Cramer-Lundberg theorem

Section 4 - Home Work 1

Student: Mehrab Atighi, mehrab.atighi@gmail.com
Lecturer: Mohammad Zokaei, Zokaei@sbu.ac.ir

Problem 1

Proof of part b of the theorem below. Embrechts et al. (1997):

Theorem 1.2.2 (Cramér–Lundberg theorem)

Consider the Cramér–Lundberg model including the net profit condition $\rho > 0$. Assume that there exists a $\nu > 0$ such that

$$\widehat{f}_I(-\nu) = \int_0^\infty e^{\nu x}\, dF_I(x) = \frac{c}{\lambda\mu} = 1 + \rho\,. \tag{1.13}$$

Then the following relations hold.

(a) *For all $u \geq 0$,*

$$\psi(u) \leq e^{-\nu u}\,. \tag{1.14}$$

(b) *If, moreover,*

$$\int_0^\infty x\, e^{\nu x}\, \overline{F}(x)\, dx < \infty\,, \tag{1.15}$$

then

$$\lim_{u\to\infty} e^{\nu u}\, \psi(u) = C < \infty\,, \tag{1.16}$$

where

$$C = \left[\frac{\nu}{\rho\mu} \int_0^\infty x\, e^{\nu x}\, \overline{F}(x)\, dx\right]^{-1}\,. \tag{1.17}$$

(c) *In the case of an exponential df $F(x) = 1 - e^{-x/\mu}$, (1.11) reduces to*

$$\psi(u) = \frac{1}{1+\rho} \exp\left\{-\frac{\rho}{\mu(1+\rho)}\, u\right\}\,, \quad u \geq 0\,. \tag{1.18}$$

Solve 1

Proof of (b). Denote $d(v) = -\psi'(v)$. Recall from (1.3) that $d(v)$ can be expressed via the random walk generated by $X_i = \xi_i - 1$. Then
$\delta(u) = P(S(t) - ct < u \forall t > 0)$
$= P(\sum_{k=1}^{n}(X_k - cY_k) \leq u \forall n \geq 1)$
and we now that $S(t) - ct = \sum_{k=1}^{n}(X_k - cY_k)$ so we have:
$= P(\sum_{k=2}^{n} X_k - cY_k) \leq u + cY_1, -X_1 \forall n \geq 2, X_1 - cY_1 \leq u)$
and we set $S\prime(t) = \sum_{k=2}^{n} X_k$ then we have $\sum_{k=2}^{n}(X_k - cY_k) = S\prime(t) - ct$ so we have:

$= P(S'(t) - ct < u + cY_1 - X_1 \forall t > 0, X_1 - cY_1 < u)$,
where S' is an independent copy of S. Hence
$\delta(u) = E(P(S'(t) - ct < u + cY_1 - X_1 \forall t > 0, X_1 - cY_1 < u | Y_1, X_1))$
Considering that it was supposed to do the proof as long as we can, **I did not understand from this part onwards and I tried my best to move forward, so the effort I took to reach the next step is an image in this part**. I will add and unfortunately I could not advance the proof from here on.

$$E\left[P\left(S'(t)-ct \leq u+cY_1-X_1\right) \; X_1-cY_1 \leq u\right)\Big]$$

independent

$$=E\left[P\left(S'(t)-ct \leq u+cY_1-X_1\right)\times \underbrace{P\left(X_1-cY_1 \leq u\right)}_{P(S(A)\leq u)\;\to\;1.3\;G_+(x)}\right]$$

$X_1=x$, $Y_1=s$

$$=E\left[P\left(S'(t)-ct \leq u+cY_1-X_1\right)\times P\left(S(t)\leq u+cs\right)\right]$$

$$=E\left[P\left(S'(t)-ct \leq u+cY_1-x\right)\times \int_0^{u+cs}\frac{e^{-\lambda s}\lambda}{1!}\,ds\right]$$

$$=\int_0^\infty\int_0^{u+cs}\lambda e^{-\lambda s}\times \underbrace{Pr\left(S'(t)-ct\leq u+cs-x\right)}\,dF(x)\,ds$$

$\delta(u+cs-x)=1-\psi(u+cs-x)$ $= \underbrace{u+cs-x}$ احتمال بقا با سرمایه اولیه

$$=\int_0^\infty \lambda e^{-\lambda s}\int_0^{u+cs}\delta(u+cs-x)\,dF(x)\,ds \Longrightarrow u+cs=z \Rightarrow ds=\frac{dz}{c}$$

$$=\int_0^\infty \lambda e^{-\lambda s}\int_0^{z}\delta(z-x)\,dF(x)\,\frac{dz}{c} = \int_0^\infty \frac{\lambda}{c}e^{-\lambda s}\int_0^{z}\delta(z-x)\,dF(x)\,dz$$

$$\int^u \frac{\lambda}{c}e^{-\lambda s}\int_0^{z}\delta(z-x)\,dF(x)\,dz+\int_u^\infty \frac{\lambda}{c}e^{-\lambda s}\int_0^{z}\delta(z-x)\,dF(x)\,dz$$

References

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.