# A low-level treatment of generalised quantifiers in categorical compositional distributional semantics

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**Abstract.** We show how one can formalise quantifiers in the categorical compositional distributional model of meaning. Our model is based on the generalised quantifier theory of Barwise and Cooper. We develop an abstract compact closed semantics and instantiate it in vector spaces and in relations. The former is an example for the distributional corpus-based models of language and the latter for the truth-theoretic ones.

## 1 Introduction

Vector space models of natural language are based on Firth's hypothesis that meanings of words can be deduced from the contexts in which they often occur [7]. One then fixes a context window, of for instance 5 words, and computes frequencies of how many times a words has occurred in this window with other words. These frequencies are often normalised to be better representatives of rare and very common words. These models have been applied to various language processing tasks, for instance thesauri construction [6]. Compositional distributional models of meaning extend the vector space models from words to sentences. The categorical such models [4, 1] do so by taking into account the grammatical structure of sentences and the vectors of the words in there. These models have proven successful in practical natural language tasks such as disambiguation, term/definition classification and phrase similarity, for example see [8, 9]. Nevertheless, it has been an open problem how to deal with meanings of logical words such as quantifiers and conjunctives. In this paper, we present preliminary work which aims to show how quantifiers can be deal with using the generalised quantifier theory of Barwise and Cooper [2].

According to generalised quantifier theory, the meaning of a sentence with a natural language quantifier Q such as 'Q Sbj Verb' is determined by first taking the intersection of the denotation of Sbj with the denotation of subjects of the Verb, then checking if the denotation of $Q(\text{Sbj})$ is an element of this set. The denotation of $Q$ is specified separately, for example, for $Q = \exists$, it is the set of non-empty subsets of the universe, for $Q = 2$ it is the set of subsets of the universe that have exactly two elements and so on. As a result, and for example, the meaning of a sentence "some men sleep" becomes true if the set of men who sleep is non empty.

In what follows, we work in the categorical compositional distributional model of [4]. We first present a brief preliminary account of compact closed categories and Frobenius algebras over them and review how vector spaces and relations provide instances. Then, we develop a compact closed categorical semantic for quantifiers, in terms of diagrams and morphisms of compact closed categories. We present two concrete interpretations for this abstract setting: relations and vector spaces. The former is the basis for a truth-theoretic model and the latter works for a corpus-base model of language.

Our future work includes formalising this rather low-level treatment in the setting of categorical logic, where quantifiers are proven to be adjoints to substitution.

Lack of much structure in vector spaces (and compact closed categories in general) and in particular lack of existence of pull-backs will be obvious obstacles. WHY IT IS HARD TO DO CATEGORICAL LOGIC IN VECTOR SPACES???

CONTRIBUTION: POSSIBLE TO DO GEN QUANT IN COMPACT CLOSED CATEGORIES with Frob on some objects

DIFFERENT INSTANTAITIONS TO GET DIFFERENT TYPES OF MEANING

LEAVE THE LOGIC FOR LATER work in a fragment without logical operations such as conjunction etc

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We also aim to experiment with this model on corpus-based datasets and tasks.

## 2 Preliminaries

### 2.1 Generalised Quantifier Theory in Natural Language

We briefly review the theory of generalised quantifiers in natural language [2]. The definitions and concepts are slightly revised from their original presentation so that they could fit easier within the categorical setting, introduced later on. Given a vocabulary $\Sigma$ and a set of lexical categories $S$, the words of the vocabulary are assigned lexical categories via the relation $\mathcal{X} \subseteq \Sigma \times S$, otherwise known as a lexicon or a dictionary. The expressions of the language are generated over these categories via a set of rules $\mathcal{R} \subseteq S \times S \times S$. We refer to a language so defined by the tuple $\mathcal{L}_\Sigma = (\mathcal{X}_S, \mathcal{R})$.

For the purpose of this paper, we consider the fragment of English generated over the following lexicon:

| Syntactic Category | Vocabulary |
|---|---|
| NP | John, Mary, something, $\cdots$ |
| N | cat, dog, man, $\cdots$ |
| VP | sneeze, sleep,$\cdots$ |
| V | love, kiss, $\cdots$ |
| Det | a, the, some, every, each, all, no, most, few, one, two, $\cdots$ |

with the following phrase structure description rules:

$$\begin{aligned} \text{S} &\to \text{NP VP} \\ \text{VP} &\to \text{V NP} \\ \text{NP} &\to \text{Det N} \end{aligned}$$

A model for this language is a pair $(U, [\![\ ]\!])$, where $U$ is a universal reference set and $[\![\ ]\!]$ is a function that assigns interpretations to the expressions of language. The interpretations of basic expressions are as follows:

$$\begin{aligned} np \in \text{NP} &\quad\Longrightarrow\quad [\![np]\!] \in U \\ n \in \text{N} &\quad\Longrightarrow\quad [\![n]\!] \subseteq U \\ vp \in \text{VP} &\quad\Longrightarrow\quad [\![vp]\!] \subseteq U \\ v \in \text{V} &\quad\Longrightarrow\quad [\![v]\!] \subseteq U \times U \\ d \in \text{Det} &\quad\Longrightarrow\quad [\![d]\!] \colon \mathcal{P}(U) \to \mathcal{P}\mathcal{P}(U) \end{aligned}$$

Noun phrases are interpreted as elements of the reference set, nouns and verb phrases as unary relations over it, and verbs as binary relations over it. Determiners are interpreted as *generalised quantifier*; a generalised quantifier $d$ is a map that assigns to each $A \subseteq U$, a family of subsets of $U$; examples are as follows:

$$
\begin{aligned}
[\![some]\!](A) &= \{X \subseteq U \mid X \cap A \neq \emptyset\} \\
[\![\text{E}very]\!](A) &= \{X \subseteq U \mid A \subseteq X\} \\
[\![\text{no}]\!](A) &= \{X \subseteq U \mid A \cap X = \emptyset\} \\
[\![n]\!](A) &= \{X \subseteq U \mid \mid X \cap A \mid = n\} \\
[\![\text{m}ost]\!](A) &= \{X \subseteq U \mid X \text{ contains most } A\text{'s}\} \\
[\![\text{f}ew]\!](A) &= \{X \subseteq U \mid X \text{ contains few } A\text{'s}\}
\end{aligned}
$$

Note that the interpretations of determiners are not quantifiers (in the sense of first order logic) yet. They become quantifiers when applied to a set. For instance, $[\![d]\!]$ is not a quantifier, but $[\![d]\!](A)$ is. The interpretations of determiners satisfy a property referred to by *living on* or *conservativity*. This property says that a set $X$ belongs to $[\![d]\!](A)$ iff $(X \cap A) \in [\![d]\!](A)$. This property is sometimes described by saying that 'the quantifier $[\![d]\!](A)$ lives on $A$'.
The interpretations of expressions generated by the rules is obtained by induction as follows:

- a noun phrase 'NP' generated by the rule 'NP $\rightarrow$ Det N'

$$
\begin{array}{lcccl}
[\![\text{Det N}]\!] & = & [\![d]\!]([\![n]\!]) & \text{where} & X \in [\![d]\!]([\![n]\!]) \textbf{ iff } (X \cap [\![n]\!]) \in [\![d]\!]([\![n]\!]) \\
 & & & \text{for} & d \in Det, n \in N
\end{array}
$$

- a verb phrase 'VP' generated by the rule 'VP $\rightarrow$ V NP'

$$
[\![\text{V NP}]\!] = \{x \mid [\![np]\!] \left(\{y \mid [\![v(x,y)]\!]\}\right)\} \qquad \text{for} \quad np \in NP, v \in V
$$

- a sentence 'S' generated by the rule 'S $\rightarrow$ NP VP'

$$
[\![\text{N}P\ VP]\!] = [\![vp]\!]([\![np]\!]) \qquad \text{for} \quad np \in NP, vp \in VP
$$

The *meaning* of a sentence is said to be *true* iff its semantic interpretation is non-empty and *false* otherwise that is we have:

**Definition 1.** *The meaning of a sentence in generalised quantifier theory is true iff* $[\![vp]\!]([\![np]\!]) = 1$ *and false otherwise.*

As an example, consider the meaning of a sentence with a quantified phrase at its subject position. This sentence has the form 'Det N VP' and its meaning is defined as follows:

$$
[\![\text{Det N VP}]\!] \;=\; \begin{cases} 1 & [\![vp]\!] \in [\![\text{Det N}]\!] \quad \text{for } vp \in \text{VP} \\ 0 & \text{other wise} \end{cases}
$$

By the *living on* property, the meaning of this sentence is true whenever $[\![vp]\!] \cap [\![n]\!] \in [\![\text{Det N}]\!]$. For instance, meaning of 'some cats sneeze' is true whenever $[\![\text{s}neeze]\!] \cap [\![\text{m}en]\!] \in [\![\text{some men}]\!]$. That is, whenever the set of things that sneeze and are men is a non-empty set. Similarly, the meaning of the sentence 'five men sneeze' is true whenever the set of things that sneeze and are men has five elements and so on.

As another example, consider the meaning of a sentence with a quantified phrase at its object position. This sentence has the form 'NP V Det N' and its meaning is defined as follows:

$$
[\![\text{NP V Det N}]\!] \;=\; \begin{cases} 1 & [\![np]\!] \in \{x \mid \{y \mid [\![v(x,y)]\!]\} \in [\![\text{Det N}]\!]\} \quad \text{for } v \in \text{V}, np \in \text{NP} \\ 0 & \text{otherwise} \end{cases}
$$

That is, the meaning of this sentence is true whenever $\{y \mid [\![v([\![np]\!], y)]\!]\} \in [\![\text{Det N}]\!]$, which by the *living on* property is the case whenever $\{y \mid [\![v([\![np]\!], y)]\!]\} \cap [\![n]\!] \in [\![\text{Det N}]\!]$, for $n$ in N. For instance, meaning of 'John kissed some cats' is true whenever $\{y \mid [\![\text{kiss}([\![John]\!], y)]\!]\} \cap [\![cats]\!] \in [\![\text{some cats}]\!]$. That is, whenever, the set of things that are kissed by John and are cats is a non-empty set. Similarly, the sentence 'John kissed five cats' is true whenever the set of things that are kissed by John and are cats has five elements and so on.

### 2.2 Type-Logical Grammar

In this section we show how to syntactically analyse the sentences of the previous fragment in a pregroup type logic.

A pregroup algebra $P = (P, \leq, \cdot, (-)^r, (-)^l)$ is a partially ordered monoid where every element has a left and a right adjoint, that is, for every element $p \in P$, there are two elements $p^l, p^r \in P$, referred to as its left and right adjoint, and these satisfy the following four inequalities

$$p \cdot p^r \leq 1 \leq p^r \cdot p \qquad p^l \cdot p \leq 1 \leq p \cdot p^l$$

A pregroup grammar over a vocabulary $\Sigma$ is a pregroup algebra $P_{\mathcal{B}}$, freely generated over a set of atomic logical types $\mathcal{B}$, together with a type-dictionary $\beta \subseteq \Sigma \times P_{\mathcal{B}}$, which assigns to each word of the vocabulary a type from the pregroup. We denote this grammar by the tuple $\mathcal{G}_{\Sigma} = (\beta, P_{\mathcal{B}})$.

For the vocabulary and the fragment of language described in the previous section, the set of atomic types are $\{m, n, s\} \in \mathcal{B}$, where $m$ is a noun phrase, $n$ is a noun, and $s$ is a declarative sentence. The dictionary is as follows:

| Words | Logical Types |
|---|---|
| John, Mary, something, $\cdots$ | $m$ |
| cat, dog, man, $\cdots$ | $n$ |
| sneeze, sleep,$\cdots$ | $m^r \cdot s$ |
| love, kiss, $\cdots$ | $m^r \cdot s \cdot m^l$ |
| a, the, some, every, each, all, no, most, few, one, two, $\cdots$ | $m \cdot n^l$ |

We assign atomic types $m$ and $n$ to the words in the lexical categories NP and N. The type assigned to the words in VP is $m^r \cdot s$, this means that words in the lexical category VP input an argument of type $m$ and they have to be to the right of that argument, then output a sentence of type $s$. The type assigned to the words in the lexicon item V is $m^r \cdot s \cdot m^l$, this means that these words input two arguments of type $m$ and they have to be to the right of one and to the left of the other, then output a sentence. Finally, the type assigned to the words in the lexicon item Det is $m \cdot n^l$, which means that these words input an argument of the type $n$ and they have to be to the left of that argument, then output a phrase of type $m$.

The grammatical reductions of the language are modelled by the partial ordering of the pregroup grammar. As an example, consider a sentence with a quantified phrase in its subject position, e.g. 'some cats sneeze', the grammatical reduction of this sentence is as follows:

$$\begin{array}{lll} \text{some} & \text{cats} & \text{sneeze} \\ (m \cdot n^l) & \cdot n & \cdot (m^r \cdot s) \leq m \cdot 1 \cdot (m^r \cdot s) = m \cdot (m^r \cdot s) \leq 1 \cdot s = s \end{array}$$

Here, first 'some' inputs 'cats' and output a noun phrase of type $m$, then the verb inputs $m$ and outputs a sentence. As another example, consider a sentence with a quantified phrase in its object position, e.g. 'John kissed some cats' the grammatical reduction of this sentence is as follows:

$$\begin{array}{ccccccc} \text{John} & \text{kissed} & \text{some} & \text{cats} & & \\ m & \cdot(m^r \cdot s \cdot m^l) & \cdot(m \cdot n^l) & \cdot n & \leq & 1 \cdot (s \cdot m^l) \cdot m \cdot 1 = (s \cdot m^l) \cdot m \leq s \cdot 1 = s \end{array}$$

Here, again first 'some' inputs 'cats' and outputs a noun phrase, at the same time the verb inputs 'John' and outputs a verb phrase of type $s \cdot m^l$, which then inputs the $m$ from the phrase 'some cats' and outputs a sentence.

Whereas the elements of lexical categories are syntactic, the types of a pregroup algebra have logical information in them. This information comes from the rules of the grammar of the language; in a lexical approach such information is encoded in the rules of the language. There are standard methods that transfer the lexical categories and rules of a language to logical type [**?**]. For the fragment of the language considered in this paper, we suffice to giving the following translation $t\colon S \to P_{\mathcal{B}}$ between the lexical categories and the pregroup grammar types:

| Syntactic Category | Logical Type |
|---|---|
| NP | $m$ |
| N | $n$ |
| VP | $m^r \cdot s$ |
| V | $m^r \cdot s \cdot m^l$ |
| Det | $m \cdot n^l$ |

### 2.3 Category Theoretic and Diagrammatic Definitions and Axioms

This subsection briefly reviews compact closed categories and Frobenius algebras. For a formal presentation, see [10, 11]. A compact closed category, $\mathcal{C}$, has objects $A, B$; morphisms $f\colon A \to B$; a monoidal tensor $A \otimes B$ that has a unit $I$, that is we have $A \otimes I \cong I \otimes A \cong A$. Furthermore, for each object $A$ there are two objects $A^r$ and $A^l$ and the following morphisms:

$$A \otimes A^r \xrightarrow{\epsilon^r_A} I \xrightarrow{\eta^r_A} A^r \otimes A \qquad A^l \otimes A \xrightarrow{\epsilon^l_A} I \xrightarrow{\eta^l_A} A \otimes A^l$$

These morphisms satisfy the following equalities, sometimes referred to as the *yanking* equalities, where $1_A$ is the identity morphism on object $A$:

$$(1_A \otimes \epsilon^l_A) \circ (\eta^l_A \otimes 1_A) = 1_A \qquad (\epsilon^r_A \otimes 1_A) \circ (1_A \otimes \eta^r_A) = 1_A$$
$$(\epsilon^l_A \otimes 1_A) \circ (1_{A^l} \otimes \eta^l_A) = 1_{A^l} \qquad (1_{A^r} \otimes \epsilon^r_A) \circ (\eta^r_A \otimes 1_{A^r}) = 1_{A^r}$$

These express the fact the $A^l$ and $A^r$ are the left and right adjoints, respectively, of $A$ in the 1-object bicategory whose 1-cells are objects of $\mathcal{C}$.

A Frobenius algebra in a monoidal category $(\mathcal{C}, \otimes, I)$ is a tuple $(X, \delta, \iota, \mu, \zeta)$ where, for $X$ an object of $\mathcal{C}$, the triple $(X, \delta, \iota)$ is an internal comonoid; i.e. the following are coassociative and counital morphisms of $\mathcal{C}$:

$$\delta\colon X \to X \otimes X \qquad\qquad \iota\colon X \to I$$

Moreover $(X, \mu, \zeta)$ is an internal monoid; i.e. the following are associative and unital morphisms:

$$\mu\colon X \otimes X \to X \qquad\qquad \zeta\colon I \to X$$

And finally the $\delta$ and $\mu$ morphisms satisfy the following *Frobenius condition*:

$$(\mu \otimes 1_X) \circ (1_X \otimes \delta) \;=\; \delta \circ \mu \;=\; (1_X \otimes \mu) \circ (\delta \otimes 1_X)$$

Informally, the comultiplication $\delta$ dispatches the information contained in one object into two objects, and the multiplication $\mu$ unifies the information of two objects into one.

**Pregroup Algebras.** A pregroup algebra $P = (P, \leq, \cdot, (-)^l, (-)^r)$ is a compact closed category whose objects are the elements of the set $p \in P$ are the objects of the category and the partial ordering between the elements are the morphisms. That is, for $p, q \in P$, we have that $p \to q$ is a morphism of the category iff $p \leq q$ in the partial order. The tensor product of the category is the monoid multiplication, whose unit is 1, and the adjoints of objects are the adjoints of the elements of the algebra. The epsilon and eta morpshism are thus as follows:

$$p \cdot p^r \xrightarrow{\epsilon_p^r} 1 \xrightarrow{\eta_p^r} p^r \cdot p \qquad p^l \cdot p \xrightarrow{\epsilon_p^l} 1 \xrightarrow{\eta_p^l} p \cdot p^l$$

The yanking equalities directly follow from the preroup inequalities on the adjoints. A pregroup with Frobenius structure on it becomes degenerate. To see this, suppose we have such an algebra on the object $p$ of such a pregroup. Then the unit morphism of the internal comonoid of this algebra becomes the parietal ordering $\iota \colon p \leq 1$; taking the right adjoints of both sides of this inequality will yield $1 = 1^r \leq p^r$, and by the multiplying both sides of this with $p$ we will obtain $p \leq p \cdot p^r$, which by adjunction results in $p \leq p \cdot p^r \leq 1$, hence we have $p \leq 1$ and also $1 \leq p$, thus $p$ must be equal to 1. That is, assuming that we have a Frobenius algebra on an object will mean that that object is 1.

**Finite Dimensional Vector Spaces.** These structures together with linear maps form a compact closed category, which we refer to as $\mathrm{FdVect}$. Finite dimensional vector spaces $V, W$ are objects of this category; linear maps $f \colon V \to W$ are its morphisms with composition being the composition of linear maps. The tensor product $V \otimes W$ is the linear algebraic tensor product, whose unit is the scalar field of vector spaces; in our case this is the field of reals $\mathbb{R}$. Here, there is a naturual isomorphism $V \otimes W \cong W \otimes V$. As a result of the symmetry of the tensor, the two adjoints reduce to one and we obtain the isomorphism $V^l \cong V^r \cong V^*$, where $V^*$ is the dual space of $V$. When the basis vectors of the vector spaces are fixed, it is further the case that we have $V^* \cong V$.

Given a basis $\{r_i\}_i$ for a vector space $V$, the epsilon maps are given by the inner product extended by linearity; i.e. we have:

$$\epsilon^l = \epsilon^r \colon V \otimes V \to \mathbb{R} \quad \text{given by} \quad \sum_{ij} c_{ij}\, \psi_i \otimes \phi_j \quad \mapsto \quad \sum_{ij} c_{ij} \langle \psi_i \mid \phi_j \rangle$$

Similarly, eta maps are defined as follows:

$$\eta^l = \eta^r \colon \mathbb{R} \to V \otimes V \quad \text{given by} \quad 1 \;\mapsto\; \sum_i r_i \otimes r_i$$

Any vector space $V$ with a fixed basis $\{\overrightarrow{v_i}\}_i$ has a Frobenius algebra over it, explicitly given as follows, where $\delta_{ij}$ is the Kronecker delta.

$$\begin{aligned}
\delta \colon V \to V \otimes V & \quad \text{given by} \quad \overrightarrow{v_i} \mapsto \overrightarrow{v_i} \otimes \overrightarrow{v_i} \\
\mu \colon V \otimes V \to V & \quad \text{given by} \quad \overrightarrow{v_i} \otimes \overrightarrow{v_j} \mapsto \delta_{ij} \overrightarrow{v_i} \\
\iota \colon V \to \mathbb{R} & \quad \text{given by} \quad \overrightarrow{v_i} \mapsto 1 \\
\zeta \colon \mathbb{R} \to V & \quad \text{given by} \quad 1 \mapsto \sum_i \overrightarrow{v_i}
\end{aligned}$$

**Relations**. Another important example of a compact closed category is $\mathrm{Rel}$, the cateogry of sets and relations. Here, $\otimes$ is cartesian product with the singleton set as its unit $I = \{\star\}$, and $^*$ is identity on objects. Closure reduces to the fact that a relation between sets $A \times B$ and $C$ is equivalently a relation between $A$ and $B \times C$. Given a set $S$ with elements $s_i, s_j \in S$, the epsilon and eta maps are given as follows:

$$
\begin{aligned}
\epsilon^l = \epsilon^r &: S \times S \to \{\star\} \quad \text{given by} \quad \{((s_i, s_j), \star) \mid s_i, s_j \in S, s_i = s_j\} \\
\eta^l = \eta^r &: \{\star\} \to S \times S \quad \text{given by} \quad \{(\star, (s_i, s_j)) \mid s_i, s_j \in S, s_i = s_j\}
\end{aligned}
$$

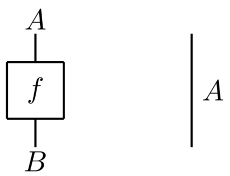
Every object in $\mathrm{Rel}$ has a Frobenius algebra over it given by the diagonal and codiagonal relations, as described below:

$$
\begin{aligned}
\delta &: S \to S \times S \quad \text{given by} \quad \{(s_i, (s_j, s_k)) \mid s_i, s_j, s_k \in S, s_i = s_j = s_k\} \\
\mu &: S \times S \to S \quad \text{given by} \quad \{((s_i, s_j), s_k) \mid s_i, s_j, s_k \in S, s_i = s_j = s_k\} \\
\iota &: S \to \{\star\} \quad \text{given by} \quad \{(s_i, \star) \mid s_i \in S\} \\
\zeta &: \{\star\} \to S \quad \text{given by} \quad \{(\star, s_i) \mid s_i \in S\}
\end{aligned}
$$

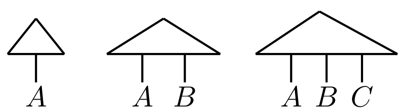
For the details of verifying that for each of the two examples above, the corresponding conditions hold see [3].

### 2.4 String Diagrams

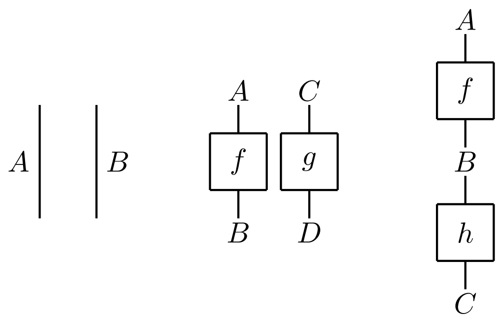
The framework of compact closed categories and Frobenius algebras comes with a complete diagrammatic calculus that visualises derivations, and which also simplifies the categorical and vector space computations. Morphisms are depicted by boxes and objects by lines, representing their identity morphisms. For instance a morphism $f\colon A \to B$, and an object $A$ with the identity arrow $1_A\colon A \to A$, are depicted as follows:



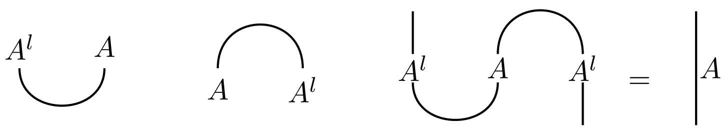

Morphisms from $I$ to objects are depicted by triangles with strings emanating from them. In concrete categories, these morphisms represent elements within the objects. For instance, an element $a$ in $A$ is represented by the morphism $a : I \to A$ and depicted by a triangle with one string emanating from it. The number of strings of such triangles depict the tensor rank of the element; for instance, the diagrams for $a \in A, a' \in A \otimes B$, and $a'' \in A \otimes B \otimes C$ are as follows:




The tensor products of the objects and morphisms are depicted by juxtaposing their diagrams side by side, whereas compositions of morphisms are depicted by putting one on top of the other; for instance the object $A \otimes B$, and the morphisms $f \otimes g$ and $f \circ h$, for $f\colon A \to B, g\colon C \to D$, and $h\colon B \to C$, are depicted as follows:

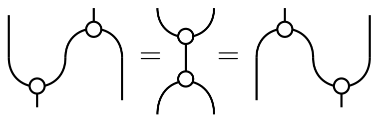



The $\epsilon$ maps are depicted by cups, $\eta$ maps by caps, and yanking by their composition and straightening of the strings. For instance, the diagrams for $\epsilon^l\colon A^l \otimes A \to I$, $\eta\colon I \to A \otimes A^l$ and $(\epsilon^l \otimes 1_A) \circ (1_A \otimes \eta^l) = 1_A$ are as follows:

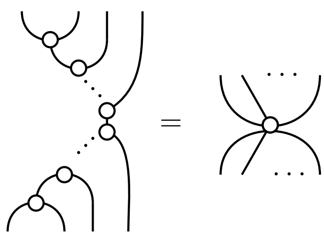



As for Frobenius algebras, the diagrams for the monoid and comonoid morphisms are as follows:


with the Frobenius condition being depicted as:




The defining axioms guarantee that any picture depicting a Frobenius computation can be reduced to a normal form that only depends on the number of input and output strings of the nodes, independent of the topology. These normal forms can be simplified to so-called ‘spiders’:

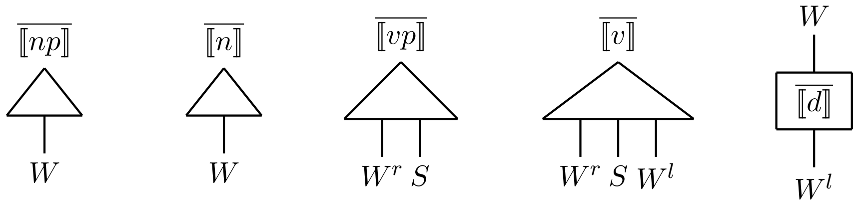



## 3 Abstract Compact Closed Semantics

An abstract compact closed categorical model for the language $\mathcal{L}_\Sigma = (\mathcal{X}_S, \mathcal{R})$ is a tuple $(\mathcal{C}_{W,S}, \overline{[\![\;]\!]})$ where $\mathcal{C}$ is a compact closed category with two distinguished objects $W$ and $S$ where $W$ has a Frobenius algebra on it and $\overline{[\![\;]\!]}$ is a function that assigns morphisms from this category to expression of the language. The interpretations of the basic expressions are as follows:

$$
\begin{aligned}
(np, \mathrm{NP}) \in \mathcal{X}_S &\quad \Longrightarrow \quad \overline{[\![np]\!]} \colon I \to W \\
(n, \mathrm{N}) \in \mathcal{X}_S &\quad \Longrightarrow \quad \overline{[\![n]\!]} \colon I \to W \\
(vp, \mathrm{VP}) \in \mathcal{X}_S &\quad \Longrightarrow \quad \overline{[\![vp]\!]} \colon I \to W^r \otimes S \\
(v, \mathrm{V}) \in \mathcal{X}_S &\quad \Longrightarrow \quad \overline{[\![v]\!]} \colon I \to W^r \otimes S \otimes W^l \\
(d, \mathrm{Det}) \in \mathcal{X}_S &\quad \Longrightarrow \quad \overline{[\![d]\!]} \colon W \to W
\end{aligned}
$$

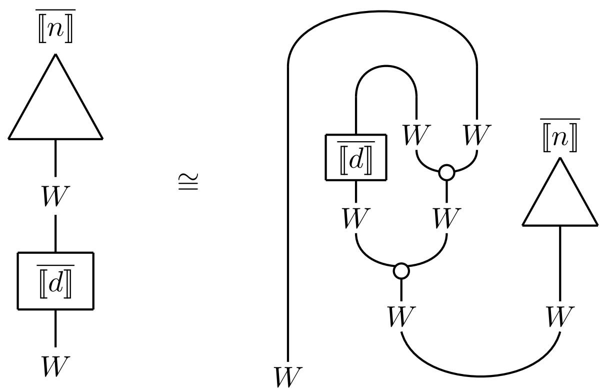
The diagrammatic semantics of the above interpretations are as follows:



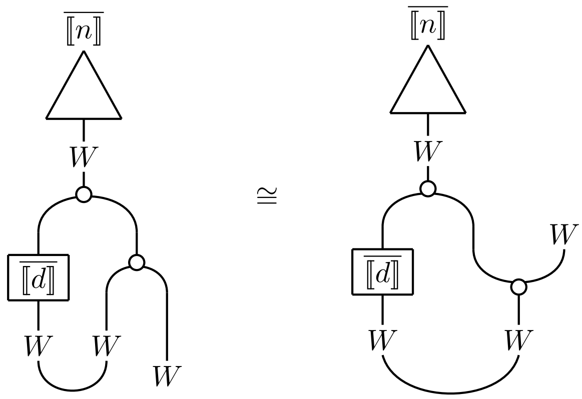

Noun phrases and nouns are elements within the object $W$; the former is a singleton element and the latter not necessarily so. The abstract language and its diagrammatic representation do not have means of distinguishing the two; when we instantiate these to concrete categories the difference between them becomes evident. Verb phrases are elements within the object $W^r \otimes S$; the intuition behind this representation is that in a compact closed category we have that $W^r \otimes S \cong W \to S$, where $W^r \to S = Hom(W, S)$ is an internal hom object of the category, coming from its monoidal closeness. Hence, we are modelling verb phrases as morphisms with input $W$ and output $S$. Similarly, verbs are elements within the object $W^r \otimes S \otimes W^r$, equivalent to morphisms $W \otimes W \to S$ with pairs of input from $W$ and output $S$.

The interpretations of the expressions generated by the rules are defined by induction as follows:

- $\overline{[\![\text{Det N}]\!]} := \overline{[\![d]\!]} \circ \overline{[\![n]\!]}$, where $\overline{[\![d]\!]} \circ \overline{[\![n]\!]} \cong (\epsilon_W \otimes 1_W) \circ (\overline{[\![d]\!]} \otimes \mu_W) \circ \sigma_W \circ \overline{[\![n]\!]}$. Diagrammatically, we have:

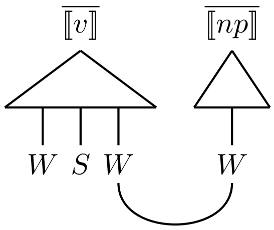



This condition expresses the ‘living on property. Intuitively, by using the axioms of compact closed categories and Frobenius algebras, the right hand side diagram above simplifies to the following left hand side diagram below, which in turn is equivalent to the right hand side diagram below:

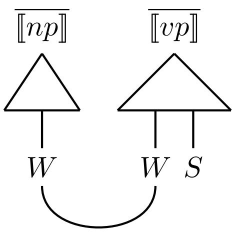


According to the diagram on the right hand side, semantics of $\overline{[\![d]\!]} \circ \overline{[\![n]\!]}$ is an element of $W$ which is equivalent to the element obtained by making a copy (via the Frobenius map $\delta_W$) of the noun in $W$, applying the determiner to one copy and taking the intersection of the other copy (via the Frobenius map $\mu_W$) with $W$.

– $\overline{[\![\text{V NP}]\!]} := (1_W \otimes 1_S \otimes \epsilon_W) \circ (\overline{[\![v]\!]} \otimes \overline{[\![np]\!]})$. Diagrammatically, we have:

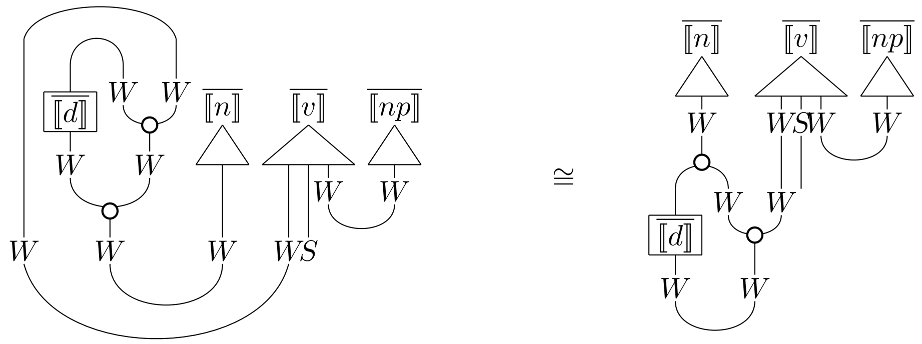


– $\overline{[\![\text{NP VP}]\!]} := (\epsilon_W \otimes 1_S) \circ (\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]})$. Diagrammatically, we have:



In the abstract setting the meaning and semantic interpretation of sentences are the same: they both are represented by the object $S$. In the next section we show how to instantiate this setting to a concrete relational setting where meaning can be defined to be true or false. Here, we provide semantics interpretations for sentences with a quantified phrase at their subject and object position.

The interpretation of a sentence with a quantified phrase in subject position and its simplified forms are as follows:

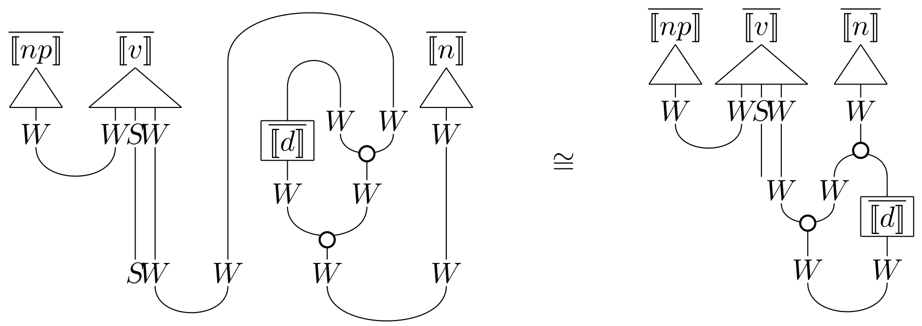


The symbolic representation of the simplified diagram above is as follows:

$$(\epsilon_W \otimes 1_S) \circ (\overline{[\![d]\!]} \otimes \mu_W \otimes 1_S) \circ (\delta_W \otimes 1_{W \otimes S} \otimes \epsilon_W) \circ (\overline{[\![n]\!]} \otimes \overline{[\![v]\!]} \otimes \overline{[\![np]\!]})$$

Intuitively, the determiner first makes a copy of the subject (via the Frobenius $\delta$ map), so now we have two copies of the subject. One of these is being unified with the subject argument of the verb (via the Frobenius $\mu$ map). In set-theoretic terms this is the intersection of the interpretations of subject and subjects-of-verb. The other copy is being inputted to the determiner map $\overline{[\![d]\!]}$ and will produce a modified noun based on the meaning of the determiner. The last step is the application of the unification to the output of $\overline{[\![d]\!]}$. Set theoretically, this step will decide whether the intersection of the subject-of-verb and the noun belongs to the interpretation of the quantified noun.

A sentence with a quantified phrase in object position is generated by the rule 'NP V Det N'. Its diagrammatic meaning and its simplified form are as follows:

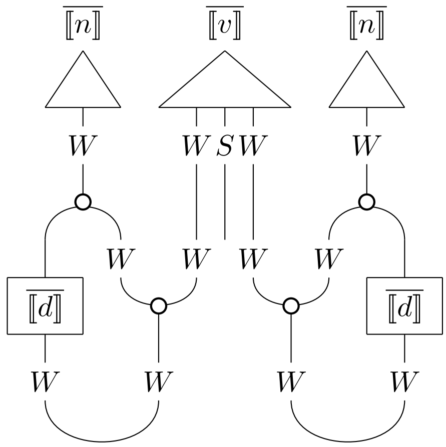


The symbolic representation of the simplified diagram above is as follows:

$$(1_S \otimes \epsilon_W) \circ (1_S \otimes \mu_W \otimes \overline{[\![d]\!]}) \circ (\epsilon_W \otimes 1_{S \otimes W} \otimes \delta_W) \circ (\overline{[\![np]\!]} \otimes \overline{[\![v]\!]} \otimes \overline{[\![n]\!]})$$

Intuitively, the determiner first makes a copy of the object (via the Frobenius $\delta$ map), so now we have two copies of the object. One of these is being unified with the object argument of the verb (via the Frobenius $\mu$ map). In set-theoretic terms this is the intersection of the interpretations of object and objects-of-verb. The other copy is being inputted to the determiner map $Det$ and will produce a modified noun based on the meaning of the determiner. The last step is the application of the unification to the output of $Det$. Set theoretically, this step will decide whether the intersection of the object of the verb and the noun belongs to the interpretation of the quantified noun.

Putting the two cases together, the semantic interpretation of a sentence with quantified phrases both at subject and at an object position has the following simplified form:




The symbolic representation of the above diagram is as follows:

$$(\epsilon_W \otimes 1_S \otimes \epsilon_W) \circ (\overline{[\![d]\!]} \otimes \otimes \mu_W \otimes 1_S \otimes \mu_W \otimes \overline{[\![d]\!]}) \circ (\delta_W \otimes 1_{W \otimes S \otimes W} \otimes \delta_W) \circ (\overline{[\![n]\!]} \otimes \overline{[\![v]\!]} \otimes \overline{[\![n]\!]})$$

## 4 Truth Theoretic Interpretation

Given the set-theoretical model $(U, [\![\ ]\!])$ of a language $\mathcal{L}_\Sigma$, a concrete relational instantiation of the abstract compact closed categorical interpretation is provided by the tuple $(\mathcal{C}_{\mathcal{PP}(U),\{\star\}}, \overline{[\![\ ]\!]})$, defined as follows:

– For a word with a lexical category N,NP, and VP, that is, for $s \in \{$N, NP, VP$\}$ and any $w \in \Sigma$ such that $(w, s) \in \mathcal{X}_\Sigma$, we have
$$\overline{[\![w]\!]} \in \mathcal{PP}(U)$$
represented by the morphism $\{\star\} \to \mathcal{PP}(U)$.
– For words with lexical category V, we have
$$\overline{[\![w]\!]} \in \mathcal{PP}(U) \times \mathcal{PP}(U)$$
Since in $Rel$, the tensor product is the cartesian product, the above is an element of $\mathcal{PP}(U) \otimes \mathcal{PP}(U)$, represented by the morphism $\{\star\} \to \mathcal{PP}(U) \otimes \mathcal{PP}(U)$.
– For a word with the lexical category Det, that is a $d \in \Sigma$ such that $(d, Det) \in \mathcal{X}$ and a $w \in \Sigma$ such that $(w,$ N$) \in \mathcal{X}$, we have
$$\overline{[\![d]\!]}(\overline{[\![w]\!]}) \subseteq \mathcal{PP}(U) \times \mathcal{PP}(U)$$
This is a relation on the powerset of powerset of $U$; it encodes the generalised quantifier map $[\![d]\!]$ in the form of a relation.

In this instantiation, the sentence space is the unit of the tensor, hence meaning of a sentence with a quantified phrase at its subject position simplifies as follows:

$$\epsilon_{\mathcal{PP}(U)}(\overline{[\![d]\!]} \otimes \mu_{\mathcal{PP}(U)})(\delta_{\mathcal{PP}(U)} \otimes 1_{\mathcal{PP}(U)} \otimes \epsilon_{\mathcal{PP}(U)})(\overline{[\![n]\!]} \otimes \overline{[\![v]\!]} \otimes \overline{[\![np]\!]})$$

**Discussion about the choice of $\mathcal{PP}(U)$ as the atomic object**. The reason one should not take $U$ to be the atomic object is that then one has to work with morphisms of the type $I \to U$, which denote elements of $U$, whereas meanings of words are subsets of $U$. The choice of $\mathcal{P}(U)$ fails because although now we have morphisms of the type $I \to \mathcal{P}(U)$, which do denote subsets of $U$, the application of the Frobenius $\mu\colon \mathcal{P}(U) \times \mathcal{P}(U)$ on them will yield unwanted results, for instance for two subsets $\{m_1, m_2\}$ and $\{m_2, m_3\}$, one wants $\mu$ to return their intersection that is $\{m_3\}$, whereas here it will return the empty set. The choice of $\mathcal{PP}(U)$ is the right choice here, as shown below in the examples and corresponding propositions.

**Example (I): Intransitive Verb.** As a truth-theoretic example, suppose $U = \{m_1, m_2, c_1\}$, from which we have two male individuals $m_1, m_2$ and a cat individual $c_1$. Suppose further that the verb 'sneeze' applies to individuals $m_1$ and $c_1$. Consider the following embedding:

$$\overline{[\![\text{men}]\!]} =\downarrow_{\neq\emptyset} \{m_1, m_2\} \qquad \overline{[\![\text{cat}]\!]} =\downarrow_{\neq\emptyset} \{c_1\} \qquad \overline{[\![\text{sneeze}]\!]} =\downarrow_{\neq\emptyset} \{m_1, c_1\}$$

So we have:

$$\overline{[\![\text{men}]\!]} = \{\{m_1\}, \{m_2\}, \{m_1, m_2\}\} \qquad \overline{[\![\text{cat}]\!]} = \{\{c_1\}\} \qquad \overline{[\![\text{sneeze}]\!]} = \{\{m_1\}, \{c_1\}, \{m_1, c_1\}\}$$

For quantifiers, we set:

$$\overline{[\![d]\!]}(\overline{[\![w]\!]}) := [\![d]\!]([\![w]\!])$$

As examples of quantified phrases we have:

$$Some\Big([\![\text{men}]\!]\Big) = \{\{m_1\}, \{m_2\}, \{m_1, m_2\}\} \qquad All\Big([\![\text{man}]\!]\Big) = \{\{m_1, m_2\}\} \qquad No\Big([\![\text{man}]\!]\Big) = \{\emptyset\}$$

The goal is to compute the meaning of "some men sneeze". In the first step of computation we obtain (the subscripts are dropped, they are always $\mathcal{PP}(U)$):

$$(\delta \otimes 1)\Big(\overline{[\![\text{men}]\!]} \otimes \overline{[\![\text{sneeze}]\!]}\Big) =$$
$$\{(\{m_1\}, \{m_1\}), (\{m_2\}, \{m_2\}), (\{m_1, m_2\}, \{m_1, m_2\})\} \otimes \{\{m_1\}, \{c_1\}, \{m_1, c_1\}\}$$

In the second step, we obtain:

$$\Big(\overline{[\![Some]\!]} \otimes \mu\Big)\Big(\{(\{m_1\}, \{m_1\}), (\{m_2\}, \{m_2\}), (\{m_1, m_2\}, \{m_1, m_2\})\} \otimes \{\{m_1\}, \{c_1\}, \{m_1, c_1\}\}\Big) =$$
$$\overline{[\![Some]\!]}\Big(\{\{m_1\}, \{m_2\}, \{m_1, m_2\}\}\Big) \otimes \mu\Big(\{\{m_1\}, \{m_2\}, \{m_1, m_2\}\} \otimes \{\{m_1\}, \{c_1\}, \{m_1, c_1\}\}\Big) =$$
$$[\![Some]\!]\Big(\{m_1, m_2\}\Big) \otimes \{\{m_1\}\} =$$
$$\{\{m_1\}, \{m_2\}, \{m_1, m_2\}\} \otimes \{\{m_1\}\}$$

In the last step, we obtain the following:

$$\epsilon\Big(\{\{m_1\}, \{m_2\}, \{m_1, m_2\}\} \otimes \{\{m_1\}\}\Big) = \{\star\}$$

Hence, the meaning of the sentence is true. For the sentence "all men sneeze", one applies $(\overline{[\![all]\!]} \otimes \mu)$ to the result of the first step as above. The second and third steps of computation are as follows:

$$\epsilon\Big(\overline{[\![all]\!]}\Big(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\Big)\otimes\mu\Big(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{c_1\},\{m_1,c_1\}\}\Big)\Big)=$$
$$\epsilon\Big(\{\{m_1,m_2\}\}\otimes\{\{m_1\}\}\Big)=\emptyset$$

So the meaning of this sentence is false. For the sentence of 'No men sneeze', we have the following:

$$\epsilon\Big(\overline{[\![no]\!]}\Big(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\Big)\otimes\mu\Big(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{c_1\},\{m_1,c_1\}\}\Big)\Big)=$$
$$\epsilon\Big(\{\emptyset\}\otimes\{\{m_1\}\}\Big)=\emptyset$$

**Example (II): Transitive Verb.** Suppose both of the male individuals love the cat. We set:

$$\overline{[\![\text{love}]\!]}:=\downarrow_{\neq\emptyset} Dom(\text{love})\times\downarrow_{\neq\emptyset} Codom(\text{love})$$

Hence we have:

$$\overline{[\![\text{love}]\!]}:=\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\times\{\{c_1\}\}$$

Meaning of the sentence 'Some men love cats' is computed as follows. In the first step we compute:

$$\begin{aligned}(\sigma\otimes 1\otimes\epsilon)(\overline{[\![\text{men}]\!]}\otimes\overline{[\![\text{love}]\!]}\otimes\overline{[\![\text{cats}]\!]})&=\\ \{(\{m_1\},\{m_1\}),(\{m_2\},\{m_2\}),(\{m_1,m_2\},\{m_1,m_2\})\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\epsilon(\{\{c_1\}\}\otimes\{\{c_1\}\})&=\\ \{(\{m_1\},\{m_1\}),(\{m_2\},\{m_2\}),(\{m_1,m_2\},\{m_1,m_2\})\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\star\}&=\\ \{(\{m_1\},\{m_1\}),(\{m_2\},\{m_2\}),(\{m_1,m_2\},\{m_1,m_2\})\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\}&\end{aligned}$$

In the second step, we apply $\overline{[\![Some]\!]}\otimes\mu$ to the above and compute:

$$\begin{aligned}\overline{[\![Some]\!]}(\{\{m_1\},\{m_2\},\{m_1,m_2\}\})\otimes\mu(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})&=\\ [\![\text{Some}]\!](\{m_1,m_2\})\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\}&=\\ \{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\}&\end{aligned}$$

In the final step, we apply $\epsilon$ to the above and compute:

$$\epsilon(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})=\{\star\}$$

So the sentence is true. For "all men love cats", the first step of the computation is as above. For the second and third steps we compute:

$$\begin{aligned}\epsilon\Big(\overline{[\![all]\!]}(\{\{m_1\},\{m_2\},\{m_1,m_2\}\})\otimes\mu(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})\Big)&=\\ \epsilon(\{\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})=\{\star\}&\end{aligned}$$

So this sentence is also true. Whereas for "no men love cats" we have:

$$\begin{aligned}\epsilon\Big(\overline{[\![no]\!]}(\{\{m_1\},\{m_2\},\{m_1,m_2\}\})\otimes\mu(\{\{m_1\},\{m_2\},\{m_1,m_2\}\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})\Big)&=\\ \epsilon(\{\emptyset\}\otimes\{\{m_1\},\{m_2\},\{m_1,m_2\}\})=\emptyset&\end{aligned}$$

So the meaning of this sentence is also false.

## 5 Justification

Recalling that, as shown in [5, 3], the Frobenius $\mu$ map is the analog of set-theoretic intersection and the compact closed epsilon map is the analog of set-theoretic application, it is not hard to show that the truth-theoretic interpretation of the compact closed semantics of quantified sentences provides us with the same meaning as their generalised quantifier semantics. In what follows, we make this formal as follows.

**Definition 2.** *The meaning of a sentence in a concrete relational instantiation of the compact closed categorical interpretation is true iff $\epsilon_{\mathcal{P}\mathcal{P}(U)}(\overline{[\![np]\!]} \otimes \overline{[\![vp]\!]}) = \{\star\}$ and is false otherwise.*

*Proof.* That the interpretation of atoms is ??? and that the rules preserve truth?

**Lemma 1.** *Using the non-empty down set embedding, the meaning of a sentence with a quantified phrase at its subject position becomes equivalent to the following*

$$\{\star \mid D_k = A_i = B_j, D_k \in [\![d]\!]([\![N]\!]), A_i \in \mathcal{P}_{\neq\emptyset}([\![N]\!]), B_j \in \mathcal{P}_{\neq\emptyset}([\![VP]\!])\}$$

*Proof.* The non-empty down set embedding means that we have the following:

$$\overline{[\![n]\!]} = \{A_i \mid A_i \in \mathcal{P}_{\neq\emptyset}([\![n]\!])\} \qquad \overline{[\![d]\!]}\Big(\overline{[\![n]\!]}\Big) = \{D_o \mid D_o \in [\![d]\!]([\![n]\!])\}$$

$$\overline{[\![np]\!]} = \{C_l \mid C_l \in \mathcal{P}_{\neq\emptyset}([\![np]\!])\} \qquad \overline{[\![v]\!]} = \{(B_j, B_k) \mid B_j, B_k \in \mathcal{P}_{\neq\emptyset}([\![v]\!])\}$$

The meaning of a sentence with a quantified subject is computed in three steps. In the first step, we obtain:

$$(\delta_N \otimes 1_N)\Big(\overline{[\![\mathrm{N}]\!]} \otimes \overline{[\![\mathrm{VP}]\!]}\Big) = \{(A_i, A_i) \mid A_i \in \mathcal{P}([\![\mathrm{N}]\!])\} \otimes \{B_j \mid B_j \in \mathcal{P}_{\neq\emptyset}([\![\mathrm{VP}]\!])\}$$

In the second step, we obtain:

$$\begin{aligned}(Det \otimes \mu_N)\Big(\{(A_i, A_i) \mid A_i \in \mathcal{P}_{\neq\emptyset}([\![\mathrm{Sbj}]\!])\} \otimes \{B_j \mid B_j \in \mathcal{P}([\![\mathrm{Verb}]\!])\}\Big) &= \\ Det\Big(\{A_i \mid A_i \in \mathcal{P}_{\neq\emptyset}([\![\mathrm{Sbj}]\!])\}\Big) \otimes \{A_i \mid A_i = B_j, A_i \in \mathcal{P}([\![\mathrm{Sbj}]\!]), B_j \in \mathcal{P}([\![\mathrm{Verb}]\!])\} &= \\ \{D_k \mid D_k \in Det([\![\mathrm{N}]\!])\} \otimes \{A_i \mid A_i = B_j, A_i \in \mathcal{P}_{\neq\emptyset}([\![\mathrm{N}]\!]), B_j \in \mathcal{P}([\![\mathrm{VP}]\!])\}&\end{aligned}$$

In the final step, we obtain:

$$\begin{aligned}\epsilon\Big(\{D_k \mid D_k \in Det([\![\mathrm{Sbj}]\!])\} \otimes \{A_i \mid A_i = B_j, A_i \in \mathcal{P}([\![\mathrm{Sbj}]\!]), B_j \in \mathcal{P}([\![\mathrm{Verb}]\!])\}\Big) &= \\ \{\star \mid D_k = A_i, D_k \in Det([\![\mathrm{N}]\!]), A_i = B_j, A_i \in \mathcal{P}_{\neq\emptyset}([\![\mathrm{N}]\!]), B_j \in \mathcal{P}([\![\mathrm{VP}]\!])\}&\end{aligned}$$

**Definition 3.** *The compact closed meaning of the sentence "Det N VP" is true in $Rel$ if and only if*

$$\epsilon \circ (Det \otimes \mu) \circ (\sigma \otimes 1_N)\Big(\overline{[\![N]\!]} \otimes \overline{[\![VP]\!]}\Big) = \{\star\}$$

**Proposition 1.** *The compact closed meaning of a quantified sentence computed in $Rel$ is true if and only if its generalised quantifier meaning is true.*

*Proof.* For the right to left direction, suppose the generalised quantifier meaning of the sentence is true, that is $[\![\text{Sbj}]\!] \cap [\![\text{Verb}]\!] \in Det([\![\text{Sbj}]\!]$ and consider the case when $Det([\![Sbj]\!]) \neq \{\emptyset\}$. We have to show that $\epsilon \circ (Det \otimes \mu) \circ (\sigma \otimes 1_N)\left(\overline{[\![\text{Sbj}]\!]} \otimes \overline{[\![\text{Verb}]\!]}\right) = \{\star\}$. For this, we need to show that there is a set $G$ equal to $D_k = A_i = B_j$ such that $G$ is in $Det([\![\text{Sbj}]\!])$ and $\mathcal{P}([\![\text{Sbj}]\!])$ and $\mathcal{P}([\![\text{Verb}]\!])$. Take $G$ to be $[\![\text{Sbj}]\!] \cap [\![\text{Verb}]\!]$. Then, it is a subset of $[\![\text{Sbj}]\!]$, hence an element of $\mathcal{P}([\![\text{Sbj}]\!])$, a subset of $[\![\text{Verb}]\!]$, hence an element of $\mathcal{P}([\![\text{Verb}]\!])$, and an element of $Det([\![\text{Sbj}]\!]$.

For the left to right direction, suppose the compact closed meaning of the sentence is true in $Rel$, and consider the case when $Det([\![Sbj]\!]) \neq \{\emptyset\}$. Then we have subsets $D_k = A_i = B_j$ such that $D_k \in Det([\![Sbj]\!])$, $A_i \in \mathcal{P}([\![Sbj]\!])$, and $B_j \in \mathcal{P}([\![Verb]\!])$. Pick an arbitrary such subset, e.g. $G$, such that it equal to $D_k = A_i = B_j$; we have that $G \in \mathcal{P}([\![Sbj]\!])$, hence $G \subseteq [\![Sbj]\!]$; and that $G \in \mathcal{P}([\![Verb]\!])$, hence $G \subseteq [\![Verb]\!]$. It thus follows that $G \subseteq [\![Sbj]\!] \cap [\![Verb]\!]$. At the same time $G \in Det([\![Sbj]\!])$, hence the generalised quantifier meaning is also true.

## 6 Concrete Corpus Interpretation

INSTANTIATE THE EXACT SAME RELATIONAL MODEL, I.E. WITH SETS OF INDIVIDUALS AS BASIS VECTORS ETC.

In a concrete vector space model, built from a corpus using distributional methods, we assume that vector meaning of the subject is

$\sum_i C_i \overrightarrow{n}_i \in N$ and the linear map corresponding to the verb is

$\sum_{jk} C_{jk} \overrightarrow{n}_j \otimes \overrightarrow{s}_k \in N \otimes S$. For $\overrightarrow{n}_i$ a basis vector of $N$, we define the map $Det$ as follows:

$$Det(\overrightarrow{n}_i) = \Phi\{\overrightarrow{w} \in N \mid d(\overrightarrow{n}_i, \overrightarrow{w}) = \alpha\} \tag{1}$$

where we have:

- $\phi$ is a linear average function such as the arithmetic or weighted mean.
- $\alpha$ indicates how close $\overrightarrow{w}$ is to the $\overrightarrow{n}_i$ and depends on the quantifier expressed by $Det$.

The intuitive reading of the above is that $Det$ of a word $\overrightarrow{n}_i$ is a linear combination, e.g. average, of all the words that are $\alpha$-close to $\overrightarrow{n}_i$. In other words, the average of all the words whose distance from $\overrightarrow{n}_i$ is $\alpha$. For instance, if $Det$ is 'few', then $\alpha$ is a small number (closer to 0 than to 1), indicating that we are taking the average of vectors that are not so close to $\overrightarrow{n}_i$. If $Det$ is 'most', then $\alpha$ will be a large number (closer to 1 than to 0), indicating that we are taking the average of vectors that are close to $\overrightarrow{n}_i$. The distance $\alpha$ can be learnt from a corpus using a relevant task. This will extend to any other (non-basis) word by linearity.

The underlying idea here is that the quantitative way of quantifying in set-theoretic models, which depends on the cardinality of the quantified sets, is now transformed into a geometric way of quantifying where the meaning of the quantified phrase depends on its geometric distance with other words. Hence, a quantified phrase such as 'few cats' returns a representative noun (obtained by taking the average of all such nouns) that is far from vector of 'cat' in the semantic space. This representative noun shares 'few' properties with 'cat'. A quantified phrase such as 'most cats' returns a representative noun that is close the the vector of 'cat' and stands for a noun that shares 'most' of the properties of 'cat'.

With this instantiation, the meaning of "Q Sbj Verb" is obtained by computing the following:

$$(\epsilon_N \otimes 1_S) \circ (Det \otimes \mu_N \otimes 1_S) \circ (\delta_N \otimes 1_{N \otimes S}) \left(\overrightarrow{\text{N}} \otimes \overrightarrow{\text{VP}}\right)$$

In the first step of computation we have:

$$(\delta_N \otimes 1_{N \otimes S})\Big(\sum_i C_i \overrightarrow{n}_i \otimes \sum_{jk} C_{jk} \overrightarrow{n}_j \otimes \overrightarrow{s}_k\Big) = (\sum_i C_i \overrightarrow{n}_i \otimes \overrightarrow{n}_i) \otimes (\sum_{jk} C_{jk} \overrightarrow{n}_j \otimes \overrightarrow{s}_k)$$

In the second step we obtain:

$$(Det \otimes \mu_N \otimes 1_S)\Big((\sum_i C_i \overrightarrow{n}_i \otimes \overrightarrow{n}_i) \otimes (\sum_{jk} C_{jk} \overrightarrow{n}_j \otimes \overrightarrow{s}_k)\Big) = \sum_{ijk} C_i C_{jk} Det(\overrightarrow{n}_i) \otimes \mu(\overrightarrow{n}_i \otimes \overrightarrow{n}_j) \otimes \overrightarrow{s}_k$$
$$= \sum_{ijk} C_i C_{jk} Det(\overrightarrow{n}_i) \otimes \delta_{ij} \overrightarrow{n}_i \otimes \overrightarrow{s}_k$$

The final step is as follows:

$$(\epsilon_N \otimes 1_S)\Big(\sum_{ijk} C_i C_{jk} Det(\overrightarrow{n}_i) \otimes \delta_{ij} \overrightarrow{n}_i \otimes \overrightarrow{s}_k\Big) = \sum_{ijk} C_i C_{jk} \langle Det(\overrightarrow{n}_i) \mid \delta_{ij} \overrightarrow{n}_i \rangle \overrightarrow{s}_k$$

**Example.** As a distributional example, take $N$ to be the two dimensional space with the basis $\{\overrightarrow{n}_1, \overrightarrow{n}_2\}$ and $S$ be the two dimensional space with the basis $\{\overrightarrow{s}_1, \overrightarrow{s}_2\}$. Suppose the linear expansion of

$$\overrightarrow{\text{N}} := C_1 \overrightarrow{n}_1 + C_2 \overrightarrow{n}_2$$

and the linear expansion of

$$\overrightarrow{\text{VP}} := C_{11}(\overrightarrow{n}_1 \otimes \overrightarrow{s}_1) + C_{12}(\overrightarrow{n}_1 \otimes \overrightarrow{s}_2) + C_{21}(\overrightarrow{n}_2 \otimes \overrightarrow{s}_1) + C_{22}(\overrightarrow{n}_2 \otimes \overrightarrow{s}_2)$$

Suppose further the following for the interpretation of the determiner:

$$Det(\overrightarrow{\text{Sbj}}) = Det(C_1 \overrightarrow{n}_1 + C_2 \overrightarrow{n}_2) = Det(C_1 \overrightarrow{n}_1) + Det(C_2 \overrightarrow{n}_2) = C'_1 \overrightarrow{n}_1 + C'_2 \overrightarrow{n}_2 \qquad (2)$$

Then the result of the first step of the computation of a meaning vector for the sentence 'Q Sbj Verb' is:

$$(C_1 \overrightarrow{n}_1 + C_2 \overrightarrow{n}_2) \otimes (C_{11} \overrightarrow{n}_1 \otimes \overrightarrow{s}_1 + C_{12} \overrightarrow{n}_1 \otimes \overrightarrow{s}_2 + C_{21} \overrightarrow{n}_2 \otimes \overrightarrow{s}_1 + C_{22} \overrightarrow{n}_2 \otimes \overrightarrow{s}_2)$$

In the second step of computation we obtain:

$$C_1 C_{11} Det(\overrightarrow{n}_1) \overrightarrow{n}_1 \otimes \overrightarrow{s}_1 + C_1 C_{12} Det(\overrightarrow{n}_1) \overrightarrow{n}_1 \otimes \overrightarrow{s}_2 + C_2 C_{21} Det(\overrightarrow{n}_2) \overrightarrow{n}_2 \otimes \overrightarrow{s}_1 + C_2 C_{22} Det(\overrightarrow{n}_2) \overrightarrow{n}_2 \otimes \overrightarrow{s}_2$$

Since $Det$ is a linear map, the above is equal to the following:

$$Det(C_1 \overrightarrow{n}_1)(C_{11} \overrightarrow{n}_1 \otimes \overrightarrow{s}_1 + C_{12} \overrightarrow{n}_1 \otimes \overrightarrow{s}_2) + Det(C_2 \overrightarrow{n}_2)(C_{21} \overrightarrow{n}_2 \otimes \overrightarrow{s}_1 + C_{22} \overrightarrow{n}_2 \otimes \overrightarrow{s}_2)$$

According to the expansion of the assumption in equation 2, the above is equivalent to the following:

$$(C'_1 \overrightarrow{n}_1)(C_{11} \overrightarrow{n}_1 \otimes \overrightarrow{s}_1 + C_{12} \overrightarrow{n}_1 \otimes \overrightarrow{s}_2) + (C'_2 \overrightarrow{n}_2)(C_{21} \overrightarrow{n}_2 \otimes \overrightarrow{s}_1 + C_{22} \overrightarrow{n}_2 \otimes \overrightarrow{s}_2)$$

Substituting this in the last step of the computation provide us with the following vector in $S$:

$$C'_1 C_{11} \overrightarrow{s}_1 + C'_1 C_{12} \overrightarrow{s}_2 + C'_2 C_{21} \overrightarrow{s}_2 + C'_2 C_{22} \overrightarrow{s}_2 \qquad (3)$$

**Small Corpus-Based Witness.** A large scale experimentation for this model constitutes work in progress. For the sake of providing intuitions for the above symbolic constructions, we provide a couple of corpus-based witnesses here. In the distributional models, the most natural instantiation of the distance $d$ in equation 1 is the co-occurrence distance. For a noun '*n*' and determiners '*few*' and '*most*', we define these as generally as follows:

few($n$) = $Avg\{$nouns that share few properties with $n\}$
most($n$) = $Avg\{$nouns that share most properties with $n\}$

For the purpose of this toy-example, the above can be instantiated in the simplest possible way as follows:

few($n$) = $Avg\{$nouns that co-occurred with $n$ few times$\}$
most($n$) = $Avg\{$nouns that co-occurred with $n$ most times$\}$

In this case, a sample query from the online *Reuter News Corpus*, with at most 100 outputs per query, provides the following instantiations:

few(*dogs*) = $Avg\{$bike, drum, snails$\}$
most(*dogs*) = $Avg\{$cats, pets,birds, puppies$\}$
few(*cats*) = $Avg\{$fluid, needle, care$\}$
most(*cats*) = $Avg\{$dogs, birds, rats, feces$\}$

A cosine-based similarity measure over this corpus results in the fact that any of the words in the 'most($n$)' set are more similar to '$n$' than any of the words in the 'few($n$)' set. This is indeed because the words in the former set are geometrically closer to '$n$' than the words in the latter set, since they have co-occurred with them more. This is the first advantage of our model over a distributional model, where words such as 'few' and 'most' are treated as noise and hence meanings of phrase such as 'few cats', 'most cats', and 'cats' become identical (and similarly for any other noun). Moreover, in our setting we can establish that 'most cats' and 'most dogs' have similar meanings, because of the over lap of their determiner sets. A larger corpus and a more thorough statistical analysis will let us achieve more, that for instance, 'few cats' and 'few dogs' also have similar meanings.

At the level of sentence meanings, compositional distributional models do not interpret determiners (e.g. see the model of [12]). As a result, meanings of sentences such as 'cats sleep', 'most cats sleep' and 'few cats sleep' will become identical; meanings of sentences 'most cats sleep' and 'few dogs snooze' become very close, since 'cats' and 'dogs' often occur in the same context and so do 'sleep' and 'snooze'. In our setting, equation 3 tells us that these sentences have different meanings, since their quantified subjects have different meanings. To see this, take $\overrightarrow{\text{cats}} = C_1\overrightarrow{n}_1 + C_2\overrightarrow{n}_2$, where as $few(\text{cats}) = C'_1\overrightarrow{n}_1 + C'_2\overrightarrow{n}_2$ and $most(\text{cats}) = C''_1\overrightarrow{n}_1 + C''_2\overrightarrow{n}_2$. Instantiating these in equation 3 provides us with the following three different vectors:

$$\begin{aligned}
\overrightarrow{\text{cats sleep}} &= C_1C_{11}\overrightarrow{s}_1 + C_1C_{12}\overrightarrow{s}_2 + C_2C_{21}\overrightarrow{s}_2 + C_2C_{22}\overrightarrow{s}_2\\
\overrightarrow{\text{few cats sleep}} &= C'_1C_{11}\overrightarrow{s}_1 + C'_1C_{12}\overrightarrow{s}_2 + C'_2C_{21}\overrightarrow{s}_2 + C'_2C_{22}\overrightarrow{s}_2\\
\overrightarrow{\text{most cats sleep}} &= C''_1C_{11}\overrightarrow{s}_1 + C''_1C_{12}\overrightarrow{s}_2 + C''_2C_{21}\overrightarrow{s}_2 + C''_2C_{22}\overrightarrow{s}_2
\end{aligned}$$

On the other hand, we have that 'most cats sleep' and 'most dogs snooze' have close meanings, one which is close to 'pets sleep'. This is because, their quantified subjects and their verbs have similar meanings, that is we have:

$$\begin{cases} most(\overrightarrow{\text{dogs}}) \sim most(\overrightarrow{\text{cats}}) \sim \overrightarrow{\text{pets}} \\ \overrightarrow{\text{snooze}} \sim \overrightarrow{\text{sleep}} \end{cases} \quad \Longrightarrow \text{ most cats sleep} \sim \text{most dogs snooze} \sim \text{pets sleep}$$

At the same time, 'few cats sleep' and 'most dogs snooze' have a less-close meaning, since their quantified subjects have different meanings, that is:

$$most(\overrightarrow{\text{dogs}}) \not\sim few(\overrightarrow{\text{cats}}) \implies \overrightarrow{\text{most dogs snooze}} \not\sim \overrightarrow{\text{few cats sleep}}$$

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