EXERCISE 3.4

Integration by Parts

If u and v are function of x, then

$$\int uv\,dx = u\int v dx - \int\left(\int v dx\right)\cdot u'\,dx$$

Question # 1(i)

Let $I = \int x\sin x\,dx$ | $u = x$, $v = \sin x$

Integration by parts

$$I = x\cdot(-\cos x) - \int(-\cos x)\cdot(1)\,dx$$
$$= -x\cos x + \int\cos x\,dx$$
$$= -x\cos x + \sin x + c$$

Question # 1(ii)

Let $I = \int \ln x\,dx$ | $u = \ln x$, $v = 1$

$$= \int \ln x\cdot 1\,dx$$

Integrating by parts

$$I = \ln x\cdot x - \int x\cdot\frac{1}{x}\,dx$$
$$= x\ln x - \int dx$$
$$= x\ln x - x + c$$

Question # 1(iii)

Let $I = \int x\ln x\,dx$ | $u = \ln x$, $v = x$

Integrating by parts

$$I = \ln x\cdot\frac{x^2}{2} - \int\frac{x^2}{2}\cdot\frac{1}{x}dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$
$$= \frac{x^2}{2}\ln x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$
$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

Question # 1(iv)

Let $I = \int x^2\ln x\,dx$ | $u = \ln x$, $v = x^2$

Do yourself

Question # 1(v)

Let $I = \int x^3\ln x\,dx$ | $u = \ln x$, $v = x^3$

Do yourself

Question # 1(vi)

Let $I = \int x^4\ln x\,dx$ | $u = \ln x$, $v = x^4$

Do yourself

Question # 1(vii)

Let $I = \int \tan^{-1}x\,dx$

$$= \int\tan^{-1}x\cdot 1\,dx$$

$u = \tan^{-1}x$, $v = 1$

Integrating by parts

$$I = \tan^{-1}x\cdot x - \int x\cdot\frac{1}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= x\tan^{-1}x - \frac{1}{2}\ln\left|1+x^2\right| + c$$

Question # 1(viii)

Let $I = \int x^2\sin x\,dx$ | $u = x^2$, $v = \sin x$

Integrating by parts

$$I = x^2(-\cos x) - \int(-\cos x)\cdot 2x\,dx$$
$$= -x^2\cos x + 2\int x\cos x\,dx$$

$u = x$, $v = \cos x$

Again integrating by parts

$$I = -x^2\cos x + 2\left(x\sin x - \int\sin x\,(1)\,dx\right)$$
$$= -x^2\cos x + 2x\sin x - 2(-\cos x) + c$$
$$= -x^2\cos x + 2x\sin x + 2\cos x + c$$

Question # 1(ix)

Let $I = \int x^2\tan^{-1}x\,dx$ | $u = \tan^{-1}x$, $v = x^2$

Integrating by parts

$$I = \tan^{-1}x\cdot\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int\left(x - \frac{x}{1+x^2}\right)dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x\,dx + -\frac{1}{3}\int\frac{x}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\cdot\frac{x^2}{2} + -\frac{1}{3}\cdot\frac{1}{2}\int\frac{2x}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + -\frac{1}{6}\int\frac{\frac{d}{dx}(1+x^2)}{1+x^2}dx$$
$$= \frac{x^3}{3}\tan^{-1}x - \frac{x^2}{6} + -\frac{1}{6}\ln\left|1+x^2\right| + c$$

Question # 1(x)

Let $I = \int x\tan^{-1}x\,dx$ | $u = \tan^{-1}x$, $v = x$

Integrating by parts

$$I = \frac{x^2}{2}\tan^{-1}x - \int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(\frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}\right)dx$$
$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}\int dx+\frac{1}{2}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c \quad \textit{Ans.}$$

Question # 1(xi)

Let $I=\int x^3\tan^{-1}x\,dx$ $\qquad u=\tan^{-1}x$, $v=x^3$

Integrating by parts

$$I=\tan^{-1}x\cdot\frac{x^4}{4}-\int\frac{x^4}{4}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int\frac{x^4}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x$$

$$-\frac{1}{4}\int\left(x^2-1+\frac{1}{1+x^2}\right)dx$$

$$\begin{array}{r|l} & x^2-1 \\ \hline 1+x^2 & x^4 \\ & \underline{-x^4-x^2} \\ & -x^2 \\ & \underline{\mp 1 \mp x^2} \\ & 1 \end{array}$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int x^2dx+\frac{1}{4}\int dx-\frac{1}{4}\int\frac{1}{1+x^2}dx$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\frac{x^3}{3}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{x^3}{12}+\frac{1}{4}x-\frac{1}{4}\tan^{-1}x+c$$

Question # 1(xii)

Let $I=\int x^3\cos x\,dx$ $\qquad u=x^3$, $v=\cos x$

Do yourself as Question # 1(viii). Integrate by parts three times.

Question # 1(xiii)

$$I=\int\sin^{-1}x\,dx$$

$$=\int\sin^{-1}x\cdot 1\,dx \qquad u=\sin^{-1}x,\ v=1$$

Integrating by parts

$$I=\sin^{-1}x\cdot x-\int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=x\sin^{-1}x-\int(1-x^2)^{-\frac{1}{2}}(x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\int(1-x^2)^{-\frac{1}{2}}\frac{d}{dx}(1-x^2)\,dx$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c$$

$$=x\sin^{-1}x+\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{1/2}+c$$

$$=x\sin^{-1}x+\sqrt{1-x^2}+c$$

Question # 1(xiv)

Let $I=\int x\sin^{-1}x\,dx$ $\qquad u=\sin^{-1}x$, $v=x$

Integrating by parts

$$I=\sin^{-1}x\cdot\frac{x^2}{2}-\int\frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{1-x^2}}dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\frac{1-x^2}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\left(\sqrt{1-x^2}-\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\int\sqrt{1-x^2}dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx$$

$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}I_1-\frac{1}{2}\sin^{-1}x \ \ldots\ldots (i)$$

Where $I_1=\int\sqrt{1-x^2}dx$

Put $x=\sin\theta \quad\Rightarrow\quad dx=\cos\theta\,d\theta$

$$\Rightarrow\ I_1=\int\sqrt{1-\sin^2\theta}\ \cos\theta\ d\theta$$

$$=\int\sqrt{\cos^2\theta}\ \cos\theta\ d\theta$$

$$=\int\cos^2\theta\ d\theta \quad =\int\left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\int(1+\cos 2\theta)\ d\theta$$

$$=\frac{1}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+c$$

$$=\frac{1}{2}\left[\theta+\frac{2\sin\theta\cos\theta}{2}\right]+c$$

$$=\frac{1}{2}\left[\theta+\sin\theta\sqrt{1-\sin^2\theta}\right]+c$$

$$=\frac{1}{2}\left[\sin^{-1}x+x\sqrt{1-x^2}\right]+c$$

Using value of I_1 in (i)

$$I=\frac{x^2}{2}\sin^{-1}x+\frac{1}{2}\left[\frac{1}{2}\left(\sin^{-1}x+x\sqrt{1-x^2}+c\right)\right]-\frac{1}{2}\sin^{-1}x$$

$$=\frac{x^2}{2}\sin^{-1}x+\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c-\frac{1}{2}\sin^{-1}x$$

$$\Rightarrow\ I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{4}\sin^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+\frac{1}{2}c$$

Question # 1(xv)

Let $I=\int e^x\sin x\cos x\,dx$ $\qquad u=e^x$, $v=\sin 2x$

$$=\frac{1}{2}\int e^x\cdot 2\sin x\cos x\ dx$$

$$=\frac{1}{2}\int e^x\sin 2x\ dx \qquad \because\ \sin 2x=2\sin x\cos x$$

Integrating by parts

$$I=\frac{1}{2}\left[e^x\cdot\frac{-\cos 2x}{2}-\int\frac{-\cos 2x}{2}\cdot e^x dx\right]$$

$$=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\int e^x\cos 2x\ dx$$

Again integrating by parts

$$I=-\frac{1}{4}e^x\cos 2x+\frac{1}{4}\left(e^x\cdot\frac{\sin 2x}{2}-\int\frac{\sin 2x}{2}\ e^x\right)$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4}\left(e^x \cdot \frac{\sin 2x}{2} - \frac{1}{2}\int e^x \sin 2x \right)$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{4}\left(e^x \cdot \frac{\sin 2x}{2} - I \right) + c$$

$$= -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x - \frac{1}{4} I + c$$

$$\Rightarrow \quad I + \frac{1}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$$

$$\Rightarrow \quad \frac{5}{4} I = -\frac{1}{4} e^x \cos 2x + \frac{1}{8} e^x \sin 2x + c$$

$$\Rightarrow \quad I = -\frac{1}{5} e^x \cos 2x + \frac{1}{10} e^x \sin 2x + \frac{4}{5} c$$

Question # 1(xvi)

Let $I = \int x \sin x \cos x \, dx$

$$= \frac{1}{2}\int x \cdot 2\sin x \cos x \; dx$$

$$= \frac{1}{2}\int x \cdot \sin 2x \; dx \qquad \begin{array}{|l} u = x \\ v = \sin 2x \end{array}$$

Integrating by parts

$$I = \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2} \right) - \int \left(\frac{-\cos 2x}{2} \right)(1) dx \right]$$

Now do yourself

Question # 1(xvii)

Let $I = \int x \cos^2 x \, dx$

$$= \int x \left(\frac{1 + \cos 2x}{2} \right) dx$$

$$= \frac{1}{2}\int x\left(1 + \cos 2x\right) dx$$

$$= \frac{1}{2}\int x\, dx + \frac{1}{2}\int x \cos 2x dx \qquad \begin{array}{|l} u = x \\ v = \cos 2x \end{array}$$

$$= \frac{1}{2} \cdot \frac{x^2}{2} + \frac{1}{2}\left[x \; \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} \cdot (1) dx \right]$$

$$= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4}\int \sin 2x \; dx$$

$$= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x - \frac{1}{4}\left(\frac{-\cos 2x}{2} \right) + c$$

$$= \frac{x^2}{4} + \frac{1}{4} x \cdot \sin 2x + \frac{1}{8} \cos 2x + c$$

Question # 1(xviii)

Let $I = \int x \sin^2 x \; dx$

$$= \int x \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int x\left(1 - \cos 2x\right) \; dx$$

$$= \frac{1}{2} \int x \; dx - \frac{1}{2} \int x \; \cos 2x \; dx \qquad \begin{array}{|l} u = x \\ v = \cos 2x \end{array}$$

Integrating by parts

$$I = \frac{1}{2} \; \frac{x^2}{2} \; - \; \frac{1}{2}\left[\; x \cdot \frac{\sin 2x}{2} - \int \frac{\sin 2x}{2} . (1)\, dx \; \right]$$

$$= \frac{x^2}{4} \; - \; \frac{1}{4} x \sin 2x + \frac{1}{4}\int \sin 2x \, dx$$

$$= \frac{x^2}{4} \; - \; \frac{1}{4} x \sin 2x + \frac{1}{4}\left(\frac{-\cos 2x}{2} \right) + c$$

$$= \frac{x^2}{4} \; - \; \frac{1}{4} x \sin 2x - \frac{1}{8} \cos 2x + c$$

Question # 1(xix)

Let $I = \int (\ln x)^2 dx$

$$= \int \left(\ln x\right)^2 \cdot 1 \; dx \qquad \begin{array}{|l} u = \left(\ln x\right)^2 \\ v = 1 \end{array}$$

Integrating by parts

$$I = \left(\ln x\right)^2 \cdot x - \int x \cdot 2\left(\ln x\right) \cdot \frac{1}{x} \; dx$$

$$= x\left(\ln x\right)^2 - 2\int \left(\ln x\right) \; dx$$

Again integrating by parts

$$I = x\left(\ln x\right)^2 - 2\left[\; \ln x \cdot x - \int x \cdot \frac{1}{x} \; dx \; \right]$$

$$= x\left(\ln x\right)^2 - 2x \ln x + 2\int dx$$

$$= x\left(\ln x\right)^2 - 2x \ln x + 2x + c$$

Question # 1(xx)

Let $I = \int \ln(\tan x) \; \sec^2 x \, dx \qquad \begin{array}{|l} u = \ln(\tan x) \\ v = \sec^2 x \end{array}$

Integrating by parts

$$I = \ln(\tan x) \cdot \tan x - \int \tan x \cdot \frac{1}{\tan x} \cdot \sec^2 x \; dx$$

$$= \tan x \ln\left(\tan x\right) - \int \sec^2 x \; dx$$

$$= \tan x \ln\left(\tan x\right) - \tan x + c$$

Question # 1(xxi)

Let $I = \int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx \qquad \begin{array}{|l} u = \sin^{-1} x \\ v = \left(1 - x^2\right)^{-\frac{1}{2}} (-2x) \end{array}$

$$= \int \sin^{-1} x \cdot \frac{1}{\sqrt{1 - x^2}} (x)\, dx$$

$$= -\frac{1}{2}\int \sin^{-1} x \cdot \left(1 - x^2\right)^{-\frac{1}{2}} (-2x)\, dx$$

Integrating by parts

$$I = -\frac{1}{2}\left[\sin^{-1} x \cdot \frac{(1 - x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} - \int \frac{(1 - x^2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \cdot \frac{1}{\sqrt{1 - x^2}} dx \right]$$

$$= -\frac{1}{2}\left[\sin^{-1} x \cdot \frac{(1 - x^2)^{\frac{1}{2}}}{\frac{1}{2}} - \int \frac{(1 - x^2)^{\frac{1}{2}}}{\frac{1}{2}} \cdot \frac{1}{\sqrt{1 - x^2}} \; dx \right]$$

$$= -\frac{1}{2}\left[\; 2(1 - x^2)^{\frac{1}{2}} \sin^{-1} x - 2\int \; dx \right]$$

$$= -\sqrt{1 - x^2} \; \sin^{-1} x + \int dx$$

$$= -\sqrt{1 - x^2} \; \sin^{-1} x + x + c$$

$$= x - \sqrt{1 - x^2} \; \sin^{-1} x + c$$

Question # 2(i)

Let $I = \int \tan^4 x dx$

$$= \int \tan^2 x \cdot \tan^2 x \; dx$$

$$= \int \tan^2 x\left(\sec^2 x - 1\right)\, dx$$
$$= \int \left(\tan^2 x \sec^2 x - \tan^2 x\right)\, dx$$
$$= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx$$
$$= \int \tan^2 x \frac{d}{dx}(\tan x)\, dx - \int (\sec^2 x - 1)\, dx$$
$$= \frac{\tan^{2+1} x}{2+1} - \int \sec^2 x\, dx - \int dx$$
$$= \frac{1}{3}\tan^3 x - \tan x - x + c$$

Question # 2(ii)

Let $I = \int \sec^4 x\, dx$
$$= \int (\sec^2 x)\cdot(\sec^2 x)\, dx$$
$$= \int (1+\tan^2 x)\cdot(\sec^2 x)\, dx$$
$$= \int \sec^2 x\, dx + \int \tan^2 x \sec^2 x\, dx$$
$$= \tan x + \int (\tan x)^2 \frac{d}{dx}(\tan x)\, dx$$
$$= \tan x + \frac{\tan^3 x}{3} + c$$

Question # 2(iii)

Let $I = \int e^x \sin 2x \cos x\, dx$
$$= \frac{1}{2}\int e^x \left(2\sin 2x \cos x\right) dx$$
$$= \frac{1}{2}\int e^x \left(\sin(2x+x) + \sin(2x-x)\right) dx$$
$$= \frac{1}{2}\int e^x \left(\sin 3x + \sin x\right) dx$$
$$= \frac{1}{2}\int e^x \sin 3x\, dx + \frac{1}{2}\int e^x \sin x\, dx$$
$$= \frac{1}{2} I_1 + \frac{1}{2} I_2 \;\ldots\ldots\ldots\; (i)$$

Where $I_1 = \int e^x \sin 3x\, dx$ and $I_2 = \int e^x \sin x\, dx$

Solve I_1 and I_2 as in Q # 1(xv) and put value of I_1 and I_2 in (i).

Question # 2(iv)

$$I = \int \tan^3 x \cdot \sec x\, dx$$
$$= \int \tan^2 x \cdot \tan x \cdot \sec x\, dx$$
$$= \int \left(\sec^2 x - 1\right)\cdot \sec x \tan x\, dx$$

Put $t = \sec x \quad \Rightarrow \quad dt = \sec x \tan x\, dx$

So $I = \int \left(t^2 - 1\right) dt$
$$= \frac{t^3}{3} - t + c$$
$$= \frac{\sec^3 x}{3} - \sec + c$$

Question # 2(v)

Let $I = \int x^3 e^{5x} dx$ $\qquad$ $u = x^3$, $v = e^x$

Integrating by parts
$$I = x^3 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot 3x^2 dx$$
$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\int x^2 e^{5x} dx \qquad u = x^2,\ v = e^x$$

Again integrating by parts
$$I = \frac{1}{5}x^3 e^{5x} - \frac{3}{5}\left[x^2 \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot 2x\, dx\right]$$
$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\int x e^{5x}\, dx$$

Again integrating by parts
$$I = \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{25}\left[x \cdot \frac{e^{5x}}{5} - \int \frac{e^{5x}}{5}\cdot (1) dx\right]$$
$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x e^{5x} - \frac{6}{125}\int e^{5x}\, dx$$
$$= \frac{1}{5}x^3 e^{5x} - \frac{3}{25}x^2 e^{5x} + \frac{6}{125}x e^{5x} - \frac{6}{125}\cdot\frac{e^{5x}}{5} + c$$
$$= \frac{e^{5x}}{5}\left(x^3 - \frac{3}{5}x^2 + \frac{6}{25}x - \frac{6}{125}\right) + c$$

Question 2(vi)

Let $I = \int e^{-x} \sin 2x\, dx$ $\qquad$ $u = e^{-x}$, $v = \sin 2x$

Integrating by parts
$$I = e^{-x}\cdot \frac{-\cos 2x}{2} - \int \frac{-\cos 2x}{2}\cdot e^{-x}(-1)\, dx$$
$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\, dx$$

Again integrating by parts
$$I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{2}\left[e^{-x}\cdot\frac{\sin 2x}{2} - \int \frac{\sin 2x}{2}\cdot e^{-x}(-1)\, dx\right]$$
$$= -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}\int e^{-x}\sin 2x\, dx$$
$$\Rightarrow\; I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x - \frac{1}{4}I + c$$
$$\Rightarrow\; I + \frac{1}{4}I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x + c$$
$$\Rightarrow\; \frac{5}{4}I = -\frac{1}{2}e^{-x}\cos 2x - \frac{1}{4}e^{-x}\sin 2x + c$$
$$\Rightarrow\; I = -\frac{2}{5}e^{-x}\cos 2x - \frac{1}{5}e^{-x}\sin 2x + \frac{4}{5}c$$
$$= -\frac{1}{5}e^{-x}\left(2\cos 2x + \sin 2x\right) + \frac{4}{5}c$$

Question # 2(vii)

Let $I = \int e^{2x}\cdot\cos 3x\, dx$

Do yourself as above

Question # 2(viii)

$I = \int \operatorname{cosec}^3 x\, dx$ $\qquad$ $u = \operatorname{cosec} x$, $v = \operatorname{cosec}^2 x$
$$= \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x\, dx$$

Integrating by parts
$$I = \csc x\left(-\cot x\right) i \int \left(-\cot x\right)\left(-\csc x \cot x\right) dx$$
$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x\, dx$$
$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x\left(\operatorname{cosec}^2 x - 1\right) dx$$
$$= -\operatorname{cosec} x \cot x - \int \left(\operatorname{cosec}^3 x - \operatorname{cosec} x\right) dx$$

$$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x\, dx + \int \operatorname{cosec} x\, dx$$
$$= -\operatorname{cosec} x \cot x - I + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow\ I + I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow\ 2I = -\operatorname{cosec} x \cot x + \ln\left|\operatorname{cosec} x - \cot x\right| + c$$
$$\Rightarrow\ I = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln\left|\csc x - \cot x\right| + \frac{1}{2}c$$

Question # 3

Let $I = \int e^{ax} \sin bx\ dx$ | $u = e^{ax}$, $v = \sin bx$

Integrating by parts

$$I = e^{ax}\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)\cdot e^{ax}(a)\,dx$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\int e^{ax}\cos bx\ dx$$

Again integrating by parts

$$I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left[e^{ax}\frac{\sin bx}{b} - \int \frac{\sin bx}{b}\cdot e^{ax} a\,dx\right]$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}\int e^{ax}\sin bx\,dx$$
$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx - \frac{a^2}{b^2}I + c_1$$
$$\Rightarrow\ I + \frac{a^2}{b^2}I = -\frac{e^{ax}\cos bx}{b} + \frac{a}{b^2}e^{ax}\sin bx + c_1$$
$$\Rightarrow\ \left(\frac{b^2 + a^2}{b^2}\right)I = \frac{e^{ax}}{b^2}\left(-b\cos bx + a\sin bx\right) + c_1$$
$$\Rightarrow\ \left(b^2 + a^2\right)I = e^{ax}\left(a\sin bx - b\cos bx\right) + b^2 c_1$$

Put $a = r\cos\theta$ & $b = r\sin\theta$

Squaring and adding

$$a^2 + b^2 = r^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$\Rightarrow\ a^2 + b^2 = r^2(1) \quad \Rightarrow\ r = \sqrt{a^2 + b^2}$$

Also

$$\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} \quad \Rightarrow\ \frac{b}{a} = \tan\theta$$
$$\Rightarrow\ \theta = \tan^{-1}\frac{b}{a}$$

So

$$\left(b^2 + a^2\right)I = e^{ax}\left(r\cos\theta\sin bx - r\sin\theta\cos bx\right) + b^2 c_1$$
$$\left(b^2 + a^2\right)I = e^{ax} r\left(\sin bx\cos\theta - \cos bx\sin\theta\right) + b^2 c_1$$
$$\Rightarrow\ \left(a^2 + b^2\right)I = e^{ax} r\sin\left(bx - \theta\right) + b^2 c_1$$

Putting value of r and θ

$$\left(a^2 + b^2\right)I = e^{ax}\sqrt{a^2 + b^2}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + b^2 c_1$$
$$\Rightarrow I = \frac{\sqrt{a^2 + b^2}}{(a^2 + b^2)}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + \frac{b^2}{a^2 + b^2}c_1$$
$$\Rightarrow\ I = \frac{1}{\sqrt{a^2 + b^2}}e^{ax}\sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$$

Where $c = \frac{b^2}{a^2 + b^2}c_1$

Question # 4(i)

Let $I = \int \sqrt{a^2 - x^2}\ dx$
$= \int \sqrt{a^2 - x^2}\cdot 1\ dx$ | $u = \sqrt{a^2 - x^2}$, $v = 1$

Integrating by parts

$$I = \sqrt{a^2 - x^2}\cdot x - \int x\cdot\frac{1}{2}\left(a^2 - x^2\right)^{-\frac{1}{2}}\cdot(-2x)\ dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{-x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\ dx$$
$$= x\sqrt{a^2 - x^2} - \int \frac{a^2 - x^2 - a^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\ dx$$
$$= x\sqrt{a^2 - x^2} - \int\left(\frac{a^2 - x^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}} - \frac{a^2}{\left(a^2 - x^2\right)^{\frac{1}{2}}}\right)dx$$
$$= x\sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2}\ dx + \int \frac{a^2}{\sqrt{a^2 - x^2}}dx$$
$$\Rightarrow\ I = x\sqrt{a^2 - x^2} - I + a^2\int \frac{1}{\sqrt{a^2 - x^2}}dx$$
$$\Rightarrow\ I + I = x\sqrt{a^2 - x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$
$$\Rightarrow\ 2I = x\sqrt{a^2 - x^2} + a^2 Sin^{-1}\frac{x}{a} + c$$
$$\Rightarrow\ I = \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 Sin^{-1}\frac{x}{a} + \frac{1}{2}c$$

Review

- $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c$
- $\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left|x + \sqrt{x^2 + a^2}\right| + c$

Question # 4(ii)

Let $I = \int \sqrt{x^2 - a^2}\ dx$
$= \int \sqrt{x^2 - a^2}\cdot 1\ dx$ | $u = \sqrt{x^2 - a^2}$, $v = 1$

Integrating by parts

$$I = \sqrt{x^2 - a^2}\cdot x - \int x\cdot\frac{1}{2}\left(x^2 - a^2\right)^{-\frac{1}{2}}\cdot(2x)\ dx$$
$$= x\sqrt{x^2 - a^2} - \int \frac{x^2}{\left(x^2 - a^2\right)^{\frac{1}{2}}}\ dx$$
$$= x\sqrt{x^2 - a^2} - \int \frac{x^2 - a^2 + a^2}{\left(x^2 - a^2\right)^{\frac{1}{2}}}\ dx$$
$$= x\sqrt{x^2 - a^2} - \int\left(\frac{x^2 - a^2}{\left(x^2 - a^2\right)^{\frac{1}{2}}} + \frac{a^2}{\left(x^2 - a^2\right)^{\frac{1}{2}}}\right)dx$$
$$= x\sqrt{x^2 - a^2} - \int \sqrt{x^2 - a^2}\ dx - \int \frac{a^2}{\sqrt{x^2 - a^2}}dx$$
$$\Rightarrow\ I = x\sqrt{x^2 - a^2} - I - a^2\int \frac{1}{\sqrt{x^2 - a^2}}dx$$

$$\Rightarrow\ I+I = x\sqrt{x^2-a^2} - a^2\ln\left|\,x+\sqrt{x^2-a^2}\,\right| + c$$

$$\because\ \int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|\,x+\sqrt{x^2-a^2}\,\right| + c$$

$$\Rightarrow\ 2I = x\sqrt{x^2-a^2} - a^2\ln\left|\,x+\sqrt{x^2-a^2}\,\right| + c$$

$$\Rightarrow\ I = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|\,x+\sqrt{x^2-a^2}\,\right| + \frac{1}{2}c$$

Question # 4(iii)

Let $\quad I = \int \sqrt{4-5x^2}\ dx$

$$= \int \sqrt{4-5x^2}\cdot 1\ dx$$

Integrating by parts

$$I = \sqrt{4-5x^2}\cdot x - \int x\cdot\frac{1}{2}(4-5x^2)^{-\frac{1}{2}}\cdot(-10x)\ dx$$

$$= \sqrt{4-5x^2}\cdot x - \int \frac{-5x^2}{(4-5x^2)}\,dx$$

$$= \sqrt{4-5x^2}\cdot x - \int \frac{4-5x^2-4}{(4-5x^2)}\,dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\left(\frac{4-5x^2}{(4-5x^2)^{\frac{1}{2}}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}}\right)dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\left((4-5x^2)^{\frac{1}{2}} - \frac{4}{(4-5x^2)^{\frac{1}{2}}}\right)dx$$

$$= \sqrt{4-5x^2}\cdot x - \int\sqrt{4-5x^2}\,dx + 4\int\frac{1}{\sqrt{4-5x^2}}\,dx$$

$$\Rightarrow\ I = \sqrt{4-5x^2}\cdot x - I + 4\int\frac{1}{\sqrt{5\left(\frac{4}{5}-x^2\right)}}\,dx$$

$$\Rightarrow\ I+I = \sqrt{4-5x^2}\cdot x + 4\int\frac{1}{\sqrt{5}\sqrt{\frac{4}{5}-x^2}}\,dx$$

$$\Rightarrow\ 2I = \sqrt{4-5x^2}\cdot x + \frac{4}{\sqrt{5}}\int\frac{1}{\sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 - x^2}}\,dx$$

$$= \sqrt{4-5x^2}\cdot x + \frac{4}{\sqrt{5}}Sin^{-1}\left(\frac{x}{2/\sqrt{5}}\right) + c_1$$

$$\because\ \int\frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}$$

$$\Rightarrow\ I = \frac{x}{2}\sqrt{4-5x^2} + \frac{4}{2\sqrt{5}}Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + \frac{1}{2}c_1$$

$$= \frac{x}{2}\sqrt{4-5x^2} + \frac{2}{\sqrt{5}}Sin^{-1}\left(\frac{\sqrt{5}\,x}{2}\right) + c$$

Where $c = \frac{1}{2}c_1$

Question # 4(iv)

Let $\ I = \int\sqrt{3-4x^2}\ dx$

Same as above.

Question # 4(v)

Same as Q # 4(ii)

Use $\int \frac{dx}{\sqrt{x^2+4}} = \ln\left|\,x+\sqrt{x^2+4}\,\right| + c$

Question # 4(vi)

Let $\ I = \int x^2 e^{ax}dx$

Do yourself as Question # 2(v)

Important Formula

Since $\quad \frac{d}{dx}\left(e^{ax}f(x)\right) = e^{ax}\frac{d}{dx}f(x) + f(x)\frac{d}{dx}e^{ax}$

$$= e^{ax}f'(x) + f(x)\cdot e^{ax}(a)$$

$$= e^{ax}\left[\ a\,f(x) + f'(x)\ \right]$$

On integrating

$$\int\frac{d}{dx}\left(e^{ax}f(x)\right)dx = \int e^{ax}\left[\ a\,f(x)+f'(x)\ \right]dx$$

$$\Rightarrow\ e^{ax}f(x) = \int e^{ax}\left[\ a\,f(x)+f'(x)\ \right]dx$$

$$\Rightarrow\ \boxed{\int e^{ax}\left[\ a\,f(x)+f'(x)\ \right]dx = e^{ax}f(x)+c}$$

Question # 5(i)

Let $\ I = \int e^x\left(\frac{1}{x}+\ln x\right)dx$

$$= \int e^x\left(\ln x + \frac{1}{x}\right)dx$$

Put $\ f(x) = \ln x \ \Rightarrow\ f'(x) = \frac{1}{x}$

So $\ I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c \quad = e^x\ln x + c$$

Question # 5(ii)

Let $\quad I = \int e^x\left(\cos x + \sin x\right)dx$

$$= \int e^x\left(\sin x + \cos x\right)dx$$

Put $\ f(x) = \sin x \ \Rightarrow\ f'(x) = \cos x$

So $\ I = \int e^x\left(f(x)+f'(x)\right)dx$

$$= e^x f(x) + c$$

$$= e^x\sin x + c$$

Question # 5(iii)

Let $\ I = \int e^{ax}\left[a\sec^{-1}x + \frac{1}{x\sqrt{x^2-1}}\right]dx$

Put $\ f(x) = \sec^{-1}x \ \Rightarrow\ f'(x) = \frac{1}{x\sqrt{x^2-1}}$

So $\ I = \int e^{ax}\left[a\,f(x)+f'(x)\right]dx$

$$= e^{ax}f(x) + c$$

$$= e^{ax}\sec^{-1}x + c$$

Question # 5(iv)

Let $\ I = \int e^{3x}\left(\frac{3\sin x - \cos x}{\sin^2 x}\right)dx$

$$= \int e^{3x}\left(\frac{3\sin x}{\sin^2 x} - \frac{\cos x}{\sin^2 x}\right)dx$$

$$= \int e^{3x}\left(3\frac{1}{\sin x} - \frac{\cos x}{\sin x \cdot \sin x}\right)dx$$

$$= \int e^{3x}\left(3\csc x - \csc x \cot x\right)dx$$

Put $f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$

$$\Rightarrow I = \int e^{3x}\left(3f(x) + f'(x)\right)dx$$

$$= e^{3x} f(x) + c$$

$$= e^{3x}\csc x + c$$

Question 5(v)

Let $I = \int e^{2x}\left(-\sin x + 2\cos x\right)dx$ **Correction*

$$= \int e^{2x}\left(2\cos x - \sin x\right)dx$$

Put $f(x) = \cos x \Rightarrow f'(x) = -\sin x$

So $I = \int e^{2x}\left(2f(x) + f'(x)\right)dx$

$$= e^{2x} f(x) + c$$

$$= e^{2x}\cos x + c$$

Question # 5(vi)

Let $I = \int \frac{xe^x}{(1+x)^2}dx$

$$= \int \frac{(1+x-1)e^x}{(1+x)^2}dx$$

$$= \int e^x\left[\frac{1+x}{(1+x)^2} - \frac{1}{(1+x)^2}\right]dx$$

$$= \int e^x\left[\frac{1}{(1+x)} - \frac{1}{(1+x)^2}\right]dx$$

Put $f(x) = \frac{1}{1+x} = (1+x)^{-1}$

$$\Rightarrow f'(x) = -(1+x)^{-2} = -\frac{1}{(1+x)^2}$$

So $I = \int e^x\left(f(x) + f'(x)\right)dx$

$$= e^x f(x) + c$$

$$= e^x\left(\frac{1}{1+x}\right) + c$$

Question # 5(vii)

Let $I = \int e^{-x}\left(\cos x - \sin x\right)dx$

$$= \int e^{-x}\left((-1)\sin x + \cos x\right)dx$$

Put $f(x) = \sin x \Rightarrow f'(x) = \cos x$

So $I = \int e^{-x}\left((-1)f(x) + f'(x)\right)dx$

$$= e^{-x} f(x) + c$$

$$= e^{-x}\sin x + c$$

Question # 5(viii)

Let $I = \int \frac{e^{m\tan^{-1}x}}{1+x^2}dx$

$$= \int e^{m\tan^{-1}x}\cdot\frac{1}{1+x^2}dx$$

Put $t = \tan^{-1}x \Rightarrow dt = \frac{1}{1+x^2}dx$

So $I = \int e^{mt}dt$

$$= \frac{e^{mt}}{m} + c$$

$$= \frac{1}{m}e^{m\tan^{-1}x} + c$$

Important Integral

Let $I = \int \tan x\,dx$

$$= \int \frac{\sin x}{\cos x}dx$$

Put $t = \cos x \Rightarrow dt = -\sin x\,dx$

$\Rightarrow -dt = \sin x\,dx$

So $I = \int \frac{-dt}{t} = -\int \frac{dt}{t}$

$$= -\ln|t| + c$$

$$= -\ln|\cos x| + c$$

$$= \ln|\cos x|^{-1} + c \qquad \because m\ln x = \ln x^m$$

$$= \ln\left|\frac{1}{\cos x}\right| + c = \ln|\sec x| + c$$

$$\Rightarrow \boxed{\int \tan x\,dx = \ln|\sec x| + c}$$

Similarly, we have

$$\boxed{\int \cot x\,dx = \ln|\sin x| + c}$$

Question # 5(ix)

Let $I = \int \frac{2x}{1-\sin x}dx$

$$= \int \frac{2x}{1-\sin x}\cdot\frac{1+\sin x}{1+\sin x}dx$$

$$= \int \frac{2x(1+\sin x)}{1-\sin^2 x}dx$$

$$= \int \frac{2x + 2x\sin x)}{\cos^2 x}dx$$

$$= \int \left(\frac{2x}{\cos^2 x} + \frac{2x\sin x}{\cos^2 x}\right)dx$$

$$= \int \frac{2x}{\cos^2 x}dx + \int \frac{2x\sin x}{\cos x\cdot\cos x}dx$$

$$= 2\int x\sec^2 x\,dx + 2\int x\sec x\tan x\,dx$$

Integrating by parts

$$I = 2\left[x\cdot\tan x - \int \tan x\cdot 1\,dx\right] + 2\left[x\cdot\sec x - \int \sec x(1)\,dx\right]$$

$$= 2\left[x\cdot\tan x - \ln|\sec x|\right] + 2\left[x\cdot\sec x - \ln|\sec x + \tan x|\right] + c$$

$$= 2x\tan x - 2\ln|\sec x| + 2x\sec x - 2\ln|\sec x + \tan x| + c$$

Question # 5(x)

Let $I = \int \frac{e^x(1+x)}{(2+x)^2}dx$

$$= \int \frac{e^x(2+x-1)}{(2+x)^2}dx$$

$$= \int e^x\left(\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right)dx$$

$$= \int e^{x}\left((2+x)^{-1}-(2+x)^{-2}\right)dx$$

Put $f(x)=(2+x)^{-1} \quad \Rightarrow \quad f'(x)=-(2+x)^{-2}$

So $I=\int e^{x}\left(f(x)+f'(x)\right)dx$

$$= e^{x} f(x)+c$$

$$= e^{x}(2+x)^{-1}+c$$

$$= \frac{e^{x}}{2+x}+c$$

Question # 15(xi)

Let $I=\int\left(\frac{1-\sin x}{1-\cos x}\right)e^{x}\,dx$

$$= \int\left(\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^{2}\frac{x}{2}}\right)e^{x}\;dx$$

$$= \int\left(\frac{1}{2\sin^{2}\frac{x}{2}}-\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^{2}\frac{x}{2}}\right)e^{x}\;dx$$

$$= \int\left(\frac{1}{2}\operatorname{cosec}^{2}\frac{x}{2}-\cot\frac{x}{2}\right)e^{x}\;dx$$

$$= \int e^{x}\left(-\cot\frac{x}{2}+\frac{1}{2}\operatorname{cosec}^{2}\frac{x}{2}\right)dx$$

Put $f(x)=-\cot\frac{x}{2} \quad \Rightarrow \quad f'(x)=\operatorname{cosec}^{2}\frac{x}{2}\cdot\frac{1}{2}$

$$\Rightarrow \quad f'(x)=\frac{1}{2}\operatorname{cosec}^{2}\frac{x}{2}$$

So $I=\int e^{x}\left(f(x)+f'(x)\right)$

$$= e^{x} f(x)+c$$

$$= e^{x}\left(-\cot\frac{x}{2}\right)+c$$

$$= -\,e^{x}\cot\frac{x}{2}+c.$$
