Exercise 1.1

Q1. (a) Given that $f(x) = x^2 - x$

i. $f(-2) = (-2)^2 - (-2) = 4 + 2 = 6$

ii. $f(0) = (0)^2 - (0) = 0$

iii. $f(x\text{-}1) = (x\text{-}1)^2 - (x\text{-}1) = x^2 - 2x + 1 - x + 1 = x^2 - 3x + 2$

iv. $f(x^2+4) = (x^2+4)^2 - (x^2+4) = x^4 + 8x^2 + 16 - x^2 - 4 = x^4 + 7x^2 + 12$

(b) Given that $f(x) = \sqrt{x+4}$

$i)\ f(-2)=\sqrt{-2+4}=\sqrt{2}$

$ii)\ f(0)=\sqrt{0+4}=\sqrt{4}=2$

$iii)\ f(x-1)=\sqrt{x-1+4}=\sqrt{x+3}$

$iv)\ f(x^2+4)=\sqrt{x^2+4+4}=\sqrt{x^2+8}$

Q2. *Given that*

$i)\quad f(x)=6x-9$

$f(a+h)=6(a+h)-9=6a+6h-9$

$f(a)=6a-9$

$$Now\quad \frac{f(a+h)-f(a)}{h}=\frac{(6a+6h-9)-(6a-9)}{h}$$

$$=\frac{6a+6h-9-6a+9}{h}=\frac{6h}{h}=6$$

$ii)\quad f(x)=\sin x\quad given$

$$\because\quad \sin\theta-\sin\varphi=2\cos\left(\frac{\theta+\varphi}{2}\right)\sin\left(\frac{\theta+\varphi}{2}\right)$$

$f(a+h)=\sin(a+h)\quad and\quad f\left(a\right)=\sin a$

$$Now\quad \frac{f(a+h)-f(a)}{h}=\frac{\sin(a+h)-\sin a}{h}$$

$$=\frac{1}{h}\left[\sin(a+h)-\sin a\right]$$

$$=\frac{1}{h}\left[2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\right]=\frac{1}{h}\left[2\cos\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]$$

$$=\frac{1}{h}\left[2\cos\left(\frac{2a}{2}+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)\right]=\frac{2}{h}\cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

iii) *Given that* $f(x)=x^3+2x^2-1$

$$f(a+h)=(a+h)^3+2(a+h)^2-1=a^3+h^3+3ah(a+h)+2(a^2+2ah+h^2)-1$$

$$=a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1$$

$$f(a)=a^3+2a^2-1$$

Now $f(a+h)-f(a)$

$$=\frac{a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-(a^3+2a^2-1)}{h}$$

$$=\frac{1}{h}\left[a^3+h^3+3a^2h+3ah^2+2a^2+4ah+2h^2-1-a^3-2a^2+1\right]$$

$$=\frac{1}{h}\left[h^3+3a^2h+3ah^2+4ah+2h^2\right]=\frac{h}{h}\left[h^2+3a^2+3ah+4a+2h\right]$$

$$=h^2+3a^2+3ah+4a+2h=h^2+3ah+2h+3a^2+4a=h^2+(3a+2)h+3a^2+4a$$

iv) *Given that* $f(x)=\cos x$

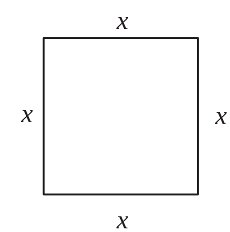
so $f(a+h)=\cos(a+h)$

and $f(a)=\cos a$

Now $\dfrac{f(a+h)-f(a)}{h}$

$$=\frac{\cos(a+h)-\cos a}{h}=\frac{1}{h}\left[-2\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)\right]=\frac{-2}{h}\sin\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)$$

Q3. (a) If 'x' unit be the side of square.



Then its perimeter P = $x+x+x+x=4x$ ………………….. (1)

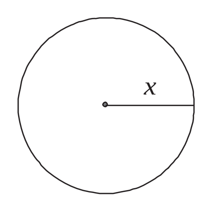
A = Area = $x\,.\,x=x^2$ …………… (2)

From (2) $x=\sqrt{A}$ putting in (1)

P = $4\sqrt{A}$

$\therefore$ P is expressed as Area

(b) Let x units be the radius of circle



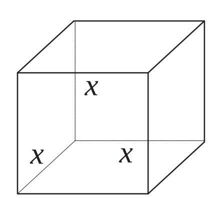
Then Area = A = πx^2 ………………….. (1)

Circumference = C = $2\pi x$ ………………….. (2)

From (2) $x=\dfrac{C}{2\pi}$ Putting in (1)

$$A=\pi\left(\frac{c}{2\pi}\right)^2=\pi\left(\frac{c^2}{4\pi^2}\right)=\frac{c^2}{4\pi}$$

$$A=\frac{c^2}{4\pi}$$ $\because$ *Area is a function of Circumference*



(c) Let x unit be each side of cube.

The Volume of Cube = $x\,.\,x\,.\,x=x^3$ ………………….. (1)

Area of base = A = x^2 ………………….. (2)

From (2) $x=\sqrt{A}$ Putting in (1)

$V = \left(\sqrt{A}\right)^3 = (A)^{3/2}$

Q5. $f(x) = x^3 - ax^2 + bx + 1$

$If \quad f(2) = -3 \quad and \quad f(-1) = 0$

$(2)^3 - a(2)^2 + b(2) + 1 = -3 \qquad (-1)^3 - a(-1)^2 + (-1) + 1 = 0$

$8 - 4a + 2b + 1 = -3 \qquad -1 - a - b + 1 = 0$

$9 - 4a + 2b = -3 \qquad -a - b = 0$

$12 - 4a + 2b = 0 \qquad a + b = 0 \quad \ldots\ldots\ldots\ldots\ldots\ (2)$

$Dividing\,by - 2$

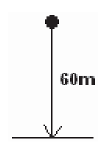
$-6 + 2a - b = 0 \ldots\ldots\ldots\ldots\ldots\ (1)$

$Solving\,(1)\,and\,(2)$

$$\begin{array}{rcl} 2a - b - 6 & = & 0 \\ a + b & = & 0 \\ \hline 3a - 6 & = & 0 \end{array}$$

$a = 2 \quad and \quad (2) \Rightarrow \ b = -a \quad \Rightarrow \quad b = -2$

Q6. $h(x) = 40 - 10x^2$



$(a) \quad x = 1\sec$

$h(1) = 40 - 10(1)^2$

$= 30m$

$(b) \quad x = 1.5\sec$

$h(1.5) = 40 - 10(1.5)^2$

$= 40 - 10(2.25) = 40 - 22.5 = 17.5m$

$(c) \quad x = 1.7\sec$

$h(1.7) = 40 - 10(1.7)^2$

$= 40 - 10(2.89) = 40 - 28.9 = 11.1m$

ii) Does the stone strike the ground = ?

$h(x) = 0$

$40 - 10x^2 = 0$

$-10x^2 = -40 \quad \Rightarrow x^2 = 4$

$x = \pm 2$

Stone strike the ground after 2 sec.

Graphs of Function

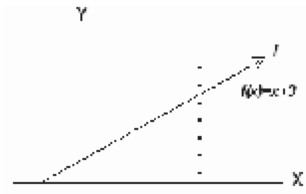
Definition:

The graph of a function f is the graph of the equation $y = f(x)$. It consists of the points in the Cartesian plane chose co-ordinates (x , y) are input - output pairs for f .

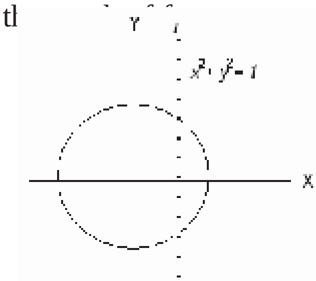
Note that not every curve we draw in the graph of a function. A function f can have only one value $f(x)$ for each x in its domain.

Vertical Line Test

No vertical line can intersect the graph of a function more than once. Thus, a circle cannot be the graph of a function. Since some vertical lines intersect the circle Twice. If 'a' is the domain of the function f , then the vertical line x = a will intersect th in the single point (a , $f(a)$).



It is graph of function



Is not a graph of function

Types of Function

ALGEBRAIC FUNCTIONS

Those functions which are defined by algebraic expressions.

1) <u>Polynomial Functions:</u>

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_1 x + a_0$$ **Is a Polynomial Function for all x where** $a_0, a_1, a_2 \ldots\ldots a_n$ **are real numbers, and exponents are non-negative integer .** a_n **is called leading coefft of p(x) of degree n, Where** $a_n \neq 0$

$\Rightarrow$ *Degree of polynomial function is the maximum power of x in equation*

$P(x) = 2x^4 - 3x^3 + 2x - 1$ $\qquad$ degree = 4

2) <u>Linear Function:</u> if the degree of polynomial fn is '1, <u>is called linear function .i.e. p(x)=ax+b</u>

or $\Rightarrow$ Degree of polynomial function is one.

$$f(x) = ax + b \qquad a \neq 0$$

$$\because \qquad y = 5x + b$$

3) <u>Identity Function:</u> For any set X, a function I: $X \to x$ of the form y = x or $f(x) = x$. Domain and range of I is x. Note. I (x)= ax +b be a linear fn if a=1,b=0 then I(x)=x or y=x is called identity fn

4) <u>Constant Function</u>: **or**

$C : X \to y$ *defined by* | $f : X \to y$ If f (x)=c, (const) then f is called constant fn

$C(x) = a \qquad \forall x \in X$ *and* $a \in y$

e.g. $\quad C : R \to R$ $\qquad$ eg y=5

$C(x) = 2$ *or* $y = 2 \qquad \forall x \in R$

5) Rational Function:

$$R(x)=\frac{P(x)}{Q(x)}$$

Both $P(x)$ *and* $Q(x)$ *are polynomial and* $Q(x)\neq 0$

e.g. $R(x)=\dfrac{3x^2+4x+1}{5x^3+2x^2+1}$

Domain of rational function is the set of all real numbers for which $Q(x)\neq 0$

6) Exponential Function:

A function in which the variable appears as exponent (power) is called an exponential function.

i) $y=a^x \;\therefore\; x\in R \qquad a>0$

ii) $y=e^x \;\therefore\; x\in R \text{ and } e=2.178$

iii) $y=2^x$ or $y=e^{xh}$

are some exponential functions.

7) Logarithmic Function:

If $x=a^y$ then $y=\log_a^x$ x >0

$\because \quad a>0 \quad a\neq 1$

'a' is called the base of Logarithemic function

Then $y=\log_a^x$ *is Logarithmic function of base 'a'*

i) *If base* $=10$ *then* $y=\log_{10}^x$

is called common Logarithm of x

ii) *If base* $=e=2.718$

$y=\log_e^x=\ln x$ *is called natural log*

8) Hyperbolic Function:

We define as

i) $y=\sinh(x)=\dfrac{e^x-e^{-x}}{2}$ Sine hyperbolic function or hyperbolic sine function

$Dom=\{x/x\in R\}$ and $Range=\{y/y\in R\}$

ii) $y=\cosh(x)=\dfrac{e^x+e^{-x}}{2}$ *is called hyperbolic cosine function* $\Rightarrow \quad x\in R, y\in[1,\infty)$

iii) $y=Tanhx=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}=\dfrac{\sinh x}{\cosh x}$ iv) $y=\coth x=\dfrac{\cosh x}{\sinh x}$

v) $y=\sec hx=\dfrac{1}{\cosh x}=\dfrac{2}{e^x+e^{-x}} \qquad x\in R$

vi) $y=\cos echx=\dfrac{1}{\sinh x}=\dfrac{2}{e^x-e^{-x}} \qquad Dom=\{x\neq 0: x\in R$

9) Inverse Hyperbolic Function: (Study in B.Sc level)

i) $y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$ *for* $\forall x \in R$

ii) $y = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$ *for* $\forall x \in R$ *and* $x > 1$

iii) $y = Tanh^{-1} x = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ $x \neq 1$ *and* $|x| < 1$

iv) $y = \operatorname{sec}h^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1-x^2}}{x}\right)$ $0 < x \leq 1$

v) $y = \coth^{-1} x = \frac{1}{2}\left|\frac{x+1}{x-1}\right|$ $\because$ $|x| > 1$

vi) $y = \cos ech^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right)$ $x \neq 0$

10) <u>Trigonometric Function:</u>

Functions	Domain(x)	Range(y)
$i)\ y = \sin x$	*All real numbers*	$-1 \leq y \leq 1$
	$\because -\infty < x + \infty$	
$ii)\ y = \cos x$	*All real numbers*	$-1 \leq y \leq 1$
	$\because -\infty < x < \infty$	
$iii)\ y = \tan x$	$x \in R - (2k+1)\frac{\pi}{2}$	$\because 'R'$ *all real numbers*
	$k \in Z$	
$iv)\ y = \cot x$	$x \in R - k\pi$	R
	$k \in Z$	
$v)\ y = \sec x$	$x \in R - (2k+1)\frac{\pi}{2}$	$R - (-1,1)$
	$k \in Z$	*or* $R - (-1 < y < 1)$
$vi)\ y = \cos ecx$	$x \in R - (k\pi)$	$R - (-1 < y < 1)$
	$k \in Z$	

11) <u>Inverse Trigonometric Functions:</u>

Function	Dom(x)	Range(y)
$y = \sin^{-1} x \Leftrightarrow x = \sin y$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$y = \cos^{-1} x \Leftrightarrow x = \cos y$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$

$y = Tan^{-1}x \Leftrightarrow x = Tany$	$x \in R$ $or\ -\infty < x < \infty$	$-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$
$y = Sec^{-1}x \Leftrightarrow x = \sec y$	$x \in R-(-1,1)$	$y \in [0,\pi]-\left\{\dfrac{\pi}{2}\right\}$
$y = Co\sec^{-1} x \Leftrightarrow x = \cos ecy$	$x \in R-(-1,1)$	$y \in \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]-\{0\}$
$y = Cot^{-1}x \Leftrightarrow x = \cot y$	$x \in R$	$0 < y < \pi$

12) <u>Explicit Function:</u>

If y is easily expressed in terms of x, then y is called an explicit function of x.

$\Rightarrow y = f(x) \qquad e.g. \qquad y = x^3 + x + 1\ \ etc.$

13) <u>Implicit Function:</u>

If x and y are so mixed up and y cannot be expressed in term of the independent variable x, Then y is called an implicit function of x. It can be written as. $\qquad f(x, y) = 0$

e.g. $\qquad x^2 + xy + y^2 = 2$ etc.

14) <u>Parametric Function:</u>

For a function y $= f(x)$ if both x& y are expressed in another variable say 't' or θ which is called a parameter of the given curve.

Such as:

i) $\qquad x = at^2 \qquad$ Parametric parabola

$\qquad y = 2at$

ii) $\qquad x = a\cos t \qquad Parametric\ equation\ of\ circle \qquad y^2 = 4a$

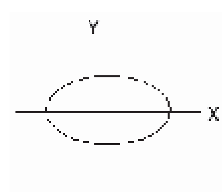
$\qquad y = a\sin t$

$\qquad x^2 + y^2 = a^2$

iii) $\qquad x = a\cos\theta \qquad Parametric\ equation\ of\ Ellipse$

$\qquad y = b\sin\theta$

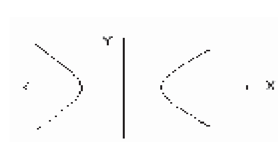
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



vi) $\qquad x = a\sec\theta \qquad Parametric\ equation\ of\ hyperbola$

$\qquad y = b\tan\theta$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$Q7.$ $\quad Parabola \quad \Rightarrow \quad y^2 = 4ax \quad$ ....(1)

$x = at^2$ ....(i)

$y = 2at$ ....(ii)

$To\, e\lim inating\ 't'\ from\,(ii) \quad t = \dfrac{y}{2a} \qquad putting\ (i)$

$$x = a\left(\frac{y}{2a}\right)^2 \quad \Rightarrow \quad x = a\left(\frac{y^2}{4a^2}\right) \quad \Rightarrow \quad x = \frac{y^2}{4a}$$

$\Rightarrow \quad y^2 = 4ax \quad which\, is\, same\, as\,(1)$

$which\, is\, equation\, of\ \ parabola.$

$ii) \quad x = a\cos\theta, \quad y = b\sin\theta$

$\Rightarrow \quad \dfrac{x}{a} = \cos\theta$...........$(i) \quad and \quad \dfrac{y}{b} = \sin\theta$..............$(ii) \qquad To\, e\lim inating\ \theta\ from\,(i)\, and\,(ii)$

$Squaring\ and\ adding\ (i)\, and\ (ii)$

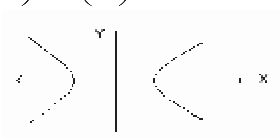
$\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{y^2}{b}\right) = 1 \qquad represent\ a\ Ellipse$

$iii) \quad x = a\sec\theta, \quad y = b\tan\theta$

$\dfrac{x}{a} = \sec\theta$...............$(i) \qquad \dfrac{y}{b} = \tan\theta$..................$(ii)$

$Squaring\ and\ Subtracting \quad (i)\, and\,(ii)$

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \sec^2\theta - \tan^2\theta \qquad \Rightarrow \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \tan^2\theta - \tan^2\theta \ \Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



Which is equation of hyperbola

$Q8. \quad (i) \quad \sinh 2x = 2\sinh x\cosh x$

$$R.H.S = 2\sinh x\cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) = 2\left(\frac{e^{2x} - e^{-2x}}{4}\right) = \frac{e^{2x} - e^{-2x}}{2}$$

$= \sinh 2x = L.H.S$

$ii) \quad \sec^2 hx = 1 - \tan^2 hx$

$$R.H.S. = 1 - \tan^2 hx = 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) = 1 - \left(\frac{e^{2x} + e^{-2x} - 2}{e^{2x} + e^{-2x} + 2}\right)$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{e^{2x} + e^{-2x} + 2} = \frac{4}{\left(e^x + e^{-x}\right)^2} = \frac{1}{\left(e^x + e^{-x}/2\right)^2}$$

$$= \frac{1}{\cosh^2 x} = \sec h^2 x = L.H.S$$

$iii)\qquad \cos eh^2 x = \coth^2 x - 1$

$$R.H.S = \coth^2 x - 1 = \left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)^2 - 1 = \frac{\left(e^x + e^{-x}\right)^2 - \left(e^x - e^{-x}\right)^2}{\left(e^x - e^{-x}\right)^2} = \frac{\left(e^{2x} + e^{-2x} + 2\right) - \left(e^{2x} + e^{-2x} - 2\right)}{\left(e^x - e^{-x}\right)^2}$$

$$= \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{\left(e^x - e^{-x}\right)^2} = \frac{4}{\left(e^x - e^{-x}\right)^2} = \frac{1}{\left(e^x - e^{-x}/2\right)^2} = \frac{1}{\sinh^2 x} = \cos ech 2x = L.H.S$$

$Q9.\qquad f(x) = x^3 + x$

$replace\, x\, by - x$

$$f(-x) = (-x)^3 + (-x) = -x^3 - x = -\left[x^3 + x\right] = -f(x)$$

$\Rightarrow \qquad f(x) = x^3 + x$ *is odd function*

$ii)\qquad f(x) = (x+2)^2$

$replace\, x\, by - x$

$$f(-x) = (-x+2)^2 \neq \pm f(x)$$

$f(x) = (x+2)^2$ *is neither even nor odd*

$iii)\qquad f(x) = x\sqrt{x^2 + 5}$

$replace\, x\, by - x$

$$f(-x) = (-x)\sqrt{(-x)^2 + 5} = -\left[x\sqrt{x^2 + 5}\right] = -f(x)$$

$f(x)$ *is odd function.*

$iv)\qquad f(x) = \frac{x-1}{x+1}$

$replace\, x\, by - x$

$$f(-x) = \frac{-x-1}{-x+1} = \frac{-(x+1)}{-(x-1)} = \frac{x+1}{x-1} \neq \pm f(x)$$

$f(x)$ *is neither even nor odd function.*

$v)\qquad f(x) = x^{\frac{2}{3}} + 6$

$replace\, x\, by - x$

$$f(-x) = (-x)^{\frac{2}{3}} + 6 = \left[(-x)^2\right]^{\frac{1}{3}} + 6 = x^{\frac{2}{3}} + 6 = f(x)$$

$f(x)$ *is an even function.*