Find norming constant of normal distribution

Section 21 - Home Work 1

Student: Mehrab Atighi, mehrab.atighi@gmail.com
Lecturer: Mohammad Zokaei, Zokaei@sbu.ac.ir

Problem 1

According to the Example 3.3.29 find norming constant of normal distribution. Embrechts et al. (1997)

Solve 1

1. Von Mises Function and Condition (3.25)

We denote by Φ the distribution function (df) and by φ the density of the standard normal distribution. First, we need to show that Φ is a von Mises function and satisfies condition (3.25).
An application of l'Hospital's rule to

$$\frac{\overline{\Phi}(x)}{x^{-1}\varphi(x)}$$

yields Mill's ratio:

$$\overline{\Phi}(x) \sim \frac{\varphi(x)}{x}$$

This implies:

$$\varphi'(x) = -x\varphi(x) < 0$$

and

$$\lim_{x\to\infty} \frac{\overline{\Phi}(x)\varphi'(x)}{\varphi^2(x)} = -1.$$

Thus, $\Phi \in \mathrm{MDA}(\Lambda)$ by Example 3.3.23 and Proposition 3.3.25.

2. Calculation of Mill's Ratio

The tail probability (survival function) is given by:

$$\overline{\Phi}(x) = 1 - \Phi(x).$$

Mill's ratio is expressed as:

$$R(x) = \frac{\overline{\Phi}(x)}{\varphi(x)}.$$

For large values of x, the Mill's ratio can be approximated as:

$$\overline{\Phi}(x) \sim \frac{\varphi(x)}{x}.$$

Derivation of Mill's Ratio

1. **Expression for the Standard Normal Density Function:** The probability density function for the standard normal distribution is:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

2. **Applying l'Hospital's Rule:** To find the asymptotic behavior of $\overline{\Phi}(x)$ for large x, we evaluate the ratio $\frac{\overline{\Phi}(x)}{\varphi(x)}$ using l'Hospital's rule.

   Since $\overline{\Phi}(x) = 1 - \Phi(x)$ and the derivative of $\Phi(x)$ is $\varphi(x)$:

$$\Phi'(x) = \varphi(x),$$

   and the derivative of $\varphi(x)$ is:

$$\varphi'(x) = -x\varphi(x).$$

   Using these derivatives:

$$\lim_{x\to\infty} \frac{\overline{\Phi}(x)}{\varphi(x)} = \lim_{x\to\infty} \frac{-\Phi'(x)}{-\varphi'(x)} = \lim_{x\to\infty} \frac{\varphi(x)}{x\varphi(x)} = \lim_{x\to\infty} \frac{1}{x}.$$

   Thus, for large x:

$$\overline{\Phi}(x) \sim \frac{\varphi(x)}{x}.$$

3. Solving for d_n and c_n

Deriving d_n

Using Proposition 3.3.28, the norming constant d_n satisfies:

$$-\ln G(d_n) = \ln n.$$

For the standard normal distribution, $G(d_n)$ represents the upper tail probability:

$$\overline{\Phi}(d_n) = \frac{1}{\sqrt{2\pi}} \int_{d_n}^{\infty} e^{-t^2/2}\, dt.$$

Using the asymptotic behavior of $\overline{\Phi}(x)$:

$$\overline{\Phi}(x) \sim \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x},$$

we have:

$$-\ln \overline{\Phi}(d_n) = \ln n$$

i.e.,

$$\ln\left(\frac{1}{\sqrt{2\pi}}\frac{e^{-d_n^2/2}}{d_n}\right) = \ln n.$$

Expanding the logarithm:

$$-\frac{d_n^2}{2} - \ln d_n - \frac{1}{2}\ln(2\pi) = \ln n.$$

Rearranging terms gives:

$$\frac{1}{2}d_n^2 + \ln d_n + \frac{1}{2}\ln(2\pi) = \ln n.$$

Now, solving this equation for d_n, we use an asymptotic expansion. Assume d_n is large, so the leading term is dominant:
1. The leading-order approximation is:

$$\frac{1}{2}d_n^2 \approx \ln n \implies d_n \approx (2\ln n)^{1/2}.$$

2. To refine this, substitute $d_n = (2\ln n)^{1/2} + \delta$ and solve for δ. Substituting into the equation:

$$\frac{1}{2}\left((2\ln n)^{1/2} + \delta\right)^2 + \ln\left((2\ln n)^{1/2} + \delta\right) + \frac{1}{2}\ln(2\pi) = \ln n.$$

Expanding and keeping terms up to δ:

$$\frac{1}{2}(2\ln n) + \delta(2\ln n)^{1/2} + \ln(2\ln n)^{1/2} + \frac{1}{2}\ln(2\pi) \approx \ln n.$$

Solving for δ:

$$\delta \approx -\frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}}.$$

Thus, the refined approximation for d_n is:

$$d_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}} + o((\ln n)^{-1/2}).$$

Deriving c_n

To derive c_n, recall:

$$R(x) = \frac{\overline{\Phi}(x)}{\varphi(x)} \sim \frac{1}{x}.$$

Thus:

$$c_n = a(d_n) \sim \frac{1}{d_n}.$$

Substituting the expansion for d_n:

$$c_n \sim \frac{1}{(2\ln n)^{1/2}}.$$

This is the leading-order term for c_n. Higher-order corrections can be included if necessary, but typically the leading-order term suffices for most asymptotic analyses.

4. Summary of Results

- The norming constant d_n is approximated as:

$$d_n = (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}} + o((\ln n)^{-1/2}).$$

- The norming constant c_n is approximated as:

$$c_n \sim \frac{1}{(2\ln n)^{1/2}}.$$

- These constants are crucial for understanding the asymptotic behavior of the standard normal distribution in extreme value theory.

Embrechts et al. (1997)

References

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.