Find norming constant of lognormal distribution

Section 21 - Home Work 2

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Problem 1

According to the Example 3.3.31 find norming constant of lognormal distribution. Embrechts et al. (1997)

Solve 1

Definition of the Lognormal Distribution

Let X be a standard normal random variable, and define the lognormal random variable as:

$$\tilde{X} = g(X) = e^{\mu+\sigma X}, \quad \mu \in \mathbb{R}, \ \sigma > 0.$$

The goal is to study the asymptotic behavior of the maximum of the lognormal distribution, $\tilde{X}$, and show that $\tilde{X} \in \text{MDA}(\Lambda)$, the maximum domain of attraction of the Gumbel distribution.

The Standard Normal Maximum

Since $X \in \text{MDA}(\Lambda)$, the maximum M_n of n standard normal random variables satisfies:

$$\lim_{n\to\infty} P\left(\frac{M_n - d_n}{c_n} \leq x\right) = \Lambda(x), \quad x \in \mathbb{R},$$

where c_n and d_n are norming constants for the standard normal distribution, given as:

$$c_n \sim \frac{1}{(2\ln n)^{1/2}}, \quad d_n \sim (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}}.$$

Now, expanding the form of d_n for large n, we have:

$$d_n \sim (2\ln n)^{1/2} - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}}.$$

We can write the expansion of d_n as:

$$d_n = (2\ln n)^{1/2}\left(1 - \frac{\ln\ln n + \ln(4\pi)}{2(2\ln n)^{1/2}}\right).$$

As $n \to \infty$, we see that $d_n \sim (2\ln n)^{1/2}$, and the correction term is of smaller order.

The Lognormal Maximum

Applying the transformation $g(x) = e^{\mu+\sigma x}$ to the maximum, the transformed maximum is:

$$\tilde{M}_n = g(M_n) = e^{\mu+\sigma M_n}.$$

To standardize $\tilde{M}_n$, we analyze the probability:

$$P\left(\tilde{M}_n \leq e^{\mu+\sigma(c_n x+d_n)}\right).$$

Taking the logarithm of both sides gives:

$$P\left(M_n \leq c_n x + d_n\right),$$

which converges to the Gumbel distribution:

$$P\left(\frac{M_n - d_n}{c_n} \leq x\right) \to \Lambda(x), \quad \text{as } n \to \infty.$$

Derivation of Norming Constants

Returning to $\tilde{M}_n$, we approximate it as:

$$\tilde{M}_n = e^{\mu+\sigma M_n} = e^{\mu+\sigma d_n} e^{\sigma c_n x + o(\sigma c_n)}.$$

Using the first-order approximation for the exponential term:

$$e^{\sigma c_n x + o(\sigma c_n)} \approx 1 + \sigma c_n x + o(\sigma c_n),$$

we find:

$$\tilde{M}_n \approx e^{\mu+\sigma d_n}\left(1 + \sigma c_n x + o(\sigma c_n)\right).$$

Thus, we approximate the norming constants:

$$\tilde{c}_n = \sigma c_n e^{\mu+\sigma d_n}, \quad \tilde{d}_n = e^{\mu+\sigma d_n}.$$

Final Results

The lognormal maximum $\tilde{M}_n$ satisfies:

$$\lim_{n\to\infty} P\left(\frac{\tilde{M}_n - \tilde{d}_n}{\tilde{c}_n} \leq x\right) = \Lambda(x), \quad x \in \mathbb{R}.$$

Thus, $\tilde{X} \in \mathrm{MDA}(\Lambda)$, with norming constants:

$$\tilde{c}_n = \sigma c_n e^{\mu+\sigma d_n}, \quad \tilde{d}_n = e^{\mu+\sigma d_n}.$$

Embrechts et al. (1997)

References

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.