EXERCISE 3.3

Question # 1

Let $I = \int \frac{-2x}{\sqrt{4-x^2}}dx$

Put $t = 4-x^2 \quad \Rightarrow \quad dt = -2x\ dx$

So $I = \int \frac{dt}{\sqrt{t}} \quad = \int (t)^{-\frac{1}{2}} dt$

$$= \frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + c \quad = \frac{(t)^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2\sqrt{t} + c \quad = 2\sqrt{4-x^2} + c$$

Important Integrals

Since $\frac{d}{dx} Tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{a^2+x^2}$

By Integrating, we have

$$Tan^{-1}\left(\frac{x}{a}\right) = \int \frac{a}{a^2+x^2}dx$$

$$= a \cdot \int \frac{1}{a^2+x^2}dx$$

$$\Rightarrow \boxed{\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a} Tan^{-1}\left(\frac{x}{a}\right)}$$

Similarly

$$\boxed{\int \frac{dx}{\sqrt{a^2-x^2}} = Sin^{-1}\frac{x}{a}}$$

$$\boxed{\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a} Sec^{-1}\frac{x}{a}}$$

Question # 2

Let $I = \int \frac{dx}{x^2+4x+13}$

$$= \int \frac{dx}{x^2+2(x)(2)+(2)^2-(2)^2+13}$$

$$= \int \frac{dx}{(x+2)^2-4+13}$$

$$= \int \frac{dx}{(x+2)^2+9} \quad = \int \frac{dx}{(x+2)^2+(3)^2}$$

Put $t = x+2 \quad \Rightarrow \quad dt = dx$

So $I = \int \frac{dt}{t^2+3^2}$

$$= \frac{1}{3} Tan^{-1}\frac{t}{3} + c$$

$$= \frac{1}{3} Tan^{-1}\frac{x+2}{3} + c$$

Question # 3

$$\int \frac{x^2}{4+x^2}dx$$

$$= \int \left(1 - \frac{4}{4+x^2}\right)dx$$

$$\begin{array}{r|l} & \ \ 1 \\ 4+x^2 & x^2 \\ & \underline{\underset{-}{x^2} \underset{-}{+4}} \\ & \ \ -4 \end{array}$$

$$= \int dx - 4\int \frac{dx}{4+x^2}$$

$$= x - 4\int \frac{dx}{(2)^2+x^2}$$

$$= x - 4 \cdot \frac{1}{2} Tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= x - 2\ Tan^{-1}\left(\frac{x}{2}\right) + c$$

Question # 4

Suppose $I = \int \frac{1}{x \ln x}dx$

$$= \int \frac{1}{\ln x} \cdot \frac{1}{x}\ dx$$

Put $t = \ln x \quad \Rightarrow \quad dt = \frac{1}{x}dx$

So $I = \int \frac{1}{t}dt \quad = \ln|t| + c$

$$= \ln|\ln x| + c$$

Question # 5

Suppose $I = \int \frac{e^x}{e^x+3}dx$

Put $t = e^x + 3 \quad \Rightarrow \quad dt = e^x dx$

So $I = \int \frac{dt}{t} = \ln|t| + c$

$$= \ln|e^x+3| + c$$

Question # 6

Let $I = \int \frac{x+b}{(x^2+2bx+c)^{\frac{1}{2}}}dx$

Put $t = x^2+2bx+c$

$\Rightarrow dt = (2x+2b)dx \quad \Rightarrow \quad dt = 2(x+b)dx$

$\Rightarrow \frac{1}{2}dt = (x+b)dx$

So $I = \int \frac{\frac{1}{2}dt}{t^{\frac{1}{2}}} \quad = \frac{1}{2}\int t^{-\frac{1}{2}}\ dt$

$$= \frac{1}{2} \cdot \frac{t^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)} + c_1 \quad = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\cancel{1}/2} + c_1$$

$$= (x^2+2bx+c)^{\frac{1}{2}} + c_1$$

$$= \sqrt{x^2+2bx+c} + c_1$$

Question # 7

Let $I = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Put $t = \tan x \quad \Rightarrow \quad dt = \sec^2 x\ dx$

So $I=\int\frac{dt}{\sqrt{t}} \quad =\int t^{-\frac{1}{2}}\,dt$

$$=\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c \quad =\frac{t^{\frac{1}{2}}}{\cancel{1}/2}+c$$

$$=2(\tan x)^{\frac{1}{2}}+c \quad =2\sqrt{\tan x}+c$$

Important Integral

$$\int\sec\theta\,d\theta=\int\frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta$$

$$=\int\frac{\sec^2\theta+\sec\theta\tan\theta}{\sec\theta+\tan\theta}d\theta$$

$$=\int\frac{\sec\theta\tan\theta+\sec^2\theta}{\sec\theta+\tan\theta}d\theta$$

Take $t=\sec\theta+\tan\theta$

$\Rightarrow\ dt=(\sec^2\theta+\sec\theta\tan\theta)d\theta$

So $\int\sec\theta\,d\theta=\int\frac{1}{t}dt$

$$=\ln|t|+c$$

$$=\ln|\sec\theta+\tan\theta|+c$$

$\Rightarrow\ \boxed{\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+c}$

Similarly

$$\boxed{\int\operatorname{cosec}\theta\,d\theta=\ln|\operatorname{cosec}\theta-\cot\theta|+c}$$

See proof at page 133

Question # 8 (a)

Let $I=\frac{dx}{\sqrt{x^2-a^2}}$

Put $x=a\sec\theta \quad\Rightarrow\quad dx=a\sec\theta\tan\theta\,d\theta$

So $I=\int\frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{(a\sec\theta)^2-a^2}}$

$$=\int\frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2(\sec^2\theta-1)}}$$

$$=\int\frac{a\sec\theta\tan\theta\,d\theta}{\sqrt{a^2\tan^2\theta}} \qquad \because\ 1+\tan^2\theta=\sec^2\theta$$

$$=\int\frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta} \quad =\int\sec\theta\,d\theta$$

$$=\ln|\sec\theta+\tan\theta|+c_1$$

$$=\ln\left|\sec\theta+\sqrt{\sec^2\theta-1}\right|+c_1$$

$$=\ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+c_1 \qquad \begin{array}{|l} x=a\sec\theta\\ \frac{x}{a}=\sec\theta\end{array}$$

$$=\ln\left|\frac{x}{a}+\sqrt{\frac{x^2-a^2}{x^2}}\right|+c_1$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$=\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+c_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|-\ln a+c_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+c$$

where $c=-\ln a+c_1$

Question # 8(b)

Let $I=\sqrt{a^2-x^2}dx$

Put $x=a\sin\theta \quad\Rightarrow\quad dx=a\cos\theta\,d\theta$

So $I=\int\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$

$$=\int\sqrt{a^2(1-\sin^2\theta)}\cdot a\cos\theta\,d\theta$$

$$=\int\sqrt{a^2\cos^2\theta}\cdot a\cos\theta\,d\theta \qquad \because\ 1-\sin^2\theta=\cos^2\theta$$

$$=\int a\cos\theta\cdot a\cos\theta\,d\theta$$

$$=a^2\int\cos^2\theta\,d\theta \qquad \because\ \cos^2\theta=\frac{1+\cos 2\theta}{2}$$

$$=a^2\int\frac{1+\cos 2\theta}{2}d\theta \quad =\frac{a^2}{2}\int(1+\cos 2\theta)d\theta$$

$$=\frac{a^2}{2}\left(\theta+\frac{\sin 2\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\frac{2\sin\theta\cos\theta}{2}\right)+c$$

$$=\frac{a^2}{2}\left(\theta+\sin\theta\sqrt{1-\sin^2\theta}\right)+c \qquad \begin{array}{|l} x=a\sin\theta\\ \frac{x}{a}=\sin\theta\\ \sin^{-1}\frac{x}{a}=\theta\end{array}$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{\frac{a^2-x^2}{a^2}}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)+c$$

$$=\frac{a^2}{2}\left(Sin^{-1}\frac{x}{a}+\frac{x}{a^2}\sqrt{a^2-x^2}\right)+c$$

$$=\frac{a^2}{2}Sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c$$

Question # 9

Let $I=\int\frac{dx}{(1+x^2)^{\frac{3}{2}}}$

Put $x=\tan\theta \quad\Rightarrow\quad dx=\sec^2\theta\,d\theta$

$$I=\int\frac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}}$$

$$=\int\frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{\frac{3}{2}}} \qquad \because\ 1+\tan^2\theta=\sec^2\theta$$

$$=\int\frac{\sec^2\theta\,d\theta}{\sec^3\theta}$$

$$=\int\frac{d\theta}{\sec\theta} \quad =\int\cos\theta\,d\theta \quad =\sin\theta+c$$

$$=\frac{\sin\theta}{\cos\theta}\cdot\cos\theta+c \quad =\tan\theta\cdot\frac{1}{\sec\theta}+c$$

$$=\tan\theta\cdot\frac{1}{\sqrt{1+\tan^2\theta}}+c$$

$$= \frac{x}{\sqrt{1+x^2}} + c \qquad \because \; x = \tan\theta$$

Question # 10

Let $I = \int \frac{1}{(1+x^2)\, Tan^{-1}x} dx$

$$= \int \frac{1}{Tan^{-1}x} \cdot \frac{1}{(1+x^2)} dx$$

Put $t = Tan^{-1}x \quad \Rightarrow \quad dt = \frac{1}{1+x^2} dx$

So $I = \int \frac{1}{t} dt \quad = \ln|t| + c$

$$= \ln\left| Tan^{-1}x \right| + c$$

Question # 11

Let $I = \int \sqrt{\frac{1+x}{1-x}}\, dx$

Put $x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta\, d\theta$

So $I = \int \sqrt{\frac{1+\sin\theta}{1-\sin\theta}} \cdot \cos\theta\, d\theta$

$$= \int \sqrt{\frac{1+\sin\theta}{1-\sin\theta} \cdot \frac{1+\sin\theta}{1+\sin\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \sqrt{\frac{(1+\sin\theta)^2}{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \sqrt{\frac{(1+\sin\theta)^2}{\cos^2\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \frac{1+\sin\theta}{\cos\theta} \cdot \cos\theta\, d\theta \quad = \int (1+\sin\theta)\, d\theta$$

$$= \theta - \cos\theta + c$$

$$= \theta - \sqrt{1-\sin^2\theta} + c \qquad \because \; x = \sin\theta$$

$$= Sin^{-1}x - \sqrt{1-x^2} + c \qquad \therefore \; \sin^{-1}x = \theta$$

Question # 12

Let $I = \int \frac{\sin\theta}{1+\cos^2\theta} d\theta$

Put $t = \cos\theta$

$\Rightarrow \quad dt = -\sin\theta\, d\theta \quad \Rightarrow \quad -dt = \sin\theta\, d\theta$

So $I = \int \frac{-dt}{1+t^2} \quad = -\int \frac{dt}{1+t^2}$

$$= -\tan^{-1}t + c$$

$$= -\tan^{-1}(\cos\theta) + c$$

Question # 13

Let $I = \int \frac{ax}{\sqrt{a^2 - x^4}}\, dx$

$$= a\int \frac{x}{\sqrt{a^2 - x^4}}\, dx$$

Put $t = x^2$ then $t^2 = x^4$

$$dt = 2x\, dx \quad \Rightarrow \quad \frac{1}{2} dt = x \cdot dx$$

So $I = a\int \frac{\frac{1}{2} dt}{\sqrt{a^2 - t^2}}$

$$= \frac{a}{2} \int \frac{dt}{\sqrt{a^2 - t^2}}$$

$$= \frac{a}{2} Sin^{-1}\left(\frac{t}{a}\right) + c \qquad \because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a}$$

$$= \frac{a}{2} Sin^{-1}\left(\frac{x^2}{a}\right) + c$$

Question # 14

Let $I = \int \frac{dx}{\sqrt{7 - 6x - x^2}}$

$$= \int \frac{dx}{\sqrt{-\left(x^2 + 6x - 7\right)}}$$

$$= \int \frac{dx}{\sqrt{-\left(x^2 + 2(3)(x) + (3)^2 - (3)^2 - 7\right)}}$$

$$= \int \frac{dx}{\sqrt{-\left((x+3)^2 - 16\right)}}$$

$$= \int \frac{dx}{\sqrt{16 - (x+3)^2}}$$

Put $t = x + 3 \quad \Rightarrow \quad dx = dt$

So $I = \frac{dt}{\sqrt{16 - t^2}} \quad = \int \frac{dx}{\sqrt{(4)^2 - (t)^2}}$

$$= Sin^{-1}\left(\frac{t}{4}\right) + c$$

$$= Sin^{-1}\left(\frac{x+3}{4}\right) + c$$

Question # 15

Let $I = \int \frac{\cos x}{\sin x \cdot \ln\sin x} dx$

$$= \int \frac{1}{\ln\sin x} \cdot \frac{\cos x}{\sin x} dx$$

Put $t = \ln\sin x \quad \Rightarrow \quad dt = \frac{1}{\sin x} \cdot \cos x\; dx$

So $I = \int \frac{1}{t} dt \quad = \ln|t| + c$

$$= \ln\left|\ln\sin x\right| + c$$

Question # 16

Let $I = \int \cos x \left(\frac{\ln\sin x}{\sin x}\right) dx$

$$= \int \ln\sin x \cdot \frac{\cos x}{\sin x}\; dx$$

Put $t = \ln\sin x \quad \Rightarrow \quad dt = \frac{1}{\sin x} \cdot \cos x\; dx$

No do yourself

Question # 17

Let $I = \int \frac{x\, dx}{4 + 2x + x^2}$

$$= \frac{1}{2} \int \frac{2x\; dx}{x^2 + 2x + 4}$$

+ing and −ing 2 in numerator.

$$\Rightarrow \quad I = \frac{1}{2} \int \frac{(2x+2) - 2}{x^2 + 2x + 4}\; dx$$

$$=\frac{1}{2}\int\left(\frac{2x+2}{x^2+2x+4}-\frac{2}{x^2+2x+4}\right)dx$$
$$=\frac{1}{2}\int\frac{2x+2}{x^2+2x+4}\,dx-\frac{1}{2}\int\frac{2}{x^2+2x+4}\,dx$$
$$=\frac{1}{2}\int\frac{\frac{d}{dx}\left(x^2+2x+4\right)}{x^2+2x+4}\,dx-\frac{2}{2}\int\frac{dx}{x^2+2x+1+3}$$
$$=\frac{1}{2}\ln\left|x^2+2x+4\right|-\int\frac{dx}{(x+1)^2+\left(\sqrt{3}\right)^2}$$
$$=\frac{1}{2}\ln\left|x^2+2x+4\right|-\frac{1}{\sqrt{3}}Tan^{-1}\frac{x+1}{\sqrt{3}}+c$$

Question # 18

Let $I=\int\frac{x}{x^4+2x^2+5}\,dx$

Put $t=x^2$ then $t^2=x^4$

$$dt=2x\,dx \quad\Rightarrow\quad \frac{1}{2}dt=x\,dx$$

So $I=\int\frac{\frac{1}{2}dt}{t^2+2t+5}\quad=\frac{1}{2}\int\frac{dt}{t^2+2t+1+4}$

$$=\frac{1}{2}\int\frac{dt}{(t+1)^2+(2)^2}$$
$$=\frac{1}{2}\cdot\frac{1}{2}Tan^{-1}\left(\frac{t+1}{2}\right)+c$$
$$=\frac{1}{4}Tan^{-1}\left(\frac{x^2+1}{2}\right)+c$$

Question # 19

Let $I=\int\left[\cos\left(\sqrt{x}-\frac{x}{2}\right)\right]\times\left(\frac{1}{\sqrt{x}}-1\right)\,dx$

Put $t=\sqrt{x}-\frac{x}{2}$

$$\Rightarrow\ dt=\left(\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{2}\right)dx \quad\Rightarrow\ dt=\frac{1}{2}\left(\frac{1}{\sqrt{x}}-1\right)dx$$
$$\Rightarrow\ 2\,dt=\left(\frac{1}{\sqrt{x}}-1\right)dx$$

So $I=\int\cos t\cdot 2\,dt$

$$=2\int\cos t\,dt$$
$$=2\sin t+c$$

Question # 20

Let $I=\int\frac{x+2}{\sqrt{x+3}}\,dx$

Put $t=x+3$ then $t-3=x$

$\Rightarrow\ dt=dx$

So $I=\int\frac{t-3+2}{\sqrt{t}}dx$

$$=\int\frac{t-1}{(t)^{\frac{1}{2}}}\,dx\quad=\int\left(\frac{t}{(t)^{\frac{1}{2}}}-\frac{1}{(t)^{\frac{1}{2}}}\right)dx$$
$$=\int\left((t)^{\frac{1}{2}}-(t)^{-\frac{1}{2}}\right)dx$$
$$=\frac{(t)^{\frac{1}{2}+1}}{\frac{1}{2}+1}-\frac{(t)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+c\quad=\frac{(t)^{\frac{3}{2}}}{3/2}-\frac{(t)^{\frac{1}{2}}}{1/2}+c$$
$$=\frac{2(x+3)^{\frac{3}{2}}}{3}-2(x+3)^{\frac{1}{2}}+c$$
$$=\frac{2(x+3)^{\frac{3}{2}}}{3}-2\sqrt{x+3}+c$$

Question # 21

Let $I=\int\frac{\sqrt{2}}{\sin x+\cos x}dx$

$$=\int\frac{1}{\frac{1}{\sqrt{2}}\left(\sin x+\cos x\right)}dx$$
$$=\int\frac{1}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x}dx$$

Put $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$

So $I=\int\frac{1}{\sin\frac{\pi}{4}\cdot\sin x+\cos\frac{\pi}{4}\cdot\cos x}dx$

$$=\int\frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx\quad=\int\sec\left(x-\frac{\pi}{4}\right)dx$$
$$=\ln\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|+c$$

Question # 22

Let $I=\int\frac{dx}{\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x}$

$\because\ \cos\frac{\pi}{3}=\frac{1}{2}\quad\&\quad\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$\therefore\ I=\int\frac{dx}{\cos\frac{\pi}{3}\sin x+\sin\frac{\pi}{3}\cos x}$

$$=\int\frac{dx}{\sin\left(x+\frac{\pi}{3}\right)}\quad=\int\text{cosec}\left(x+\frac{\pi}{3}\right)dx$$
$$=\ln\left|\text{cosec}\left(x+\frac{\pi}{3}\right)-\cot\left(x+\frac{\pi}{3}\right)\right|+c$$