In the name of God

Producer:

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Subject:

Monte carlo method for Normal-Cuashy bayes estimator

Date:

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Supervisor:

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Issue:

For the normal-Cauchy Bayes estimator

$$\delta(x) = \frac{\int_{-\infty}^{+\infty} \frac{\theta}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta}{\int_{-\infty}^{+\infty} \frac{1}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta}$$

solve the following questions when x = 0, 2, 4.

a. Plot the integrands, and use Monte Carlo integration based on a Cauchy simulation to calculate the integrals.

b. Monitor the convergence with the standard error of the estimate. Obtain three digits of accuracy with probability .95.

c. Repeat the experiment with a Monte Carlo integration based on a normal simulation and compare both approaches.

Solve:

We know that according to the Monte carlo methods we have:

$$\delta(x) = \frac{\int_{-\infty}^{+\infty} \frac{\theta}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta}{\int_{-\infty}^{+\infty} \frac{1}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta} = \frac{\pi \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{\theta}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta}{\pi \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{1}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta}$$

step1 :we set the f(θ), the caushy normal distrubtion Caushu(0,1)

$$f(\theta) = \frac{1}{\pi(1+\theta^2)}$$

step2 :we set the other functions infront of the integral,g(x)

so we have:

$$\delta(x) = \frac{\pi \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{1}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} \theta d\theta}{\pi \int_{-\infty}^{+\infty} \frac{1}{\pi} \frac{1}{1+\theta^2} e^{-\frac{(x-\theta)^2}{2}} d\theta} = \frac{\pi \int_{-\infty}^{+\infty} f(\theta) * e^{-\frac{(x-\theta)^2}{2}} \theta d\theta}{\pi \int_{-\infty}^{+\infty} f(\theta) * e^{-\frac{(x-\theta)^2}{2}} d\theta}$$

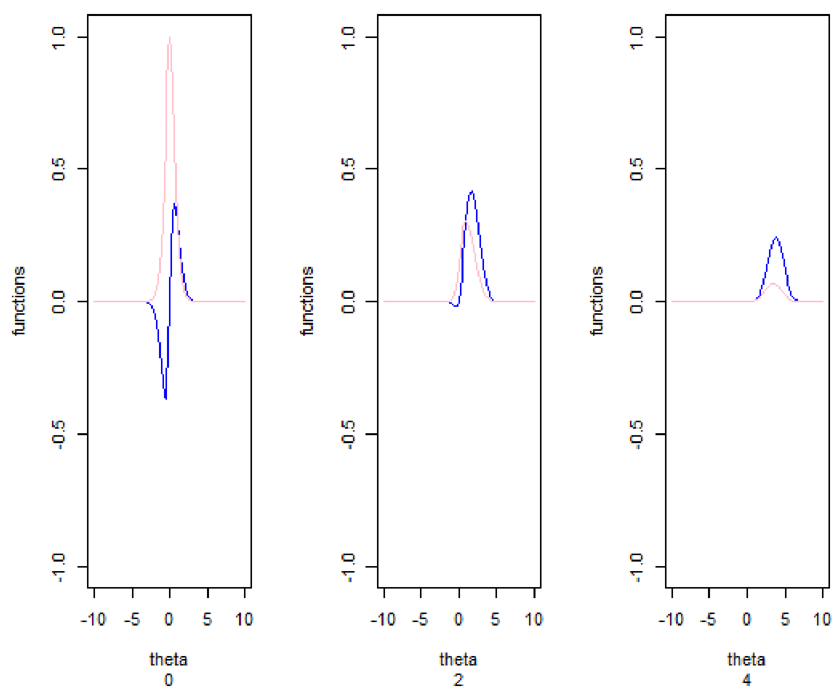
so now we set the $g_1(x,\theta) = e^{-\frac{(x-\theta)^2}{2}} \theta$, $g_2(x,\theta) = e^{-\frac{(x-\theta)^2}{2}}$

then:

$$\delta(x) = \frac{\pi \int_{-\infty}^{+\infty} f(\theta) g_1(x,\theta) d\theta}{\pi \int_{-\infty}^{+\infty} f(\theta) g_2(x,\theta) d\theta} \overset{M.c}{\Longleftrightarrow} \frac{\pi E_f[g_1(x,\theta)]}{\pi E_f[g_2(x,\theta)]} \Rightarrow \delta(x) = \frac{E_f[g_1(x,\theta)]}{E_f[g_2(x,\theta)]}$$

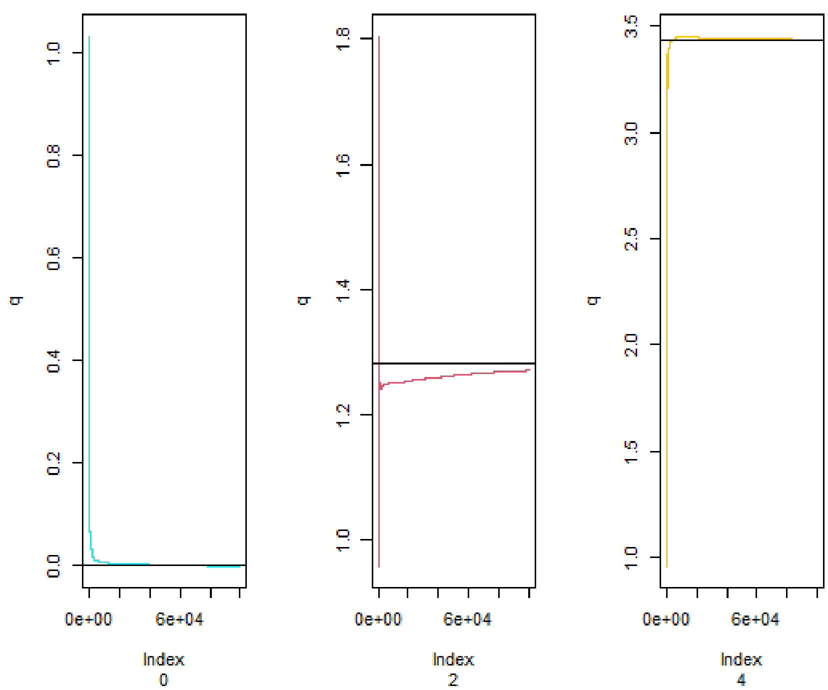
For $\delta(x)$ we have:

```
> rm(list=ls())
> memory.limit(size=999999999)
[1] 1e+09
> x=c(0,2,4)
> N=10^5
> g1<-c();g2<-c()
> g.1<-c();g.2<-c()
> delta<-matrix(c(rep(0,3*N)),ncol = N)
> theta<-rcauchy(N)
> par(mfrow=c(1,3))
> for(i in 1:3){
+   f.1=function(theta){ theta/(1+(theta)^2)*exp(-(x[i]-theta)^2/2)}
+   f.2=function(theta){ 1/(1+(theta)^2)*exp(-(x[i]-theta)^2/2)}
+   plot(f.1,xlab="theta",ylab="functions",type ="l" ,xlim = c(-10,10),ylim=c(-1,1) ,col="blue" , sub=x[i])
+   plot(f.2,add=TRUE,type="l" , col="pink" , xlim = c(-10,10))}
```




```
+   for(i in 1:N){
+   g1[i]<-theta[i]*exp(-(1/2*(x[j]-theta[i])^2))
+   g2[i]<-1*exp(-(1/2*(x[j]-theta[i])^2))
+   delta[j,i]<-mean(g1[1:i])/mean(g2[1:i])}
+   print(paste("the delta(x) value with x=",x[j],"is:",mean(g1)/mean(g2)))}
[1] "the delta(x) value with x= 0 is: -0.00233317554413187"
[1] "the delta(x) value with x= 2 is: 1.27888732761474"
[1] "the delta(x) value with x= 4 is: 3.42736175946439"
```

```
>  o<-c()
> for(j in 1:3){
+   g.1<-function(t){(t/(1+(t^2)))*exp(-((1/2)*(x[j]-t)^2))}
+   g.2<-function(t){(1/(1+(t^2)))*exp(-((1/2)*(x[j]-t)^2))}
+   print(o[j]<-(integrate(g.1,-Inf,Inf)$val)/(integrate(g.2,-Inf,Inf)$val))}
[1] 0
[1] 1.282195
[1] 3.435062
>  for(j in 1:3){
+   r<-which((delta[j,]=="NaN"))
+   for(w in 1:length(r)){
+     delta[j,r[w]]<-o[j]}}
> (sd<-apply(delta,1,sd))
[1] 0.01037757 0.01340714 0.02649938
> par(mfrow=c(1,3))
> for(j in 1:3){
+   q<-cumsum(delta[j,])/1:N
+   plot(q,type="l" ,sub = x[j],col=85*j,xlim=c(0,N),ylim=c(min(q),max(q)))
+   abline(h=o[j] ,col="black")}
```



Now we want to make it for normal:

$$\delta(x)=\frac{\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{(x-\theta)^2}{2}}d\theta}{\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{(x-\theta)^2}{2}}d\theta}=\frac{\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{(x^2+\theta^2-2x\theta)}{2}}d\theta}{\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{(x^2+\theta^2-2x\theta)}{2}}d\theta}$$

$$=\frac{\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{x^2}{2}}e^{-\frac{\theta^2}{2}}e^{\theta x}\,d\theta}{\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{x^2}{2}}e^{-\frac{\theta^2}{2}}e^{\theta x}\,d\theta}=\frac{\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{x^2}{2}}e^{-\frac{\theta^2}{2}}e^{\theta}e^{x}d\theta}{\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{x^2}{2}}e^{-\frac{\theta^2}{2}}e^{\theta}e^{x}d\theta}$$

$$=\frac{\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}e^{-\frac{x^2}{2}}d\theta}{\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}e^{-\frac{x^2}{2}}d\theta}=\frac{\sqrt{2\pi}\int_{-\infty}^{+\infty}\frac{\theta}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}*\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d\theta}{\sqrt{2\pi}\int_{-\infty}^{+\infty}\frac{1}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}*\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d\theta}$$

Now if $if\ X\sim Normal(0,1), f_X(x)=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx.$

If we set $h_1(\theta)=\frac{\theta}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}$, $h_2(\theta)=\frac{1}{1+\theta^2}e^{-\frac{\theta^2}{2}+\theta x}$

So We have :

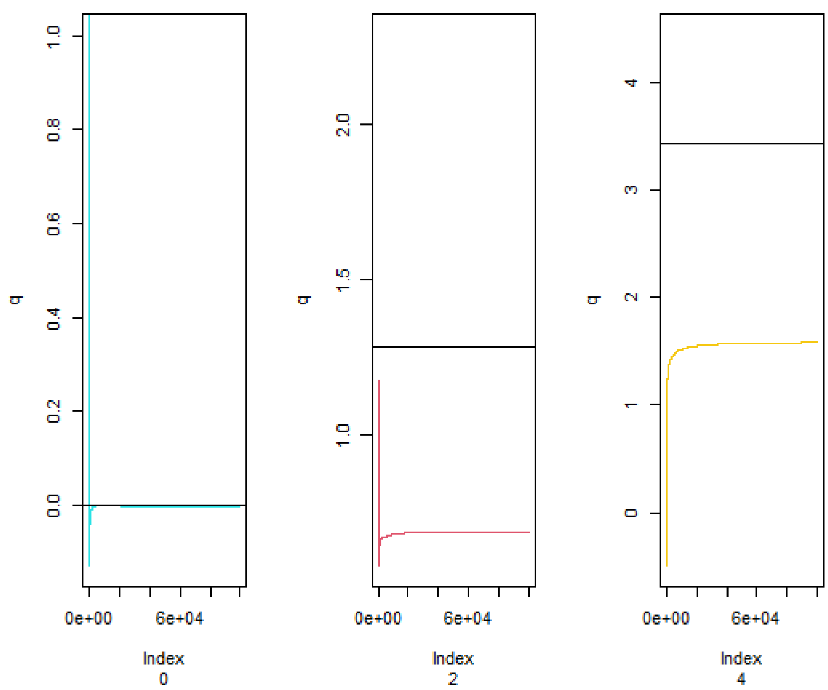
$$\delta(x)=\frac{\sqrt{2\pi}\int_{-\infty}^{+\infty}h_1(\theta)*f_X(x)d\theta}{\sqrt{2\pi}\int_{-\infty}^{+\infty}h_2(\theta)*f_X(x)d\theta}\overset{M.C}{\Longrightarrow}\delta(x)\cong\frac{\sqrt{2\pi}E_x[h_1(\theta)]}{\sqrt{2\pi}E_x[h_2(\theta)]}=\frac{E_x[h_1(\theta)]}{E_x[h_2(\theta)]}$$

The two front function plot is same with cuashy plot.(so I don't show it again.)

Now we want to estimate $\delta(x)$,(according to up)with Normal-standard density function:

```
> #normal part:
> rm(list=ls())
> x=c(0,2,4)
> N=10^5
> h1<-c();h2<-c()
> h.1<-c();h.2<-c()
> delta2<-matrix(c(rep(0,3*N)),ncol = N)
> for(i in 1:3){
+   f.1=function(theta){ theta/(1+(theta)^2)*exp(-(x[i]-theta)^2/2)}
+   f.2=function(theta){ 1/(1+(theta)^2)*exp(-(x[i]-theta)^2/2)}
+   plot(f.1,xlab="theta",ylab="functions",type ="l" ,xlim = c(-10,10),ylim=c(-1,1) ,col=85 , sub=x[i])
+   plot(f.2,add=TRUE,type="l" , col="purple" , xlim = c(-10,10))}
> for( j in 1:3){theta<-rnorm(N,mean = 0,sd=1)
+   for(i in 1:N){
+     h1[i]<-(((theta[i])/(1+(theta[i])^2))*(exp((-1/2*(theta[i])^2)+(theta[i]*x[j]))))
+     h2[i]<-(((1)/(1+(theta[i])^2))*(exp((-1/2*(theta[i])^2)+(theta[i]*x[j]))))
+     delta2[j,i]<-mean(h1[1:i])/mean(h2[1:i])}
+   print(paste("the delta(x) value with x=",x[j],"is:",mean(h1)/mean(h2)))}
[1] "the delta(x) value with x= 0 is: 0.000377986859489263"
[1] "the delta(x) value with x= 2 is: 0.685799582291693"
[1] "the delta(x) value with x= 4 is: 1.58736740680525"
```

```
> o<-c()
> for(j in 1:3){
+   h.1<-function(t){(t/(1+(t^2)))*exp(-((1/2)*(x[j]-t)^2))}
+   h.2<-function(t){(1/(1+(t^2)))*exp(-((1/2)*(x[j]-t)^2))}
+   print(o[j]<-(integrate(h.1,-Inf,Inf)$val)/(integrate(h.2,-Inf,Inf)$val))}
[1] 0
[1] 1.282195
[1] 3.435062
> for(j in 1:3){
+   r<-which((delta2[j,]=="NaN"))
+   for(w in 1:length(r)){
+     delta2[j,r[w]]<-o[j]}}
> (sd<-apply(delta2,1,sd))
[1] 0.007764371 0.007211865 0.036078876
> par(mfrow=c(1,3))
> for(j in 1:3){
+   q<-cumsum(delta2[j,])/1:N
+   plot(q,type="l" ,sub = x[j],col=85*j,xlim=c(0,N),ylim=c(min(q),o[j]+1))
+   abline(h=o[j] ,col="black")}
```



Conclusion: the estiamate for $\delta(x)$ with normal-standard distrubtion is not good but the Caushy estimate is good for that,we can see that in estimate with cuashy random variable converges very quickly to $\delta(x)$.