CHORDS AND ARCS

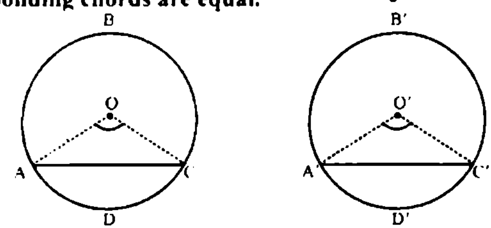
In this unit, students will learn how to:

Prove the following theorems along with corollaries and apply them to solve appropriate problems.

- If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.
- If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
- Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centres).
- If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.

THEOREM 1

11.1 (i) If two arcs of a circle (or of congruent circles) are congruent then the corresponding chords are equal.



Given:

ABCD and A'B'C'D' are two congruent circles with centres O and O' respectively. So that $m\,\widehat{ADC} = m\,\widehat{A'D'C'}$

To prove:

$m\,\overline{AC} = m\,\overline{A'C'}$

Construction:

Join O with A, O with C, O' with, A' and O' with C'.

So that we can form Δ OAC and O' A' C'.

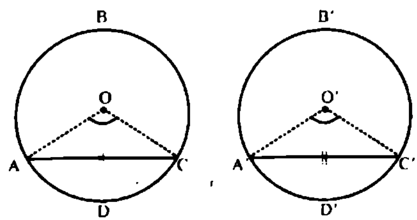
Proof:

Statements	Reasons
In two equal circles ABCD and A'B'C'D' with centres O and O' respectively.	Given
$m\widehat{ADC} = \widehat{A'D'C'}$	Given
$m\angle AOC = m\angle A'O'C'$	Central angles subtended by equal arcs of the equal circles.
Now in $\Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\angle AOC = m\angle A'O'C'$	Already Proved
$m\overline{OC} = m\ \overline{O'C'}$	Radii of equal circles.
$\Delta AOC \cong \Delta A'O'C'$	S.A.S ≅ S.A.S
and in particular $m\overline{AC} = m\overline{A'C'}$	
Similarly we can prove the theorem in the same circle.	

THEOREM 2

Converse of Theorem 1

11.1 (ii) If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major or semi-circular) are congruent.
In equal circles or in the same circle, if two chords are equal, they cut off equal arcs.



Given:

ABCD and A'B'C'D' are two congruent circles with centres O and O' respectively.

So that chord $m\overline{AC} = m\overline{A'C'}$.

To prove:

$m\widehat{ADC} = m\widehat{A'D'C'}$

Construction:

Join O with A, O with C, O' with A' and O' with C".

Proof:

Statements	Reasons

In $\quad \Delta AOC \leftrightarrow \Delta A'O'C'$	
$m\overline{OA} = m\overline{O'A'}$	Radii of equal circles
$m\overline{OC} = m\overline{O'C'}$	Radii of equal circles
$m\overline{AC} = m\overline{A'C'}$	Given
$\Delta AOC \cong \Delta A'O'C'$	S.S.S ≅ S.S.S
$\Rightarrow \quad m\angle AOC = m\angle A'O'C'$	
Hence $m\widehat{ADC} = m\widehat{A'D'C'}$	Arcs corresponding to equal central angles.

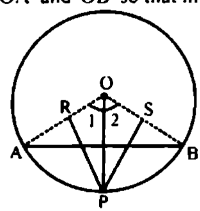
Example 1:

A point P on the circumference is equidistant from the radii $\overline{OA}$ and $\overline{OB}$.

Prove that $m\widehat{AP} = m\widehat{BP}$

Given:

AB is the chord of a circle with centre O. Point P on the circumference of the circle is equidistant from the radii $\overline{OA}$ and $\overline{OB}$ so that $m\overline{PR} = m\overline{PS}$.



To prove:

$m\widehat{AP} = m\widehat{BP}$

Construction:

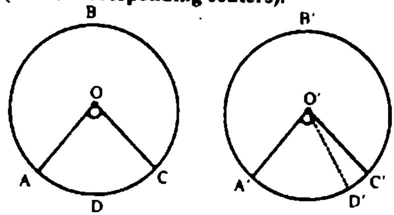
Join O with P. Write $\angle 1$ and $\angle 2$ as shown in the figure.

Proof:

Statements	Reasons
In $\quad \angle$rt ΔOPR and $\angle$rt ΔOPS	
$m\overline{OP} = m\overline{OP}$	Common
$m\overline{PR} = m\overline{PS}$	Point P is equidistance from radii (Given)
$\therefore \quad \Delta OPR \cong \Delta OPS$	(In $\angle$rtΔ' H.S ≅ H.S)
So $\quad m\angle 1 = m\angle 2$	Central angles of a circle.
$\Rightarrow \quad$ Chord AP ≅ Chord BP	
Hence $m\widehat{AP} = m\widehat{BP}$	Arcs corresponding to equal chords in a circle.

THEOREM 3

11.1 (iii) Equal chords of a circle (or of congruent circles) subtend equal angles at the centre (at the corresponding centers).



Given:

ABC and A'B'C' are two congruent circles with centres O and O' respectively.
So that $\overline{AC} = \overline{A'C'}$

To prove:

$\angle AOC \cong \angle A'O'C'$

Construction:

Let if possible $m\angle AOC \neq m\angle A'O'C'$ then consider $\angle AOC \cong \angle A'O'D'$

Proof:

Statements		Reasons
$\angle AOC \cong \angle A'O'D'$		Construction
$\overset{\frown}{AC} \cong \overset{\frown}{A'D'}$	(i)	Arcs subtended by equal Central angles in congruent circles
$\overline{AC} \cong \overline{A'D'}$	(ii)	Using Theorem 1
But $\overline{AC} = \overline{A'C'}$	(iii)	Given
$\therefore$ $\overline{A'C'} = \overline{A'D'}$		Using (ii) and (iii)
Which is only possible, if C' concides with D'.		
Hence $m\angle A'O'C = m\angle A'O'D'$	(iv)	
But $m\angle AOC = m\angle A'O'D'$	(v)	Construction
$\Rightarrow$ $m\angle AOC = m\angle A'O'C'$		Using (iv) and (v)

Corollary 1.

In congruent circles or in the same circle, if central angles are equal then corresponding sectors are equal.

Corollary 2.

In congruent circles or in the same circle, unequal arcs will subtend unequal central angles.

Example 1:

The internal bisector of a central angle in a circle bisects an arc on which it stands.

Solution:

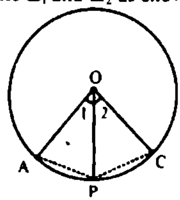
In a circle with centre O. OP is an internal bisector of central angle AOB.

To prove:

$\overset{\frown}{AP} \cong \overset{\frown}{BP}$

Construction:

Draw $\overline{AP}$ and $\overline{BP}$, then write $\angle_1$ and $\angle_2$ as shown in the figure.



Proof:

Statements		Reasons
In	$\Delta OAP \leftrightarrow \Delta OBP$	
	$m\overline{OA} = m\overline{OB}$	Radii of the same circle
	$m\angle 1 = m\angle 2$	Given OP as an angle bisector of $\angle AOB$
and	$m\overline{OP} = m\overline{OP}$	Common
	$\Delta OAP \cong \Delta OBP$	(S.A.S $\cong$ S.A.S)
Hence	$\overline{AP} \cong \overline{BP}$	Arcs corresponding to equal chords in a circle.
$\Rightarrow$	$\overset{\frown}{AP} \cong \overset{\frown}{BP}$	

Example 2:

In a circle if any pair of diameters are $\perp$ to each other then the lines joining its ends in order, form a square.

Given:

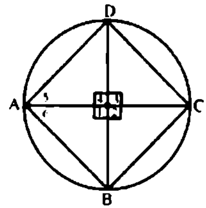
$\overline{AC}$ and $\overline{BD}$ are two perpendicular diameters of a circle with centre O. So ABCD is a quadrilateral.

To prove:

ABCD is a square

Construction:

Write $\angle 2$, $\angle 3$, $\angle 4$, $\angle 5$ and $\angle 6$ as shown, in the, figure.

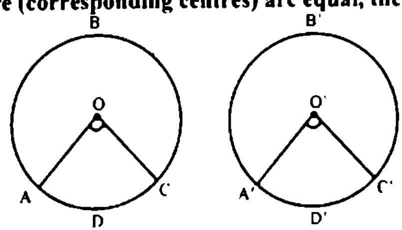


Proof:

Statements	Reasons
$\overline{AC}$ and $\overline{AD}$ are two $\perp$ diameters of a circle with centre O	Given
$m\angle 1 = m\angle 2 - m\angle 3 = m\angle 4 = 90^\circ$	Pair of diameters, are $\perp$ to each other.
$m\widehat{AB} = m\widehat{BC} = m\widehat{CD} = m\widehat{DA}$	Arcs opposite to the equal central angles in a circle.
$\Rightarrow$ $m\overline{AB} = m\overline{BC} = m\overline{CD} = m\overline{DA}$ (i)	Chords corresponding to equal arcs.
Moreover $m\angle A = m\angle 5 + m\angle 6$ $= 45^\circ + 45^\circ = 90^\circ$ (ii)	
Similarly $m\angle B = m\angle C = m\angle D = 90^\circ$ (iii)	
Hence ABCD is a square	Using (i), (ii) and (iii).

THEOREM 4

11.1 (iv) If the angles subtended by two chords of a circle (or congruent circles) at the centre (corresponding centres) are equal, the chords are equal.



Given:

ABCD and A'B'C'D' are two congruent circles with centres O and O' respectively.

So that $m\angle AOC = m\angle A'O'C'$

To prove:

$m\widehat{AC} = m\widehat{A'C'}$

Proof:

Statements	Reasons
Since ABCD and A'B'C'D' are two congruent circles with centres O and O' respectively.	Given
Place the circle ABCD on the circle A'B'C'D'	
So that point O falls on O' (i)	
Also $m\angle AOC = m\angle A'O'C'$	Given
$m\overline{OA} = m\overline{O'A'}$ (ii)	Radii for congruent circles
and $m\overline{OC} = m\overline{O'C'}$ (iii)	Radii for congruent circles
So point A will coincide with A' and point C will Coincide with C'	Using (i), (ii) and (iii)
Now every point on $\overset{\frown}{ADC}$ or on $\overset{\frown}{A'D'C'}$ is equidistant from the centres O and O' respectively.	
Hence $\overset{\frown}{ADC}$ coincides with $\overset{\frown}{A'D'C'}$.	Using theorem 1
or $m\overset{\frown}{AC} = m\overset{\frown}{A'C'}$	
i.e., $m(\overset{\frown}{ADC}) = m(\overset{\frown}{A'D'C'})$	

SOLVED EXERCISE 11.1

1. **In a circle two equal chords AB and CD intersect each other. Prove that mAD = mBC.**

Solution:

Given:

In a circle having centre at O.

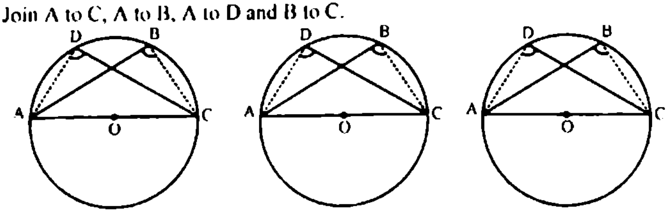
$m\overline{OB} \cong m\overline{CD}$

To prove:

$m\overline{AD} = m\overline{BC}$

Construction:

Join A to C, A to B, A to D and B to C.



(a) (b) (c)