Homogeneous Poisson process

Section1 - HW1

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Problem 1

A consequence of the Cramer-Lundberg model definition is that $(N(t))$ is a homogeneous Poisson process with intensity $\lambda > 0$. Hence (EKM97):

(a) Prove the following

(i) $P(N(t) = k) = \exp^{-\lambda t} \frac{(\lambda t)^k}{k!}$ $k = 0, 1, 2, \cdots$.

Solve 1

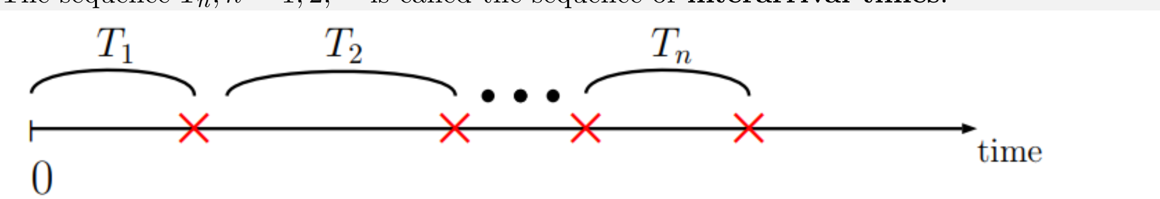
we know that Any counting process N(t) must satisfy:

1. N(t) $\geq$ 0;
2. N(t) is integer valued;
3. if s<t, then N(s) $\leq$ N(t);
4. For any s<t, N(t) $\leq$ N(s) equals the number of events that occur in the interval $(s, t]$

Consider a Poisson process:

1. Denote the time of the first event by T_1.
2. For any n>1, let T_n denote the elapsed time between the $(n-1)$st and the nth event.

The sequence $T_n, n = 1, 2, \cdots$is called the sequence of **interarrival times**.



and we know that $T_n, n = 1, 2, \cdots$ are independent identically distributed (iid) exponential random variables with parameter λ.

$$P(T_1 > t) = P(N(t) = 0) = \exp^{\lambda t} \rightarrow T_1 \sim \text{EXP}(\lambda)$$

The total waiting time for n occurrences of the event has a Gamma distribution (with parameters (n, λ)
This implies that

$$E(S_n) = \frac{n}{\lambda} Var(S_n) = \frac{n}{\lambda^2}$$

Let

$$N(t) = \max\{n \geq 0 : T_1 + \cdots + T_n \leq t\}$$

Then $N(t), t \geq$ is a Poisson process with rate λ. for show the above relation we have:
Fix an integer $n \geq 0$. Then $S_n = T_1 + \cdots + T_n \sim \Gamma(n, \lambda)$ and it is independent of T_{n+1}.
By definition of

$$\begin{aligned}
P(N(t) = n) &= P(S_n \leq t, S_n + T_{n+1} > t) \\
&= \int_0^t \int_{t-s}^{\infty} f_{S_n}(s) f_{T_{n+1}}(x)\, dx\, ds \\
&= \int_0^t P(T_{n+1} > t - s) f_{S_n}(s)\, ds \\
&= \int_0^t e^{-\lambda(t-s)} \frac{\lambda(\lambda s)^{n-1} e^{-\lambda s}}{(n-1)!}\, ds \\
&= \frac{(\lambda t)^n e^{-\lambda t}}{n!}.
\end{aligned}$$

This shows that $N(t) \sim Pois(\lambda t)$ The homogeneous Poisson process is a type of stochastic process that models events occurring randomly over time. Let's go through a step-by-step proof of some fundamental properties of a **homogeneous Poisson process** $N(t)$, with rate $\lambda > 0$. (Che19)
Another way to solve this question is that:

1. Definition
   A **Poisson process** $N(t)$ with rate $\lambda > 0$ is defined as a stochastic process with the following properties:

   1. $N(0) = 0$ (the process starts at 0).

   2. **Independent increments**: The number of events that occur in disjoint time intervals are independent.

   3. **Stationary increments**: The probability of k events occurring in any time interval of length t depends only on t , not on where the interval starts, and is given by the Poisson distribution:

   $$P(N(t+s) - N(s) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

2. Proof: $N(t) \sim \text{Poisson}(\lambda t)$

   We will prove that the number of events $N(t)$ in a time interval $[0, t]$ follows a Poisson distribution with parameter λt.

   Step 1: Small time intervals approximation

   Divide the time interval $[0, t]$ into n small sub-intervals of length $\Delta t = \frac{t}{n}$. For large n, each sub-interval is short, and we assume that:

   1. The probability of one event occurring in a sub-interval is approximately $\lambda \Delta t$.

   2. The probability of more than one event occurring in a sub-interval is negligible, i.e., $O(\Delta t^2)$.

Thus, for each sub-interval $[t_i, t_{i+1}]$:

$$P(1 \text{ event in } [t_i, t_{i+1}]) \approx \lambda\Delta t, \quad P(\text{no event in } [t_i, t_{i+1}]) \approx 1 - \lambda\Delta t.$$

Step 2: Approximation for the total number of events

Let $N_n(t)$ represent the number of events in the n sub-intervals. Since the intervals are independent, $N_n(t)$ is the sum of n independent Bernoulli random variables, where the probability of an event in each sub-interval is $\lambda\Delta t$.

The expected number of events in $[0, t]$ is:

$$E[N_n(t)] = n \cdot \lambda\Delta t = \lambda t$$

.

As $n \to \infty$, the sum of these Bernoulli trials converges to a Poisson distribution with mean λt (this follows from the **Poisson limit theorem**).

3. Step 3: Deriving the Poisson distribution From the Poisson limit theorem, we conclude that as $\Delta t \to 0$ (or equivalently $n \to \infty$), the number of events $N(t)$ in $[0, t]$ converges to a Poisson random variable with parameter λt, i.e.,

$$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, 2, \ldots$$

Conclusion:
We have shown that the number of events in a homogeneous Poisson process over a time interval $[0, t]$ follows a Poisson distribution with mean λt, confirming the definition of the homogeneous Poisson process.

References

[Che19] Guangliang Chen. Poisson processes, 2019.

[EKM97] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997. For insurance and finance.