Homogeneous Poisson process

Section 3 - Home Work 1

Student: Mehrab Atighi, mehrab.atighi@gmail.com
Lecturer: Mohammad Zokaei, Zokaei@sbu.ac.ir

Problem 1

Proof of the Expected Value of the Risk Process with Small o Term. Embrechts et al. (1997):

(a) Prove the following

(i) $E[U(t)] = u + (c - \lambda\mu)t(1 + o(1))$

Solve 1

To prove the expected value of the risk process $U(t)$ in a renewal model, we start with the definition:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where:

1. u is the initial reserve.
2. c is the premium rate.
3. $N(t)$ is the number of claims up to time t, which follows a renewal process.
4. X_i are the individual claim amounts, assumed to be i.i.d. random variables with mean μ.

Steps to Prove the Expected Value

1. **Define the Renewal Process:**

   Let $\{X_i\}$ be the interarrival times between claims, which are i.i.d. with distribution function F_T and mean λ^{-1}.

   The counting process $N(t)$ represents the number of claims by time t.

2. **Risk Process:**

   The risk process $U(t)$ is given by:

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

3. **Expected Value of the Risk Process:**

   To find the expected value $E[U(t)]$, we use the linearity of expectation:

$$E[U(t)] = E[u + ct - \sum_{i=1}^{N(t)} X_i]$$

$$E[U(t)] = u + ct - E[\sum_{i=1}^{N(t)} X_i]$$

4. **Expected Value of the Sum of Claims:**

   The expected value of the sum of claims up to time t is:

   $$E[\sum_{i=1}^{N(t)} X_i] = E[N(t)] \cdot E[X]$$

   Using the renewal reward theorem, we know that for large t:

   $$E[N(t)] \approx \frac{t}{\lambda}$$

   Therefore:

   $$E[\sum_{i=1}^{N(t)} X_i] \approx \frac{t}{\lambda} \cdot \mu$$

5. **Incorporating the Small o Term**

   The renewal reward theorem also implies that there is a small error term $o(1)$ that goes to zero as t grows large. Thus, we can write:

   $$E[N(t)] = \frac{t}{\lambda}(1 + o(1))$$

   Therefore:

   $$E[\sum_{i=1}^{N(t)} X_i] = \frac{t}{\lambda} \cdot \mu(1 + o(1))$$

6. **Combining Results:**

   Substituting back into the expression for $E[U(t)]$:

   $$E[U(t)] = u + ct - \frac{t}{\lambda} \cdot \mu(1 + o(1))$$

   Simplifying, we get:

   $$E[U(t)] = u + ct - \frac{\mu}{\lambda}t(1 + o(1))$$

   Since ($\lambda = \frac{1}{\mu}$), we have:

   $$E[U(t)] = u + (c - \lambda\mu)t(1 + o(1))$$

Conclusion:
The expected value of the risk process $U(t)$ in a renewal model, as t approaches infinity, is given by:

$$E[U(t)] = u + (c - \lambda\mu)t(1 + o(1))$$

This incorporates the small o term, which represents a small error that diminishes as t becomes large. Schmidli (2018); Geng et al. (2024); Asmussen (2000, 2003); Asmussen and Glynn (2010)

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