Project Report: Numerical Solution of the 2D Energy Equation

Objective

The goal of this project was to numerically solve the 2D incompressible energy equation for a given velocity field. The project was divided into several parts, focusing on validating the flux and source integrals against their exact solutions, and then solving the energy equation using various numerical methods. The final part of the project involved analyzing the thermal development length in a channel flow.

Problem Description

The 2D energy equation for an incompressible flow is given by:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{Re \cdot Pr}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + S$$

where:

- T is the temperature,
- u and v are the velocity components in the x and y directions, respectively,
- Re is the Reynolds number,
- Pr is the Prandtl number,
- S is the source term.

The velocity field is fully developed and given by:

$$u(x, y) = 6\overline{u}y(1 - y), \quad v(x, y) = 0$$

The domain is a rectangular channel with a length of 5 units and a height of 1 unit. The boundary conditions are:

$$u(x,y) = 6\bar{u}y(1-y)$$
$$v(x,y) = 0$$
$$\begin{cases} \bar{u} = 3 \\ T_{up} = 1 \\ T_{bottom} = 0 \end{cases}$$

The parameters used are:

- Re=25Re=25,
- Pr=0.7Pr=0.7,
- Ec=0.1Ec=0.1 (Eckert number).

Project Structure

The project was divided into six main parts:

1. **Flux Integral Validation**: Implementation and validation of the flux integral against its exact solution.
2. **Source Term Validation**: Implementation and validation of the source term against its exact solution.
3. **Implicit Time-Stepping Method**: Implementation of an implicit time-stepping method using approximate factorization.
4. **Stability and Accuracy Analysis**: Analysis of the stability and accuracy of the implicit method.
5. **Realistic Problem**: Solving the energy equation for a realistic problem with different mesh sizes and estimating the thermal development length.
6. **Comparison of Explicit and Implicit Methods**: Comparison of explicit (RK-2 and RK-4) and implicit methods in terms of computational efficiency and accuracy.

Part 1: Flux Integral Validation

The first part of the project focused on implementing and validating the flux integral of the energy equation. The flux integral was discretized using second-order central differences and compared with the exact solution for a given temperature and velocity distribution.

Exact Solution for Flux Integral

The exact solution for the flux integral is given by:

$$\oint \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - \frac{1}{Re \cdot Pr}\nabla^2 T \right) ds =$$

$$u_0 T_0 \pi \cos(2\pi x) y \sin(\pi y) + v_0 T_0 \pi x \cos(\pi x) \cos(2\pi y) + \frac{2T_0\pi^2 \cos(\pi x) \sin(\pi y)}{Re \cdot Pr}$$

Results

The numerical flux integral was computed for different mesh sizes (10x10 and 20x20) and compared with the exact solution. The error norms (L1, L2, and L∞) were calculated, and the order of accuracy was found to be approximately 2, as expected for a second-order discretization.

Table 1: Error Norms for Flux Integral

Mesh Size	L1 Norm	L2 Norm	L∞ Norm
10x10	1.687010368	2.201034439	7.093929267
20x20	0.429178702	0.559481352	1.887223671

Part 2: Source Term Validation

The second part focused on implementing and validating the source term of the energy equation. The source term was discretized using second-order central differences and compared with the exact solution.

Exact Solution for Source Term

The exact solution for the source term is given by:

$$S = \frac{Ec}{Re}\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2\right]$$

Results

The numerical source term was computed for different mesh sizes (10x10 and 20x20) and compared with the exact solution. The error norms were calculated, and the order of accuracy was found to be approximately 2.

Table 2: Error Norms for Source Term

Mesh Size	L1 Norm	L2 Norm	L∞ Norm
10x10	0.021310198	0.028861542	0.075339503
20x20	0.005390472	0.007306837	0.019861354

Part 3: Implicit Time-Stepping Method

The third part involved implementing an implicit time-stepping method using approximate factorization to solve the energy equation. The method was based on the Alternating Direction Implicit (ADI) approach, which decomposes the problem into two tridiagonal systems that can be solved efficiently using the Thomas algorithm.

Methodology

The implicit method was implemented using the following steps:

1. Discretize the energy equation using second-order central differences.
2. Apply approximate factorization to decompose the system into two tridiagonal systems.
3. Solve the tridiagonal systems using the Thomas algorithm.
4. Update the temperature field iteratively until convergence.

Results

The implicit method was validated by comparing the numerical solution with the exact solution for a fully developed temperature profile. The error norms were calculated, and the order of accuracy was found to be approximately 2.

Table 3: Error Norms for Implicit Method

Mesh Size	L1 Norm	L2 Norm	L∞ Norm
25x10	0.006851232	0.009599684	0.019006652
50x20	0.001989532	0.002756128	0.005850452

Part 4: Stability and Accuracy Analysis

The fourth part focused on analyzing the stability and accuracy of the implicit method. The method was tested for different time steps, and the maximum stable time step was determined. The results showed that the method is stable for a wide range of time steps, and the accuracy is maintained as long as the time step is sufficiently small.

Part 5: Realistic Problem

The fifth part involved solving the energy equation for a realistic problem with different mesh sizes (10x25 and 20x50). The goal was to estimate the thermal development length in the channel. The heat flux along the bottom wall was computed, and the thermal development length was estimated to be between 30 to 35 meters.

Results

The heat flux along the bottom wall was plotted for different mesh sizes, and the results showed that the flow reaches thermal development after approximately 30 meters.

Part 6: Comparison of Explicit and Implicit Methods

The final part of the project involved comparing the performance of explicit (RK-2 and RK-4) and implicit methods. The explicit methods were found to be faster but required smaller time steps for stability. The implicit method, while more computationally expensive, allowed for larger time steps and was more stable.

Results

The CPU time for each method was compared, and the results showed that the explicit methods (RK-2 and RK-4) were significantly faster than the implicit method. However, the implicit method was more stable and allowed for larger time steps.

Table 4: CPU Time Comparison

Method	Mesh Size	CPU Time (seconds)
Implicit (Approximate Factorization)	25x10	42.24
Implicit (Approximate Factorization)	50x20	524.19
RK-2	25x10	1.60
RK-2	50x20	3.40
RK-4	25x10	2.65
RK-4	50x20	3.41

Conclusion

The project successfully implemented and validated the flux and source integrals, and solved the 2D energy equation using both explicit and implicit methods. The implicit method was found to be more stable but computationally expensive, while the explicit methods were faster but required smaller time steps. The thermal development length in the channel was estimated to be between 30 to 35 meters. The results demonstrated the importance of choosing the appropriate numerical method based on the problem requirements and computational constraints.

Future Work

- Explore parallel computing techniques to further reduce the computational cost of the implicit method.
- Investigate adaptive mesh refinement to improve accuracy in regions of interest without increasing the computational cost globally.
- Extend the analysis to more complex geometries and boundary conditions.

This report provides a detailed overview of the project, including the methods used, results obtained, and conclusions drawn. The project successfully demonstrated the trade-offs between accuracy, computational cost, and method efficiency in solving the 2D energy equation numerically.

Following titles are the titles of my code for the current project. I coded this problem in python in following sections:

1. Importing Required Libraries

- Load necessary Python libraries for numerical computation and visualization.

2. Flux Integral Validation

2.1 Numerical Discretization of the Flux Integral

- Discretization using second-order central differences.

2.2 Analytical Solution of the Flux Integral

- Derivation and implementation of the exact flux integral solution.

2.3 Error Norms for Flux Integral Approximation

- Compute L1,L2,L∞L_1, L_2, L_{\infty}L1,L2,L∞ norms to assess accuracy.

2.4 Determination of Convergence Order for Flux Integral

- Assess numerical order of accuracy by refining the mesh.

3. Source Term Validation

3.1 Numerical Discretization of the Source Term

- Central differencing scheme for source term evaluation.

3.2 Analytical Solution for the Source Term

- Derivation and implementation of the exact source term solution.

3.3 Error Norms for Source Term Approximation

- Compute accuracy metrics (L1,L2,L∞L_1, L_2, L_{\infty}L1,L2,L∞ norms).

4. Implicit Time-Stepping Method for Energy Equation

4.1 Implementation of the Thomas Algorithm

- Efficient tridiagonal solver for the implicit scheme.

4.2 Boundary Condition Definition for the Implicit Solver

- Apply Dirichlet boundary conditions for temperature.

4.3 **Implicit Discretization of the Energy Equation**

- Formulation using an Alternating Direction Implicit (ADI) approach.

4.4 **Analytical Solution for the Energy Equation (T_exact Definition)**

- Obtain the theoretical solution for validation.

4.5 **Numerical vs. Analytical Solution: Error Analysis**

- Compute deviation metrics between exact and numerical solutions.

4.6 **Order of Accuracy Assessment for the Implicit Method**

- Evaluate the method's convergence properties.

4.7 **Visualization: Temperature Contours and Profiles**

- Generate contour plots comparing numerical and analytical solutions.

5. Simulation of Developing Flow in a Rectangular Channel

5.1 **Boundary Condition Definition for Developing Flow**

- Implement velocity-dependent temperature boundary conditions.

5.2 **Incorporation of Realistic Boundary Conditions in the Implicit Solver**

- Modify the numerical scheme to accommodate developing flow.

5.3 **Determination of Thermal Entry Length via Wall Heat Flux Analysis**

- Plot the heat flux distribution along the channel walls.

6. Explicit Time-Stepping Methods for Energy Equation

6.1 **Runge-Kutta 2nd Order (RK2) Discretization**

- Implement the RK2 explicit method for time integration.

6.2 **Runge-Kutta 4th Order (RK4) Discretization**

- Implement the RK4 explicit method for improved accuracy.

6.3 **Thermal Entry Length Estimation Using Explicit Methods**

- Analyze wall heat flux profiles for entry length determination.

7. Computational Performance Analysis

7.1 **CPU Time Comparison: Implicit vs. Explicit Methods**

- Benchmark computational efficiency of Implicit, RK2, and RK4 methods.