Relative Fractional Packing Number and Its Properties

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Abstract

The concept of the *relative fractional packing number* between two graphs G and H, initially introduced in [1], serves as an upper bound for the ratio of the zero-error Shannon capacity of these graphs. Defined as:

$$\sup_{W} \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)}$$

where the supremum is computed over all arbitrary graphs and $\boxtimes$ denotes the strong product of graphs.

This article delves into various critical theorems regarding the computation of this number. Specifically, we address its $\mathcal{NP}$-hardness and the complexity of approximating it. Furthermore, we develop a conjecture for a necessary and sufficient conditions for this number to be less than one, we also validate this conjecture for specific graph families. Additionally, we present miscellaneous concepts and introduce a generalized version of the independence number, that gives insights that could significantly contribute to the study of the relative fractional packing number.

1 Introduction

1.1 Preliminaries

In this section, we introduce the necessary concepts and background knowledge related to our study.

Let G be a simple graph, we denote the independence number of G by $\alpha(G)$. Computing the independence number is a known $\mathcal{NP}$-Hard problem [3], and it is even Poly-APX-complete [2]. Two sets of vertices, S and T, are said to be disconnected if and only if their intersection is empty and there are no edges between them.

A graph G is called vertex-transitive or point-symmetric if and only if for any pair of vertices $u, v \in V(G)$, there exists an automorphism π of G such that $\pi(u) = v$. Note that an automorphism for a graph G is just an isomorphism from G to itself.

The strong product of two graphs G and H, denoted as $G \boxtimes H$, is a graph product commonly utilized in combinatorics and information theory. The vertex set of $G \boxtimes H$ is the Cartesian product of the vertex sets of G and H. Two vertices (u_1, v_1) and (u_2, v_2) in $G \boxtimes H$ are adjacent if and only if any of the following conditions hold:

- $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$
- $u_1 = u_2$ and $v_1v_2 \in E(H)$
- $u_1u_2 \in E(G)$ and $v_1 = v_2$

Using this product, the Shannon zero-error capacity of a graph G is defined as $\theta(G)$, which is obtained as the limit of $\sqrt[n]{\alpha(G^{\boxtimes n})}$ as n approaches infinity. The Shannon zero-error capacity plays a crucial role in evaluating the zero-error capacity of a channel represented by the graph G [6].

Calculating the Shannon zero-error capacity efficiently remains a challenge, and even for small graphs like C_7, the exact value is unknown. Lovász showed that $\theta(C_5) = \sqrt{5}$. To the best of our knowledge, the Lovász number of a graph [7] and the Haemers' number of a graph [4] are the most commonly used upper bounds for the Shannon capacity of a graph.

The Integer Programming formulation of the independence number of a graph G is as follows:

$$\begin{array}{ll} \text{maximize} & \sum_{v \in V} x_v \\ \text{subject to} & \sum_{v \in C} x_v \leq 1 \; \forall C \in \mathcal{C} \\ & x \geq 0 \end{array}$$

Here, $\mathcal{C}$ represents the set of all cliques in the graph G. The optimal value of the relaxed linear program of the optimization problem above is known as the fractional packing number, denoted by $\alpha^*(G)$.

In [5], Hales demonstrated that:

$$\alpha^*(G) = \sup_W \frac{\alpha(G \boxtimes W)}{\alpha(W)}$$

Here, the supremum is taken over all arbitrary graphs W, additionally, Hales proved that this supremum it is indeed a maximum. He also constructed a maximizer for this term.

Motivated by this, Alipour and Gohari [1] introduced the concept of the relative fractional packing number of two graphs. It is defined as follows:

$$\sup_W \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)}$$

They provided a linear programming formulation to compute this number and established several properties relating to bounding the ratios of different graph properties. We delve deeper into this topic in Section 3.

1.2 Our Contribution

In this paper, we focus on the fractional packing number and its significance. In our research, we aim to explore the applications of this concept and establish its connections with other graph invariants, especially, the Shannon number and the independence number of graphs. Throughout this paper, we present a collection of theorems and properties. Our objective is to provide valuable insights that can contribute to the discovery of new findings related to this graph quantity.

One of the primary objectives of our study was to investigate the computational complexity of computing and approximating the fractional packing number. By drawing parallels between this quantity and the independence number of a graph, we were able to establish that computing the fractional packing number is a $\mathcal{NP}$-Hard problem. Furthermore, we demonstrated that it is also Poly-APX-hard, showing the challenge to develope efficient approximation algorithms.

In our research, we specifically examined the case when both graphs involved in calculating the fractional packing number are cycles and we were able to calculate the exact value of this quantity in this case. By employing a generalized technique we derived lower bounds for this quantity in the general case. We then mentioned some tight cases for this lower-bound.

Section 3 of this paper is dedicated to the investigation of scenarios where the fractional packing number is below one. We aimed to identify necessary and sufficient conditions for this phenomenon and formulated a conjecture to address this concept. Additionally, we provided evidence for the validity of this conjecture by proving its soundness for well-known families of graphs, including perfect graphs and cycles.

Lastly, we introduced a generalized concept of the independence number and established a set of theorems concerning this quantity. We also highlighted the close relationship between this generalized independence number and the fractional packing number, suggesting the potential utilization of the independence number in future studies of the relative fractional packing number.

2 Calculating the Relative Fractional Packing Number

This section mentions some results regarding the calculating the $\alpha^*(G|H)$. firstly, we discuss some theorems about the complexity of calculating the relative fractional packing number and its hardness of approximation. Moreover, we will calculate this number exactly when both graphs G and H are a cycle. Finally, we will generalize our previous technique to get some bounds on this number.

2.1 Hardness Results

Theorem 1. *The problem of computing $\alpha^*(G|H)$ is $\mathcal{NP}$-Hard.*

Proof. We will reduce the problem of finding the independence number of a graph to this problem, therefore, showing this problem is $\mathcal{NP}$-Hard.

Imagine graph G is a single vertex graph. In this case we would have:

$$\alpha^*(G|H) = \max_W \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)} = \max_W \frac{\alpha(W)}{\alpha(H \boxtimes W)} = \frac{1}{\inf\limits_W \frac{\alpha(H \boxtimes W)}{\alpha(W)}} = \frac{1}{\alpha(H)}$$

Where the last equality is because $\alpha(H \boxtimes W) \geq \alpha(H) \times \alpha(W)$. $\square$

One can easily verify that the above reduction is in fact an approximation-preserving reduction, therefore, since the problem of finding the independence number is Poly-APX-hard, the factional packing number is also Poly-APX-hard.

Remark 1. *The relative fractional packing number can't be approximated to a constant factor in polynomial time unless $\mathcal{P} = \mathcal{NP}$.*

2.2 Relative Fractional Packing Number for Two Cycles

In this section, we are going to calculate the fractional packing number when both graphs G and H are cycles, but first, we are going to mention a lemma, that is proved at [1] which helps us on these calculations.

Lemma 1. *If graph G is vertex-transitive with n vertices, then for any arbitrary graph H we have:*

$$\alpha^*(G|H) = \frac{n}{\alpha(G^c \boxtimes H)}$$

Since cycles are vertex transitive graphs, we apply this lemma to calculate their relative fractional packing number.

Before stating the theorem we are going to define the projection of an Independence set to a vertex as follows:

Definition 1. *for an independence set I of the graph $G \boxtimes H$ and a vertex $v \in G$, we define the projection of I to v as the set $\{u \in H \mid (v,u) \in I\}$. The definition is similar when projecting to a vertex in H.*

Theorem 2. *For two numbers $n\,, m \geq 3$, let $f(n,m)$ be the following function:*

$$f(n,m) = \begin{cases} \frac{n}{m} & m \text{ is even} \\ \frac{n}{m-1} & n \text{ is even and } m \\ \frac{n}{m} & n \text{ and } m \text{ are odd and } n \leq m \\ \frac{n}{m-1} & n \text{ and } m \text{ are odd and } m < n \end{cases}$$

Now, for any two cycles C_n and C_m we have $\alpha^*(C_n, C_m) = f(n,m)$.

Proof. Since even cycles are perfect graphs if the first case happens, then we have

$$\alpha^*(C_n, C_m) = \frac{n}{\alpha(C_n^c \boxtimes C_m)} = \frac{n}{\alpha(C_n^c) \times \alpha(C_m)}$$

now $\alpha(C_n^c) = 2$ and $\alpha(C_m) = \frac{m}{2}$ therefore,

$$\frac{n}{\alpha(C_n^c) \times \alpha(C_m)} = \frac{n}{m}$$

The second case goes similarly, if C_n is an even cycle, then it would be a perfect graph and by the perfect graph theorem [8], C_n^c will also be a perfect graph so the calculations goes as follows:

$$\alpha^*(C_n, C_m) = \frac{n}{\alpha(C_n^c \boxtimes C_m)} = \frac{n}{\alpha(C_n^c) \times \alpha(C_m)}$$

now this time since H is odd $\alpha(C_m) = \frac{m-1}{2}$ therefore,

$$\frac{n}{\alpha(C_n^c) \times \alpha(C_m)} = \frac{n}{m-1}$$

For the two latter cases imagine that u is an arbitrary vertex in C_m, then the size of the projection of any maximum independent set of $C_n^c \boxtimes C_m$ to u is at most two because the largest possible independent set in C_n^c has size two. Now if for a vertex u we have $|A_u| = 2$, then the set A_u should contain two adjacent vertices in C_n but since two adjacent vertices in C_n are a dominating set in C_n^c, then the neighbors of u in G_m have empty projections, therefore the average projection size is less than or equal to one meaning than $\alpha(C_n^c \boxtimes C_m) \leq m$. additionally its obvious that $\alpha(C_n^c \boxtimes C_m) \geq \alpha(C_n^c) \times \alpha(C_m) \geq m-1$. So we only have to check which of these two possibilities happens. when n and m are both odd and $n \leq m$, then we can have the following independent set of $C_n^c \boxtimes C_m$ with size m.

Imagine vertices of C_n are $u_1, \ldots, u_n$ and vertices of C_m are $v_1, \ldots, v_m$ in order. Now the claimed independent set for $C_n^c \boxtimes C_m$ is as follows:

$$A = \{(u_1, v_1), \ldots, (u_n, v_n), (u_1, v_{n+1}), (u_2, v_{n+2}), (u_1, v_{n+3}), \ldots, (u_2, v_m)\}$$

you can easily verify this is a valid independent set which means $\alpha(C_n^c \boxtimes C_m) = m$ and therefore, $\alpha^*(C_n, C_m) = \frac{n}{m}$.

For the final case we show that its impossible to have an independent set for $C_n^c \boxtimes C_m$ with size m. Imagine such an independent set A exists, now based on previous reasoning, for any vertex $v \in c_m$, size of the projection of A to v is exactly one, we show this projection with the set A_v. Additionally, for any two adjacent vertices v_i and v_{i+1} in H, because their projections should be disconnected, if A_{v_i} contains the vertex u_j, $A_{v_{i+1}}$ either contains u_{j+1} or u_{j-1} Now we label the edges of C_m with the following rule. Label the edge between v_i and v_{i+1} in C_m with a $+1$ if $A_{u_{i+1}}$ contains v_{j+1}, otherwise, put a -1 on that edge. Now if we take the summation of all the number on the edges of C_m we should get 0 because we are looping from a number back to itself in a cycle. However, this summation consists of m $+1$ or -1 modulo n. Basic number theory tells us that this summation can't be equal to 0 modulo n when $m < n$ and both n and m are odd. This implies that $\alpha(C_n^c \boxtimes C_m) = m - 1$ which means in this case $\alpha^*(C_n|C_m) = \frac{n}{m-1}$. □

3 necessary and Sufficient Conditions for Relative Fractional Packing Number to Be Bellow One

As showed in [1], the relative fractional packing number can serve as an upper-bound on the ratio of many different graph properties, in fact, they showed the following theorem.

Theorem 3. *Let X(G) be any of the followings: independence number, the zero-error Shannon capacity, the fractional packing number, the Lovász number of a graph G. Then we have:*

$$\frac{X(G)}{X(H)} \leq \alpha^*(G|H)$$

The above theorem mentions one important usage for the relative fractional packing number. If we find a necessary and sufficient condition on graphs G and H so that $\alpha^*(G|H) \leq 1$, the above theorem mentions that that we have found a sufficient condition for $X(G) \leq X(H)$ for all choices of X.

In this section we are going to propose a conjecture for such a condition. Additionally, we will prove this conjecture for some famous families of graphs.

Definition 2. *for a graph G we define Expand(G) as the set of all graphs that can be obtained by a sequence of the following three operations on G Iteratively.*

1. *Remove a vertex v from G.*

2. *add a new edge to G.*

3. *replace a vertex v by a clique of a arbitrarily size.*

The latter operations just puts a clique of size k instead of v and connects these k new vertices to all neighbors of v.

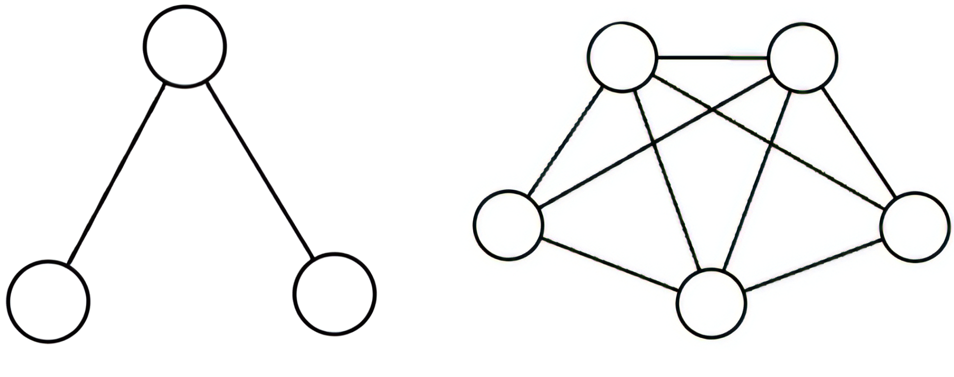

Figure 1: The graph P_3 before and after replacing the middle vertex with a clique of size 3 using the third operation.

You can trivially verify that all the above operations, when applied to a graph G, will not increase $\alpha(G \boxtimes W)$ for any graph choice of W, therefore, if we have two graphs G and H such that $G \in expand(H)$, then we can conclude $\alpha(G \boxtimes W) \leq \alpha(H \boxtimes W)$ for any graph W which means $\alpha^*(G|H) \leq 1$.

Remark 2. *Note that by the above operations, one can merge (contract) any two vertices v and u in G. We only have to delete one of them, for instance v, by applying the first operation, and then, add new edges between u and the neighbors of v by applying the second rule.*

Theorem 4. *From the definition of expand, it's trivial that $\alpha^*(G|H) \leq 1$ if $G \in expand(H)$.*

It's time to mention the main conjecture.

Conjecture 1. *$\alpha^*(G|H) \leq 1$ if and only if $G \in expand(H)$.*

Now, we will validate this conjecture when G belongs to certain specific classes of graphs. Firstly, we examine the case where G is a cycle.

Theorem 5. *If G is a Cycle and H is an arbitrary graph, then $\alpha^*(G|H) \leq 1$ if and only if $G \in Expand(H)$.*

Proof. According to Theorem 4, one direction of the theorem is true. We will now assume $\alpha^*(G|H) \leq 1$ and prove that $G \in \text{Expand}(H)$.

Since G is a cycle, it is also vertex-transitive. Therefore, by Lemma 1, we deduce:

$$\alpha^*(G|H) = \frac{n_G}{\alpha(G^c \boxtimes H)}$$

So $\alpha^*(G|H) \leq 1$ means that $n_G \leq \alpha(G^c \boxtimes H)$.

Recall the definition of projection from Definition 1, for any vertex u in G or H, we name the projection from the maximum independent set of $G^c \boxtimes H$ to u as A_u. Now we have:

$$\alpha(G^c \boxtimes H) = \sum_{u \in G} |A_u| \geq n_G$$

We split the rest of the proof into two cases.

Case 1 $\forall u \in G \ |A_u| \geq 1$:

For any vertex v of H, A_v is an independent set in G^c. Therefore, it's a clique in G. Since G is a cycle, this clique is either the empty set, a single vertex, or a single edge. Now we try to construct G from H by applying the expansion operations. Firstly, replace every vertex v in H with a clique of size $|A_v|$ to obtain H'. Imagine the vertices of these cliques are labeled by the members of A_v. If A_v is empty, just delete the vertex v from H.

If two vertices in H' labeled by u_i and u_j are connected by an edge, then either for a vertex v in H we have $u_i, u_j \in A_v$, or there exist two connected vertices $v_i, v_j \in H$, such that $u_i \in A_{v_i}$ and $u_j \in A_{v_j}$. Since these projections were from an independent set of $G^c \boxtimes H$, therefore for both cases, u_i must be disconnected from u_j in G^c. Consequently, there is an edge between them in G. By the above observation, we can deduce that if there is an edge between two vertices labeled by u_i and u_j in H', then there is an edge between u_i and u_j in G.

Finally, using Remark 2, we will merge all the vertices who have the same label to obtain H''. By the assumption of this case and the statement above, H'' is a spanning subgraph of G. Now, by adding the missing edges of G to H'', we will finish the construction of G, and the theorem is proved for this case.

Case 2 $\exists u \in G \ |A_u| = 0$: Imagine such u exists. Order the vertices of G clockwise from u_1 to u_{n_G}, setting $u = u_1$. Consider these following independent sets in G:

$$S = \{u_2, u_4, \ldots, u_{n_G - 1}\} \quad T = \{u_3, u_5, \ldots, u_{n_G}\}$$

Since $\sum_{u \in G} |A_u| \geq n_G$ by the symmetry we can assume $\sum_{u \in S} |A_u| \geq \lceil \frac{n_G}{2} \rceil$. Now similar to the previous case we try to construct G from H by applying the expansion operations.

Firstly, delete every vertex v from H that $A_v \bigcap S = \emptyset$ to obtain H' because S was an independent set in G, it's a clique in G^c. Now since are projections are coming from an independent set $G^c \boxtimes H$, therefore, all the remaining vertices in H' are disconnected and H' is an independent set.

Because of our assumption about the set S, H' has more vertices than $\lceil \frac{n_G}{2} \rceil$. Replace $\lfloor \frac{n_G}{2} \rfloor$ of these vertices by a clique of size two (an edge) to obtain H''. H'' is $\lfloor \frac{n_G}{2} \rfloor$ disjoint edges (and one single vertex if n_G is odd), therefore it's a spanning subgraph of G.

Now, by adding the missing edges of H'' to it, we complete the construction of G, and the theorem is also proved for this case. $\square$

It is not hard to see that the proof in **Case 1** was completely independent from the fact that G is a cycle. Therefore, if one can prove that the second case won't happen for a certain family of graphs, they have indeed proved the correctness of Conjecture 1 for that certain family of graphs.

Finally, we are going to investigate the correctness of Conjecture 1 when G is a perfect graph. We will apply a similar technique of using the projections to construct G by applying expansion operations to H.

But first, we are going to mentions couple of well-known theorems regarding the perfect Graphs.

Theorem 6 (Rosenfeld [9])**.** *If G is a perfect graph, then for any arbitrary graph H we have:*

$$\alpha(G \boxtimes H) = \alpha(G) \times \alpha(H)$$

Theorem 7 (Lovász [8])**.** *A graph G is perfect if and only if G^c is perfect.*

The above theorem is know as the perfect graph theorem.

Theorem 8 (Alipour and Gohari [1])**.** *If G is a perfect graph and H is an arbitrary graph, then $\alpha^*(G|H) = \frac{\alpha(G)}{\alpha(H)}$.*

Theorem 9. *If G is a perfect graph and H is an arbitrary graph, then $\alpha^*(G|H) \leq 1$ if and only if $G \in Expand(H)$.*

Proof. Based on Theorem 8, we know that if G is a perfect graph, then $\alpha^(G|H) = \frac{\alpha(G)}{\alpha(H)}$. Therefore, $\alpha^(G|H) \leq 1$ implies that $\alpha(G) \leq \alpha(H)$. Additionally, it implies that for any arbitrary graph W, we have:

$$\frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)} \leq 1$$

By setting $W = G^c$, we obtain:

$$\frac{n_G}{\alpha(H \boxtimes G^c)} \leq \frac{\alpha(G \boxtimes G^c)}{\alpha(H \boxtimes G^c)} \leq 1 \longrightarrow n_G \leq \alpha(H \boxtimes G^c)$$

Using the above inequalities, we will attempt to construct G from H by applying the expansion operations.

By the Perfect Graph Theorem, we know that G^c is also perfect. Therefore, we have $n_G \leq \alpha(H) \times \alpha(G^c)$. Consider the maximum independent set of $H \boxtimes G^c$ as the Cartesian product of the maximum independent sets of H and G^c.

We apply the technique used in the previous proof for cycles and replace vertices of H with cliques of the size equal to the cardinality of their projection, resulting in the graph H'. Due to the selection of the maximum independent set, H' is $\alpha(W)$ disjoint cliques, each of size $\alpha(G^c)$.

Now we will prove a lemma that is critical for the remainder of our proof.

Lemma 2. *If G is a perfect graph, then its vertices can be partitioned to $\alpha(G)$ disjoint cliques.*

Proof. We know that if a graph G is perfect, then for every induced subgraph of G, the maximum clique size is equal to the chromatic number. Moreover, by the perfect graph theorem, we know that if G is perfect, then its complement G^c is also perfect. So the maximum clique size of G^c, which is the same as the independence number of G, is equal to the chromatic number of G^c.

If we view vertices of the same color as a set, then coloring a graph is just partitioning its vertices into disjoint independent sets. Therefore, a coloring of G^c is equivalent to partitioning the vertices of G into cliques. Because the chromatic number of G^c is equal to $\alpha(G)$, this partition consists of $\alpha(G)$ cliques. Hence, the lemma is proven. □

Now, using the above theorem, we can observe that since cliques of G^c are, in fact, independent sets of G, these cliques have a maximum size of $\alpha(G^c)$. This implies that we can remove some vertices from H' to obtain H'' such that H'' precisely represents the clique partition of G.

Therefore, H'' is a spanning subgraph of G, and by adding the remaining edges to H'', we will complete the construction of G from H. □

Corollary 1. *From the above theorems we can see that the Conjecture 1 is true for many famous class of graphs such as bipartite, cycles, stars and friendship graphs.*

4 Open Problems and Future Directions

We have demonstrated the truth of Conjecture 1 for various classes of graphs. However, proving or disproving its correctness for all classes of graphs remains an intriguing future research direction. One interesting special case to investigate is when G is vertex-transitive. Due to the symmetry in vertex-transitive graphs, one can reasonably assume that Case 2 of the proof for Theorem 5 will not occur. Consequently, it appears probable that this conjecture holds true for this particular class of graphs.

Next, we have established that the calculation of the relative fractional packing number is $\mathcal{NP}$-Hard. Nonetheless, we have successfully determined its exact value when both graphs G and H are cycles. One can try to generalize this proof and compute the relative fractional packing number for other scenarios and obtain the relative fractional packing number for different classes of graphs.

Lastly, while investigating the properties of the relative fractional packing number, we were able to prove the following theorem:

Theorem 10. *For a graph G and a fractional number $\alpha(G) \leq a \leq \alpha^*(G)$, there exists a graph W_a such that:*

$$\frac{\alpha(G \boxtimes W_a)}{\alpha(W_a)} = a$$

The Lovász number of a graph G satisfies $\alpha(G) \leq \nu(G) \leq \alpha^*(G)$. By applying theorem 10, we conclude that there exists a graph W_ν such that $\frac{\alpha(G \boxtimes W_\nu)}{\alpha(W_\nu)} = \nu(G)$. This observation suggests the potential existence of an algorithmic method to construct the graph W_ν. Similar to the independence number, which is defined by an Integer Program, and the fractional packing number, which is defined by a Linear Program, the Lovász number is defined by a Semi Definite

Program. Therefore, if such a property exists, it could potentially be deduced by analyzing the constraints and optimal solutions of the associated Semi Definite Program.

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Appendix A

4.1 Generalized Independence Number of a graph

Definition 3. *For a graph G and an integer k, we define $\alpha_k(G)$ as the maximum number of vertices that can be selected from G such that, from any clique in G, at most k vertices are chosen. It is important to note that a vertex may be selected multiple times.*

Remark 3. *You can see that if $k = 1$, the above definition is the same as the normal independence number of a graph.*

Remark 4. *If we define α_k^* similar to the fractional packing number, then we have $\alpha_k^*(G) = k\alpha^*(G)$ because the linear program to compute $\alpha_k^*(G)$ is basically k times the linear program to calculate $\alpha^*(G)$.*

Note that Remark 4 does not hold for the integral version, meaning that we don't necessarily have equality in $\alpha_k(G) \geq k\alpha(G)$. A counterexample for this equality occurs when $k = 2$ and $G = C_5$. In this case, we have $2 \cdot \alpha(C_5) = 4$, but $\alpha_2(C_5) = 5$.

Theorem 11. *The generalized independence number is superadditive, meaning that for a graph G and two integral numbers k_1 and k_2, we have $\alpha_{k_1}(G) + \alpha_{k_2}(G) \leq \alpha_{k_1+k_2}(G)$.*

Proof. It is easy to see because if x is a solution to the integer program for finding $\alpha_{k_1}(G)$ and y is a solution to the integer program for finding $\alpha_{k_2}(G)$, then $x + y$ is a solution to the integer program for finding $\alpha_{k_1+k_2}(G)$. □

Now, we show we can calculate the fractional packing number using the generalized version of the independence number.

Theorem 12. *We have $\alpha_k(G \boxtimes W) \leq \alpha^*(G)\alpha_k(W)$.*

Proof. The relaxed linear program (LP) for calculating the fractional packing number is given by:

$$\begin{array}{ll} \text{maximize} & \sum\limits_{u \in V} x_u \\ \text{subject to} & \sum\limits_{u \in C} x_u \leq 1 \quad \forall\, \textit{Clique } C \\ & x \geq 0 \end{array} \tag{1}$$

Let A be a maximum k-generalized independent set in $G \boxtimes W$. For a vertex $u \in G$, we define A_u as the projection of A onto vertex u, i.e., $A_u = v \in W, |, (u, v) \in A$. It can be easily seen that A_u is still a k-generalized independent set, and for a set of vertices in a clique C in G, the union of their projections is also a k-generalized independent set.

Now, let us set $x_v = \frac{|A_u|}{\alpha_k(W)}$ for all vertices v in W. By the above arguments, we can conclude that x is a feasible solution for the LP 1. Hence, we have:

$$\sum_{u \in V} x_u = \sum_{u \in V} \frac{|A_u|}{\alpha_k(W)} = \frac{\alpha_k(G \boxtimes W)}{\alpha_k(W)} \leq \alpha^*(G)$$

Thus, we have shown that $\alpha_k(G \boxtimes W) \leq \alpha^*(G)\alpha_k(W)$, as desired. □

Remark 5. *If we set W to be a graph with a single vertex, then the above inequality says that $\alpha_k(G) \leq k\alpha^*(G)$.*

Theorem 13. $\sup\limits_{\textit{Graph } W} \frac{\alpha_k(G \boxtimes W)}{\alpha_k(W)} = \alpha^*(G)$.

Proof. Based on Theorem 12, we have one side of the inequality. For the other side, we need to show that there exists a graph W for which equality holds. Let x^* be the optimal solution to the LP 1, and let N be a large integer such that for any vertex $i \in G$, the number $n_i := x_i^* N$ is an integer.

Consider the graph $W = G^c(n_1, n_2, \ldots, n_{|V|})$ where each vertex u_i is repeated n_i times. We claim that this graph maximizes the expression $\frac{\alpha_k(G \boxtimes W)}{\alpha_k(W)}$, and equality holds for this graph.

To find $\alpha_k(G \boxtimes W)$, we can pick vertex u n_u times. Since for any clique C, we have $\sum\limits_{u \in C} n_u = N \sum\limits_{u \in C} x^* \leq N$, this new set is a feasible solution for $\alpha_k(G \boxtimes W)$, and its value is $\sum\limits_{u \in V} n_u = N \sum\limits_{u \in V} x^* = N\alpha^*(G)$. Now, by Remark 5, this is the optimal solution. Therefore, we have shown that $\sup\limits_{\text{Graph } W} \frac{\alpha_k(G \boxtimes W)}{\alpha_k(W)} = \alpha^*(G)$. □

Theorem 14. *For any graph G, there exists an integer k such that $\alpha_k(G) = k\alpha^*(G)$.*

Proof. Define x^* and N as in the previous theorem. We claim that $\alpha_N(G) = N\alpha^*(G)$. To find $\alpha_N(G)$, we can select vertex u n_u times. Since for any clique C, we have $\sum_{u \in C} n_u = N \sum_{u \in C} x^* \leq N$, this new set is a feasible solution for $\alpha_N(G)$, and its value is $\sum_{u \in V} n_u = N \sum_{u \in V} x^* = N\alpha^*(G)$.

Now, by Remark 5, we know that this is the optimal solution, and equality holds. Therefore, we have found an integer $k = N$ with the desired property. Hence, we have shown that for any graph G, there exists an integer k such that $\alpha_k(G) = k\alpha^*(G)$. □

Theorem 15. *For a collection of graphs $G_1, \ldots, G_k$, there exists an integer k such that for every i, $\alpha_k(G_i) = k\alpha^*(G_i)$.*

Proof. Using Theorem 14 and the superadditivity of the generalized independent number, we can see that if we set $k = N_1 \times \cdots \times N_k$, where N_i is a large integer such that multiplying the optimal solution of the LP for graph G_i by N_i gives an integer answer, then we have the desired property. □

Theorem 16. *For any graph G, we have $\lim_{n\to\infty} \frac{\alpha_n(G)}{n} = \alpha^*(G)$.*

Proof. First, set N to be the large number for which Theorem 14 holds. Now, consider a number $M = Nq + r$, where $q, r \in \mathbb{N}$ and $r < N$. By the superadditivity of the generalized independence number, we have $(M - N)\alpha^*(G) \leq qN\alpha^*(G) \leq qN\alpha^*(G) + \alpha_r(G) \leq \alpha_M(G)$. Additionally, using Remark 5, we have $\alpha_M(G) \leq M\alpha^*(G)$.

By applying the sandwich theorem, we can conclude:

$$\lim_{n\to\infty} \frac{(n-N)\alpha^*(G)}{n} \leq \lim_{n\to\infty} \frac{\alpha_n(G)}{n} \leq \lim_{n\to\infty} \frac{n\alpha^*(G)}{n}$$

Since N is a constant, we can simplify further and state that $\lim_{n\to\infty} \frac{\alpha_n(G)}{n} = \alpha^*(G)$. Thus, the theorem is proved.

The similarity of the above theorem with the definition of the fractional chromatic number is really interesting. If we take the dual of the integer program used to calculate the k-generalized independent number of a graph G, it corresponds to covering the vertices of G with cliques such that each vertex is covered by at least k cliques. If we assign different colors to each vertices in different cliques, it becomes a k-fold coloring of the vertices of G^c, which is one of the ways we define the fractional chromatic number. □