The Cramér-Lundberg Model in Risk Theory with Non-Homogeneous Claims

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Abstract

This document provides an overview of the Cramér-Lundberg model, a cornerstone in risk theory, with extensions to non-homogeneous claim arrival processes. It includes mathematical foundations, implementation details, and connections to simulations using the Shiny application. The focus is on analyzing risk processes under Lognormal and Exponential claim size distributions, emphasizing how parameter variations influence the variance and behavior of the risk process.

1 Introduction

The Cramér-Lundberg model is a fundamental stochastic model in risk theory, used to describe the surplus of an insurance company over time. The surplus process is defined as:

$$R(t) = u + ct - S(t), \quad t \geq 0,$$

where:

- u is the initial reserve,
- c is the premium rate,
- $S(t)$ is the aggregate claim amount by time t.

The goal is often to study the probability of ruin, which occurs when $R(t) < 0$ for some $t \geq 0$.

In this extension, we consider a time-dependent claim arrival process $N(t)$, where the rate of claims may vary over time. Consequently, the distribution of the number of claims and their aggregate amount becomes time-dependent.

2 Mathematical Formulation

2.1 Aggregate Claims

The aggregate claims $S(t)$ are modeled as:

$$S(t) = \sum_{i=1}^{N(t)} X_i,$$

where:

- $N(t)$ is a non-homogeneous process representing the number of claims up to time t, potentially modeled using a Negative Binomial distribution,
- X_i are independent and identically distributed random variables representing claim sizes.

The expected value and variance of $S(t)$ depend on the time-dependent rate of $N(t)$, denoted as $\lambda(t)$, and the distribution of X_i.

2.2 Risk Process

The risk process is given by:

$$R(t) = u + ct - \sum_{i=1}^{N(t)} X_i.$$

If $\mathbb{E}[X_i] = \mu$, the expected value of the risk process is:

$$\mathbb{E}[R(t)] = u + ct - \mathbb{E}[N(t)]\mu.$$

The variance of $R(t)$ includes contributions from the variability in $N(t)$ and X_i, expressed as:

$$\mathrm{Var}(R(t)) = \mathrm{Var}(S(t)) = \mathrm{Var}(N(t))\mu^2 + \mathbb{E}[N(t)]\sigma_X^2.$$

2.3 Claim Arrival Process

In non-homogeneous cases, $N(t)$ may follow a distribution like the Negative Binomial. For a Negative Binomial process:

$$\mathbb{P}(N(t) = k) = \binom{k + r - 1}{k} (1 - p)^r p^k, \quad k \geq 0,$$

where:

- r is the dispersion parameter,
- p is the success probability.

The mean and variance of $N(t)$ are given by:

$$\mathbb{E}[N(t)] = \frac{r(1-p)}{p}, \quad \mathrm{Var}(N(t)) = \frac{r(1-p)}{p^2}.$$

2.4 Probability of Ruin

The probability of ruin $\psi(u)$ remains a key metric, but its calculation requires numerical methods for non-homogeneous $N(t)$. The approximation:

$$\psi(u) \approx Ce^{-\gamma u},$$

may still apply under certain assumptions, with γ determined from $\mathbb{E}[e^{-\gamma X}]$.

3 Claim Size Distributions

3.1 Lognormal Distribution

The Lognormal distribution is defined as:

$$X \sim \text{Lognormal}(\mu, \sigma),$$

with probability density function:

$$f_X(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right), \quad x > 0.$$

The mean and variance are given by:

$$\mathbb{E}[X] = e^{\mu + \frac{\sigma^2}{2}}, \quad \text{Var}(X) = \left(e^{\sigma^2} - 1\right) e^{2\mu + \sigma^2}.$$

As σ increases, the variance of X grows exponentially, leading to greater variability in the risk process.

3.2 Exponential Distribution

The Exponential distribution is defined as:

$$X \sim \text{Exponential}(\lambda),$$

with probability density function:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0.$$

The mean and variance are given by:

$$\mathbb{E}[X] = \frac{1}{\lambda}, \quad \text{Var}(X) = \frac{1}{\lambda^2}.$$

As λ decreases, the variance increases, leading to heavier-tailed behavior and greater variability in claim sizes.

4 Implementation in Shiny

The Shiny application simulates the risk process under Lognormal and Exponential distributions, incorporating the non-homogeneous claim arrival process. Parameters such as r, p, $\lambda(t)$, μ, σ, and λ are adjustable via the interface, directly affecting the variance and behavior of the risk process [Embrechts et al., 1997].

4.1 Simulation Algorithm

The simulation involves:

1. Generating claim arrival times using a non-homogeneous process like Negative Binomial.
2. Sampling claim sizes from the specified distribution.
3. Calculating $S(t)$ and $R(t)$ for each time step.

For instance, increasing r or decreasing p in the Negative Binomial distribution increases the variance of $N(t)$, impacting the risk process.

4.2 Key Code Snippets

The function to simulate the risk process is implemented as follows:

```
simulate_risk_process <- function(u, c, t_max, claim_size_dist, size, prob) {
t <- seq(0, t_max)
num_claims <- rnbinom(1, size = size, prob = prob)
claim_times <- sort(runif(num_claims, 0, t_max))
claim_sizes <- claim_size_dist(num_claims)
S_t <- rep(0, length(t))
for (i in 1:num_claims) {
S_t[t >= claim_times[i]] <- S_t[t >= claim_times[i]] + claim_sizes[i]
}
RU <- u + c * t - S_t
return(data.frame(time = t, RU = RU))
}
```

5 Results and Visualization

The Shiny application provides:

- Time series plots of $R(t)$ for multiple simulations.
- Comparison of risk processes under Lognormal and Exponential distributions.
- Analysis of how parameter changes in $N(t)$ and claim size distributions affect variance.

Sample plots include:

- Risk processes simulated for Lognormal distribution.
- Risk processes simulated for Exponential distribution.
- Combined comparison plot.

6 Conclusion

The Cramér-Lundberg model with non-homogeneous claim arrival processes extends the classical framework, allowing for greater flexibility and realism in modeling risk. Parameters such as r, p, $\lambda(t)$, μ, σ, and λ significantly influence the variance and tail behavior of the claim size distributions. Using Shiny, it is possible to visualize these effects, facilitating deeper insights into insurance risk management.

References

[Embrechts et al., 1997] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.