Relationship between the negative binomial distribution and the gamma
Section 6 - Home Work 2

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Problem 1

Consider equation (1.35) with $F \in S$. Fix $t > 0$, and suppose that the sequence $(p_t(n))$ satisfies:

$$\sum_{n=0}^{\infty} (1+\epsilon)^n p_t(n) < \infty$$

for some $\epsilon > 0$. We need to show that $G_t \in S$ and that

$$\overline{G_t}(x) \sim EN(t)\overline{F}(x), \quad x \to \infty.$$

Embrechts et al. (1997):

Solve 1

The negative binomial distribution describes the probability of having k failures before achieving r successes in a sequence of independent and identically distributed Bernoulli trials. This distribution can be elegantly expressed using the gamma function.

Negative Binomial Distribution

The probability mass function (PMF) of the negative binomial distribution for a random variable X is given by:

$$P(X = k) = \binom{r+k-1}{k} (1-p)^r p^k$$

where:

- r is the number of successes,
- p is the probability of success,
- k is the number of failures.

Using the Gamma Function

Using the gamma function, which is defined as $\Gamma(n) = (n-1)!$, the PMF can be rewritten. The binomial coefficient can be expressed using the gamma function:

$$\binom{r+k-1}{k} = \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}$$

Therefore, the PMF becomes:

$$P(X=k) = \frac{\Gamma(r+k)}{\Gamma(r)\Gamma(k+1)}(1-p)^r p^k$$

Simplification

To simplify further, let us transform the variables and assume a new parameter $n = k$:

$$P(X=n) = \frac{\Gamma(r+n)}{\Gamma(r)\Gamma(n+1)}(1-p)^r p^n$$

Using the property of the gamma function $\Gamma(n+1) = n!$, we have:

$$P(X=n) = \frac{\Gamma(r+n)}{\Gamma(r)n!}(1-p)^r p^n$$

Now, we can see that:

$$\Gamma(r+n) = (r+n-1)\cdot(r+n-2)\cdots(r)\cdot\Gamma(r)$$

When $n \to \infty$, this can be approximated by $n^{r-1}\cdot\Gamma(r)$, thus:

$$P(X=n) \approx \frac{(1-p)^r p^n n^{r-1}\Gamma(r)}{\Gamma(r)n!}$$

Now, We can see that:

$$P(X=n) \approx \frac{(1-p)^r \cdot n^{r-1} \cdot p^n}{\Gamma(r)}$$

This is the relationship between the negative binomial distribution and the gamma. Now we can use it in the book example.

References

Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin. For insurance and finance.