In the name of God

Producer:
Mehrab Atighi

Subject:
Different of Importance sampling M.c
and classical M.c

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Supervisor:
Dr.Seyed Noorullah Mousavi

Issue: For the computation of the expectation E*f*[*h*(*X*)] when *f* is the normal pdf and $h(x) = exp(-\frac{(x-3)^2}{2}) + exp(-\frac{(x-6)^2}{2})$:

a. Show that E*f*[*h*(*X*)] can be computed in closed form and derive its value.

b. Construct a regular Monte Carlo approximation based on a normal *N* (0, 1) sample of size Nsim=10^3 and produce an error evaluation.

c. Compare the above with an importance sampling approximation based on an importance function *g* corresponding to the *U*(-8, -1) distribution and a sample of size Nsim=10^3. (Warning: This choice of *g* does not provide a converging approximation of E*f*[*h*(*X*)]!)

d.do the part c with expontional distrubtion function and show the standard devation,error.

solve:

a)

We know that if $X \sim N(0,1), f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, and according to the issue we know that:

$$h(x)\left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right) dx$$

$$= \int_{-\infty}^{\infty} f(x)h(x)dx \overset{\text{M.c}}{\Longrightarrow} E_f[h(x)]$$

Now we want to solve this antegral with close form method:

$$E_f[(x)] = \int_{-\infty}^{\infty} f(x)h(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right) dx$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-6)^2}{2}}\right) dx$$

$$=$$

$$= \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}\right) dx + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-6)^2}{2}}\right) dx$$

If we set $g(x) = e^{-\frac{x^2}{2}}$, $h_1(x) = \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}\right) = \text{Normal}(3,1)$, $h_2(x) = \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-6)^2}{2}}\right) = \text{Normal}(6,1)$.

so we have:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}\right) dx = \int_{-\infty}^{\infty} g(x).\, h_1(x)dx$$

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-6)^2}{2}}\right) dx = \int_{-\infty}^{\infty} g(x).\, h_2(x)dx$$

We now that the normal distrubtion($h_1(x), h_2(x)$) antegral is 1.

$$\int_{-\infty}^{\infty} g(x).\, h_1(x)dx = \int_{-\infty}^{\infty} g(x).\, 1dx = \int_{-\infty}^{\infty} g(x)dx$$

$$\int_{-\infty}^{\infty} g(x).\, h_2(x)dx = \int_{-\infty}^{\infty} g(x).\, 1dx = \int_{-\infty}^{\infty} g(x)dx$$

So we should find the area under the g(x) curve.

$$I = \int_{-\infty}^{\infty} g(x)dx = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx$$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dxdy$$

Now if we set $x = r\cos(\theta), y = r\sin(\theta) \rightarrow x^2 + y^2 = (r\cos(\theta))^2 + (r\sin(\theta))^2$

$$= r^2(\cos^2(\theta)) + r^2(\sin^2(\theta)) = r^2(\cos^2(\theta) + \sin^2(\theta)) = r^2(1) = r^2$$

$$|j| = drdr\theta, rdrd\theta \approx dxdy$$

$$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dxdy = \int_{0}^{2\pi}\int_{0}^{\infty} e^{-\frac{r^2}{2}} rdrd\theta$$

If we set $u = -\frac{r^2}{2} \rightarrow du = -rdr$.

$$I^2 \int_{0}^{2\pi}\int_{0}^{\infty} e^{u}\, dud\theta \overset{*\frac{-1}{-1}}{\Longleftrightarrow} -\int_{0}^{2\pi}\int_{0}^{\infty} -e^{-u}\, dud\theta = -\int_{0}^{2\pi} [-e^{-u}]_0^{\infty}\, d\theta = -\int_{0}^{2\pi} [0-(-1)]\, d\theta$$

$$= \int_{0}^{2\pi} d\theta = [\theta]_0^{2*\pi} = [2\pi - 0] = 2\pi$$

So we have:

$$I^2 = 2\pi \rightarrow I = \sqrt{2\pi}$$

$$E_f[(x)] = \int_{-\infty}^{\infty} g(x).\, h_1(x)dx + \int_{-\infty}^{\infty} g(x).\, h_2(x)dx = 2\int_{-\infty}^{\infty} g(x)dx = 2 * \sqrt{2\pi}$$

b)

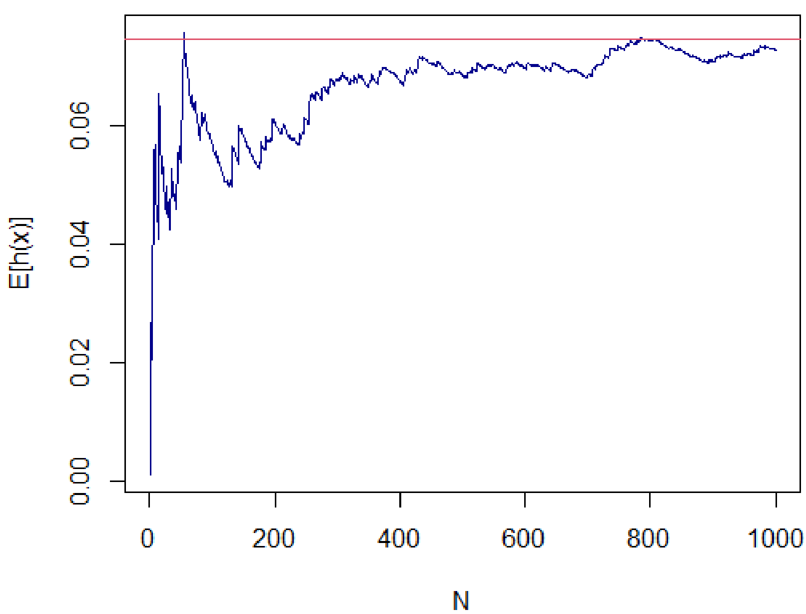
if we want to use classical monte carlo method for $E_f[(x)]$ we have:

$$E_f[(x)] = \int_{-\infty}^{\infty} g(x).h_1(x)dx + \int_{-\infty}^{\infty} g(x).h_2(x)dx \overset{M.c}{\Longrightarrow} E_{h1}[g(x)] + E_{h2}[g(x)]$$

$$= \lim_{n\to\infty} \frac{g(x_{i.h1})}{n} + \lim_{n\to\infty} \frac{g(x_{i.h2})}{n} \quad \text{for } i = 1,2,3,\dots,n$$

Now we can simulate it in R:

```
> g<-function(x){exp(-(x^2)/2)}
> N=10^3
> H1<-rnorm(N,mean = 3,sd=1)
> H2<-rnorm(N,mean = 6,sd=1)
> E<-(cumsum(g(H1))/1:N)+(cumsum(g(H2))/1:N)
> plot(E , type = "l",col="Blue4")
> abline(h=a , col="850")
```



Our estimate for $E_f[h(x)]$ *is*:

```
> E[N]
[1] 0.07275448
```

The real value is:

```
> a
[1] 0.07453205
```

Now we can find the our monte carlo estimate error:

```
> abs(a-E[N])
[1] 0.001777571
```

now we can find the standard deviation:

```
> sd(E)
[1] 0.007590212
```

c)

we know that if $U \sim \mathrm{unifrom}(-8,-1)$, then $f_U(u) = \frac{1}{-1-(-8)} = \frac{1}{7}$.

Now we want to use the Importance sampling Monte carlo .

$$E_f[(x)] = \int_{-\infty}^{\infty} f(x)h(x)\,dx = \int_{-\infty}^{\infty} \frac{f(x)h(x)}{g(x)} g(x)dx = E_g\left[\frac{f(x)h(x)}{g(x)}\right];$$

According to the question part c we should use $g(x) = f_U(u)$ in our formula, so we have:

$$E_f[(x)] = \int_{-\infty}^{\infty} f(x)h(x)\,dx = \int_{-\infty}^{\infty} \frac{f(x)h(x)}{f(u)} f(u)dx = \int_{-8}^{-1} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right) * \frac{7}{7} dx$$

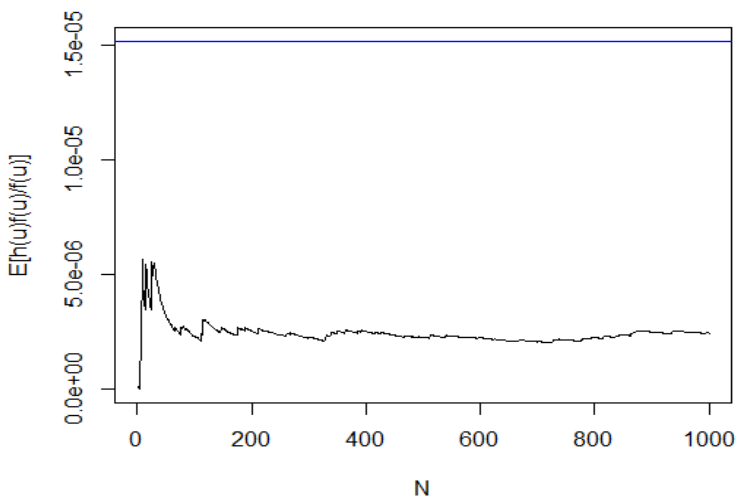
$$= 7\int_{-8}^{-1} \frac{\left[\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right)\right]}{7} dx$$

$$= 7\int_{-8}^{-1} \frac{\left[e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}\right)\right]}{7} dx + 7\int_{-8}^{-1} \frac{\left[e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-6)^2}{2}}\right)\right]}{7} dx$$

$$\overset{M.c}{\Longrightarrow} 7E_u\left[\frac{f(u)h_1(u)}{f(u)}\right] + 7E_u\left[\frac{f(u)h_2(u)}{f(u)}\right]$$

So now we can simulate it in R:

```
> N=10^3
> G1<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-3)^2)/(sqrt(2*pi))}
> G2<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-6)^2)/(sqrt(2*pi))}
> G<-integrate(G1,-8,-1)$val+integrate(G2,-8,-1)$val
> u<-runif(N,min=-8,max=-1)
> g1<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-3)^2)/(7*sqrt(2*pi))}
> g2<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-6)^2)/(7*sqrt(2*pi))}
> R<-7*((cumsum(g1(u))/1:N)+(cumsum(g2(u))/1:N))
> plot(R, type = "l" , ylim =c(min(R),G), xlab="N" , ylab = "E[h(u)f(u)/f(u)]")
> abline(h=G ,col="Blue")
```



Our estimate for $7\mathrm{E_u}\left[\frac{\mathrm{f(u)h_1(u)}}{\mathrm{f(u)}}\right] + 7\mathrm{E_u}\left[\frac{\mathrm{f(u)h_2(u)}}{\mathrm{f(u)}}\right]$ is:

```
> R[N]
[1] 2.449683e-06
```

The real value is:

```
> G
[1] 1.516476e-05
```

Now we can find the our monte carlo estimate error:

```
> abs(G-R[N])
[1] 1.271508e-05
```

now we can find the standard deviation:

```
> sd(R)
[1] 4.939516e-07
```

d)

we know that ifP~E(λ), $f_P(p) = \frac{1}{\lambda} e^{-\frac{x}{\lambda}}$

the importance sampling minte carlo method formula is:

$$E_f[(x)] = \int_{-\infty}^{\infty} f(x)h(x)\,dx = \int_{-\infty}^{\infty} \frac{f(x)h(x)}{g(x)} g(x)dx = E_g\left[\frac{f(x)h(x)}{g(x)}\right];$$

According to our qurstion pard d, we should set the $g(x) = f_P(p)$.so we have:

$$E_f[(x)] = \int_{-\infty}^{\infty} f(x)h(x)\,dx = \int_{-\infty}^{\infty} \frac{f(x)h(x)}{f(u)} f(u)dx$$

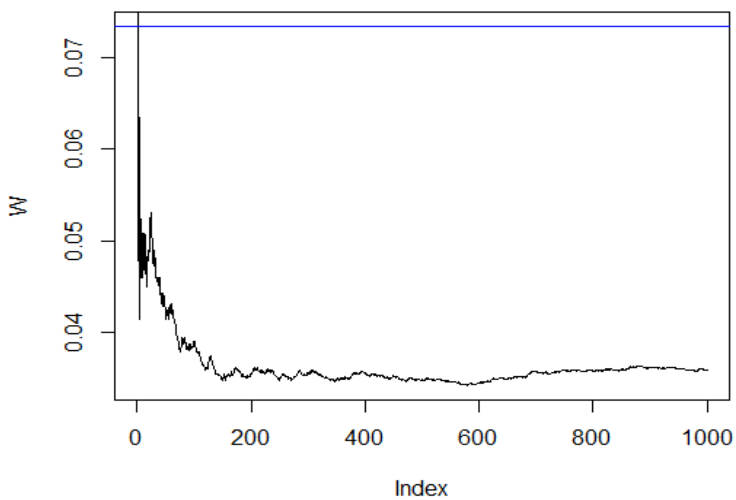
$$= \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left(e^{\frac{(x-3)^2}{2}} + e^{-\frac{(x-6)^2}{2}}\right) * \frac{\frac{1}{\lambda} e^{-\frac{x}{\lambda}}}{\frac{1}{\lambda} e^{-\frac{x}{\lambda}}} dx$$

$$\int_{0}^{\infty} \frac{\left[e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-3)^2}{2}}\right)\right]}{\frac{1}{\lambda} e^{-\frac{x}{\lambda}}} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx + \int_{0}^{\infty} \frac{\left[e^{-\frac{x^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{\frac{(x-6)^2}{2}}\right)\right]}{\frac{1}{\lambda} e^{-\frac{x}{\lambda}}} \frac{1}{\lambda} e^{-\frac{x}{\lambda}} dx$$

$$\overset{M.c}{\Longrightarrow} E_u\left[\frac{f(p)h_1(p)}{f(p)}\right] + E_u\left[\frac{f(p)h_2(p)}{f(p)}\right]$$

Now we simulate in R:(here we set $\lambda = 1$)

```
> N=10^3
> F1<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-3)^2)/(sqrt(2*pi))}
> F2<-function(x){exp(-(x^2)/2)*exp(-1/2*(x-6)^2)/(sqrt(2*pi))}
> F<-integrate(F1,0,Inf)$val+integrate(F2,0,Inf)$val
> q<-rexp(N,rate =1)
> f1<-function(x){(1/exp(-x/2))*exp(-(x^2)/2)*exp(-1/2*(x-3)^2)/(sqrt(2*pi))}
> f2<-function(x){(1/exp(-x/2))*exp(-(x^2)/2)*exp(-1/2*(x-6)^2)/(sqrt(2*pi))}
> W<-((cumsum(f1(q))/1:N)+(cumsum(f2(q))/1:N))
> plot(W, type = "l" , ylim =c(min(W),F))
> abline(h=F,col="Blue")
```



Our estimate for $E_u\left[\frac{f(p)h_1(p)}{f(p)}\right] + E_u\left[\frac{f(p)h_2(p)}{f(p)}\right]$

```
> W[N]
[1] 0.03597266
```
The real value is:
```
> F
[1] 0.07335276
```
Now we can find the our monte carlo estimate error:
```
> abs(F-W[N])
[1] 0.0373801
```
now we can find the standard deviation:
```
> sd(W)
[1] 0.003373489
```

End.