Season 4: Stability Analysis, State Feedback, and LQR

Learning Outcomes:

- Analyze system stability using eigenvalues, Lyapunov theory, and BIBO criteria
- Solve Lyapunov equations for continuous and discrete systems
- Test matrix definiteness using Cholesky decomposition
- Design state feedback controllers using pole placement
- Apply LQR (Linear Quadratic Regulator) for optimal control
- Understand and solve Riccati equations
- Design digital controllers using discretization

Prerequisites: Seasons 1-3

MATLAB Version: R2025b

Toolboxes Required: Control System Toolbox

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```
close all; clear; clc;
rng(0);
```

Section 4.1: Stability - Eigenvalue Analysis

Mathematical Background

For continuous-time system $\dot{x} = Ax$:

- **Stable**: all eigenvalues have Re(λ) < 0
- **Marginally stable**: eigenvalues on imaginary axis, none with Re(λ) > 0
- **Unstable**: at least one eigenvalue with Re(λ) > 0

For discrete-time system $x[k+1] = Ax[k]$:

- **Stable**: all eigenvalues have λ < 1
- **Unstable**: at least one eigenvalue with λ ≥ 1

Example 1: Stable system

```
A_stable = [-1  0;
             0 -2];

eig_stable = eig(A_stable);
fprintf('Example 1: Stable system\n');
```

```
Example 1: Stable system
```

```
fprintf('A =\n'); disp(A_stable);
```

```
A =
    -1     0
     0    -2
```

```
fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f ', eig_stable);
```

```
-2.0000 -1.0000
```

```
fprintf('\n');

if all(real(eig_stable) < 0)
    fprintf('✓ STABLE (all eigenvalues have negative real part)\n\n');
else
    fprintf('✗ UNSTABLE\n\n');
end
```

```
✓ STABLE (all eigenvalues have negative real part)
```

```

```

Example 2: Unstable system

```
A_unstable = [1   0;
              0  -3];

eig_unstable = eig(A_unstable);
fprintf('Example 2: Unstable system\n');
```

```
Example 2: Unstable system
```

```
fprintf('A =\n'); disp(A_unstable);
```

```
A =
     1     0
     0    -3
```

```
fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f ', eig_unstable);
```

```
-3.0000 1.0000
```

```
fprintf('\n');

if all(real(eig_unstable) < 0)
    fprintf('√ STABLE\n\n');
else
    fprintf('✗ UNSTABLE (eigenvalue at %.4f has positive real part)\n\n', ...
        eig_unstable(find(real(eig_unstable) > 0, 1)));
end
```

```
✗ UNSTABLE (eigenvalue at 1.0000 has positive real part)
```

Example 3: Marginally stable (oscillatory)

```
A_marginal = [0  1;
             -1  0];

eig_marginal = eig(A_marginal);
fprintf('Example 3: Marginally stable (oscillatory)\n');
```

```
Example 3: Marginally stable (oscillatory)
```

```
fprintf('A =\n'); disp(A_marginal);
```

```
A =
     0     1
    -1     0
```

```
fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f%+.4fi ', real(eig_marginal), imag(eig_marginal));
```

```
0.0000+0.0000i 1.0000-1.0000i
```

```
fprintf('\n');
fprintf('Pure imaginary eigenvalues → Marginally stable (oscillatory)');
```

```
Pure imaginary eigenvalues → Marginally stable (oscillatory)
```

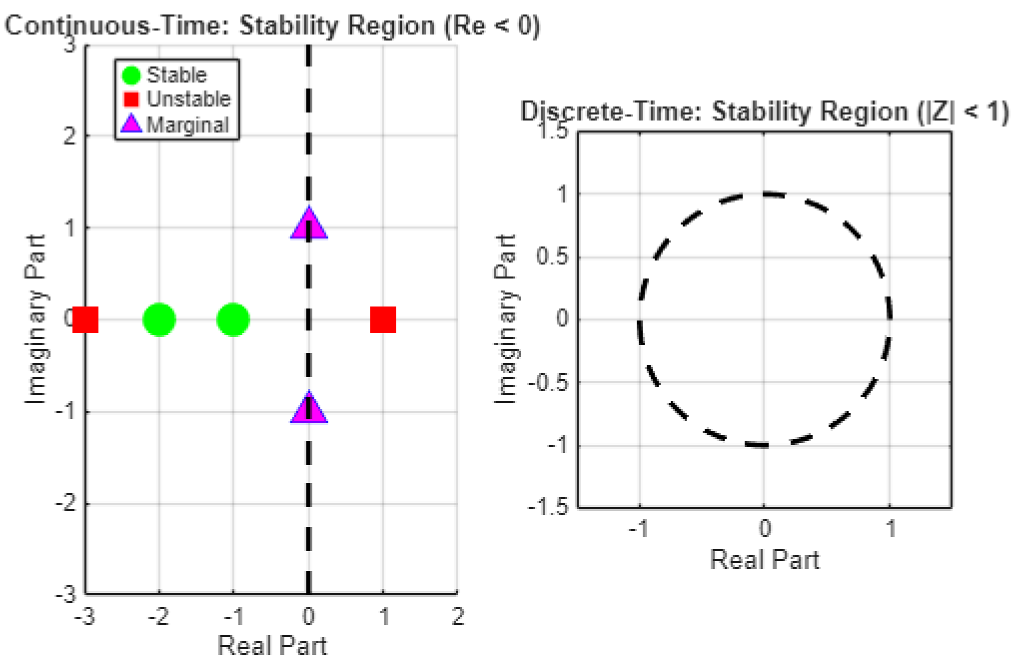
Visualize eigenvalues on complex plane

```
figure('Name', 'Eigenvalue Locations and Stability');
subplot(1,2,1);  % Continuous-time
hold on;
plot(real(eig_stable), imag(eig_stable), 'go', 'MarkerSize', 12, ...
    'MarkerFaceColor', 'g', 'DisplayName', 'Stable');
plot(real(eig_unstable), imag(eig_unstable), 'rs', 'MarkerSize', 12, ...
    'MarkerFaceColor', 'r', 'DisplayName', 'Unstable');
plot(real(eig_marginal), imag(eig_marginal), 'b^', 'MarkerSize', 12, ...
```

```
    'MarkerFaceColor', 'magenta', 'DisplayName', 'Marginal');

% Draw stability boundary (imaginary axis)
plot([0 0], [-3 3], 'k--', 'LineWidth', 2, 'HandleVisibility', 'off');
xlim([-3 2]);
ylim([-3 3]);
grid on;
xlabel('Real Part');
ylabel('Imaginary Part');
title('Continuous-Time: Stability Region (Re < 0)');
legend('Location', 'best');

subplot(1,2,2);  % Discrete-time stability region (unit circle)
theta = linspace(0, 2*pi, 100);
plot(cos(theta), sin(theta), 'k--', 'LineWidth', 2);
hold on;
grid on;
axis equal;
xlim([-1.5 1.5]);
ylim([-1.5 1.5]);
xlabel('Real Part');
ylabel('Imaginary Part');
title('Discrete-Time: Stability Region (|Z| < 1)');
```



Section 4.2: Lyapunov Stability Theory

Mathematical Background

Lyapunov's Direct Method:

For $\dot{x} = Ax$, if ∃ positive definite matrix P such that:

$$A^T P + PA = -Q$$

where Q > 0, then the system is asymptotically stable.

Lyapunov Equation: $A^T P + PA + Q = 0$

```
A_lyap = [-1  0.5;
          -0.5 -2];

fprintf('System matrix A:\n');
```

```
System matrix A:
```

```
disp(A_lyap);
```

```
   -1.0000    0.5000
   -0.5000   -2.0000
```

```


fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f ', eig(A_lyap));
```

```
-1.5000 -1.5000
```

```
fprintf('\n\n');
```

Solve Lyapunov equation: A'P + PA + Q = 0

```
Q = eye(2);  % Choose Q = I (positive definite)
```

Solving Lyapunov equation A^TP + PA + Q = 0 with Q = I:

```
fprintf('Solution P:\n');
```

```
Solution P:
```

```
P = lyap(A_lyap', Q);
disp(P);
```

```
    0.4815    0.0370
    0.0370    0.2593
```

Verify solution

```
residual = A_lyap'*P + P*A_lyap + Q; % must be near zero
fprintf('Residual A^TP + PA + Q:\n');
```

```
Residual A^TP + PA + Q:
```

```
disp(residual);
```

```
   1.0e-15 *

         0   -0.0278
   -0.0278   -0.2220
```

```
fprintf('Residual norm: %.2e\n\n', norm(residual));
```

```
Residual norm: 2.25e-16
```

Check if P is positive definite

```
eig_P = eig(P);
fprintf('Eigenvalues of P: ');
```

```
Eigenvalues of P:
```

```
fprintf('%.4f ', eig_P);
```

```
0.2532 0.4875
```

```
fprintf('\n');

if all(eig_P > 0)
    fprintf('✓ P is positive definite → System is STABLE\n\n');
else
    fprintf('✗ P is not positive definite → System is UNSTABLE\n\n');
end
```

```
✓ P is positive definite → System is STABLE
```

Try with unstable system

```
A_unstable_lyap = [0.5  2;
                   0  1];

fprintf('Testing unstable system:\n');
```

```
Testing unstable system:
```

```
fprintf('A =\n'); disp(A_unstable_lyap);
```

```
A =
    0.5000    2.0000
         0    1.0000
```

```
fprintf('Eigenvalues:\n');disp(eig(A_unstable_lyap));
```

```
Eigenvalues:
    0.5000
    1.0000
```

The "try ... catch" block is used to handle errors safely.

MATLAB first runs the code inside the "try" section.

If everything runs correctly, it skips the "catch" section.

But if an error happens (for example, the Lyapunov equation cannot be solved), MATLAB immediately jumps to the "catch" section instead of stopping the program.

This lets you handle the error gracefully — for example, by printing a message instead of crashing the script. use " help try" to read more.

```
try
    P_unstable = lyap(A_unstable_lyap', Q);
    fprintf('P computed:\n');
    disp(P_unstable);
    fprintf('Eigenvalues of P: ');
    fprintf('%.4f ', eig(P_unstable));
    fprintf('\n');
catch ME
    fprintf('√ Lyapunov equation has no solution\n');
end
```

```
P computed:
   -1.0000    1.3333
    1.3333   -3.1667
Eigenvalues of P:
-3.8013 -0.3654
```

The Lyapunov equation always has a formal solution,but only for stable A will P be positive definite.

Negative eigenvalues of P confirm the system is unstable.

Section 4.3: Matrix Definiteness and Cholesky Decomposition

Concept Overview

A symmetric matrix M is:

- **Positive definite**: x^T M x > 0 for all x ≠ 0
- **Positive semi-definite**: x^T M x ≥ 0

Cholesky Test: M > 0 □ M = L L^T exists (L lower triangular)

Positive definite matrix

```
M_pd = [4  1;
        1  3];

fprintf('Matrix M:\n');
```

```
Matrix M:
```

```
disp(M_pd);
```

```
     4     1
     1     3
```

Test using Cholesky decomposition

```
try
    L = chol(M_pd, 'lower');
    fprintf('Cholesky decomposition succeeded:\n');
    fprintf('L (lower triangular):\n');
    disp(L);
    fprintf('Verification L*L^T:\n');
    disp(L*L');
    fprintf('✓ Matrix is POSITIVE DEFINITE\n\n');
catch
    fprintf('✗ Matrix is NOT positive definite\n\n');
end
```

```
Cholesky decomposition succeeded:
L (lower triangular):
    2.0000         0
    0.5000    1.6583
Verification L*L^T:
     4     1
     1     3
✓ Matrix is POSITIVE DEFINITE
```

Check eigenvalues

```
fprintf('Eigenvalues of M: ');
```

```
Eigenvalues of M:
```

```
fprintf('%.4f ', eig(M_pd));
```

```
2.3820 4.6180
```

```
fprintf('All positive → positive definite\n\n');
```

```
All positive → positive definite
```

Not positive definite matrix

```
M_nd = [1  2;
        2  1];

fprintf('Testing semi-definite matrix:\n');
```

```
Testing semi-definite matrix:
```

```
fprintf('M =\n'); disp(M_nd);
```

```
M =
     1     2
     2     1
```

```

try
    L_nd = chol(M_nd, 'lower');
```

```
    fprintf('√ Positive definite\n');
catch
    fprintf('✗ NOT positive definite (Cholesky failed)\n');
end
```

```
✗ NOT positive definite (Cholesky failed)
```

```
fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f ', eig(M_nd));
```

```
-1.0000 3.0000
```

```
fprintf('(Has negative eigenvalue)\n\n');
```

```
(Has negative eigenvalue)
```

Section 4.4: BIBO Stability

Mathematical Background

Bounded-Input Bounded-Output (BIBO) Stability:

System is BIBO stable if bounded input produces bounded output.

For LTI systems: BIBO stable □ all poles have Re(s) < 0

Create transfer functions

```
num1 = [1];
den1 = [1 3 2];  % Poles at s = -1, -2

num2 = [1];
den2 = [1 -1 -2];  % Poles at s = -2, +1

fprintf('Transfer Function 1:\n');
```

```
Transfer Function 1:
```

```
G1 = tf(num1, den1)
```

```
G1 =

        1
  -------------
  s^2 + 3 s + 2

Continuous-time transfer function.
Model Properties
```

```
poles1 = pole(G1);
fprintf('Poles: ');
```

```
Poles:
```

```
fprintf('%.4f ', poles1);
```

```
-2.0000 -1.0000
```

```
fprintf('\n');
if all(real(poles1) < 0)
    fprintf('✓ BIBO STABLE\n\n');
else
    fprintf('✗ BIBO UNSTABLE\n\n');
end
```

```
✓ BIBO STABLE
```

```

fprintf('Transfer Function 2:\n');
```

```
Transfer Function 2:
```

```
G2 = tf(num2, den2)
```

```
G2 =

       1
  -----------
  s^2 - s - 2

Continuous-time transfer function.
Model Properties
```

```
poles2 = pole(G2);
fprintf('Poles: ');
```

```
Poles:
```

```
fprintf('%.4f ', poles2);
```

```
2.0000 -1.0000
```

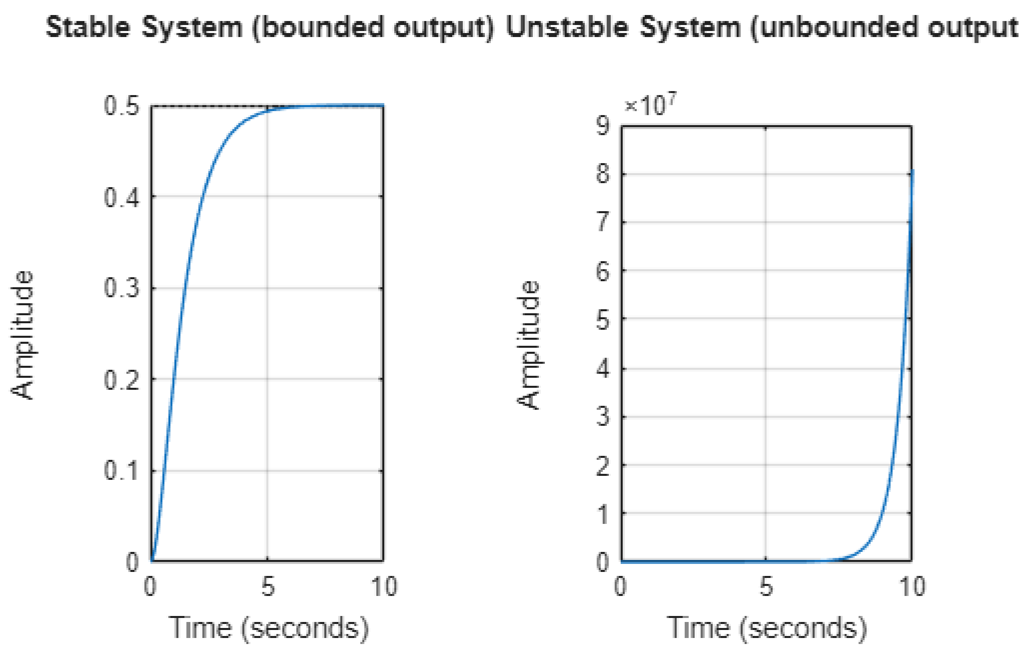
```
fprintf('\n');
if all(real(poles2) < 0)
    fprintf('✓ BIBO STABLE\n\n');
else
    fprintf('✗ BIBO UNSTABLE (pole at s = %.2f)\n\n', poles2(real(poles2)>0));
end
```

```
✗ BIBO UNSTABLE (pole at s = 2.00)
```

Demonstrate with step response

```
figure('Name', 'BIBO Stability Comparison');
subplot(1,2,1);
step(G1, 10);
```

```
grid on;
title('Stable System (bounded output)');
subplot(1,2,2);
try
    step(G2, 10);
    grid on;
    title('Unstable System (unbounded output)');
catch
    fprintf('Unstable system produces unbounded output\n');
end
```



Section 4.5: State Feedback and Pole Placement

Mathematical Background

State feedback: $u = -Kx + r$

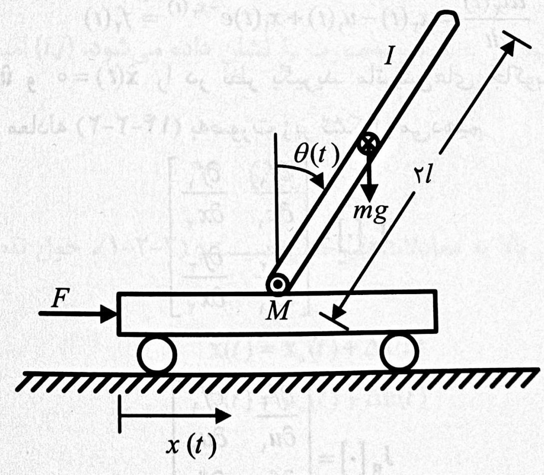
Closed-loop system: $\dot{x} = (A - BK)x + Br$

Goal: Choose K to place closed-loop poles at desired locations

Requirement: System must be controllable

$$(m+M)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^{2}\sin\theta=F \qquad (۱۸\text{-}۲\text{-}۲)$$

$$\left(I+ml^{2}\right)\ddot{\theta}+ml\ddot{x}\cos\theta-mgl\sin\theta=0 \qquad (۱۹\text{-}۲\text{-}۲)$$



شکل ۲-۱ آونگ وارونه

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```
syms m M l I g real
syms x dx th dth F real

X  = [x; dx; th; dth];
u  = sym('F','real');

syms ddx ddth real

eq1 = (m+M)*ddx + m*l*ddth*cos(th) - m*l*dth^2*sin(th) - F == 0;
eq2 = (I + m*l^2)*ddth + m*l*ddx*cos(th) - m*g*l*sin(th) == 0;

S = solve([eq1, eq2],[ddx, ddth],'ReturnConditions',false);
ddx_expr  = simplify(S.ddx);
ddth_expr = simplify(S.ddth);

f = [ dx;
      ddx_expr;
      dth;
      ddth_expr ];

% Linearize: at upright equilibrium (x=0,dx=0,th=0,dth=0,u=0)
A_sym = jacobian(f, X);
B_sym = jacobian(f, u);

x_eq = [0; 0; 0; 0];
u_eq = 0;

A = simplify(subs(A_sym, [X; u], [x_eq; u_eq]));
```

```
B = simplify(subs(B_sym, [X; u], [x_eq; u_eq]));

C = eye(4);
D = zeros(4,1);

disp('A ='); disp(A);
```

A =

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\,l^2\,m^2}{M\,m\,l^2 + \mathrm{I}\,m + \mathrm{I}\,M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{g\,l\,m\,(M+m)}{M\,m\,l^2 + \mathrm{I}\,m + \mathrm{I}\,M} & 0 \end{pmatrix}$$

```
disp('B ='); disp(B);
```

B =

$$\begin{pmatrix} 0 \\ \frac{m\,l^2 + \mathrm{I}}{M\,m\,l^2 + \mathrm{I}\,m + \mathrm{I}\,M} \\ 0 \\ -\frac{l\,m}{M\,m\,l^2 + \mathrm{I}\,m + \mathrm{I}\,M} \end{pmatrix}$$

```
disp('C ='); disp(C);
```

```
C =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

```
disp('D ='); disp(D);
```

```
D =
     0
     0
     0
     0
```

```
m_val = 0.6;      % pendulum mass [kg]
M_val = 0.5;      % cart mass [kg]
l_val = 0.3;      % COM distance [m]
I_val = 0.006;    % pendulum inertia about COM [kg·m^2]
g_val = 9.81;     % gravity [m/s^2]

A_pend = double(subs(A, [m M l I g], [m_val M_val l_val I_val g_val]));
B_pend = double(subs(B, [m M l I g], [m_val M_val l_val I_val g_val]));
```

```
disp('A (numeric) ='); disp(A_pend);
```

```
A (numeric) =
         0    1.0000         0         0
         0         0   -9.4596         0
         0         0         0    1.0000
         0         0   57.8089         0
```

```
disp('B (numeric) ='); disp(B_pend);
```

```
B (numeric) =
         0
    1.7857
         0
   -5.3571
```

```
fprintf('Inverted pendulum system (4 states):\n');
```

```
Inverted pendulum system (4 states):
```

```
fprintf('Open-loop eigenvalues: ');
```

```
Open-loop eigenvalues:
```

```
fprintf('%.4f ', eig(A_pend));
```

```
0.0000 0.0000 7.6032 -7.6032
```

```
fprintf('\n');
fprintf('System is UNSTABLE (positive eigenvalues)\n\n');
```

```
System is UNSTABLE (positive eigenvalues)
```

Check controllability

```
C_ctrb = ctrb(A_pend, B_pend);
if rank(C_ctrb) == size(A_pend, 1)
    fprintf('✓ System is controllable → Pole placement possible\n\n');
    fprintf('The controllability rank is: %d',rank(C_ctrb));
    fprintf('The rank is: %d',size(A_pend,1));
else
    fprintf('✗ System not controllable → Cannot place all poles\n\n');
    fprintf('The controllability rank is: %d',rank(C_ctrb));
    fprintf('The rank is: %d',size(A_pend,1));
end
```

```
✓ System is controllable → Pole placement possible
The controllability rank is: 4
The rank is: 4
```

Desired pole locations (all in left half-plane for stability)

```
poles_desired = [-1, -1.5, -2, -2.5];
```

```
fprintf('Desired closed-loop poles: ');
```

```
Desired closed-loop poles:
```

```
fprintf('%.4f ', poles_desired);
```

```
-1.0000 -1.5000 -2.0000 -2.5000
```

Design state feedback using place

```
K = place(A_pend, B_pend, poles_desired);

fprintf('State feedback gain K:\n');
```

```
State feedback gain K:
```

```
disp(K);
```

```
   -0.1427   -0.3663  -14.1519   -1.4288
```

Closed-loop system

```
A_cl = A_pend - B_pend*K;
fprintf('Closed-loop eigenvalues: ');
```

```
Closed-loop eigenvalues:
```

```
fprintf('%.4f ', eig(A_cl));
```

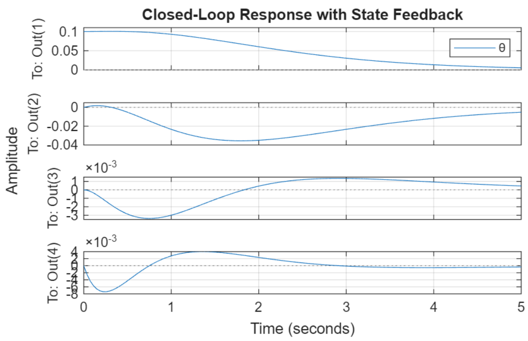
```
-2.5000 -2.0000 -1.5000 -1.0000
```

Verify poles are at desired locations

```
pole_error = sort(eig(A_cl)) - sort(poles_desired)';
fprintf('Pole placement error: %.2e\n\n', norm(pole_error));
```

```
Pole placement error: 3.16e-13
```

```

% Simulate closed-loop response
sys_cl = ss(A_cl, B_pend, eye(4), 0);
x0 = [0.1; 0; 0; 0];  % Initial angle perturbation

figure('Name', 'Pole Placement - Inverted Pendulum');
initial(sys_cl, x0, 5);
grid on;
title('Closed-Loop Response with State Feedback');
legend('θ', 'dθ/dt', 'x', 'dx/dt');
```




```
Warning: Ignoring extra legend entries.
```

Section 4.6: Ackermann's Formula

Mathematical Background

For single-input systems (m=1):

$$K = [0\ \ 0\ \ \ldots\ \ 0\ \ 1]\mathscr{C}^{-1}\alpha(A)$$

where α(s) is desired characteristic polynomial

```
A_acker = [0  1; -2 -3];
B_acker = [0; 1];

fprintf('Second-order system:\n');
```

```
Second-order system:
```

```
fprintf('A =\n'); disp(A_acker);
```

```
A =
     0     1
    -2    -3
```

```
fprintf('B =\n'); disp(B_acker);
```

```
B =
     0
     1
```

```

% Desired poles
p1 = -4;
p2 = -5;
fprintf('Desired poles: %.1f, %.1f\n', p1, p2);
```

```
Desired poles: -4.0, -5.0
```

```

% Using acker function
K_acker = acker(A_acker, B_acker, [p1 p2]);
fprintf('Gain using acker(): K = [%.4f  %.4f]\n', K_acker);
```

```
Gain using acker(): K = [18.0000  6.0000]
```

```
% Using place function for comparison
K_place = place(A_acker, B_acker, [p1 p2]);

fprintf('Gain using place():  K = [%.4f  %.4f]\n', K_place);
```

```
Gain using place():  K = [18.0000  6.0000]
```

```
fprintf('Difference: %.2e\n\n', norm(K_acker - K_place));
```

```
Difference: 7.32e-15
```

```
% Verify closed-loop poles
A_cl_acker = A_acker - B_acker*K_acker;
fprintf('Closed-loop poles: ');
```

```
Closed-loop poles:
```

```
fprintf('%.4f ', eig(A_cl_acker));
```

```
-4.0000 -5.0000
```

```
fprintf('\n\n');
```

Section 4.8: LQR - Linear Quadratic Regulator

Mathematical Background

(This topic is covered in Season 8 (Optimal Control) of Control Modern Fundamentals by A. Khaki Sedigh, which is not covered in class. Still useful to know and use.)

LQR Problem: Minimize cost function

$$J = \int_0^\infty (x^T Q x + u^T R u) dt$$

Solution: $u = -Kx$ where $K = R^{-1} B^T P$

P solves Continuous Algebraic Riccati Equation (CARE):

$$A^T P + PA - PBR^{-1} B^T P + Q = 0$$

```
A_lqr = [0  1; -1 -0.5];
B_lqr = [0; 1];
C_lqr = [1 0];
D_lqr = 0;

fprintf('System:\n');
```

```
System:
```

```
fprintf('A =\n'); disp(A_lqr);
```

```
A =
         0    1.0000
   -1.0000   -0.5000
```

```
fprintf('B =\n'); disp(B_lqr);
```

```
B =
     0
     1
```

```

% Design parameters
Q_lqr = diag([10, 1]);  % Penalize position more than velocity
R_lqr = 1;              % Input cost

fprintf('Cost matrices:\n');
```

```
Cost matrices:
```

```
fprintf('Q (state cost) =\n'); disp(Q_lqr);
```

```
Q (state cost) =
    10     0
     0     1
```

```
fprintf('R (input cost) = %.1f\n\n', R_lqr);
```

```
R (input cost) = 1.0
```

```

% Solve LQR
[K_lqr, S, poles_lqr] = lqr(A_lqr, B_lqr, Q_lqr, R_lqr);

fprintf('LQR Results:\n');
```

```
LQR Results:
```

```
fprintf('Optimal gain K = [%.4f  %.4f]\n', K_lqr);
```

```
Optimal gain K = [2.3166  1.9255]
```

```
fprintf('Solution to Riccati equation S:\n'); disp(S);
```

```
Solution to Riccati equation S:
    7.5446    2.3166
    2.3166    1.9255
```

```
fprintf('Closed-loop poles: ');
```

```
Closed-loop poles:
```

```
fprintf('%.4f ', poles_lqr);
```

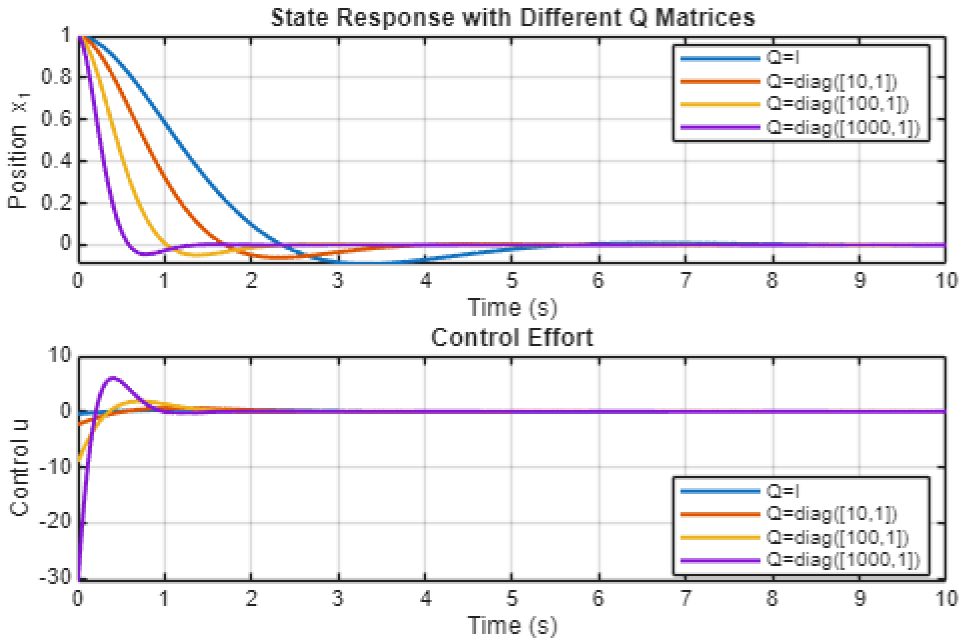
```
-1.2128 -1.2128
```

```
% Verify S satisfies CARE: A'S + SA - SBR^(-1)B'S + Q = 0
CARE_residual = A_lqr'*S + S*A_lqr - S*B_lqr*(R_lqr\B_lqr')*S + Q_lqr;
fprintf('CARE residual norm: %.2e\n\n', norm(CARE_residual));
```

```
CARE residual norm: 4.00e-15
```

Compare responses with different Q matrices

```
Q_values = {eye(2), diag([10,1]), diag([100,1]), diag([1000,1])};
Q_labels = {'Q=I', 'Q=diag([10,1])', 'Q=diag([100,1])', 'Q=diag([1000,1])'};

figure('Name', 'LQR - Effect of Q Matrix');
x0 = [1; 0];

for i = 1:length(Q_values)
    [K_temp, ~, ~] = lqr(A_lqr, B_lqr, Q_values{i}, R_lqr);
    A_cl_temp = A_lqr - B_lqr*K_temp;
    sys_cl_temp = ss(A_cl_temp, zeros(2,1), C_lqr, 0);

    [y, t, x] = initial(sys_cl_temp, x0, 10);

    subplot(2,1,1);
    plot(t, x(:,1), 'LineWidth', 1.5, 'DisplayName', Q_labels{i});
    hold on;

    % Compute control effort
    u = -K_temp * x';
    subplot(2,1,2);
    plot(t, u, 'LineWidth', 1.5, 'DisplayName', Q_labels{i});
    hold on;
end

subplot(2,1,1);
grid on;
xlabel('Time (s)');
ylabel('Position x_1');
title('State Response with Different Q Matrices');
legend('Location', 'best');

subplot(2,1,2);
grid on;
xlabel('Time (s)');
ylabel('Control u');
title('Control Effort');
legend('Location', 'best');
```




```
fprintf('Observation: Larger Q₁ → faster response but higher control effort\n\n');
```

```
Observation: Larger Q₁ → faster response but higher control effort
```

Section 4.9: Solving Riccati Equation Directly

Using care() and dare() for CARE and DARE

```
A_ricc = [0 1; -2 -3];
B_ricc = [0; 1];
Q_ricc = eye(2);
R_ricc = 1;

fprintf('Continuous-time Algebraic Riccati Equation (CARE):\n');
```

```
Continuous-time Algebraic Riccati Equation (CARE):
```

```
fprintf('A^TP + PA - PBR^(-1)B^TP + Q = 0\n\n');
```

```
A^TP + PA - PBR^(-1)B^TP + Q = 0
```

```
% Solve using care
[P_care, ~, K_care] = care(A_ricc, B_ricc, Q_ricc, R_ricc);

fprintf('Solution P (using care):\n');
```

```
Solution P (using care):
```

```
disp(P_care);
```

```
    1.2361    0.2361
    0.2361    0.2361
```

```
fprintf('Optimal gain K = R^(-1)B^TP:\n');
```

```
Optimal gain K = R^(-1)B^TP:
```

```
disp(K_care);
```

```
    0.2361    0.2361
```

```
% Compare with lqr
[K_lqr_comp, P_lqr_comp] = lqr(A_ricc, B_ricc, Q_ricc, R_ricc);

fprintf('Comparison with lqr():\n');
```

```
Comparison with lqr():
```

```
disp('P using CARE:'); disp(P_care);
```

```
P using CARE:
    1.2361    0.2361
    0.2361    0.2361
```

```
disp('P using LQR:'); disp(P_lqr_comp);
```

```
P using LQR:
    1.2361    0.2361
    0.2361    0.2361
```

```
fprintf('Difference in P: %.2e\n', norm(P_care - P_lqr_comp));
```

```
Difference in P: 2.24e-16
```

```
disp('K using CARE:'); disp(K_care);
```

```
K using CARE:
    0.2361    0.2361
```

```
disp('K using LQR:'); disp(K_lqr_comp);
```

```
K using LQR:
    0.2361    0.2361
```

```
fprintf('Difference in K: %.2e\n\n', norm(K_care - K_lqr_comp));
```

```
Difference in K: 1.76e-16
```

Discrete-time Riccati equation

There is a function for Discrete-time Riccati equation (DARE) which you can use by utilizing:

```
[P_dare, ~, K_dare] = dare(A_d, B_d, Q_ricc, R_ricc);
```

for the sake of staying on topic, we will skip it.

Section 4.10: Summary and Key Takeaways

Key Concepts Covered:

1. Stability analysis: eigenvalues, Lyapunov, BIBO
2. Lyapunov equations: `lyap`, `dlyap`
3. Matrix definiteness: Cholesky decomposition
4. State feedback: pole placement with `place`, `acker`
5. Optimal control: LQR design with `lqr`, `dlqr`
6. Riccati equations: `care`, `dare`
7. Digital control: discretization and discrete LQR

MATLAB Functions Mastered:

`eig`, `lyap`, `dlyap`, `chol`, `place`, `acker`, `lqr`, `dlqr`, `care`, `dare`, `c2d`, `d2c`, `pole`

Design Trade-offs:

- Q matrix: penalizes state deviation (larger = faster response)
- R matrix: penalizes control effort (larger = less aggressive)
- Pole placement: direct control over dynamics
- LQR: optimal balance between performance and effort