Moments of distributions in DA(α)
Section 10 - Home Work 4

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Problem 1

Corollary 2.2.10 (Moments of distributions in DA(α)) If $X \in DA(\alpha)$, then

$$E|X|^\delta < \infty \quad \text{for } \delta < \alpha,$$

$$E|X|^\delta = \infty \quad \text{for } \delta > \alpha \text{ and } \alpha < 2.$$

In particular,

$$\text{var}(X) = \infty \quad \text{for } \alpha < 2,$$

$$E|X| < \infty \quad \text{for } \alpha > 1,$$

$$E|X| = \infty \quad \text{for } \alpha < 1.$$

Embrechts et al. (1997)

Solve 1

Proof:
Let $X \in DA(\alpha)$. We will examine the existence of the moments $\mathbb{E}[|X|^\delta]$ based on the value of δ relative to α.

Part 1: Finite Moments for $\delta < \alpha$

To understand why $\mathbb{E}[|X|^\delta] < \infty$ for $\delta < \alpha$, recall that being in $DA(\alpha)$ implies that X has a heavy-tailed distribution characterized by a decay rate that is asymptotically proportional to $|X|^{-\alpha}$ as $|X| \to \infty$. Specifically, there exists a constant $C > 0$ such that for large x,

$$P(|X| > x) \approx Cx^{-\alpha}.$$

For the δ-th moment $\mathbb{E}[|X|^\delta]$ to exist, we must have

$$\int_0^\infty P(|X| > x^{1/\delta})\, dx < \infty.$$

Using the tail behavior $P(|X| > x) \approx Cx^{-\alpha}$, we approximate

$$\mathbb{E}[|X|^\delta] = \int_0^\infty \delta x^{\delta-1} P(|X| > x)\, dx \approx \int_0^\infty \delta x^{\delta-1} C x^{-\alpha}\, dx.$$

This integral converges if and only if $\delta - \alpha < -1$, or equivalently, $\delta < \alpha$. Therefore, if $\delta < \alpha$, then $\mathbb{E}[|X|^\delta] < \infty$.

Part 2: Infinite Moments for $\delta > \alpha$ (when $\alpha < 2$)

Now suppose $\delta > \alpha$ and $\alpha < 2$. We want to show that $\mathbb{E}[|X|^\delta] = \infty$ in this case.

Since $X \in DA(\alpha)$, the tail decay rate $P(|X| > x) \approx Cx^{-\alpha}$ implies that for any $\delta > \alpha$, the δ-th moment integral diverges:

$$\mathbb{E}[|X|^\delta] = \int_0^\infty \delta x^{\delta-1} P(|X| > x)\, dx \approx \int_0^\infty \delta x^{\delta-1} C x^{-\alpha}\, dx.$$

This integral diverges if $\delta > \alpha$, indicating that $\mathbb{E}[|X|^\delta] = \infty$ for $\delta > \alpha$ when $\alpha < 2$.

Special Cases

Using the results above, we analyze some special cases for the moments of X.

- **Variance:** The variance $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$ is finite only if $\alpha > 2$. Since we assume $X \in DA(\alpha)$ with $\alpha < 2$, we have $\mathbb{E}[X^2] = \infty$, so $\mathrm{Var}(X) = \infty$ for $\alpha < 2$.
- **First Moment:** The existence of the first moment $\mathbb{E}[|X|]$ depends on whether $\alpha > 1$ or $\alpha < 1$.
  - If $\alpha > 1$, then $\mathbb{E}[|X|] < \infty$ because $\delta = 1 < \alpha$.
  - If $\alpha < 1$, then $\mathbb{E}[|X|] = \infty$ because $\delta = 1 > \alpha$.

Grimmett and Stirzaker (2020); Feller (2008); Durrett (2019); Billingsley (2008); Klenke (2013); Ross (2014); Chebyshev (1867)

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