EXERCISE 4.5

◈ *Homogenous 2^{nd} Degree Equation*

Every homogenous second degree equation

$$ax^2 + 2hxy + by^2 = 0$$

represents straight lines through the origin.

Consider the equations are $y = m_1x$ and $y = m_2x$

$$\Rightarrow \quad m_1x - y = 0 \quad \text{and} \quad m_2x - y = 0$$

Taking product

$$(m_1x - y)(m_2x - y) = 0$$

$$\Rightarrow \quad m_1m_2x^2 - m_1xy - m_2xy + y^2 = 0$$

$$\Rightarrow \quad m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0 \quad \ldots\ldots\ldots. (i)$$

Also we have

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow \quad \frac{a}{b}x^2 + \frac{2h}{b}xy + y^2 = 0 \qquad \div\text{ing by } b$$

$$\Rightarrow \quad \frac{a}{b}x^2 - \left(-\frac{2h}{b}\right)xy + y^2 = 0$$

Comparing it with (i), we have

$$\boxed{m_1m_2 = \frac{a}{b}} \quad \text{and} \quad \boxed{m_1 + m_2 = -\frac{2h}{b}}$$

Let θ be the angles between the lines then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2}$$

$$= \frac{\sqrt{(m_1 - m_2)^2}}{1 + m_1m_2} = \frac{\sqrt{m_1^2 + m_2^2 - 2m_1m_2}}{1 + m_1m_1}$$

$$= \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 - 4m_1m_2}}{1 + m_1m_1} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1m_2}}{1 + m_1m_1}$$

$$= \frac{\sqrt{\left(-\frac{2h}{b}\right)^2 - 4\frac{a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2}{b^2} - \frac{4a}{b}}}{1 + \frac{a}{b}} = \frac{\sqrt{\frac{4h^2 - 4ab}{b^2}}}{\frac{b + a}{b}}$$

$$\Rightarrow \quad \tan\theta = \frac{\sqrt{4(h^2 - ab)}}{b + a} \quad \Rightarrow \quad \boxed{\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}}$$

Question # 1

$$10x^2 - 23xy - 5y^2 = 0 \quad \ldots\ldots\ldots\ldots (i)$$

$$\Rightarrow \quad 10x^2 - 25xy + 2xy - 5y^2 = 0$$

$$\Rightarrow \quad 5x(2x - 5y) + y(2x - 5y) = 0 \quad \Rightarrow \quad (2x - 5y)(5x + y) = 0$$

$$\Rightarrow \quad 2x - 5y = 0 \quad \text{and} \quad 5x + y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

So $a = 10$, $2h = -23 \Rightarrow h = -\dfrac{23}{2}$, $b = -5$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(-\frac{23}{2}\right)^2 - (10)(-5)}}{10-5} = \frac{2\sqrt{\frac{529}{4} + 50}}{5}$$

$$= \frac{2\sqrt{729/4}}{5} = \frac{2\left(27/2\right)}{5} = \frac{27}{5}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{27}{5}\right) = 79^\circ 31'$$

Hence acute angle between the lines $= 79^\circ 31'$

Question # 2 & 3 *Do yourself as above*

Question # 4

$$2x^2 + 3xy - 5y^2 = 0 \;\ldots\ldots\ldots\ldots\; (i)$$

$$\Rightarrow 2x^2 + 5xy - 2xy - 5y^2 = 0$$

$$\Rightarrow x(2x+5y) - y(2x+5y) = 0$$

$$\Rightarrow (2x+5y)(x-y) = 0$$

$$\Rightarrow 2x+5y = 0 \text{ and } x - y = 0$$

are the required lines.

Comparing eq. (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$$\Rightarrow a = 2 \quad , \quad 2h = 3 \Rightarrow h = \frac{3}{2} \quad , \quad b = -5$$

Let θ be angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\left(\frac{3}{2}\right)^2 - (2)(-5)}}{2-5} = \frac{2\sqrt{\frac{9}{4} + 10}}{-3}$$

$$= -\frac{2\sqrt{49/4}}{3} = -\frac{2\left(7/2\right)}{3} = -\frac{7}{3}$$

$$\Rightarrow -\tan\theta = \frac{7}{3}$$

$$\Rightarrow \tan(180 - \theta) = \frac{7}{3} \qquad \because \tan(180-\theta) = -\tan\theta$$

$$\Rightarrow 180 - \theta = \tan^{-1}\left(\frac{7}{3}\right) \qquad \Rightarrow 180 - \theta = 66^\circ 48'$$

$$\Rightarrow \theta = 180 - 66^\circ 48' = 113^\circ 12'$$

Hence acute angle between the lines $= 180 - 113^\circ 12' = 66^\circ 48'$

Question # 5

Do yourself as above

Question # 6

$$x^2 + 2xy\sec\alpha + y^2 = 0 \;\ldots\ldots\ldots\; (i)$$

$\div$ing by y^2

$$\frac{x^2}{y^2} + \frac{2xy\sec\alpha}{y^2} + \frac{y^2}{y^2} = 0$$

$$\Rightarrow \left(\frac{x}{y}\right)^2 + 2\sec\alpha\left(\frac{x}{y}\right) + 1 = 0$$

This is quadric equation in $\frac{x}{y}$ with $a = 1$, $b = 2\sec\alpha$, $c = 1$

$$\frac{x}{y} = \frac{-2\sec\alpha \pm \sqrt{(2\sec\alpha)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-2\sec\alpha \pm \sqrt{4\sec^2\alpha - 4}}{2(1)} = \frac{-2\sec\alpha \pm \sqrt{4(\sec^2\alpha - 1)}}{2}$$

$$= \frac{-2\sec\alpha \pm \sqrt{4\tan^2\alpha}}{2} \qquad \because\; 1 + \tan^2\alpha = \sec^2\alpha$$

$$= \frac{-2\sec\alpha \pm 2\tan\alpha}{2}$$

$$\Rightarrow \frac{x}{y} = -\sec\alpha \pm \tan\alpha$$

$$= -\frac{1}{\cos\alpha} \pm \frac{\sin\alpha}{\cos\theta} = \frac{-1 \pm \sin\alpha}{\cos\alpha}$$

$$\Rightarrow \frac{x}{y} = \frac{-1 + \sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{x}{y} = \frac{-1 - \sin\alpha}{\cos\alpha}$$

$$\Rightarrow x\cos\alpha = (-1 + \sin\alpha)y \quad \text{and} \quad x\cos\alpha = (-1 - \sin\alpha)y$$

$$\Rightarrow x\cos\alpha - (-1 + \sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha - (-1 - \sin\alpha)y = 0$$

$$\Rightarrow x\cos\alpha + (1 - \sin\alpha)y = 0 \quad \text{and} \quad x\cos\alpha + (1 + \sin\alpha)y = 0$$

are required equations of lines.

Now comparing (i) with

$$ax^2 + 2hxy + by^2 = 0$$

$a = 1$, $2h = 2\sec\alpha \Rightarrow h = \sec\alpha$, $b = 1$b

If θ is angle between lines then

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$= \frac{2\sqrt{\sec^2\alpha - (1)(1)}}{1 + 1} = \frac{2\sqrt{\sec^2\alpha - 1}}{2} = \sqrt{\tan^2\alpha}$$

$$\Rightarrow \tan\theta = \tan\alpha \quad \Rightarrow \theta = \alpha$$

Question # 7

Given: $x^2 - 2xy\tan\alpha - y^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$a = 1 ,\quad 2h = -2\tan\alpha \;\Rightarrow\; h = -\tan\alpha ,\quad b = -1$$

$$m_1 + m_2 = -\frac{2h}{b}$$

$$= -\frac{-2\tan\alpha}{-1}$$

$$\Rightarrow m_1 + m_2 = -2\tan\alpha$$

$$\& \quad m_1 m_2 = \frac{a}{b} = \frac{1}{-1} \Rightarrow m_1 m_2 = -1$$

Now slopes of lines $\perp$ar to given lines are $-\dfrac{1}{m_1}$ and $-\dfrac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$

$$\Rightarrow \ m_1 y = -x \quad \& \quad m_2 y = -x$$

$$\Rightarrow \ x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$

$$\Rightarrow \ x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$

$$\Rightarrow \ x^2 + (-2\tan\alpha)xy + (-1)y^2 = 0$$

$$\Rightarrow \ x^2 - 2xy\tan\alpha - y^2 = 0$$

Question # 8

Given: $ax^2 + 2hxy + by^2 = 0$

Suppose m_1 and m_2 are slopes of given lines then

$$m_1 + m_2 = -\frac{2h}{b} \quad \& \quad m_1 m_2 = \frac{a}{b}$$

Now slopes of lines $\perp$ar to given lines are $-\dfrac{1}{m_1}$ and $-\dfrac{1}{m_2}$, then their equations are

$$y = -\frac{1}{m_1}x \quad \& \quad y = -\frac{1}{m_2}x \qquad \text{(Passing through origin)}$$

$$\Rightarrow \ m_1 y = -x \quad \& \quad m_2 y = -x$$

$$\Rightarrow \ x + m_1 y = 0 \quad \& \quad x + m_2 y = 0$$

Their joint equation:

$$(x + m_1 y)(x + m_2 y) = 0$$

$$\Rightarrow \ x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$$

$$\Rightarrow \ x^2 + \left(-\frac{2h}{b}\right)xy + \left(\frac{a}{b}\right)y^2 = 0$$

$$\Rightarrow \ bx^2 - 2hxy + ay^2 = 0$$

Question # 9

$$10x^2 - xy - 21y^2 = 0 \quad , \quad x + y + 1 = 0$$

$$\Rightarrow \ 10x^2 - 15xy - 14xy - 21y^2 = 0$$

$$\Rightarrow \ 5x(2x - 3y) - 7y(2x - 3y) = 0$$

$$\Rightarrow \ (2x - 3y)(5x - 7y) = 0$$

$$\Rightarrow \ 2x - 3y = 0 \quad \& \quad 5x - 7y = 0$$

So we have equation of lines

$l_1 :\ 2x - 3y = 0$ ………..… (*i*)

$l_2 :\ 5x - 7y = 0$ ………. (*ii*)

$l_3 :\ x + y + 1 = 0$ ……… (*iii*)

Now do yourself as Q # 14 (Ex. 4.4)
