Criterion for the WLLN proof
Section 9 - Home Work 1

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Problem 1

proof the WLLN that

$$\overline{X_n} \to^P \infty.$$

Embrechts et al. (1997):

Solve 1

Proof of the Weak Law of Large Numbers using Chebyshev's Inequality

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean μ and variance σ^2. Define the sample mean $\overline{X}_n$ as follows:

$$\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

To prove the Weak Law of Large Numbers, we need to show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \geq \epsilon\right) = 0$$

First, let's consider the variance of $\overline{X}_n$:

$$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

Using the properties of variance, we have:

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

Next, we apply Chebyshev's inequality:

$$P\left(\left|\overline{X}_n - \mu\right| \geq \epsilon\right) \leq \frac{\mathrm{Var}(\overline{X}_n)}{\epsilon^2}$$

Substituting the variance of $\overline{X}_n$, we get:

$$P\left(\left|\overline{X}_n - \mu\right| \geq \epsilon\right) \leq \frac{\sigma^2/n}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

As n approaches infinity, the right-hand side of the inequality approaches zero:

$$\lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2} = 0$$

Therefore, we have:

$$\lim_{n\to\infty} P\left(\left|\overline{X}_n - \mu\right| \geq \epsilon\right) = 0$$

So we have:

$$\overline{X_n} \to^P \infty.$$

This completes the proof of the Weak Law of Large Numbers using Chebyshev's inequality. Ross (2014); Grimmett and Stirzaker (2020); Chebyshev (1867); Feller (2008); Billingsley (2008); Durrett (2019); Klenke (2013)

References

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