EXERCISE 4.6

1. If α, β are the roots of $3x^2-2x+4=0$ Then find

(i) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=?$

Sol. $3x^2-2x+4=0$

$a=3 \quad b=-2 \quad c=4$

$\alpha+\beta=-\frac{b}{a}=-\frac{(-2)}{3}=\frac{2}{3}$

$\alpha\beta=\frac{c}{a}=\frac{4}{3}$

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}=\frac{\alpha^2+\beta^2+2\alpha\beta-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$$

$$=\left[\left(\frac{2}{3}\right)^2-2\times\left(\frac{4}{3}\right)\right]\Big/\left(\frac{4}{3}\right)^2$$

$$=\left(\frac{4}{9}-\frac{8}{3}\right)\times\frac{9}{16}$$

$$=\left(\frac{4-24}{9}\right)\times\frac{9}{16}=\frac{-20}{16}=\frac{-5}{4}$$

(ii) $$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\left[\left(\frac{2}{3}\right)^2-2\left(\frac{4}{3}\right)\right]\Big/\frac{4}{3}$$

$$=\left(\frac{4}{9}-\frac{8}{3}\right)\times\frac{3}{4}=\frac{4-24}{9}\times\frac{3}{4}$$

$$=\frac{-20}{9}\times\frac{3}{4}=-\frac{5}{3}$$

(iii) $$\alpha^4+\beta^4=\{(\alpha^2)^2+(\beta^2)^2+2\alpha^2\beta^2\}-2\alpha^2\beta^2$$

$$=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2$$

$$=\left[(\alpha^2+\beta^2+2\alpha\beta)-2\alpha\beta\right]^2-2(\alpha\beta)^2$$

$$=\left[(\alpha+\beta)^2-2\alpha\beta\right]^2-2(\alpha\beta)^2$$

$$=\left[\left(\frac{2}{3}\right)^2-2\left(\frac{4}{3}\right)\right]^2-2\left(\frac{4}{3}\right)^2$$

$$=\left(\frac{4}{9}-\frac{8}{3}\right)^2-\frac{32}{9}$$

$$=\left(\frac{4-24}{9}\right)^2-\frac{32}{9}=\frac{400}{81}-\frac{32}{9}$$

$$=\frac{400-288}{81}=\frac{112}{81}$$

(iv) $$\alpha^3+\beta^3=\left[\alpha^3+\beta^3+3\alpha\beta(\alpha+\beta)\right]-3\alpha\beta(\alpha+\beta)$$

$$=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)$$

$$=\left(\frac{2}{3}\right)^3-3\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)$$

$$=\frac{8}{27}-\frac{8}{3}=\frac{8-72}{27}=-\frac{64}{27}$$

(v) $$\frac{1}{\alpha^3}+\frac{1}{\beta^3}=\frac{\beta^3+\alpha^3}{\alpha^3\beta^3}$$

$$=\left[(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)\right]\Big/(\alpha\beta)^3$$

$$=\left[\left(\frac{2}{3}\right)^3-3\left(\frac{4}{3}\right)\left(\frac{2}{3}\right)\right]\Big/\left(\frac{4}{3}\right)^3$$

$$=\left(\frac{8}{27}-\frac{8}{3}\right)\times\frac{27}{64}=\left(\frac{8-72}{27}\right)\times\frac{27}{64}$$

$$=-\frac{64}{27}\times\frac{27}{64}=-1$$

(vi) $$\alpha^2-\beta^2=(\alpha+\beta)(\alpha-\beta)$$

$$=(\alpha+\beta)\sqrt{(\alpha-\beta)^2}$$

$$=(\alpha+\beta)\sqrt{(\alpha+\beta)^2-4\alpha\beta}$$

$$=\left(\frac{2}{3}\right)\sqrt{\left(\frac{2}{3}\right)^2-4\left(\frac{4}{3}\right)}$$

$$=\frac{2}{3}\sqrt{\frac{4}{9}-\frac{16}{3}}=\frac{2}{3}\sqrt{\frac{4-48}{9}}$$

$$=\frac{2}{3}\sqrt{\frac{-44}{9}}=\frac{4\sqrt{11}\,i}{3}$$

2. If α, β are the roots of $x^2-px-p-c=0$ Then prove That $(1+\alpha)(1+\beta)=1-c$

Sol. $x^2-px-p-c=0$

$A=1 \quad B=-p \quad C=-p-c$

$\alpha+\beta=-\frac{B}{A}=-\frac{(-p)}{1}=p$

$\alpha\beta=\frac{C}{A}=\frac{-p-c}{1}=-p-c$

$$L.H.S=(1+\alpha)(1+\beta)$$

$$=1+\beta+\alpha+\alpha\beta$$

$$=1+p-p-c=1-c=R.H.S$$

3. Find condition.........

(i) $x^2+px+q=0$

$a=1 \quad b=p \quad c=q$

Let $\alpha, 2\alpha$ be the roots of the eq

$S=\alpha+2\alpha=-\frac{b}{a}=-\frac{p}{1}=-p$

$3\alpha=-p \Rightarrow \alpha=\frac{-p}{3}$

$P=\alpha(2\alpha)=\frac{c}{a}=\frac{q}{1}=q$

$2\alpha^2=q \Rightarrow 2\left(-\frac{p}{3}\right)^2=q$

$\Rightarrow 2\frac{p^2}{9}=q \Rightarrow 2p^2=9q$

which is reqd condition

(ii) Square of the other

Let α, α^2 be the roots of the eq

$S = \alpha + \alpha^2 = -\frac{b}{a} = \frac{-p}{1} = -p$

$P = \alpha(\alpha^2) = \frac{c}{a} = \frac{q}{1} = q$

$(\alpha + \alpha^2)^3 = (-p)^3$

$\alpha^3 + (\alpha^2)^3 + 3\alpha\alpha^2(\alpha + \alpha^2) = -p^3$

$\alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -p^3$

$q + q^2 + 3q(-p) + p^3 = 0$

$q + q^2 - 3pq + p^3 = 0$

which is reqd condition

(iii) Additive inverse of other.

Sol. Let $\alpha, -\alpha$ be the roots of eq

$S = \alpha + (-\alpha) = \frac{-p}{1} = -p \Rightarrow p = 0$

$P = \alpha(-\alpha) = \frac{q}{1} = q \Rightarrow -\alpha^2 = q$

So $p = 0$ is the reqd condition

(iv) Multiplicative inverse of the other.

Sol Let $\alpha, \frac{1}{\alpha}$ be the roots of the eq

$S = \alpha + \frac{1}{\alpha} = \frac{-p}{1} = -p$

$P = \alpha\left(\frac{1}{\alpha}\right) = \frac{q}{1} = q$

$\Rightarrow q = 1$ is reqd condition.

4. If the roots --------

Sol. $x^2 - px + q = 0$

$a = 1 \quad b = -p \quad c = q$

Let α, β be the roots of the eq

$\alpha + \beta = -\frac{b}{a} = -\frac{(-p)}{1} = p$

$\alpha\beta = \frac{c}{a} = \frac{q}{1} = q$

By given condition

$\alpha - \beta = 1 \Rightarrow (\alpha - \beta)^2 = 1$

$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$

$\Rightarrow p^2 - 4q = 1$

$\Rightarrow p^2 = 4q + 1$

5. Find the conditions....

Sol $\frac{a}{x-a} + \frac{b}{x-b} = 5$

Multiplying both sides by $(x-a)(x-b)$ we get

$a(x-b) + b(x-a) = 5(x-a)(x-b)$

$\Rightarrow ax - ab + bx - ab = 5x^2 - 5bx - 5ax + 5ab$

$\Rightarrow -5x^2 + 6ax + 6bx - 7ab = 0$

$\Rightarrow 5x^2 - 6(a+b)x + 7ab = 0$

$A = 5 \quad B = -6(a+b) \quad C = 7ab$

Let $\alpha, -\alpha$ be the roots of the eq

$S = \alpha + (-\alpha) = -\left[-\frac{6(a+b)}{5}\right]$

$0 = \frac{6}{5}(a+b) \Rightarrow a + b = 0$

$P = \alpha(-\alpha) = \frac{7ab}{5} \Rightarrow \alpha^2 = -\frac{7ab}{5}$

So $a + b = 0$ is reqd condition

6. If the roots -------

Prove that $\sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}} = 0$

Sol $px^2 + qx + q = 0$

$a = p \quad b = q \quad c = q$

Let α, β be the roots of the eq

$\alpha + \beta = -\frac{q}{p} \qquad \alpha\beta = \frac{q}{p}$

$L.H.S = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{q}{p}}$

$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{q}{p}}$

$= \frac{-q/p}{\sqrt{q/p}} + \sqrt{\frac{q}{p}}$

$= -\sqrt{\frac{q}{p}} + \sqrt{\frac{q}{p}} = 0 = R.H.S$

7. If α, β are the --------

(i) $ax^2 + bx + c = 0$

$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$

$S = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= \left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)$

$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

$P = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}$

Required quadratic eq.

$y^2 - Sy + P = 0$

$\Rightarrow y^2 - \left(\frac{b^2 - 2ac}{a^2}\right)y + \frac{c^2}{a^2} = 0$

$\Rightarrow a^2y^2 - (b^2 - 2ac)y + c^2 = 0$

(ii) $\frac{1}{\alpha}, \frac{1}{\beta}$

$S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c}$

$P = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c}$

Reqd eq $y^2 - Sy + P = 0$

$\Rightarrow y^2 - \left(-\frac{b}{c}\right) y + \frac{a}{c} = 0$

$\Rightarrow cy^2 + by + a = 0$

(iii) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$

$S = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\beta^2+\alpha^2}{\alpha^2\beta^2}$

$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2} = \left[\left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}\right] \Big/ \left(\frac{c}{a}\right)^2$

$= \left(\frac{b^2}{a^2} - \frac{2c}{a}\right)\frac{a^2}{c^2} = \left(\frac{b^2 - 2ac}{a^2}\right)\frac{a^2}{c^2}$

$P = \frac{1}{\alpha^2}\cdot\frac{1}{\beta^2} = \frac{1}{(\alpha\beta)^2} = \frac{1}{(c/a)^2} = \frac{a^2}{c^2}$

Reqd eq $y^2 - Sy + P = 0$

$y^2 - \left(\frac{b^2 - 2ac}{c^2}\right) y + \frac{a^2}{c^2} = 0$

$\Rightarrow c^2y^2 - (b^2 - 2ac)y + a^2 = 0$

(iv) α^3, β^3

$S = \alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= \left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)$

$= \frac{-b^3}{a^3} + \frac{3bc}{a^2} = \frac{-b^3 + 3abc}{a^3}$

$P = \alpha^3\beta^3 = (\alpha\beta)^3 = \left(\frac{c}{a}\right)^3 = \frac{c^3}{a^3}$

Reqd eq $y^2 - Sy + P = 0$

$y^2 - \left(\frac{3abc - b^3}{a^3}\right) y + \frac{c^3}{a^3} = 0$

$\Rightarrow a^3y^2 - (3abc - b^3)y + c^3 = 0$

(v) $\frac{1}{\alpha^3}, \frac{1}{\beta^3}$

$S = \frac{1}{\alpha^3} + \frac{1}{\beta^3} = \frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}$

$= \frac{\alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta) - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}$

$= \left[(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)\right] / (\alpha\beta)^3$

$= \left[\left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)\right] \Big/ \left(\frac{c}{a}\right)^3$

$= \left(\frac{-b^3 + 3abc}{a^3}\right)\frac{a^3}{c^3} = \frac{3abc - b^3}{c^3}$

$P = \frac{1}{\alpha^3}\cdot\frac{1}{\beta^3} = \frac{1}{(\alpha\beta)^3} = \frac{1}{(c/a)^3} = \frac{a^3}{c^3}$

Reqd Eq $y^2 - Sy + P = 0$

$y^2 - \left(\frac{3abc - b^3}{c^3}\right) y + \frac{a^3}{c^3} = 0$

$\Rightarrow c^3y^2 - (3abc - b^3)y + a^3 = 0$

(vi) $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$

$S = \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = \alpha + \beta + \frac{\beta+\alpha}{\alpha\beta}$

$= \left(-\frac{b}{a}\right) + \frac{-b/a}{c/a} = \frac{-bc - ab}{ac}$

$P = \left(\alpha + \frac{1}{\alpha}\right)\left(\beta + \frac{1}{\beta}\right)$

$= \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}$

$= \frac{(\alpha\beta)^2 + \alpha^2 + \beta^2 + 1}{\alpha\beta}$

$= \left[(\alpha\beta)^2 + (\alpha+\beta)^2 - 2\alpha\beta + 1\right] / \alpha\beta$

$= \left[\left(\frac{c}{a}\right)^2 + \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a} + 1\right] \Big/ \frac{c}{a}$

$= \left(\frac{c^2 + b^2 - 2ac + a^2}{a^2}\right)\frac{a}{c}$

Reqd Eq $y^2 - Sy + P = 0$

$y^2 - \left(\frac{-bc - ab}{ac}\right) y + \frac{a^2 + b^2 + c^2 - 2ac}{ac} = 0$

$\Rightarrow acy^2 + b(a+c)y + b^2 + (a-c)^2 = 0$

(vii) $(\alpha-\beta)^2, (\alpha+\beta)^2$

$S = (\alpha-\beta)^2 + (\alpha+\beta)^2$

$= \alpha^2 + \beta^2 - 2\alpha\beta + (\alpha+\beta)^2$

$= \alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta + (\alpha+\beta)^2$

$= (\alpha+\beta)^2 - 4\alpha\beta + (\alpha+\beta)^2$

$= 2(\alpha+\beta)^2 - 4\alpha\beta$

$= 2\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$

$= \frac{2b^2}{a^2} - \frac{4c}{a} = \frac{2b^2 - 4ac}{a^2}$

$P = (\alpha-\beta)^2(\alpha+\beta)^2$

$= \left[(\alpha+\beta)^2 - 4\alpha\beta\right](\alpha+\beta)^2$

$= \left[\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}\right]\left(-\frac{b}{a}\right)^2$

$= \left(\frac{b^2 - 4ac}{a^2}\right)\frac{b^2}{a^2}$

Reqd Eq $y^2 - Sy + P = 0$

$\Rightarrow y^2 - \left(\frac{2b^2 - 4ac}{a^2}\right) y + \frac{b^2(b^2 - 4ac)}{a^4} = 0$

$\Rightarrow a^4y^2 - 2a^2(b^2 - 2ac)y + b^2(b^2 - 4ac) = 0$

(viii) $-\frac{1}{\alpha^3}, -\frac{1}{\beta^3}$

$S = -\frac{1}{\alpha^3} + \left(-\frac{1}{\beta^3}\right) = -\left(\frac{\beta^3 + \alpha^3}{\alpha^3\beta^3}\right)$

$$S = -\left(\frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{(\alpha\beta)^3}\right)$$

$$= -\left[\left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)\right] \Big/ \left(\frac{c}{a}\right)^3$$

$$= -\left(-\frac{b^3}{a^3} + \frac{3bc}{a^2}\right)\frac{a^3}{c^3}$$

$$= -\left[\frac{-b^3 + 3abc}{a^3}\right]\frac{a^3}{c^3} = \frac{-3abc + b^3}{c^3}$$

$$P = \left(-\frac{1}{\alpha^3}\right)\left(-\frac{1}{\beta^3}\right) = \frac{1}{(\alpha\beta)^3} = \frac{1}{(c/a)^3} = \frac{a^3}{c^3}$$

Reqd Eq $y^2 - Sy + P = 0$

$$y^2 - \left(\frac{b^3 - 3abc}{c^3}\right)y + \frac{a^3}{c^3} = 0$$

$$\Rightarrow c^3y^2 - (b^3 - 3abc)y + a^3 = 0$$

8. If α, β are the ......

Sol. $5x^2 - x - 2 = 0$

$a = 5 \quad b = -1 \quad c = -2$

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-1)}{5} = \frac{1}{5}$$

$$\alpha\beta = \frac{c}{a} = \frac{-2}{5}$$

$$S = \frac{3}{\alpha} + \frac{3}{\beta} = 3\left(\frac{\beta+\alpha}{\alpha\beta}\right)$$

$$= 3\left(\frac{1/5}{-2/5}\right) = -\frac{3}{2}$$

$$P = \left(\frac{3}{\alpha}\right)\left(\frac{3}{\beta}\right) = \frac{9}{\alpha\beta} = \frac{9}{-2/5} = -\frac{45}{2}$$

Reqd Eq $y^2 - Sy + P = 0$

$$\Rightarrow y^2 - \left(-\frac{3}{2}\right)y + \left(-\frac{45}{2}\right) = 0$$

$$\Rightarrow 2y^2 + 3y - 45 = 0$$

9. If α and β ....

Sol $x^2 - 3x + 5 = 0$

$a = 1 \quad b = -3 \quad c = 5$

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-3)}{1} = 3$$

$$\alpha\beta = \frac{c}{a} = \frac{5}{1} = 5$$

$$S = \frac{1-\alpha}{1+\alpha} + \frac{1-\beta}{1+\beta}$$

$$= \frac{(1-\alpha)(1+\beta) + (1+\alpha)(1-\beta)}{(1+\alpha)(1+\beta)}$$

$$= \frac{1+\beta-\alpha-\alpha\beta+1-\beta+\alpha-\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{2-2\alpha\beta}{1+\alpha+\beta+\alpha\beta} = \frac{2(1-\alpha\beta)}{1+(\alpha+\beta)+\alpha\beta}$$

$$S = \frac{2(1-5)}{1+3+5} = \frac{2(-4)}{9} = -\frac{8}{9}$$

$$P = \frac{(1-\alpha)(1-\beta)}{(1+\alpha)(1+\beta)} = \frac{1-\beta-\alpha+\alpha\beta}{1+\beta+\alpha+\alpha\beta}$$

$$= \frac{1-(\alpha+\beta)+\alpha\beta}{1+(\alpha+\beta)+\alpha\beta} \cdot \frac{1-3+5}{1+3+5}$$

$$= \frac{3}{9} = \frac{1}{3}$$

Reqd Eq $y^2 - Sy + P = 0$

$$\Rightarrow y^2 - \left(-\frac{8}{9}\right)y + \frac{1}{3} = 0$$

$$\Rightarrow 9y^2 + 8y + 3 = 0$$