfrac{2\sqrt{8}}{\sqrt{3}}$	$-x+y = \dfrac{-2\sqrt{8}}{\sqrt{3}}$
Adding: $2y = \dfrac{2\sqrt{8}}{\sqrt{3}} \Rightarrow y = \dfrac{2\sqrt{2}}{\sqrt{3}}$	Adding: $2y = -\dfrac{2\sqrt{8}}{3} \Rightarrow y = \dfrac{-2\sqrt{2}}{3}$

$$x + y = 0 \Rightarrow x = -y = -\frac{2\sqrt{2}}{3}$$

$$x + y = 0$$

$$\Rightarrow \quad x = -y = -\left(\frac{-2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}}{3}$$

Hence $\left(\frac{-2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}\right)$ and $\left(\frac{2\sqrt{2}}{3}, \frac{-2\sqrt{2}}{3}\right)$ are the foci of (1).

(v) $\quad 7x^2 - 6\sqrt{3}\, xy + 13y^2 - 16 = 0$

Solution. $7x^2 - 6\sqrt{3}\, xy + 13y^2 - 16 = 0 \qquad \ldots (1)$

Here $a = 7,\ b = 13,\ 2h = -6\sqrt{3}$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-6\sqrt{3}}{7 - 13} = \frac{-6\sqrt{3}}{-6} = \sqrt{3}$$

$$\Rightarrow \quad 2\theta = 60^\circ \quad \Rightarrow \quad \theta = 30^\circ$$

Equations of transformation become

$$\left.\begin{aligned} x &= X\cos 30^\circ - Y\sin 30^\circ = X.\frac{\sqrt{3}}{2} - Y.\frac{1}{2} = \frac{\sqrt{3}\,x - Y}{2} \\ y &= X\sin 30^\circ + Y\cos 30^\circ = X.\frac{1}{2} + Y\frac{\sqrt{3}}{2} = \frac{X + \sqrt{3}\,Y}{2} \end{aligned}\right] \ldots (2)$$

Substituting these expressions for x and y into (1), we get

$$7\left(\frac{\sqrt{3}\,X - Y}{2}\right)^2 - 6\sqrt{3}\left(\frac{\sqrt{3}\,X - Y}{2}\right)\left(\frac{X + \sqrt{3}\,Y}{2}\right)$$

$$+ 13\left(\frac{\sqrt{3}\,X + Y}{2}\right)^2 - 16 = 0$$

$$7\left(\frac{3X^2 - 2\sqrt{3}\,XY + Y^2}{4}\right) - 6\sqrt{3}\left(\frac{\sqrt{3}\,X^2 + 2XY - \sqrt{3}\,Y^2}{4}\right)$$

$$+ 13\left(\frac{X^2 + 2\sqrt{3}\,Y + 3Y^2}{4}\right) - 16 = 0$$

$$\frac{(21X^2 - 14\sqrt{3}\,XY + 7Y^2)}{4} - \frac{(18X^2 + 12\sqrt{3}\,XY - 18Y^2)}{4}$$

$$+ \frac{(13X^2 + 26\sqrt{3}\,XY + 39Y^2)}{4} - 16 = 0$$

$$21X^2 - 14\sqrt{3}\,XY + 7Y^2 - 18Y^2 - 12\sqrt{3}\,XY + 18Y^2 + 13X^2$$

$$+ 26\sqrt{3}\,XY + 39Y^2 - 64 = 0$$

$$16X^2 + 64Y^2 = 64$$

$$\Rightarrow \quad \frac{X^2}{4} + \frac{Y^2}{1} = 1 \qquad \ldots (3)$$

Which represents an ellipse.

From (ii), we have

$$\sqrt{3}\ X - Y = 2x \qquad \ldots (4)$$

$$X + \sqrt{3}\ Y = 2y \qquad \ldots (5)$$

Multiplying (iv) by $\sqrt{3}$, we get

$$3X - \sqrt{3}\ Y = 2\sqrt{3}\ x \qquad \ldots (6)$$

Adding (5) and (6), we get

$$4X = 2y + 2\sqrt{3}\ x \quad \Rightarrow \quad X = \frac{1}{2}(\sqrt{3}\ x + y)$$

Multiplying (5) by $\sqrt{3}$, we get

$$\sqrt{3}\ X + 3Y = 2\sqrt{3}\ Y \qquad \ldots (7)$$

Subtracting (7) from (4), we get

$$-4Y = 2\sqrt{3}\ y - 2x \quad \Rightarrow \quad Y = \frac{1}{2}\left(x - \sqrt{3}\ y\right)$$

Thus $X = \frac{1}{2}(\sqrt{3}\ x + y)$ and $Y = \frac{1}{2}\left(x - \sqrt{3}\ y\right)$

Elements of ellipse are

Centre of (3) is $X = 0,\ Y = 0$

$$X = 0 \quad \left(\frac{1}{2}(\sqrt{3}\ x + y) = 0 \quad \left(\sqrt{3}\ x + y = 0\right.\right.$$

$$Y = 0 \quad \left(\frac{1}{2}(x - \sqrt{3}\ y) = 0 \quad \left(x - \sqrt{3}\ y = 0\right.\right.$$

Solving these equations, we get $x = 0,\ y = 0$

Hence, $C\ (0, 0)$ centre of (1).

Vertices of (3) are $X = \pm a = \pm 2$ and $Y = 0$

$$X = \pm 2 \quad \Rightarrow \quad \frac{1}{2}(\sqrt{3}\ x + y) = \pm 2 \quad \Rightarrow \quad \sqrt{3}\ x + y = \pm 4$$

$$Y = 0 \quad \Rightarrow \quad \frac{1}{2}(x - \sqrt{3}\ y) = 0 \quad \Rightarrow \quad x - \sqrt{3}\ y = 0$$

$$\sqrt{3}\ x + y = 4 \qquad \ldots (4)$$

$$x - \sqrt{3}\ y = 0 \qquad \ldots (5)$$

Multiplying (4) by $\sqrt{3}$ and adding these equations, we get

$$\sqrt{3}\ x + y = -4 \qquad \ldots (6)$$

$$x - \sqrt{3}\ y = 0 \qquad \ldots (7)$$

Multiplying (6) by $\sqrt{3}$ and adding these equations, we get

$4x = 4\sqrt{3} \Rightarrow x = \sqrt{3}$

(5) $\Rightarrow \sqrt{3}\, y = x = \sqrt{3} \Rightarrow y = 1$

$4x = -4\sqrt{3} \Rightarrow x = -\sqrt{3}$

(7) $\Rightarrow \sqrt{3}\, y = x = -\sqrt{3} \Rightarrow y = -1$

$(\sqrt{3}, 1)$, $(-\sqrt{3}, -1)$, as vertices of (1)

Ecentricity: $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{4-1}}{2} = \dfrac{\sqrt{3}}{2}$

Foci of (3) are: $X = \pm\sqrt{3}$, $Y = 0$

$$X = \pm\sqrt{3} \Rightarrow \frac{1}{2}(\sqrt{3}\, x + y) = \pm\sqrt{3} \Rightarrow \sqrt{3}\, x + y = \pm 2\sqrt{3}$$

$$Y = 0 \Rightarrow \frac{1}{2}(\sqrt{3}\, y - x) = 0 \Rightarrow -x + \sqrt{3}\, y = 0$$

$\sqrt{3}\, x + y = 2\sqrt{3}$ ... (8)

$-x + \sqrt{3}\, y = 0$ ... (9)

Multiplying (9) by $\sqrt{3}$ and adding these equations, we get

$$4y = 2\sqrt{3} \Rightarrow y = \frac{\sqrt{3}}{2}$$

(9) $\Rightarrow$ x \r(3) y = \r(3) . \f(\r(3),2) = \f(3,2)

$$\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$$

$\sqrt{3}\, x + y = -2\sqrt{3}$ ... (10)

$-x + \sqrt{3}\, y = 0$ ... (11)

Multiplying (11) by $\sqrt{3}$ and adding these equations, we get

$$4y = -2\sqrt{3} \Rightarrow y = \frac{-\sqrt{3}}{2}$$

(11) $\Rightarrow$ x = \r(3) y

$= \sqrt{3}\left(\dfrac{-\sqrt{3}}{2}\right) = \dfrac{-3}{2}$

$$\left(\frac{-3}{2}, \frac{-\sqrt{3}}{2}\right)$$

Hence $\left(\dfrac{3}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(\dfrac{-3}{2}, \dfrac{-\sqrt{3}}{2}\right)$, as foci of (1).

Equation of major axis: $Y = 0 \Rightarrow \dfrac{1}{2}\left(\sqrt{3}\, y - x\right) = 0 \Rightarrow x - \sqrt{3}y = 0$

Equation of minor axis: $X = 0 \Rightarrow \dfrac{1}{2}\left(\sqrt{3}\, x + y\right) = 0 \Rightarrow \sqrt{3}x + y = 0.$

(vi) Identify. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$

Solution. $4x^2 - 4xy + 7y^2 + 12x + 6y - 9 = 0$ ... (1)

Here $a = 4$, $b = 7$, $2h = -4$, the angle θ through which axes be rotated to given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{-4}{4-7} = \frac{-4}{-3} = \frac{4}{3}$$

$$\frac{2\tan q}{1-\tan^2 q} = \frac{4}{3} \implies 6\tan\theta = 4-4\tan^2\theta$$

$$4\tan^2\theta + 6\tan\theta - 4 = 0 \implies 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3\pm\sqrt{(3)^2-4(2)(-2)}}{2(2)} = \frac{-3\pm\sqrt{9+16}}{4}$$

$$= \frac{-3\pm\sqrt{25}}{4} = \frac{-3\pm 5}{4} = -2, \frac{1}{2} \implies \tan\theta = \frac{1}{2}$$

Now $\tan\theta = \frac{1}{2} \implies$ base = 2, $\perp$ = 1, so hypotenuse = $\sqrt{4+1} = \sqrt{5}$

$\therefore$ $\sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X\cdot\frac{2}{\sqrt{5}} - Y\cdot\frac{1}{\sqrt{5}} = \frac{2X-Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\sin\theta = X\cdot\frac{1}{\sqrt{5}} + Y\cdot\frac{2}{\sqrt{5}} = \frac{X+2Y}{\sqrt{5}} \end{aligned}\right\} \ldots (2)$$

Substituting these expressions for x, and y into (i), we get

$$4\left(\frac{2X-Y}{\sqrt{5}}\right)^2 - 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) + 7\left(\frac{X+2Y}{\sqrt{5}}\right)^2$$
$$+ 12\left(\frac{2X-Y}{\sqrt{5}}\right) + 6\left(\frac{X+2Y}{\sqrt{5}}\right) - 9 = 0$$

$$\left(\frac{4X^2-4XY+Y^2}{5}\right) - 4\left(\frac{2X^2+3XY-2Y^2}{5}\right) + 7\left(\frac{X^2+4XY+4Y^2}{5}\right)$$
$$+ \frac{24X-12Y}{\sqrt{5}} + \frac{6X+12Y}{\sqrt{5}} - 9 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2$$
$$+\sqrt{5}\,(24X-12Y) + \sqrt{5}\,(6X+12Y) - 45 = 0$$

$$16X^2 - 16XY + 4Y^2 - 8X^2 - 12XY + 8Y^2 + 7X^2 + 28XY + 28Y^2 + 24\sqrt{5}\,X$$
$$-12\sqrt{5}\,Y + 6\sqrt{5}\,X + 12\sqrt{5}\,Y - 45 = 0$$

$$15X^2 + 40Y^2 + 30\sqrt{5}\,X - 45 = 0 \implies 3X^2 + 8Y^2 + 6\sqrt{5}\,X - 9 = 0$$

$$3(X^2 + 2\sqrt{5}\,X) + 8Y^2 = 9 \implies 3(X^2 + 2\sqrt{5}\,X + (\sqrt{5})^2) + 8Y^2 = 9 + 15.$$

$$3(X+\sqrt{5})^2 + 8Y^2 = 24 \implies \frac{(X+\sqrt{5})^2}{8} + \frac{Y^2}{3} = 1 \qquad (3)$$

which represents an ellipse.

From (2), we have

$$2X - Y = \sqrt{5}\ x \qquad \dots (4)$$

$$X + 2Y = \sqrt{5}\ y \qquad \dots (5)$$

Multiplying (5) by 2 and subtracting from (4), we get

$$5Y = 2\sqrt{5}\ y - \sqrt{5}\ x \implies Y = \frac{1}{\sqrt{5}}(-x + 2Y)$$

Put $Y = \frac{1}{\sqrt{5}}(-x + 2y)$ in (5), we get

$$X + \frac{2}{\sqrt{5}}(-x + 2y) = \sqrt{5}\ y \implies X = \sqrt{5}\ y - \frac{2}{\sqrt{5}}(-x + 2y)$$

$$= \sqrt{5}\ y + \frac{2}{\sqrt{5}}\ x - \frac{4}{\sqrt{5}}\ y$$

$$= \frac{1}{\sqrt{5}}(2x + y)$$

Thus $X = \frac{1}{\sqrt{5}}(2x + y)$ and $Y = \frac{1}{\sqrt{5}}(-x + 2y)$

For centre of (3) $X + \sqrt{5} = 0, Y = 0 \implies X = -\sqrt{5}, Y = 0$

$$X = -\sqrt{5} \implies \frac{1}{\sqrt{5}}(2x + y) = -\sqrt{5} \implies 2x + y = -5 \qquad (7)$$

$$Y = 0 \implies \frac{1}{\sqrt{5}}(-x + 2y) = 0 \implies -x + 2y = 0 \qquad (8)$$

Multiplying equation (8) by 2, we get

$$-2x + 4y = 0 \qquad (9)$$

Adding equation (7) and (9), we get

$$5y = -5 \implies y = -1$$

Equation. (8) $\implies x = 2y = 2(-1) = -2$

Hence $C\ (-2, -1)$ is the centre of (1)

Vertices of (3) are $X + \sqrt{5} = \pm\sqrt{8},\ Y = 0$

$$X + \sqrt{5} = \sqrt{8},\ Y = 0$$

$$X + \sqrt{5} = \sqrt{8} \implies \frac{1}{\sqrt{5}}(2x + y) + \sqrt{5} = \sqrt{8}$$

$$2x + y + 5 = \sqrt{40}$$

$$2x + y = -5 + \sqrt{40} \qquad \dots (10)$$

$$Y = 0 \quad \Rightarrow \quad -x + 2y = 0 \qquad \dots (11)$$

Multiplying (11) by 2

$$-2x + 4y = 0 \qquad \dots (12)$$

Adding equation (10) and equation (12), we get

$$5y = -5 + \sqrt{40} \quad \Rightarrow \quad y = -1 + \sqrt{\frac{8}{5}}$$

$$(12) \Rightarrow \quad x = 2y = 2\left(-1 + \sqrt{\frac{8}{5}}\right) = - + \sqrt{\frac{32}{5}}$$

Similarly, solving $X + \sqrt{5} = -\sqrt{8}$, and $Y = 0$, we get

$$x = -2 - \sqrt{\frac{32}{5}}, \; y = -1 - \sqrt{\frac{8}{5}}$$

$$\therefore \quad \left(-2 + \sqrt{\frac{32}{5}}, -1 + \sqrt{\frac{8}{5}}\right), \left(-2 - \sqrt{\frac{32}{2}}, -1 - \sqrt{\frac{8}{5}}\right)$$

are the vertices of (1)

Eccentricity:- $e = \dfrac{\sqrt{a^2 - b^2}}{a} = \dfrac{\sqrt{8-3}}{\sqrt{8}} = \dfrac{\sqrt{5}}{\sqrt{8}} = \sqrt{\dfrac{5}{8}}$

Foci of (3) are $X + \sqrt{5} = \pm\sqrt{5}$, $Y = 0$

$$X + \sqrt{5} = \sqrt{5} \quad \text{and} \quad Y = 0$$

$$X + \sqrt{5} = \sqrt{5} \quad \Rightarrow \quad X = 0$$

$$\frac{2x + y}{\sqrt{5}} = 0 \quad \Rightarrow \quad 2x + y = 0 \qquad \dots (13)$$

$$Y = 0 \quad \Rightarrow \quad x + 2y = 0 \qquad \dots (14)$$

Multiplying equation (14) by 2, we get

$$-2x + 4y = 0 \qquad \dots (15)$$

Adding (13) and (15), we get

$$5y = 0 \quad \Rightarrow \quad y = 0$$

Equation (14) $\Rightarrow$ $x = 2y = 2(0) = 0$

Similarly, solving $X + \sqrt{5} = -\sqrt{5}$ and $Y = 0$.

We get $x = -4$, $y = -2$

Thus $(0, 0), (-4, -2)$ are the foci of (1).

Equation of major axis $Y = 0 \Rightarrow \dfrac{2y - x}{\sqrt{5}} = 0 \Rightarrow x - 2y = 0$

Equation of minor axis $X = 0 \Rightarrow X + \sqrt{5} = 0 \Rightarrow X = -\sqrt{5}$

$$\Rightarrow 2x + y + 5 = 0.$$

(vii) Identify : $xy - 4x - 2y = 0$

Solution. $xy - 4x - 2y = 0$ ... (1)

Here $a = 0,\ b = 0,\ h = \dfrac{1}{2}$, the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{2\left(\frac{1}{2}\right)}{0 - 0} = \frac{1}{0} = \infty \Rightarrow 2\theta = 90^\circ \Rightarrow \theta = 45^\circ$$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos 45^\circ - Y\sin 45^\circ = X\,.\,\frac{1}{\sqrt{2}} - Y\,.\,\frac{1}{\sqrt{2}} = \frac{X - Y}{\sqrt{2}} \\ y &= X\sin 45^\circ + Y\cos 45^\circ = X\,.\,\frac{1}{\sqrt{2}} + Y\,.\,\frac{1}{\sqrt{2}} = \frac{X + Y}{\sqrt{2}} \end{aligned}\right\} \quad \ldots \quad (2)$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X - Y}{\sqrt{2}}\right)\left(\frac{X + Y}{\sqrt{2}}\right) - 4\left(\frac{X - Y}{\sqrt{2}}\right) - 2\left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$\frac{X^2 - Y^2}{2} - 4\left(\frac{X - Y}{\sqrt{2}}\right) - \left(\frac{X + Y}{\sqrt{2}}\right) = 0$$

$$X^2 - Y^2 - 4\sqrt{4}\ (X - Y) - 2\sqrt{2}\ (X + Y) = 0$$

$$X^2 - Y^2 - 4\sqrt{2}\ X + 4\sqrt{2}\ Y - 2\sqrt{2}\ X - 2\sqrt{2}\ Y = 0$$

$$X^2 - Y^2 - 6\sqrt{2}\ X + 2\sqrt{2}\ Y = 0$$

$$(X^2 - 6\sqrt{2}\ X + 18) - (y^2 - 2\sqrt{2}\ Y + 2) = 18 - 2$$

$$(X - 3\sqrt{2})^2 - (Y - \sqrt{2})^2 = 16$$

$$\frac{(X - 3\sqrt{2})^2}{16} - \frac{(Y - \sqrt{2})^2}{16} = 1 \quad \ldots \quad (3)$$

which represents a hyperbola.

From (2), we have

$$X - Y = \sqrt{2}\ x \quad \ldots \quad (4)$$

$$X + Y = \sqrt{2}\ y \quad \ldots \quad (5)$$

Adding (4) and (5), we have

$$\,\cdot = \sqrt{2}\ x + \sqrt{2}\ y \quad \Rightarrow \quad X = \frac{1}{\sqrt{2}}\ (x + y)$$

$$(4) \Rightarrow \quad Y = X - \sqrt{2}\ x = \frac{1}{\sqrt{2}}\ (x + y) - \sqrt{2}\ x = \frac{1}{\sqrt{2}}\ (-x + y)$$

Thus $X = \frac{1}{\sqrt{2}}\ (x + y)$ and $Y = \frac{1}{\sqrt{2}}\ (-x + y)$

Elements of Hyperbola:–

Centre of (4) is $X - 3\sqrt{2} = 0 \quad \Rightarrow \quad X = 3\sqrt{2}$

and $\quad Y - \sqrt{2} = 0 \quad \Rightarrow \quad Y = \sqrt{2}$

i.e., $\quad \frac{1}{\sqrt{2}}\ (x + y) = 3\sqrt{2} \quad \Rightarrow \quad x + y = 6 \qquad \ldots (6)$

and $\quad \frac{1}{\sqrt{2}}\ (-x + y) = \sqrt{2} \quad \Rightarrow \quad -x + y = 2 \qquad \ldots (7)$

Adding (6) and (7), we get

$$2y = 8 \quad \Rightarrow \quad y = 4$$

(vi) $\quad \Rightarrow \quad x = 6 - y = 6 - 4 = 2$

Hence centre of (1) is $C\ (2, 4)$.

Equation of focal axis:

$$Y - \sqrt{2} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\ (-x + y) - \sqrt{2} = 0$$

$$-x + y - 2 = 0 \quad \Rightarrow \quad x - y + 2 = 0$$

Equation of the conjugate axis:

$$X - 3\sqrt{2} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{2}}\ (x + y) - 3\sqrt{2} = 0$$

$$x + y - 6 = 0$$

Eccentricity: $e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{16 + 16}{16}} = \sqrt{\frac{32}{16}} = \sqrt{2}$

Foci of (3) $\quad X - 3\sqrt{2} = \pm 4\sqrt{2}, \quad Y - \sqrt{2} = 0$

$$X = 3\sqrt{2} \pm 4\sqrt{2}, \quad Y = \sqrt{2}$$

$$\frac{1}{\sqrt{2}}\ (x + y) = \sqrt{2}\ (3 \pm 4), \quad \frac{1}{\sqrt{2}}\ (-x + y) = \sqrt{2}$$

$$x + y = 14, -2, \quad -x + y = 2$$

$x + y = 14$ and $x + y = -2$,

$$x + y = 14$$
$$-x + y = 2$$

Adding: $2y = 16 \implies y = 8$

$x + y = 14 \implies x = 14 - y = 14 - 8 = 6$

$$x + y = -2$$
$$-x + y = 2$$

Adding: $2y = 6 \implies y = 3$

$x + y = -2 \implies x = -2 - y$

$= -2 - 0 = -2$

Foci of (1) are $(6, 8)$ and $(-2, 0)$

Vertices of (3) are $X - 3\sqrt{2} = \pm 4$, $Y - \sqrt{2} = 0$

$X = \pm 4 + 3\sqrt{2}$, $Y = \sqrt{2}$

i.e., $\frac{1}{\sqrt{2}}(x + y) = \pm 4 + 3\sqrt{2}$, $\frac{1}{\sqrt{2}}(-x + y) = \sqrt{2}$

$x + y = \pm 4\sqrt{2} + 6$, $-x + y = 2$

$$x + y = 4\sqrt{2} + 6$$
$$-x + y = 2$$

Adding: $2y = 4\sqrt{2} + 8$

$y = 2\sqrt{2} + 4$

$-x + y = 2 \implies x = y - 2$

$= 2\sqrt{2} + 4 - 2 = 2\sqrt{2} + 2$

$$x + y = -4\sqrt{2} + 6$$
$$-x + y = 2$$

Adding: $2y = -4\sqrt{2} + 8$

$\implies y = -2\sqrt{2} + 4$

$-x + y = 2 \implies x = y - 2$

$= -2\sqrt{2} + 4 - 2 = -2\sqrt{2} + 2$

Hence $\left(2\sqrt{2} + 2, 2\sqrt{2} + 4\right)$ and $\left(-2\sqrt{2} + 2, -2\sqrt{2} + 4\right)$ are vertices of (1).

(viii) Identify: $x^2 + 4xy - 2y - 6 = 0$

Solution. $x^2 + 4xy - 2y - 6 = 0$ (i)

Here $a = 1$, $b = -2$, $h = 2$, the angle θ through which axes be rotated is given by

$$\tan 2\theta = \frac{2h}{a - b} = \frac{2(2)}{1 - (-2)} = \frac{4}{3}$$

$$\frac{2\tan\theta}{1 - \tan^2\theta} = \frac{4}{3} \implies 6\tan\theta = 4 - 4\tan^2\theta$$

$$\implies 4\tan^2\theta + 6\tan\theta - 4 = 0 \implies 2\tan^2\theta + 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-3 \pm \sqrt{3^2 - 4(2)(-2)}}{2(2)} = \frac{-3 \pm \sqrt{9 + 16}}{4} = \frac{-3 \pm 5}{4}$$

$$= -2, \frac{1}{2} \implies \tan\theta = \frac{1}{2} \text{ (as } \theta \text{ is in the first quadrant)}$$

Now $\tan\theta = \frac{1}{2} \implies$ base $= 2$, $\perp = 1$, so hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{1}{\sqrt{5}}$ and $\cos\theta = \frac{2}{\sqrt{5}}$

Equations of transformations become

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X.\frac{2}{\sqrt{5}} - Y.\frac{1}{\sqrt{5}} = \frac{2X-Y}{\sqrt{5}} \\ y &= X\cos\theta + Y\cos\theta = X.\frac{1}{\sqrt{5}} + Y.\frac{2}{\sqrt{5}} = \frac{X+2Y}{\sqrt{5}} \end{aligned}\right] \text{(ii)}$$

Substituting these expressions for x and y into (i), we get

$$\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 4\left(\frac{2X-Y}{\sqrt{5}}\right)\left(\frac{X+2Y}{\sqrt{5}}\right) - \left(\frac{X+2Y}{\sqrt{5}}\right)^2 - 6 = 0$$

$$\left(\frac{4X^2-4XY+Y^2}{5}\right) + 4\left(\frac{2X^2+3XY-2Y^2}{5}\right) - 2\left(\frac{X^2+4XY+4Y^2}{5}\right) - 6 = 0$$

$$4X^2 - 4XY + Y^2 + 8X^2 + 12XY - 8Y^2 - 2X^2 - 8XY - 8Y^2 - 30 = 0$$

$$10X^2 - 15Y^2 - 30 = 0 \implies 10X^2 - 15Y^2 = 30$$

$$\frac{X^2}{3} - \frac{Y^2}{2} = 1 \qquad \dots (3)$$

which represents a hyperbola.

From (2), we have

$$2X - Y = \sqrt{5}\,x \qquad \dots (4)$$

$$X + 2Y = \sqrt{5}\,y \qquad \dots (5)$$

Multiplying (4) by 2, we get

$$4X - 2Y = 2\sqrt{5}\,x \qquad \dots (6)$$

Adding (5) and (6), we get

$$5X = 2\sqrt{5}\,x + \sqrt{5}\,y \implies X = \frac{1}{\sqrt{5}}(2x+y)$$

6) $\implies Y = 2X - \sqrt{5}\,x = \frac{2}{\sqrt{5}}(2x+y) - \sqrt{5}\,x = \frac{1}{\sqrt{5}}(-x+2y)$

Thus $\quad X = \frac{1}{\sqrt{5}}(2x+y)$ and $Y = \frac{1}{\sqrt{5}}(-x+2y)$

Elements of Hyperbola:-

Centre of (3) is $X = 0$, $Y = 0$

i.e., $\frac{1}{\sqrt{5}}(2x + y) = 0$, $\frac{1}{\sqrt{5}}(-x + 2y) = 0$

$2x + y = 0$, $-x + 2y = 0$

Solving we get $x = 0$, $y = 0$

Hence centre of (1) is $C\ (0, 0)$.

Equation of focal axis: $Y = 0 \Longrightarrow \frac{1}{\sqrt{5}}(-x+2y) = 0 \Longrightarrow x - 2y = 0$

Equation of the conjugate axis:

$$X = 0 \quad \Longrightarrow \quad \frac{1}{\sqrt{5}}(2x + y) = 0 \quad \Longrightarrow \quad 2x + y = 0$$

Eccentricity: $e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{3 + 2}{3}} = \sqrt{\frac{5}{3}}$

Foci of (3): $X = \pm\sqrt{3} . \sqrt{\frac{5}{3}}$, $Y = 0$

$\frac{1}{\sqrt{5}}(2x + y) = \pm\sqrt{5}$, $\frac{1}{\sqrt{5}}(-x + 2y) = 0$

$2x + y = \pm 5$, $x - 2y = 0$

$2x + y = 5$

$x - 2y = 0$

Solving, we get

$x = 2$, $y = 1$

$2x + y = -5$

$x - 2y = 0$

Solving, we get

$x = -2$, $y = -1$

Foci of (1) are $(2, 1)$ and $(-2, -1)$.

Vertices of (3) are $X = \pm\sqrt{3}$, $Y = 0$

i.e., $\frac{1}{\sqrt{5}}(2x + y) = \pm\sqrt{3}$, $\frac{1}{\sqrt{5}}(-x + 2y) = 0$

$2x + y = \pm\sqrt{15}$, $-x + 2y = 0$

$2x + y = \sqrt{15}$

$-x + 2y = 0$

Solving, we get

$y = \sqrt{\frac{3}{5}}$ and $x = 2\sqrt{\frac{3}{5}}$

$2x + y = -\sqrt{15}$

$-x + 2y = 0$

Solving, we get

$y = -\sqrt{\frac{3}{5}}$ and $x = -2\sqrt{\frac{3}{5}}$

Hence $\left(2\sqrt{\frac{3}{15}}, \sqrt{\frac{3}{15}}\right)$ and $\left(-2\sqrt{\frac{3}{15}}, -\sqrt{\frac{3}{15}}\right)$

are the vertices of (1).

(ix) $x^2 - 4xy - 2y^2 + 10x + 4y = 0$

Solution. $x^2 - 4xy - 2y^2 + 10x + 4y = 0 \qquad \ldots \quad (1)$

Here $a = 1$, $b = -2$, $2h = -4$ the angle θ through which axes be rotated to given by

$$\tan 2q = \frac{2h}{a-b} = \frac{-4}{1-(2)} = \frac{-4}{3}$$

$$\frac{2\tan q}{1-\tan^2 q} = -\frac{4}{3} \qquad \Rightarrow \qquad 6\tan\theta = 4\tan^2\theta = -4$$

$$4\tan^2\theta - 6\tan\theta - 4 = 0 \qquad \Rightarrow \qquad 2\tan^2\theta - 3\tan\theta - 2 = 0$$

$$\tan\theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)} = \frac{3 \pm \sqrt{9+16}}{4} = \frac{3 \pm 5}{4}$$

$$= 2, -\frac{1}{2} \quad \Rightarrow \quad \tan\theta = 2 \text{ (as } \theta \text{ is the first quadrant),}$$

Now $\tan\theta = \frac{2}{1} \Rightarrow$ base $= 1$, $\perp = 2$, so hypotenuse $= \sqrt{4+1} = \sqrt{5}$

$\therefore \quad \sin\theta = \frac{2}{\sqrt{5}}$ and $\cos\theta = \frac{1}{\sqrt{5}}$

Equations of transformation become.

$$\left.\begin{aligned} x &= X\cos\theta - Y\sin\theta = X \cdot \frac{1}{\sqrt{5}} - y \cdot \frac{2}{\sqrt{5}} = \frac{X-2Y}{\sqrt{5}} \\ y &= X\sin\theta + Y\cos\theta = X \cdot \frac{-2}{\sqrt{5}} + Y \cdot \frac{1}{\sqrt{5}} = \frac{2X+Y}{\sqrt{5}} \end{aligned}\right] \quad \text{(ii)}$$

Substituting these expressions for x and y into (1), we get

$$\left(\frac{X-2Y}{\sqrt{5}}\right)^2 - 4\left(\frac{X-2Y}{\sqrt{5}}\right)\left(\frac{2X-Y}{\sqrt{5}}\right) - 2\left(\frac{2X-Y}{\sqrt{5}}\right)^2 + 10\left(\frac{X-2Y}{\sqrt{5}}\right)$$
$$+ 4\left(\frac{2X-Y}{\sqrt{5}}\right) = 0$$

$$\left(\frac{X^2-4XY+4Y^2}{5}\right) - 4\left(\frac{2X^2-3XY-2Y^2}{5}\right) - 2\left(\frac{4X^2+4XY+Y^2}{5}\right)$$
$$+ 2\sqrt{5}(X-2Y) + 4\left(\frac{2X+Y}{\sqrt{5}}\right) = 0$$

$$X^2 - 4XY + 4Y^2 - 8X^2 + 12XY + 8Y^2 - 8X^2 - 8XY - 2Y^2 + 10\sqrt{5}\ X$$

$$- 20\sqrt{5}\ Y + 8\sqrt{5}\ X + 4\sqrt{5}\ Y = 0$$

$$-15X^2 + 10Y^2 + 18\sqrt{5}\ X - 16\sqrt{5}\ Y = 0$$

$$(10Y^2 - 16\sqrt{5}\ Y) - (15X^2 - 18\sqrt{5}\ X) = 0$$

$$10\left(Y^2 - \frac{8}{\sqrt{5}}\ Y\right) - 15\left(X^2 - \frac{6}{\sqrt{5}}\ X\right) = 0$$

$$10\left(Y^2 - \frac{8}{\sqrt{5}}\ Y + \left(\frac{4}{\sqrt{5}}\right)^2\right) - 15\left(X^2 - \frac{6}{\sqrt{5}} + \left(\frac{3}{\sqrt{5}}\right)^2\right)$$

$$= 10\left(\frac{4}{\sqrt{5}}\right)^2 - 15\left(\frac{3}{\sqrt{5}}\right)^2$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = \frac{180}{5} - \frac{135}{5}$$

$$10\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 15\left(X - \frac{3}{\sqrt{5}}\right)^2 = 32 - 27 = 5$$

$$2\left(Y - \frac{4}{\sqrt{5}}\right)^2 - 3\left(X - \frac{3}{\sqrt{5}}\right)^2 = 1$$

$$\frac{\left(Y - \frac{4}{\sqrt{5}}\right)^2}{\frac{1}{2}} - \frac{\left(X - \frac{3}{\sqrt{5}}\right)^2}{\frac{1}{3}} = 1 \qquad \dots (3)$$

which represents a hyperbola.

From (2), we have

$$X - 2Y = \sqrt{5}\ x \qquad \dots (4)$$

$$2X + Y = \sqrt{5}\ y \qquad \dots (5)$$

Solving (4) and (5), we get

$$X = \frac{x + 2y}{\sqrt{5}}\ ,\ Y = \frac{y - 2x}{\sqrt{5}}$$

Centre of (3) is $X - \frac{3}{\sqrt{5}} = 0,\ Y - \frac{4}{\sqrt{5}} = 0$

$$X - \frac{3}{\sqrt{5}} = 0 \Rightarrow X = \frac{3}{\sqrt{5}} \Rightarrow \frac{x + 2y}{\sqrt{5}} = \frac{3}{\sqrt{5}} \Rightarrow x + 2y = 3$$

and $Y - \frac{4}{\sqrt{5}} = 0 \Rightarrow Y = \frac{4}{\sqrt{5}} \Rightarrow \frac{y - 2x}{\sqrt{5}} = \frac{4}{\sqrt{5}} \Rightarrow -2x + y = 4$

Solving $x + 2y = 3$ and $-2x + y = 4$, we get

$x = -1,\ y = 2$

Hence $(-1, 2)$ is the centre of (1).

Equation of the focal axis: $X - \frac{3}{\sqrt{5}} = 0$

$$X = \frac{3}{\sqrt{5}} \implies \frac{x + 2y}{\sqrt{5}} = \frac{3}{\sqrt{5}} \implies x + 2y = 3$$

Equation of the conjugate axis: $Y - \frac{4}{\sqrt{5}} = 0$

$$Y = \frac{4}{\sqrt{5}} \implies \frac{y - 2x}{\sqrt{5}} = \frac{4}{\sqrt{5}} \implies -2x + y = 4$$

$$\textbf{Eccentricity} = e = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2}}} = \sqrt{\frac{5}{6} \cdot \frac{2}{1}} = \sqrt{\frac{5}{3}}$$

Foci of (3) $Y - \frac{4}{\sqrt{5}} = \pm \sqrt{\frac{1}{2}} \sqrt{\frac{5}{3}} = \pm \sqrt{\frac{5}{6}}, \quad X - \frac{3}{\sqrt{5}} = 0$

$$Y = \frac{4}{\sqrt{5}} \pm \sqrt{\frac{5}{6}} \qquad , \qquad X = \frac{3}{\sqrt{5}}$$

$$\frac{y - 2x}{\sqrt{5}} = \frac{4}{\sqrt{5}} \pm \sqrt{\frac{5}{6}} \qquad , \qquad \frac{x + 2y}{\sqrt{5}} = \frac{3}{\sqrt{5}}$$

$$y - 2x = 4 \pm \frac{5}{\sqrt{6}} \qquad , \qquad x + 4y = 3$$

$y - 2x = 4 + \frac{5}{\sqrt{6}}$	$y - 2x = 4 - \frac{5}{\sqrt{6}}$
$x + 2y = 3$	$x + 2y = 3$
Solving, we get	Solving, we get
$x = -1 - \frac{2}{\sqrt{6}}, \; y = 2 + \frac{1}{\sqrt{6}}$	$x = \left(-1 + \frac{2}{\sqrt{6}}, \; 2 - \frac{1}{\sqrt{6}}\right)$

Hence foci of (1) as $\left(-1 - \frac{2}{\sqrt{6}}, 2 + \frac{1}{\sqrt{6}}\right)$ and $\left(-1 + \frac{2}{\sqrt{6}}, 2 - \frac{1}{\sqrt{6}}\right)$

Vertices of (3) are $X - \frac{3}{\sqrt{5}} = 0$, $Y - \frac{4}{\sqrt{5}} = \pm \frac{1}{\sqrt{2}}$

$$X - \frac{3}{\sqrt{5}} = 0 \implies X = \frac{3}{\sqrt{5}} \implies \frac{x + 2y}{\sqrt{5}} = \frac{3}{\sqrt{5}} \implies x + 2y = 3$$

$$Y - \frac{4}{\sqrt{5}} = \pm \frac{1}{\sqrt{2}} \implies Y = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{2}}, \frac{4}{\sqrt{5}} - \frac{1}{\sqrt{2}}$$

$$\frac{y-2x}{\sqrt{5}} = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{2}} \quad \Rightarrow \quad y - 2x = 4 + \frac{\sqrt{5}}{\sqrt{2}}$$

and $$\frac{y-2x}{\sqrt{5}} = \frac{4}{\sqrt{5}} - \frac{1}{\sqrt{2}} \quad \Rightarrow \quad y - 2x = 4 - \frac{\sqrt{5}}{\sqrt{2}}$$

Solving $x + 2y = 3$ and $y - 2x = 4 + \frac{\sqrt{5}}{\sqrt{2}}$, we get

$$x = -1 - \frac{2}{\sqrt{10}}, \; y = 2 + \frac{1}{\sqrt{10}}$$

Again, Solving $x + 2y = 3$ and $y - 2x = 4 - \frac{\sqrt{5}}{2}$, we get

$$x = -1 + \frac{2}{\sqrt{10}} \quad \text{and} \quad y = 2 - \frac{1}{\sqrt{10}}$$

$\left(-1 - \frac{2}{\sqrt{10}}, 2 + \frac{1}{\sqrt{10}}\right)$ and $\left(-1 + \frac{2}{\sqrt{10}}, 2 - \frac{1}{\sqrt{10}}\right)$ are vertices of (i).

2. Show that (i) $10xy + 8x - 15y - 12 = 0$ and

(ii) $6x^2 + xy - y^2 - 21x - 8y + 9 = 0$ each represents a pair of straight lines and find an equation of each line.

Solution. (i) $10xy + 8x - 15y - 12 = 0$

Here $a = 0,\ b = 0,\ h = 5,\ g = 4,\ f = \frac{-15}{2},\ c = -12$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & \frac{-15}{2} \\ 4 & \frac{-15}{2} & -12 \end{vmatrix}$$

$$= 0 - 5(-60 + 30) + 4\left(\frac{-75}{2} - 0\right)$$

$$= 150 - 150 = 0$$

The given equation represents a degenerate conic which is a pair of lines.
The given equation is

$$10xy + 8x - 15y - 12 = 0 \Longrightarrow (10xy - 15y) + (8x - 12) = 0$$

$$\Longrightarrow \quad 5y(2x - 3) + 4(2x - 3) = 0 \Longrightarrow (2x - 3)(5y + 4) = 0$$

Equations of the lines are $2x - 3 = 0$ and $5y + 4 = 0$

Solution. (ii) $6x^2 + xy - y^2 - 21x - 8y + 9 = 0$

Here $a = 6,\ b = 1,\ h = \frac{1}{2},\ g = -\frac{21}{2},\ f = -4, c = 9$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 6 & \frac{1}{2} & \frac{-21}{2} \\ \frac{1}{2} & -1 & -4 \\ \frac{-21}{2} & -4 & 9 \end{vmatrix}$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\begin{vmatrix} 12 & 1 & -21 \\ 1 & -2 & -8 \\ -21 & -8 & 18 \end{vmatrix}$$

$$= \frac{1}{8}\ [\,12(-36-64) - 1(18-168) - 21(-8-42)\,]$$

$$= 8\,[\,12(-100) - 1(-150) - 21(-50)\,]$$

$$= 8\,[-1200 + 150 + 1050\,]$$

$$= 8\,[\,1200 - 1200\,] \quad = 8\,[\,0\,] = 8$$

Hence given equation represents a pair of lines. Further, rearranging the given equation as quadratic in y, we have

$$6x^2 + xy - y^2 - 21x - 8y + 9 = 0$$

$$-y^2 + xy - 8y + 6x^2 - 21x + 9 = 0$$

$$y^2 - xy + 8y - 6x^2 + 21x - 9 = 0$$

$$\Longrightarrow \quad y^2 - y(x-8) - 3(2x^2 - 7x + 3) = 0$$

$$\therefore \quad y = \frac{(x-8) \pm \sqrt{(3x-8)^2 + 4(1)3(2x^2 \quad 7x + 3)}}{2}$$

$$= \frac{(x-8) \pm \sqrt{x^2 - 16x + 64 + 24x^2 - 84x + 36}}{2}$$

$$= \frac{(x-8) \pm \sqrt{25x^2 - 100x + 100}}{2}$$

$$= \frac{(x-8) \pm 5\sqrt{x^2 - 4x + 4}}{2}$$

$$= \frac{(x-8) \pm 5\sqrt{(x-2)^2}}{2} \quad = \frac{(x-8) \pm 5(x-2)}{2}$$

$$= \frac{(x-8) + 5(x-2)}{2}, \quad \frac{(x-8) - 5(x-2)}{2}$$

$$= \frac{6x-18}{2}, \quad \frac{4x+2}{2} \quad = 3x - 9, \quad 2x + 1.$$

Hence, required lines are : $y = 3x - 9$, $y = 2x + 1$.

3. Find an equation of the tangent to each of the given conic at the indicated point.

(i) $3x^2 - 7y^2 + 2x - y - 48 = 0$ at $(4, 1)$

Solution. $3x^2 - 7y^2 + 2x - y - 48 = 0$ ... (1)

Differentiating (i) w.r.t. x, we have

$$6x - 14y\frac{dy}{dx} + 2 - \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 6x + 2 - (14y + 1)\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{6x + 2}{14y + 1}$$

$$m = \left.\frac{dy}{dx}\right|_{(4,1)} = \frac{6(4) + 2}{14(1) + 1} = \frac{24 + 2}{14 + 1} = \frac{26}{15}$$

Using $y - y_1 = m(x - x_1)$, required eqiuation of tangent is given by

Hence, equation of tangent at $(4, 1)$ is $y - 1 = \frac{26}{15}(x - 4)$

$$15y - 15 = 26x - 104 \quad \Rightarrow \quad 26x - 15y - 89 = 0.$$

(ii) Tangent to: $x^2 + 5xy - 4y^2 + 4 = 0$ **at** $y = -1$

Solution. $x^2 + 5xy - 4y^2 + 4 = 0$ ... (1)

To find the points, putting $y = -1$ in (1), then

$$x^2 + 5x(-1) - 4(-1)^2 + 4 = 0 \quad \Rightarrow \quad x^2 - 5x = 0$$

$$\Rightarrow \quad x(x - 5) = 0 \quad \Rightarrow \quad x = 0, 5$$

Hence there are twosuch points $(0, -1), (5, -1)$

Now equation of tangent at (x_1, y_1), repacing x^2 by xx_1, y^2 by yy_1 and xy by $xy_1 + x_1y$,

$$xx_1 + \frac{5}{2}(xy_1 + x_1y) - 4yy_1 + 4 = 0 \quad \text{... (2)}$$

Tangent at $(0, -1)$, by putting $x_1 = 0, y_1 = -1$ is

$$x(0) + \frac{5}{2}[x(-1) + (0)y] - 4y(-1) + 4 = 0$$

$$\frac{5}{2}(-x) - 4y(-1) + 4 = 0 \quad \text{or} \quad -\frac{5}{2}x + 4y + 4 = 0$$

or $\quad -5x + 8y + 8 = 0 \quad$ or $\quad 5x - 8y - 8 = 0$

Tangent at $(5, -1)$, by putting $x_1 = 0, y_1 = -1$ is

$$x(5) + \frac{5}{2}[x(-1) + (5)y] - 4y(-1) + 4 = 0$$

$$5x + \frac{5}{2}[(-x) + 5y] + 4y + 4 = 0$$

$$10x - 5x + 25y + 8y + 8 = 0$$

$$5x + 25y + 8y + 8 = 0 \quad \text{or} \quad 5x + 33y + 8 = 0$$

(iii) $\boldsymbol{x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0}$ **at** $\boldsymbol{x = 3}$

Solution. $x^2 + 4xy - 3y^2 - 5x - 9y + 6 = 0 \quad \ldots \quad (1)$

Putting $x = 3$ in (1), then

$$(3)^2 + 4(3)\,y - 3y^2 - 5(3) - 9y + 6 = 0$$

$$9 + 12y - 3y^2 - 15 - 9y + 6 = 0$$

$$-3y^2 + 3y = 0 \quad \Longrightarrow \quad 3y^2 - 3y = 0$$

$$3y\,(y - 1) = 0 \quad \Longrightarrow \quad y = 0, 1$$

The two points on the conic are $(3, 0)$, $(3, 1)$

Now equation of tangent at (x_1, y_1), repacing x^2 by xx_1, y^2 by yy_1 and $2xy$ by $xy_1 + x_1y$,

i.e. $\quad xx_1 + 2(xy_1 + x_1y) - 3yy_1 - \dfrac{5}{2}(x+x_1) - \dfrac{9}{2}(y+y_1) + 6 = 0 \quad \ldots \quad (2)$

Tangent at $(3, 0)$, by putting $x_1 = 3, y_1 = 0$ is

$$x(3) + 2\big[(x(0) + (3)y\big] - 3y(0) - \frac{5}{2}[x+(3)] - \frac{9}{2}[y+(0)] + 6 = 0$$

$$3x + 2[0 + 3y] - \frac{5}{2}[x + 3] - \frac{9}{2}[y] + 6 = 0$$

$$3x + 6y - \frac{5}{2}[x + 3] - \frac{9}{2}[y] + 6 = 0$$

$$3x + 6y - \frac{}{2}[5x + 15] - \frac{9}{2}[y] + 6 = 0$$

$$6x + 12y - 5x - 15 - 9y + 12 = 0$$

$$\boxed{x + 3y - 3 = 0}$$

Tangent at $(3, 1)$, by putting $x_1 = 3, y_1 = 0$ is

$$x(3) + 2\big[(x(1) + (3)y\big] - 3y(1) - \frac{5}{2}[x+(3)] - \frac{9}{2}[y+(1)] + 6 = 0$$

$$3x + 2[x + 3y] - 3y - \frac{5}{2}[x + 3] - \frac{9}{2}[y + 1] + 6 = 0$$

$$6x + 4[x + 3y] - 6y - 5[x + 3] - 9[y + 1] + 12 = 0$$

$$6x + 4x + 12y - 6y - 5x - 15 - 9y - 9 + 12 = 0$$

$$\boxed{5x - 3y - 12 = 0}$$