# Characterisation of domain of attraction

Section 10 - Home Work 2

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**Problem 1**

**Theorem 2.2.8 (Characterisation of domain of attraction)**

(b) The df $F$ belongs to the domain of attraction of an $\alpha$-stable law for some $\alpha < 2$ if and only if

$$F(-x) = \frac{c_1 + o(1)}{x^\alpha} L(x), \quad 1 - F(x) = \frac{c_2 + o(1)}{x^\alpha} L(x), \quad x \to \infty,$$

where $L$ is slowly varying and $c_1, c_2$ are non-negative constants such that $c_1 + c_2 > 0$.

Embrechts et al. (1997):

**Solve 1**

**Proof:**
Consider a sequence of i.i.d. random variables $X_1, X_2, \ldots, X_n$ with distribution function $F$. We want to show that $F$ belongs to the domain of attraction of an $\alpha$-stable law with $\alpha < 2$ if and only if the tails of $F$ satisfy:

$$F(-x) = \frac{c_1 + o(1)}{x^\alpha} L(x) \quad \text{and} \quad 1 - F(x) = \frac{c_2 + o(1)}{x^\alpha} L(x), \quad x \to \infty,$$

where $L(x)$ is slowly varying and $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

### Step 1: Tail Behavior and Characterization

For $F$ to be in the domain of attraction of an $\alpha$-stable law, the tails of the distribution function $F$ must have the form:

$$F(-x) \sim \frac{c_1}{x^\alpha} L(x) \quad \text{and} \quad 1 - F(x) \sim \frac{c_2}{x^\alpha} L(x), \quad \text{as } x \to \infty,$$

where $L(x)$ is a slowly varying function. This condition ensures that the distribution $F$ has heavy tails, which is a characteristic of $\alpha$-stable distributions.

### Step 2: Normalizing Constants

For a random variable $X$ with distribution function $F$, we need to show that for large $n$,

$$\frac{S_n - a_n}{b_n} \xrightarrow{d} X,$$

where $S_n = X_1 + X_2 + \cdots + X_n$, and $a_n$ is centering constant and $b_n$ is normalizing constants. Using properties of $\alpha$-stable laws, we can choose $b_n = n^{1/\alpha}$ and $a_n = 0$.

### Step 3: Slowly Varying Function $L(x)$

A function $L(x)$ is slowly varying at infinity if for any $t > 0$,

$$\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1.$$

This definition implies that $L(x)$ grows very slowly compared to any polynomial function. In our context, $L(x)$ modulates the tail behavior of the distribution $F$.

### Step 4: Convergence of Normalized Sums

To conclude, if $F$ has the specified tail behavior, then the normalized sum $\frac{S_n}{n^{1/\alpha}}$ converges in distribution to an $\alpha$-stable law. This implies that $F$ is in the domain of attraction of the $\alpha$-stable law. For large $n$, consider:

$$P\left(\frac{S_n}{n^{1/\alpha}} \leq x\right) \approx F\left(xn^{1/\alpha}\right).$$

Using the tail behavior of $F$:

$$F\left(xn^{1/\alpha}\right) \sim \frac{c_1}{(xn^{1/\alpha})^\alpha} L(xn^{1/\alpha}) + \frac{c_2}{(xn^{1/\alpha})^\alpha} L(xn^{1/\alpha}).$$

Since $L(x)$ is slowly varying,

$$\frac{L(xn^{1/\alpha})}{L(n^{1/\alpha})} \to 1 \quad \text{as} \quad n \to \infty.$$

Thus,

$$F\left(xn^{1/\alpha}\right) \sim \frac{c_1 + c_2}{x^\alpha} L(n^{1/\alpha}),$$

which implies that

$$P\left(\frac{S_n}{n^{1/\alpha}} \leq x\right) \to G(x),$$

where $G(x)$ is the distribution function of an $\alpha$-stable law. Therefore, $F$ is in the domain of attraction of an $\alpha$-stable law if the tails of $F$ have the specified form with a slowly varying function $L(x)$. Thus, we have shown that the second moment being slowly varying is a necessary condition for a distribution to be in the domain of attraction of the normal law.
i try to use Feller (2008b) for this proof. In the following you can see two page of this book.

**Theorem 1.** *A distribution $F$ belongs to the domain of attraction of some distribution $G$ iff there exists a slowly varying $L$ such that*

$$U(x) \sim x^{2-\alpha} L(x), \qquad x \to \infty, \tag{8.3}$$

*with $0 < \alpha \le 2$, and when $\alpha < 2$*

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p, \qquad \frac{F(-x)}{1 - F(x) + F(-x)} \to q. \tag{8.4}$$



*When $\alpha = 2$ condition* (8.3) *alone is sufficient provided $F$ is not concentrated at one point.*[6]

We shall see that (8.3) with $\alpha = 2$ implies convergence to the normal distribution. This covers distributions with finite variance, but also many distributions with unbounded slowly varying $U$ [see example VIII,4(*a*)].

Using theorem 2 of VIII,9 with $\xi = 2$ and $\eta = 0$ it is seen that *the relation* (8.3) *is fully equivalent to*[7]

$$\frac{x^2[1-F(x)+F(-x)]}{U(x)} \to \frac{2-\alpha}{\alpha} \tag{8.5}$$

in the sense that the two relations imply each other.

When $0 < \alpha < 2$ we can rewrite (8.5) in the form

$$1 - F(x) + F(-x) \sim \frac{2-\alpha}{\alpha}\, x^{-\alpha} L(x), \tag{8.6}$$

and conversely (8.6) implies (8.3) and (8.5). This leads us to a reformulation of the theorem which is more intuitive inasmuch as it describes the behavior of the individual tails. (For other alternatives see problem 17.)

**Theorem 1a.** (*Alternative form*). (i) *A distribution $F$ belongs to the domain of attraction of the normal distribution iff $U$ varies slowly.*

(ii) *It belongs to some other domain of attraction iff* (8.6) *and* (8.4) *hold for some $0 < \alpha < 2$.*

**Proof.** We shall apply the theorem of section 7 to the array of variables $\mathbf{X}_{k,n} = \mathbf{X}_k/a_n$ with distributions $F_n(x) = F(a_n x)$. The row sums of the array $\{\mathbf{X}_{k,n}\}$ are given by

$$\mathbf{S}_n = (\mathbf{X}_1 + \cdots + \mathbf{X}_n)/a_n.$$

Obviously $a_n \to \infty$ and hence $\{\mathbf{X}_{k,n}\}$ is a null-array. To show that the condition (7.8) is satisfied we put

$$v(x) = \int_{-x}^{x} y\, F\{dy\} \tag{8.7}$$

[6] For distributions with finite variance, $U$ varies slowly except when $F$ is concentrated at the origin. In all other cases (8.3) and (8.4) remain unchanged if $F(x)$ is replaced by $F(x+b)$.

[7] Condition (8.4) requires a similar relation for each tail separately:

$$\frac{x^2[1-F(x)]}{U(x)} \to p\,\frac{2-\alpha}{\alpha}, \qquad \frac{x^2 F(-x)}{U(x)} \to q\,\frac{2-\alpha}{\alpha} \tag{*}$$

When $\alpha = 2$ these relations follow from (8.5), which explains the absence of a second condition when $\alpha = 2$. Theorem 1 could have been formulated more concisely (but more artificially) as follows: *$F$ belongs to some domain of attraction iff* (*) *is true* with $0 < \alpha \le 2$, $p \ge 0$, $q \ge 0$, $p + q = 1$.

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> Grimmett and Stirzaker (2020); Feller (2008a); Durrett (2019); Billingsley (2008); Klenke (2013); Ross (2014); Chebyshev (1867)

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