chapter one : utility theorem

تمارین فصل اول: نظریه مطلوبیت

(1)

according to the question information we know that the Probability of X, Y are:

$X \longrightarrow$

x	400	900
$P(X=x)$	$\frac{1}{2}=0.5$	$\frac{1}{2}=0.5$

$Y \longrightarrow$

y	100	1600
$P(Y=y)$	0.6	0.4

a) for show that a Person Prefers X to Y we should use utility function so we know that : $u(w) = \sqrt{w}$ then we have:

discrete $\to E[X] = \sum x P(X=x)$

if we set $w=X \Rightarrow E[u(w)] = E[u(X)] = E[\sqrt{X}] = \left(\frac{1}{2} \times \sqrt{400}\right) + \left(\frac{1}{2} \times \sqrt{900}\right)$

$\Rightarrow E[u(X)] = 10 + 15 = 25$

discrete $\to E[Y] = \sum y P(Y=y)$

if we set $w=Y \Rightarrow E[u(w)] = E[u(Y)] = E[\sqrt{Y}] = (0.6 \times \sqrt{100}) + (0.4 \times \sqrt{1600})$

$\Rightarrow E[u(Y)] = 6 + 16 = 22$

so we have $E[u(X)] > E[u(Y)]$ and it means that a Person with $u(\cdot)$ utility function will Prefer X to Y.

b) if we want to reject this request we should use a w in our utility function that's with more expected value than X. So we have:

w is a number so $E[u(w)] = u(w)$ *

$E[u(w)] > E[u(X)] \overset{*}{\Longleftrightarrow} u(w) > E[u(X)] \to$ from Part a we have this value

$u(w) = \sqrt{w}$

$u(w) > 25 \overset{\uparrow}{\Longleftrightarrow} \sqrt{w} > 25 \xrightarrow{\text{Power } 2} w > 25^2$

c) if we want to find a new utility function we have:

for example we set $u(w) = w$

$\Rightarrow E[u(X)] = \left(\frac{1}{2} \times 400\right) + \left(\frac{1}{2} \times 900\right) = 200 + 450 = 650$

Subject | Year: | Month: | Date:

$E[u(Y)] = 0.6 \times 100 + 0.4 \times 1600 = 60 + 640 = 700$

so $700 > 650 \Leftrightarrow E[u(Y)] > E[u(X)]$

---

② from 1.10 we have this formula for calculating P^+ (maximum Premium that will be Pay). $E[u(w-X)] = u(w-P^+)$ and according to the question we have:

x	0	36
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

so we have: we will set $\underline{w=100}$.

$u(w) = \log w$; X is discrete $\rightarrow E[X] = \sum x P(X=x)$

$E[u(w-X)] = E[u(100-X)] = E[\log(100-X)] = \left(\frac{1}{2} \times \log(100-0)\right) + \left(\frac{1}{2} \times \log(100-36)\right)$

$= \frac{\log 100}{2} + \frac{\log 64}{2} = \log 100^{\frac{1}{2}} + \log 64^{\frac{1}{2}} = \log 10 + \log 8 = \log 80$

$u(w-P^+) = \log(100-P^+) \xrightarrow{A = 100 - P^+} \log(A) = \log 80 \Rightarrow e^{\log 80} = A$; $\ln A = B \Leftrightarrow e^B = A$

$\Rightarrow 80 = 100 - P^+ \Rightarrow \boxed{P^+ = 20}$

for calculating the approximation of P^+ according to the (1.18) we have

$P^+ \approx \mu + \frac{1}{2} r(w-\mu) \sigma^2$ that $E[X] = \mu$, $Var(X) = \sigma^2$ and $r(w) = -\frac{u''(w)}{u'(w)}$

so we have: $E[X] = \sum x P(X=x) = (0 \times \frac{1}{2}) + (36 \times \frac{1}{2}) = 18$

$Var(X) = E[(X-\mu)^2] = \sum (x-\mu)^2 P(X=x) = \left[\frac{1}{2}(0-18)^2\right] + \left[\frac{1}{2}(36-18)^2\right] = \frac{18^2}{2} + \frac{18^2}{2} = 18^2$

$\Rightarrow Var(X) = 324$, $u(w) = \log w \Rightarrow u'(w) = \frac{1}{w}$ and $u''(w) = \frac{-1}{w^2} \Rightarrow r(w) = -\frac{\left(-\frac{1}{w^2}\right)}{\left(\frac{1}{w}\right)}$

$\Rightarrow r(w) = +\frac{1}{w}$ so we have:

$P^+ \approx \mu + \frac{1}{2} r(w-\mu) \sigma^2 = 18 + \left(\frac{1}{2}\right)\left(\frac{1}{100-18}\right) \times 18^2 = 18 + 1.975 = 19.975$

(3) according to the question we know that $P^- = 19$ $\quad u(w) = \log(w)$

again i will rewrite x Probabilities here:

X	0	36
$P(X=x)$	$0.5 = \frac{1}{2}$	$0.5 = \frac{1}{2}$

$u(w) = E[u(w+P^- - X)]$, $u(w) = \log w$, x is discrete $\to E[X] = \sum x P(X=x)$

$\Rightarrow E[u(w+P^- - X)] = E[\log(w+19-X)] = \left(\frac{1}{2}\cdot(\log(w+19-0))\right) + \left(\frac{1}{2}\cdot(\log(w+19-36))\right)$

$= \left(\frac{1}{2}\cdot\log(w+19)\right) + \frac{1}{2}\left(\log(w-17)\right) = \frac{1}{2}\left[\log(w+19) + \log(w-17)\right] = \log w$

$\Leftrightarrow \log(w+19) + \log(w-17) = 2\log w \Leftrightarrow \log(w+19) + \log(w-17) = \log(w^2)$

exp

$\Leftrightarrow (w+19)\times(w-17) = w^2 \Rightarrow w^2 - 17w + 19w - (19\times17) = w^2$

$\Rightarrow w^2 - w^2 + 19w - 17w - (19\times17) = 0 \Leftrightarrow 2w = 19\times17 \to w = \frac{19\times17}{2} = 161.5$

---

(4) according to the (1.10) and question we have: $w = 0$ $\quad X \sim \text{Bernoulli}(\frac{1}{2})$

X	0	1
$\Rightarrow P(X=x)$	$\frac{1}{2} = 0.5$	$\frac{1}{2} = 0.5$

and $u(x) = \begin{cases} \frac{2x}{3} & x \geq -\frac{3}{4} \\ 2x+1 & -1 < x < -\frac{3}{4} \\ 3x+2 & x \leq -1 \end{cases}$

for X: $E[u(w-X)] = u(w-P^+) \overset{w=0}{\Longrightarrow} E[u(-X)] = u(-P^+)$

for $2X$: $E[u(w-2X)] = u(w-P^+) \overset{w=0}{\Longrightarrow} E[u(-2X)] = u(-P^+)$

x is discrete $\to E[X] = \sum x P(X=x)$

$\Rightarrow$ for X we have: $E[u(-X)] = u(-P^+) \Leftrightarrow (0\times\frac{1}{2}) + (-1\times\frac{1}{2}) = -\frac{1}{2} = u(-P^+)$

x is discrete $\to E[X] = \sum x P(X=x)$

$\Rightarrow$ for $2X$ we have: $E[u(-2X)] = u(-P^+) \Leftrightarrow (2\times0\times\frac{1}{2}) + (2\times-1\times\frac{1}{2}) = -1 = u(-P^+)$

if we set $x = -\frac{3}{4} \Rightarrow u(-\frac{3}{4}) = -\frac{1}{2} \Rightarrow P^+_{[X]} = \frac{3}{4}$

if we set $x = -1 \Rightarrow u(-1) = -1 \Rightarrow P^+_{[2X]} = -\frac{4}{3}$

$\Rightarrow P^+_{[2X]} = 4/3 < 2P^+_{[X]} = 2\times\frac{3}{4} = \frac{6}{4} = 1.5$

So done. ✓

⑤ we should Proof that $P^- = \frac{1}{\alpha}\log(M_x(t))$ with $u(x) = -\alpha e^{-\alpha x}$

$$E[u(W)] = E[u(W + P^- - X)] \iff E[-\alpha e^{-\alpha W}] = E[-\alpha e^{-(W+P^- - X)}] = E[-\alpha e^{-\alpha(W+P^-)} e^{+\alpha X}]$$

$$= -\alpha e^{-\alpha(W+P^-)} E[e^{\alpha X}] = -\alpha e^{-\alpha W} \iff \cancel{-\alpha e^{-\alpha W}}\, e^{-\alpha P^-} E[e^{+\alpha X}] = \cancel{-\alpha e^{-\alpha W}}$$

$$\iff e^{-\alpha P^-} + \underbrace{E[e^{\alpha X}]}_{M_x(\alpha)} = 1 \overset{\log}{\iff} -\alpha P^- + \log(M_x(\alpha)) = 0 \iff P^- = \frac{1}{\alpha}\log(M_x(\alpha))$$

for $P^+ = \frac{1}{\alpha}\log(M_x(\alpha))$ we have: $E(u(W-X)) = u(W-P^+)$

$$E[u(W-X)] = E[-\alpha e^{-\alpha(W-X)}] = E[-\alpha e^{-\alpha W} e^{\alpha X}] = -\alpha e^{-\alpha W} E[e^{\alpha X}]$$

$$= -\alpha e^{-\alpha W} M_x(\alpha) = u(W-P^+) = -\alpha e^{-\alpha(W-P^+)} \iff \cancel{-\alpha e^{-\alpha W}}\, M_x(\alpha) = \cancel{-\alpha e^{-\alpha W}}\, e^{\alpha P^+}$$

$$\Rightarrow M_x(\alpha) = e^{\alpha P^+} \overset{\log}{\iff} \log M_x(\alpha) = \alpha P^+ \Rightarrow P^+ = \frac{1}{\alpha}\log M_x(\alpha)$$

⑥ according to the question we know that: $\alpha = 0.001$, $u(x) = -\alpha e^{-\alpha x}$
and $X \sim N(400, 25000)$, $Y \sim N(420, 20000)$ $P_x^+ > P_y^+$ find α?

for $X \Rightarrow \alpha = 0.001$, $\mu = 400$, $\sigma^2 = 25000$, $M_x(\alpha) = e^{\mu\alpha + \frac{\sigma^2\alpha^2}{2}} \Rightarrow \log(M_x(\alpha)) = \mu\alpha + \frac{\sigma^2\alpha^2}{2}$

for $Y \Rightarrow \alpha = 0.001$, $\mu = 420$, $\sigma^2 = 20000$, $M_y(\alpha) = e^{\mu\alpha + \frac{\sigma^2\alpha^2}{2}} \Rightarrow \log(M_x(\alpha)) = \mu\alpha + \frac{\sigma^2\alpha^2}{2}$

$\Rightarrow$ we know that $P^{+\text{or}-} = \frac{1}{\alpha}\log(M_x(\alpha))$ for exponential utility function.

$$P_x^- = \frac{1}{0.001}\left(400\times 0.001 + \frac{(25000)(0.001)^2}{2}\right) = 412.5$$

$\Rightarrow$ Y is greater than X.

$$P_y^- = \frac{1}{0.001}\left(420\times 0.001 + \frac{(20000)(0.001)^2}{2}\right) = 430$$

if want to find α when $\bar{P}_x > \bar{P}_y$ we have

$$\cancel{\tfrac{1}{\alpha}}\left(\mu_x\alpha + \frac{\sigma_x^2\alpha^2}{2}\right) > \cancel{\tfrac{1}{\alpha}}\left(\mu_y\alpha + \frac{\sigma_y^2\alpha^2}{2}\right) \iff \alpha\mu_x - \alpha\mu_y + \frac{\sigma_x^2\alpha^2}{2} - \frac{\sigma_y^2\alpha^2}{2} > 0$$

$$\iff \alpha(\mu_x-\mu_y) + \alpha^2\left(\frac{\sigma_x^2-\sigma_y^2}{2}\right) > 0 \overset{\div\alpha}{\iff} \alpha^2\left(\frac{\sigma_x^2-\sigma_y^2}{2}\right) > \alpha\cdot-(\mu_x-\mu_y)$$

$$\iff \frac{\alpha^2}{\alpha} > \frac{-(\mu_x-\mu_y)}{-\left(\frac{\sigma_x^2-\sigma_y^2}{2}\right)} \iff \alpha > \frac{-(\mu_x-\mu_y)}{\left(\frac{\sigma_x^2-\sigma_y^2}{2}\right)}$$

according to the question we know that $\mu_x = 400$, $\sigma_x^2 = 25000$, $\mu_y = 450$, $\sigma_y^2 = 20000$

$$\Rightarrow \alpha > \frac{-(400-420)}{\left(\frac{25000-20000}{2}\right)} \Rightarrow \alpha > \frac{+20}{2500} \Rightarrow \alpha > +0.012$$

Done

and we know that $\alpha > 0$ in exponential distribution So $\Rightarrow$ $\alpha > 0.008$

(7) we know that $\bar{P}_{[x]} = \frac{1}{\alpha}\log(M_x(\alpha))$, $\bar{P}_{[2x]} = \frac{1}{\alpha}\log(M_{2x}(\alpha))$

recursive Proof:

$$\bar{P}_{[2x]} > 2\bar{P}_{[x]} \iff \frac{1}{\alpha}\log(M_{2x}(\alpha)) > \frac{2}{\alpha}\log(M_x(\alpha))$$

$$\overset{\times\alpha}{\iff} \log(M_{2x}(\alpha)) > 2\log(M_x(\alpha)) \iff \log(M_{2x}(\alpha)) > \log(M_x(\alpha))^2$$

$$\iff \log E[e^{2\alpha x}] > \log E[e^{\alpha x}]^2 \overset{\exp}{\iff} E[e^{2\alpha x}] > E[e^{\alpha x}]^2 \overset{z = e^{\alpha x}}{\iff} E[z^2] > E[z]^2$$

$$\iff Var(z) > 0 \quad \checkmark$$

But i think we can solve it with Jensen inequality theorem

Subject Year: Month: Date:

(3) $0 \leq d \leq 1$ $\quad E[(S-d)_+] = \frac{1}{3}(1-d)^3$

we know that $\Pi_X'(d) = F_X(d) - 1 = \left(E[(S-d)_+]\right)' = \left(\frac{1}{3}(1-d)^3\right)'$

$\Rightarrow \Pi_X'(d) = (-1)(\frac{1}{3})(3)(1-d)^2 = -(1-d)^2 \Rightarrow F_X(d) = 1-(1-d)^2$

$F_X'(d) = 0-(-2)(1-d)(-1) = 2-2d = 2(1-d)$