Relative Fractional Packing Number and Its Properties

Mehrshad Taziki

Abstract

The concept of the *relative fractional packing number* between two graphs G and H, initially introduced in [1], serves as an upper bound for the ratio of the zero-error Shannon capacity of these graphs. Defined as:

$$\sup_{W} \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)}$$

where the supremum is computed over all arbitrary graphs and $\boxtimes$ denotes the strong product of graphs.

This article delves into various critical theorems regarding the computation of this number. Specifically, we address its $\mathcal{NP}$-hardness and the complexity of approximating it. Furthermore, we develop a conjecture for a necessary and sufficient conditions for this number to be less than one, we also validate this conjecture for specific graph families. Additionally, we present miscellaneous concepts and introduce a generalized version of the independence number, that gives insights that could significantly contribute to the study of the relative fractional packing number.

1 Introduction

1.1 Preliminaries

In this section, we introduce the necessary concepts and background knowledge related to our study.

Let G be a simple graph, we denote the independence number of G by $\alpha(G)$. Computing the independence number is a known $\mathcal{NP}$-Hard problem [3], and it is even Poly-APX-complete [2]. Two sets of vertices, S and T, are said to be disconnected if and only if their intersection is empty and there are no edges between them.

The strong product of two graphs G and H, denoted as $G \boxtimes H$, is a graph product commonly utilized in combinatorics and information theory. The vertex set of $G \boxtimes H$ is the Cartesian product of the vertex sets of G and H. Two vertices (u_1, v_1) and (u_2, v_2) in $G \boxtimes H$ are adjacent if and only if any of the following conditions hold:

- $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$
- $u_1 = u_2$ and $v_1v_2 \in E(H)$
- $u_1u_2 \in E(G)$ and $v_1 = v_2$

Using this product, the Shannon zero-error capacity of a graph G is defined as $\theta(G)$, which is obtained as the limit of $\sqrt[n]{\alpha(G^{\boxtimes n})}$ as n approaches infinity. The Shannon zero-error capacity plays a crucial role in evaluating the zero-error capacity of a channel represented by the graph G [6].

Calculating the Shannon zero-error capacity efficiently remains a challenge, and even for small graphs like C_7, the exact value is unknown. Lovász showed that $\theta(C_5) = \sqrt{5}$. To the best of our knowledge, the Lovász number of a graph [7] and the Haemers' number of a graph [4] are the most commonly used upper bounds for the Shannon capacity of a graph.

The Integer Programming formulation of the independence number of a graph G is as follows:

$$\begin{aligned} \text{maximize} \quad & \sum_{v \in V} x_v \\ \text{subject to} \quad & \sum_{v \in C} x_v \leq 1 \; \forall C \in \mathcal{C} \\ & x \geq 0 \end{aligned}$$

Here, $\mathcal{C}$ represents the set of all cliques in the graph G. The optimal value of the relaxed linear program of the optimization problem above is known as the fractional packing number, denoted by $\alpha^*(G)$.

In [5], Hales demonstrated that:

$$\alpha^*(G) = \sup_W \frac{\alpha(G \boxtimes W)}{\alpha(W)}$$

Here, the supremum is taken over all arbitrary graphs W, additionally, Hales proved that this supremum it is indeed a maximum. He also constructed a maximizer for this term.

Motivated by this, Alipour and Gohari [1] introduced the concept of the relative fractional packing number of two graphs. It is defined as follows:

$$\sup_W \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)}$$

They provided a linear programming formulation to compute this number and established several properties relating to bounding the ratios of different graph properties. We delve deeper into this topic in Section 3.

1.2 Our Contribution

In this paper, we focus on the fractional packing number and its significance. In our research, we aim to explore the applications of this concept and establish its connections with other graph invariants, especially, the Shannon number and the independence number of graphs. Throughout this paper, we present a collection of theorems and properties that. Our objective is to provide valuable insights that can contribute to the discovery of new findings related to this graph quantity.

One of the primary objectives of our study was to investigate the computational complexity of computing and approximating the fractional packing number. By drawing parallels between this quantity and the independence number of a graph, we were able to establish that computing the fractional packing number is a $\mathcal{NP}$-Hard problem. Furthermore, we demonstrated that it is also Poly-APX-hard, showing the challenge to develope efficient approximation algorithms.

In our research, we specifically examined the case when both graphs involved in calculating the fractional packing number are cycles and we were able to calculate the exact value of this quantity in this case. By employing a generalized technique we derived lower bounds for this quantity in the general case. We then mentioned some tight cases for this lower-bound.

Section 3 of this paper is dedicated to the investigation of scenarios where the fractional packing number is below one. We aimed to identify necessary and sufficient conditions for this phenomenon and formulated a conjecture to address this concept. Additionally, we provided evidence for the validity of this conjecture by proving its soundness for well-known families of graphs, including perfect graphs and cycles.

Lastly, we introduced a generalized concept of the independence number and established a set of theorems concerning this quantity. We also highlighted the close relationship between this generalized independence number and the fractional packing number, suggesting the potential utilization of the independence number in future studies of the relative fractional packing number.

2 Calculating the Relative Fractional Packing Number

This section mentions some results regarding the calculating the $\alpha^*(G|H)$. firstly, we discuss some theorems about the complexity of calculating the relative fractional packing number and its hardness of approximation. Moreover, we will calculate this number exactly when both graphs G and H are a cycle. Finally, we will generalize our previous technique to get some bounds on this number.

2.1 Hardness Results

Theorem 1. *The problem of computing $\alpha^*(G|H)$ is $\mathcal{NP}$-Hard.*

Proof. We will reduce the problem of finding the independence number of a graph to this problem, therefore, showing this problem is $\mathcal{NP}$-Hard.

Imagine graph G is a single vertex graph. In this case we would have:

$$\alpha^*(G|H) = \max_W \frac{\alpha(G \boxtimes W)}{\alpha(H \boxtimes W)} = \max_W \frac{\alpha(W)}{\alpha(H \boxtimes W)} = \frac{1}{\inf_W \frac{\alpha(H \boxtimes W)}{\alpha(W)}} = \frac{1}{\alpha(H)}$$

Where the last equality is because $\alpha(H \boxtimes W) \geq \alpha(H) \times \alpha(W)$. $\square$

One can easily verify that the above reduction is in fact an approximation-preserving reduction, therefore, since the problem of finding the independence number is Poly-APX-hard, the factional packing number is also Poly-APX-hard.

Remark 1. *The relative fractional packing number can't be approximated to a constant factor in polynomial time unless $\mathcal{P} = \mathcal{NP}$.*

2.2 Relative Fractional Packing Number for Two Cycles

In this section, we are going to calculate the fractional packing number when both graphs G and H are cycles, but first, we are going to mention a lemma, that is proved at [1] which helps us on these calculations.

Lemma 1. *If graph G is vertex-transitive with n vertices, then for any arbitrary graph H we have:*

$$\alpha^*(G|H) = \frac{n}{\alpha(G^c \boxtimes H)}$$

Since cycles are vertex transitive graphs, we apply this lemma to calculate their relative fractional packing number.

Before stating the theorem we are going to define the projection of an Independence set to a vertex as follows:

Definition 1. *for an independence set I of the graph $G \boxtimes H$ and a vertex $v \in G$, we define the projection of I to v as the set $\{u \in H \mid (v, u) \in I\}$. The definition is similar when projecting to a vertex in H.*

Theorem 2. *For two numbers $n\,, m \geq 3$, let $f(n,m)$ be the following function:*

$$f(n,m) = \begin{cases} \frac{n}{m} & m \text{ is even} \\ \frac{n}{m-1} & n \text{ is even and } m \\ \frac{n}{m} & n \text{ and } m \text{ are odd and } n \leq m \\ \frac{n}{m-1} & n \text{ and } m \text{ are odd and } m < n \end{cases}$$

now for any two cycles C_n and C_m we have $\alpha^*(C_n, C_m) = f(n,m)$.

Proof. Since even cycles are perfect graphs if the first case happens, then we have

$$\alpha^*(C_n, C_m) = \frac{n}{C_n^c \boxtimes C_m)} = \frac{n}{\alpha(C_n^c) \times \alpha(C_m)}$$

now $\alpha(C_n^c) = 2$ and $\alpha(C_m) = \frac{m}{2}$ therefore,

$$\frac{n}{\alpha(C_n^c) \times \alpha(C_m)} = \frac{n}{m}$$

The second case goes similarly, if C_n is an even cycle, then it would be a perfect graph and by the perfect graph theorem [8], C_n^c will also be a perfect graph so the calculations goes as follows:

$$\alpha^*(C_n, C_m) = \frac{n}{\alpha(C_n^c \boxtimes C_m)} = \frac{n}{\alpha(C_n^c) \times \alpha(C_m)}$$

now this time since H is odd $\alpha(C_m) = \frac{m-1}{2}$ therefore,

$$\frac{n}{\alpha(C_n^c) \times \alpha(C_m)} = \frac{n}{m-1}$$

For the two latter cases imagine that u is an arbitrary vertex in C_m then the size of the projection of any maximum independent set of $C_n^c \boxtimes C_m$ to u is at most two because the largest possible independent set in C_n^c has size two. Now if for a vertex u we have $|A_u| = 2$ then the set A_u should contain two adjacent vertices in C_n but since two adjacent vertices in C_n are a dominating set in C_n^c then the neighbors of u in G_m have empty projections, therefore the average projection size is less than or equal to one meaning than $\alpha(C_n^c \boxtimes C_m) \leq m$. additionally its obvious that $\alpha(C_n^c \boxtimes C_m) \geq \alpha(C_n^c) \times \alpha(C_m) \geq m-1$. So we only have to check which of these two possibilities happens. when n and m are both odd and $n \leq m$ then we can have the following independent set of $C_n^c \boxtimes C_m$ with size m.

Imagine vertices of C_n are $u_1, \ldots, u_n$ and vertices of C_m are $v_1, \ldots, v_m$ in order. Now the claimed independent set for $C_n^c \boxtimes C_m$ is as follows:

$$A = \{(u_1, v_1), \ldots, (u_n, v_n), (u_1, v_{n+1}), (u_2, v_{n+2}), (u_1, v_{n+3}), \ldots, (u_2, v_m)\}$$

you can easily verify this is a valid independent set which means $\alpha(C_n^c \boxtimes C_m) = m$ and therefore, $\alpha^*(C_n, C_m) = \frac{n}{m}$.

For the final case we show that its impossible to have an independent set for $C_n^c \boxtimes C_m$ with size m. Imagine such an independent set A exists, now based on previous reasoning, for any vertex $v \in c_m$, size of the projection of A to v is exactly one, we show this projection with the set A_v. Additionally, for any two adjacent vertices v_i and v_{i+1} in H, because their projections should be disconnected then if A_{v_i} contains the vertex u_j, $A_{v_{i+1}}$ either contains u_{j+1} or u_{j-1} Now we label the edges of C_m with the following rule. Label the edge between v_i and v_{i+1} in C_m with a $+1$ if $A_{u_{i+1}}$ contains v_{j+1}, otherwise, put a -1 on that edge. Now if we take the summation of all the number on the edges of C_m we should get 0 because we are looping from a number back to itself in a cycle. However, this summation consists of m $+1$ or -1 modulo n. Basic number theory tells us that this summation can't be equal to 0 modulo n when $m < n$ and both n and m are odd. This implies that $\alpha(C_n^c \boxtimes C_m) = m-1$ which means in this case $\alpha^*(C_n|C_m) = \frac{n}{m-1}$. □

3 Necessarily and Sufficient Conditions for Relative Fractional Packing Number to Be Bellow One

As showed in [1], the relative fractional packing number can serve as an upper-bound on the ratio of many different graph properties, in fact, they showed the following theorem.

Theorem 3. *Let X(G) be any of the followings: independence number, the zero-error Shannon capacity, the fractional packing number, the Lovász number of a graph G. Then we have:*

$$\frac{X(G)}{X(H)} \leq \alpha^*(G|H)$$

The above theorem mentions one important usage for the relative fractional packing number. If we find a necessarily and sufficient condition on graphs G and H so that $\alpha^*(G|H) \leq 1$, the above theorem mentions that that we have found a sufficient condition for $X(G) \leq X(H)$ for all choices of X.

In this section we are going to propose a conjecture for such a condition. Additionally, we will prove this conjecture for some famous families of graphs.

Definition 2. *for a graph G we define $Expand(G)$ as the set of all graphs that can be obtained by a sequence of the following three operations on G Iteratively.*

1. *Remove a vertex v from G.*

2. *add a new edge to G.*

3. *replace a vertex v by a clique of a arbitrarily size.*

The latter operations just puts a clique of size k instead of v and connects these k new vertices to all neighbors of v.

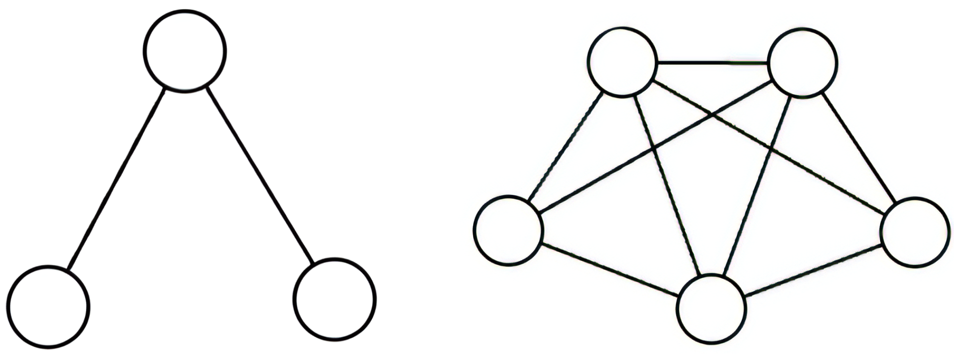

Figure 1: The graph P_3 before and after replacing the middle vertex with a clique of size 3 using the third operation.

You can trivially verify that all the above operations, when applied to a graph G, will not increase $\alpha(G \boxtimes W)$ for any graph choice of W, therefore, if we have two graphs G and H such that $G \in expand(H)$, then we can conclude $\alpha(G \boxtimes W) \leq \alpha(H \boxtimes W)$ for any graph W which means $\alpha^*(G|H) \leq 1$.

Remark 2. *Note that by the above operations, one can merge (contract) any two vertices v and u in G.*

we only have to delete one of them, for instance v, by applying the first operation, and then, add new edges between u and the neighbors of v by applying the second rule.

Now it's time to mention the main conjecture.

Conjecture 1. $\alpha^*(G|H) \leq 1$ *if and only if* $G \in expand(H)$.

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