Season 3: Controllability, Observability, and Realization

Learning Outcomes:

- Test controllability and observability using rank and PBH criteria
- Understand and compute controllability/observability gramians
- Perform Kalman decomposition to identify structural properties
- Transform systems to canonical forms
- Compute minimal realizations
- Apply balanced realization for model reduction

Prerequisites: Season 1 & 2 (Mathematical Foundations, State-Space Modeling)

MATLAB Version: R2025b

Toolboxes Required: Control System Toolbox, Symbolic Math Toolbox

Table of Contents


```
close all; clear; clc;
rng(0);
```

Section 3.1: Controllability - Rank Test

Mathematical Background

A system (A,B) is controllable if the **controllability matrix** has full rank:

$$\mathscr{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

rank(C) = n □ system is controllable

Physical Meaning: Can reach any state from any initial state using input u(t)

Example 1: Fully controllable system

```
A1 = [0 1; -2 -3];
B1 = [0; 1];
```

```
fprintf('Example 1: Second-order system\n');
```

```
Example 1: Second-order system
```

```
fprintf('A =\n'); disp(A1);
```

```
A =
     0     1
    -2    -3
```

```
fprintf('B =\n'); disp(B1);
```

```
B =
     0
     1
```

```
% Compute controllability matrix
C_ctrb1 = ctrb(A1, B1);
rank_c1 = rank(C_ctrb1);
n1 = size(A1, 1);

fprintf('Controllability matrix C = [B AB]:\n');
```

```
Controllability matrix C = [B AB]:
```

```
disp(C_ctrb1);
```

```
     0     1
     1    -3
```

```
fprintf('Rank: %d, System order: %d\n', rank_c1, n1);
```

```
Rank: 2, System order: 2
```

```
if rank_c1 == n1
    fprintf('✓ System is CONTROLLABLE\n\n');
else
    fprintf('✗ System is NOT controllable\n\n');
end
```

```
✓ System is CONTROLLABLE
```

Example 2: Uncontrollable system

```
A2 = [1 1; 4 -2];
B2 = [0; 0];

fprintf('Example 2: Uncontrollable system\n');
```

```
Example 2: Uncontrollable system
```

```
fprintf('A =\n'); disp(A2);
```

```
A =
     1     1
     4    -2
```

```
fprintf('B =\n'); disp(B2);
```

```
B =
     0
     0
```

```

C_ctrb2 = ctrb(A2, B2);
rank_c2 = rank(C_ctrb2);
n2 = size(A2, 1);

fprintf('Controllability matrix C = [B AB]:\n');
```

```
Controllability matrix C = [B AB]:
```

```
disp(C_ctrb2);
```

```
     0     0
     0     0
```

```
fprintf('Rank: %d, System order: %d\n', rank_c2, n2);
```

```
Rank: 0, System order: 2
```

```

if rank_c2 == n2
    fprintf('✓ System is CONTROLLABLE\n\n');
else
    fprintf('✗ System is NOT controllable (rank deficiency: %d)\n\n', n2 - rank_c2);
end
```

```
✗ System is NOT controllable (rank deficiency: 2)
```

Example 3: Multi-input system

```
A3 = [0 1 0; 0 0 1; -1 -2 -3];
B3 = [0 1; 0 0; 1 0];

fprintf('Example 3: Multi-input system (3 states, 2 inputs)\n');
```

```
Example 3: Multi-input system (3 states, 2 inputs)
```

```
fprintf('A =\n'); disp(A3);
```

```
A =
     0     1     0
     0     0     1
    -1    -2    -3
```

```
fprintf('B =\n'); disp(B3);
```

```
B =
     0     1
     0     0
     1     0
```

```


C_ctrb3 = ctrb(A3, B3);
rank_c3 = rank(C_ctrb3);
n3 = size(A3, 1);

fprintf('Controllability matrix C = [B AB]:\n');disp(C_ctrb3);
```

```
Controllability matrix C = [B AB]:
     0     1     0     0     1     0
     0     0     1     0    -3    -1
     1     0    -3    -1     7     3
```

```
fprintf('Controllability matrix size: %dx%d\n', size(C_ctrb3, 1), size(C_ctrb3, 2));
```

```
Controllability matrix size: 3x6
```

```
fprintf('Rank: %d, System order: %d\n', rank_c3, n3);
```

```
Rank: 3, System order: 3
```

```


if rank_c3 == n3
    fprintf('✓ System is CONTROLLABLE\n\n');
else
    fprintf('✗ System is NOT controllable\n\n');
end
```

```
✓ System is CONTROLLABLE
```

Section 3.2: Observability - Rank Test

Mathematical Background

A system (A,C) is observable if the observability matrix has full rank:

$$O = \begin{bmatrix} C \\ \mathrm{CA} \\ \vdots \\ \mathrm{CA}^{N-1} \end{bmatrix}$$

rank(O) = n system is observable

Physical Meaning: Can determine initial state from output measurements

Example 1: Fully observable system

```
C1 = [1 0];  % Measure first state
```

```
fprintf('Example 1: Measure first state only\n');
```

```
Example 1: Measure first state only
```

```
fprintf('A =\n'); disp(A1);
```

```
A =
     0     1
    -2    -3
```

```
fprintf('C =\n'); disp(C1);
```

```
C =
     1     0
```

```

O_obsv1 = obsv(A1, C1);
rank_o1 = rank(O_obsv1);

fprintf('Observability matrix O = [C; CA]:\n');
```

```
Observability matrix O = [C; CA]:
```

```
disp(O_obsv1);
```

```
     1     0
     0     1
```

```
fprintf('Rank: %d, System order: %d\n', rank_o1, n1);
```

```
Rank: 2, System order: 2
```

```

if rank_o1 == n1
    fprintf('✓ System is OBSERVABLE\n\n');
else
    fprintf('✗ System is NOT observable\n\n');
end
```

```
✓ System is OBSERVABLE
```

Example 2: Unobservable system

```
A4 = [1 1; 4 -2];
C4 = [-1 1; 1 -1];  % Both rows are linearly dependent

fprintf('Example 2: Linearly dependent output measurements\n');
```

```
Example 2: Linearly dependent output measurements
```

```
fprintf('A =\n'); disp(A4);
```

```
A =
     1     1
     4    -2
```

```
fprintf('C =\n'); disp(C4);
```

```
C =
    -1     1
     1    -1
```

```
O_obsv2 = obsv(A4, C4);
rank_o2 = rank(O_obsv2);
n4 = size(A4, 1);

fprintf('Observability matrix O = [C; CA]:\n');
```

```
Observability matrix O = [C; CA]:
```

```
disp(O_obsv2);
```

```
    -1     1
     1    -1
     3    -3
    -3     3
```

```
fprintf('Rank: %d, System order: %d\n', rank_o2, n4);
```

```
Rank: 1, System order: 2
```

```
if rank_o2 == n4
    fprintf('✓ System is OBSERVABLE\n\n');
else
    fprintf('✗ System is NOT observable (rank deficiency: %d)\n\n', n4 - rank_o2);
end
```

```
✗ System is NOT observable (rank deficiency: 1)
```

Section 3.3: PBH (Popov-Belevitch-Hautus) Test

Mathematical Background

Controllability PBH Test:

(A,B) is controllable □ rank([λI-A B]) = n for all eigenvalues λ

Observability PBH Test:

(A,C) is observable □ rank([λI-A; C]) = n for all eigenvalues λ

```
A_pbh = [1 1; 4 -2];
B_pbh = [1; -2];
C_pbh = [1 0];

fprintf('System matrices:\n');
```

```
System matrices:
```

```
fprintf('A =\n'); disp(A_pbh);
```

```
A =
     1     1
     4    -2
```

```
fprintf('B =\n'); disp(B_pbh);
```

```
B =
     1
    -2
```

```
fprintf('C =\n'); disp(C_pbh);
```

```
C =
     1     0
```

```

% Get eigenvalues
eigenvalues = eig(A_pbh);
fprintf('Eigenvalues: ');
```

```
Eigenvalues:
```

```
fprintf('%.4f ', eigenvalues);
```

```
2.0000 -3.0000
```

```
fprintf('\n\n');
```

PBH controllability test

```
fprintf('PBH Controllability Test:\n');
```

```
PBH Controllability Test:
```

```
fprintf('Checking rank([λI-A, B]) for each eigenvalue λ\n\n');
```

```
Checking rank([λI-A, B]) for each eigenvalue λ
```

```

n_pbh = size(A_pbh, 1);
controllable_pbh = false;

for i = 1:length(eigenvalues)
    lambda = eigenvalues(i);
    M = [lambda*eye(n_pbh) - A_pbh, B_pbh];
    r = rank(M);
    fprintf('λ = %.4f: rank([λI-A, B]) = %d ', lambda, r);
    if r < n_pbh
        fprintf('✗ FAILS\n');
        controllable_pbh = false;
    else
        fprintf('✓\n');
```

```
        controllable_pbh = true;
    end
end
```

```
λ = 2.0000: rank([λI-A, B]) = 2
✓
λ = -3.0000: rank([λI-A, B]) = 2
✓
```

```
if controllable_pbh
    fprintf('\nConclusion: System is CONTROLLABLE\n\n');
else
    fprintf('\nConclusion: System is NOT CONTROLLABLE\n\n');
end
```

```
Conclusion: System is CONTROLLABLE
```

PBH observability test

```
fprintf('PBH Observability Test:\n');
```

```
PBH Observability Test:
```

```
fprintf('Checking rank([λI-A; C]) for each eigenvalue λ\n\n');
```

```
Checking rank([λI-A; C]) for each eigenvalue λ
```

```
observable_pbh = false;

for i = 1:length(eigenvalues)
    lambda = eigenvalues(i);
    M = [lambda*eye(n_pbh) - A_pbh; C_pbh];
    r = rank(M);
    fprintf('λ = %.4f: rank([λI-A; C]) = %d ', lambda, r);
    if r < n_pbh
        fprintf('✗ FAILS\n');
        observable_pbh = false;
    else
        fprintf('✓\n');
        observable_pbh = true;
    end
end
```

```
λ = 2.0000: rank([λI-A; C]) = 2
✓
λ = -3.0000: rank([λI-A; C]) = 2
✓
```

```
if observable_pbh
    fprintf('\nConclusion: System is OBSERVABLE\n\n');
```

```
else
    fprintf('\nConclusion: System is NOT OBSERVABLE\n\n');
end
```

```
Conclusion: System is OBSERVABLE
```

Section 3.4: Controllability and Observability Gramians

Mathematical Background

Controllability Gramian: $W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$

Solves: $AW_c + W_cA^T + BB^T = 0$ (Lyapunov equation)

Observability Gramian: $W_o = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$

Solves: $A^TW_o + W_oA + C^TC = 0$

Properties:

- System is controllable □ $W_c > 0$ (positive definite)
- System is observable □ $W_o > 0$ (positive definite)

```
A_gram = [0 1; -2 -3];
B_gram = [0; 1];
C_gram = [1 0];

fprintf('System:\n');
```

```
System:
```

```
fprintf('A =\n'); disp(A_gram);
```

```
A =
     0     1
    -2    -3
```

```
fprintf('B =\n'); disp(B_gram);
```

```
B =
     0
     1
```

```
fprintf('C =\n'); disp(C_gram);
```

```
C =
     1     0
```

Controllability gramian

```
Wc = gram(ss(A_gram, B_gram, C_gram, 0), 'c');
```

```
fprintf('Controllability Gramian Wc:\n');
```

```
Controllability Gramian Wc:
```

```
disp(Wc);
```

```
    0.0833    0.0000
    0.0000    0.1667
```

```

% Check positive definiteness
eig_Wc = eig(Wc);
fprintf('Eigenvalues of Wc: ');
```

```
Eigenvalues of Wc:
```

```
fprintf('%.4f ', eig_Wc);
```

```
0.0833 0.1667
```

```
fprintf('\n');

if all(eig_Wc > 0)
    fprintf('Wc is positive definite → System is CONTROLLABLE\n\n');
else
    fprintf('Wc is NOT positive definite → System is NOT controllable\n\n');
end
```

```
Wc is positive definite → System is CONTROLLABLE
```

Observability gramian

```
Wo = gram(ss(A_gram, B_gram, C_gram, 0), 'o');
fprintf('Observability Gramian Wo:\n');
```

```
Observability Gramian Wo:
```

```
disp(Wo);
```

```
    0.9167    0.2500
    0.2500    0.0833
```

```

eig_Wo = eig(Wo);
fprintf('Eigenvalues of Wo: ');
```

```
Eigenvalues of Wo:
```

```
fprintf('%.4f ', eig_Wo);
```

```
0.0141 0.9859
```

```
fprintf('\n');

if all(eig_Wo > 0)
    fprintf('Wo is positive definite → System is OBSERVABLE\n\n');
```

```
else
    fprintf('Wo is NOT positive definite → System is NOT observable\n\n');
end
```

```
Wo is positive definite → System is OBSERVABLE
```

Section 3.5: Kalman Decomposition

Concept Overview

Kalman decomposition separates a system into four subsystems:

1. Controllable and observable
2. Controllable but not observable
3. Not controllable but observable
4. Not controllable and not observable

```
A_kd = [-1  0   0   0;
         0  -2  0   0;
         0   0  -3  0;
         0   0   0  -4];

B_kd = [1; 1; 0; 0];  % States 1,2 controllable; 3,4 not controllable
C_kd = [1  0  1  0];  % States 1,3 observable; 2,4 not observable

fprintf('Original system (4 states):\n');
```

```
Original system (4 states):
```

```
fprintf('A =\n'); disp(A_kd);
```

```
A =
    -1     0     0     0
     0    -2     0     0
     0     0    -3     0
     0     0     0    -4
```

```
fprintf('B =\n'); disp(B_kd);
```

```
B =
     1
     1
     0
     0
```

```
fprintf('C =\n'); disp(C_kd);
```

```
C =
     1     0     1     0
```

```

% Check controllability and observability
Cc = ctrb(A_kd, B_kd);
Oc = obsv(A_kd, C_kd);
```

```
fprintf('Controllability rank: %d / %d\n', rank(Cc), size(A_kd,1));
```

```
Controllability rank: 2 / 4
```

```
fprintf('Observability rank: %d / %d\n\n', rank(Oc), size(A_kd,1));
```

```
Observability rank: 2 / 4
```

```
% Create state-space model
sys_full = ss(A_kd, B_kd, C_kd, 0);
sys_min = minreal(sys_full);
```

```
3 states removed.
```

```
fprintf('Minimal realization (removes uncontrollable/unobservable modes):\n');
```

```
Minimal realization (removes uncontrollable/unobservable modes):
```

```
fprintf('Original system: %d states\n', size(sys_full.A, 1));
```

```
Original system: 4 states
```

```
fprintf('Minimal realization: %d states\n', size(sys_min.A, 1));
```

```
Minimal realization: 1 states
```

```
fprintf('Reduction: %d states removed\n\n', size(sys_full.A,1) - size(sys_min.A,1));
```

```
Reduction: 3 states removed
```

```
% Compare transfer functions
fprintf('Transfer function (original):\n');
```

```
Transfer function (original):
```

```
[num_full, den_full] = ss2tf(A_kd, B_kd, C_kd, 0)
```

```
num_full = 1×5
     0     1     9    26    24
den_full = 1×5
     1    10    35    50    24
```

```
fprintf('Transfer function (minimal):\n');
```

```
Transfer function (minimal):
```

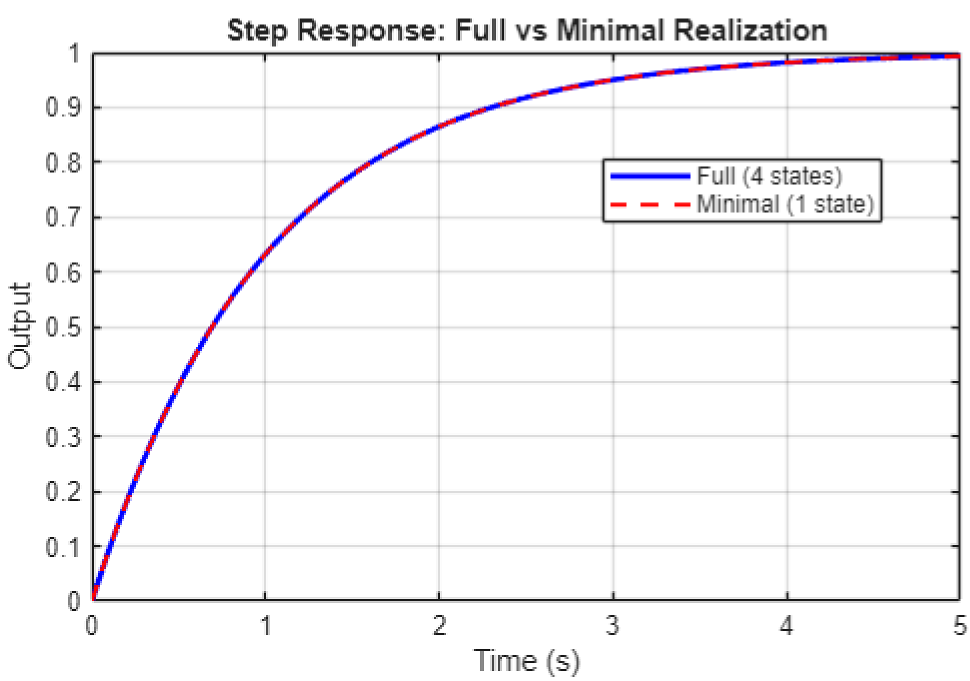
```
[num_min, den_min] = ss2tf(sys_min.A, sys_min.B, sys_min.C, sys_min.D)
```

```
num_min = 1×2
     0     1
den_min = 1×2
     1     1
```

```
% fprintf('Transfer function (original):\n');
% disp(tf(num_full, den_full));
%
% fprintf('Transfer function (minimal):\n');
% disp(tf(num_min, den_min));

% Verify step responses are identical
figure('Name', 'Kalman Decomposition - Step Response Comparison');
t = 0:0.01:5;
[y_full, ~] = step(sys_full, t);
[y_min, ~] = step(sys_min, t);

plot(t, y_full, 'b-', 'LineWidth', 2, 'DisplayName', 'Full (4 states)');
hold on;
plot(t, y_min, 'r--', 'LineWidth', 1.5, 'DisplayName', 'Minimal (1 state)');
grid on;
xlabel('Time (s)');
ylabel('Output');
title('Step Response: Full vs Minimal Realization');
legend('Location', 'best');
```




```
fprintf('Step response difference (max error): %.2e\n\n', max(abs(y_full - y_min)));
```

```
Step response difference (max error): 0.00e+00
```

Section 3.6: Canonical Forms

Concept Overview

Canonical forms are standard state-space representations:

- **Controllable canonical form -** easy controller design
- **Observable canonical form -** easy observer design
- **Modal (diagonal) canonical form -** decoupled dynamics

Function: `canon(sys, 'type')`

```
% Original system
A_orig = [0 1; -6 -5];
B_orig = [0; 1];
C_orig = [1 0];
D_orig = 0;

sys_orig = ss(A_orig, B_orig, C_orig, D_orig);

fprintf('Original system:\n');
```

```
Original system:
```

```
fprintf('A =\n'); disp(A_orig);
```

```
A =
     0     1
    -6    -5
```

```
fprintf('B =\n'); disp(B_orig);
```

```
B =
     0
     1
```

Controllable canonical form (companion form)

```
[sys_ctrl, T_ctrl] = canon(sys_orig,'companion');

fprintf('Controllable Canonical Form:\n');
```

```
Controllable Canonical Form:
```

```
fprintf('A_c =\n'); disp(sys_ctrl.A);
```

```
A_c =
     0    -6
     1    -5
```

```
fprintf('B_c =\n'); disp(sys_ctrl.B);
```

```
B_c =
     1
     0
```

```
fprintf('Transformation matrix T:\n'); disp(T_ctrl);
```

```
Transformation matrix T:
     5     1
     1     0
```

Observable canonical form

Observable companion via dual + map back

```
% 1) Dual system
sys_dual = ss(A_orig.', C_orig.', B_orig.', D_orig);

% 2) Put the dual in controllable companion
[sys_dual_c, T_dual] = canon(sys_dual, 'companion');

% 3) Map back to observable companion of the original
A_o = sys_dual_c.A.';     % transpose back
B_o = sys_dual_c.C.';     % note the swap
C_o = sys_dual_c.B.';
D_o = D_orig;

% 4) Transformation for the original system (so that T_o^{-1} A T_o = A_o)
T_o = inv(T_dual.');      % T_o = (T_dual^{-1})^T

%% Display
fprintf('Observable Canonical Form:\n');
```

```
Observable Canonical Form:
```

```
disp('A_o ='); disp(A_o);
```

```
A_o =
     0     1
    -6    -5
```

```
disp('B_o ='); disp(B_o);
```

```
B_o =
     0
     1
```

```
disp('C_o ='); disp(C_o);
```

```
C_o =
     1     0
```

```
disp('Transformation matrix T_o ='); disp(T_o);
```

```
Transformation matrix T_o =
     1     0
     0     1
```

```


%% Verify
tol = 1e-10;
ok = norm(T_o\A_orig*T_o - A_o, 'fro') < tol && ...
     norm(T_o\B_orig - B_o)      < tol   && ...
     norm(C_orig*T_o - C_o)      < tol;
fprintf('Verification passed: %d\n', ok);
```

```
Verification passed: 1
```

```

```

Modal (diagonal) form

```
[sys_modal, T_modal] = canon(sys_orig, 'modal');

fprintf('\nModal Canonical Form:\n');
```

```
Modal Canonical Form:
```

```
fprintf('A_m (diagonal with eigenvalues):\n');
```

```
A_m (diagonal with eigenvalues):
```

```
disp(sys_modal.A);
```

```
   -2.0000         0
         0   -3.0000
```

```
fprintf('B_m =\n'); disp(sys_modal.B);
```

```
B_m =
    2.8284
    3.6056
```

Verify eigenvalues are preserved

```
fprintf('Eigenvalues (original): ');
```

```
Eigenvalues (original):
```

```
fprintf('%.4f ', eig(A_orig));
```

```
-2.0000 -3.0000
```

```
fprintf('\n');
fprintf('Eigenvalues (modal): ');
```

```
Eigenvalues (modal):
```

```
fprintf('%.4f ', eig(sys_modal.A));
```

```
-3.0000 -2.0000
```

```
fprintf('\n\n');
```

Section 3.7: State-Space Transformations

Mathematical Background

Similarity transformation: $\bar{A} = T^{-1}AT$, $\bar{B} = T^{-1}B$, $\bar{C} = CT$

Transfer function is invariant: $C(sI - A)^{-1}B = \bar{C}(sI - \bar{A})^{-1}\bar{B}$

```
A_tf = [0 1; -2 -3];
B_tf = [0; 1];
C_tf = [1 0];
D_tf = 0;

% Custom transformation matrix
T = [1 2; 0 1];

fprintf('Original system:\n');
```

```
Original system:
```

```
fprintf('A =\n'); disp(A_tf);
```

```
A =
     0     1
    -2    -3
```

```
fprintf('B =\n'); disp(B_tf);
```

```
B =
     0
     1
```

```
fprintf('C =\n'); disp(C_tf);
```

```
C =
     1     0
```

```


fprintf('\nTransformation matrix T:\n');
```

```
Transformation matrix T:
```

```
disp(T);
```

```
     1     2
     0     1
```

```


% Transform using ss2ss
sys_original = ss(A_tf, B_tf, C_tf, D_tf);
sys_transformed = ss2ss(sys_original, T); % ss2ss performs the similarity
transformation

fprintf('Transformed system (using ss2ss):\n');
```

```
Transformed system (using ss2ss):
```

```
fprintf('A_new =\n'); disp(sys_transformed.A);
```

```
A_new =
    -4     3
    -2     1
```

```
fprintf('B_new =\n'); disp(sys_transformed.B);
```

```
B_new =
     2
     1
```

```
fprintf('C_new =\n'); disp(sys_transformed.C);
```

```
C_new =
     1    -2
```

```

% Manual transformation for verification
A_new_manual = T * A_tf / T;
B_new_manual = T * B_tf;
C_new_manual = C_tf / T;

fprintf('\nManual transformation verification:\n');
```

```
Manual transformation verification:
```

```
fprintf('A_new (manual) =\n'); disp(A_new_manual);
```

```
A_new (manual) =
    -4     3
    -2     1
```

```
fprintf('Error in A: %.2e\n', norm(sys_transformed.A - A_new_manual));
```

```
Error in A: 0.00e+00
```

```

% Verify transfer functions are identical
fprintf('\nTransfer function (original):\n');
```

```
Transfer function (original):
```

```
tf_orig = tf(sys_original)
```

```
tf_orig =

        1
  -------------
  s^2 + 3 s + 2

Continuous-time transfer function.
Model Properties
```

```
fprintf('Transfer function (transformed):\n');
```

```
Transfer function (transformed):
```

```
tf_trans = tf(sys_transformed)
```

```
tf_trans =

        1
  -------------
  s^2 + 3 s + 2
```

```
Continuous-time transfer function.
Model Properties
```

Section 3.8: Balanced Realization

Concept Overview

Balanced realization transforms system so that:

- Controllability gramian = Observability gramian = Σ (diagonal)
- Diagonal elements (Hankel singular values) indicate state importance
- Enables systematic model reduction

Function: `balreal(sys)`

```
% Create a higher-order system
A_bal = [-1  0   0.5  0;
          0  -2  0    0.3;
          0  0   -5   0;
          0  0   0    -10];

B_bal = [1; 0.5; 0.2; 0.1];
C_bal = [1 1 0.5 0.1];
D_bal = 0;

sys_unbal = ss(A_bal, B_bal, C_bal, D_bal);

fprintf('Original system (4 states):\n');
```

```
Original system (4 states):
```

```
% Compute gramians before balancing
Wc_before = gram(sys_unbal, 'c');
Wo_before = gram(sys_unbal, 'o');

fprintf('Controllability gramian eigenvalues:\n');
```

```
Controllability gramian eigenvalues:
```

```
disp(sort(eig(Wc_before), 'descend'));
```

```
    0.5759
    0.0081
    0.0004
    0.0000
```

```
fprintf('Observability gramian eigenvalues:\n');
```

```
Observability gramian eigenvalues:
```

```
disp(sort(eig(Wo_before), 'descend'));
```

```
    0.7656
    0.0218
    0.0010
    0.0000
```

```
% Balanced realization
[sys_bal, g, T_bal] = balreal(sys_unbal);

fprintf('\nHankel singular values (importance of each state):\n');
```

```
Hankel singular values (importance of each state):
```

```
disp(g);
```

```
    0.6317
    0.0139
    0.0006
    0.0000
```

```
fprintf('Balanced system:\n');
```

```
Balanced system:
```

```
sys_bal
```

```
sys_bal =

  A = 
             x1       x2       x3       x4
   x1    -1.216  -0.4986  -0.1482  0.02552
   x2   -0.4986   -2.419   -1.353   0.2421
   x3   -0.1482   -1.353   -5.013    1.698
   x4   0.02552   0.2421    1.698   -9.352

  B = 
              u1
   x1      -1.24
   x2    -0.2597
   x3   -0.07558
   x4      0.013

  C = 
              x1        x2        x3        x4
   y1      -1.24   -0.2597  -0.07558     0.013

  D = 
       u1
   y1   0

Continuous-time state-space model.
Model Properties
```

```
% Gramians of balanced system should be equal and diagonal
Wc_after = gram(sys_bal, 'c');
Wo_after = gram(sys_bal, 'o');
```

```
fprintf('Controllability gramian (balanced):\n');
```

```
Controllability gramian (balanced):
```

```
disp(Wc_after);
```

```
    0.6317   -0.0000    0.0000    0.0000
   -0.0000    0.0139    0.0000    0.0000
    0.0000    0.0000    0.0006    0.0000
    0.0000    0.0000    0.0000    0.0000
```

```

fprintf('Observability gramian (balanced):\n');
```

```
Observability gramian (balanced):
```

```
disp(Wo_after);
```

```
    0.6317    0.0000   -0.0000   -0.0000
    0.0000    0.0139   -0.0000    0.0000
   -0.0000   -0.0000    0.0006    0.0000
   -0.0000    0.0000    0.0000    0.0000
```

```

fprintf('Difference between gramians: %.2e\n', norm(Wc_after - Wo_after));
```

```
Difference between gramians: 7.36e-15
```

Model reduction: keep only significant states

If a Hankel singular value is very small, that state contributes little

```
threshold = 0.1 * g(1);  % Keep states with σ > 10% of largest
n_keep = sum(g > threshold);

fprintf('\nModel reduction threshold: %.4f\n', threshold);
```

```
Model reduction threshold: 0.0632
```

```
fprintf('States to keep: %d out of %d\n', n_keep, length(g));
```

```
States to keep: 1 out of 4
```

```

% Extract reduced model
A_red = sys_bal.A(1:n_keep, 1:n_keep);
B_red = sys_bal.B(1:n_keep, :);
C_red = sys_bal.C(:, 1:n_keep);
D_red = sys_bal.D;

sys_reduced = ss(A_red, B_red, C_red, D_red);

fprintf('\nReduced system:\n');
```

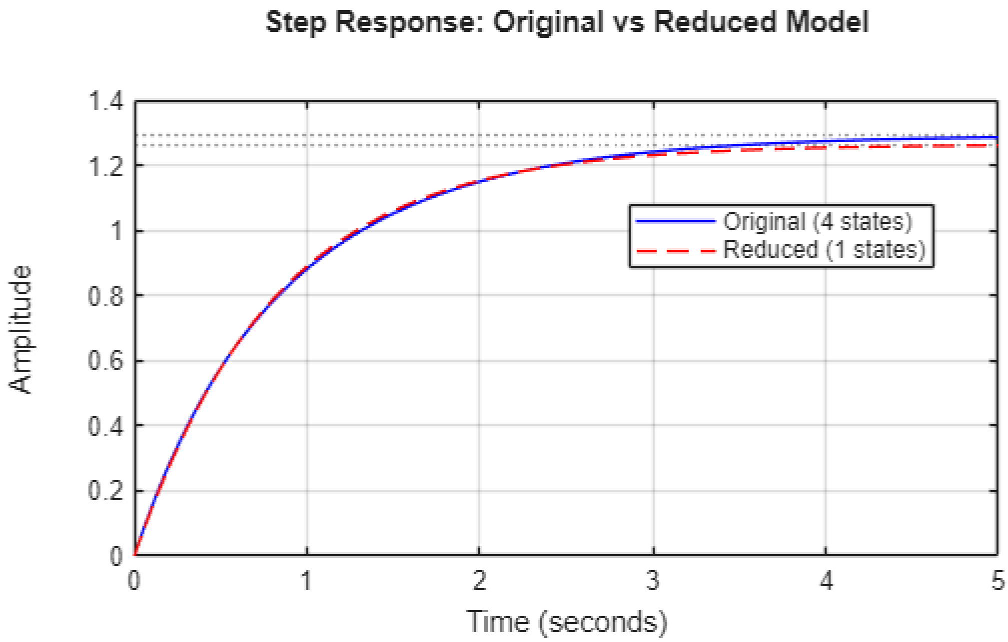
```
Reduced system:
```

```
disp(sys_reduced);
```

```
  ss with properties:

                A: -1.2162
                B: -1.2396
                C: -1.2396
                D: 0
                E: []
          Offsets: []
           Scaled: 0
        StateName: {''}
        StatePath: {''}
        StateUnit: {''}
    InternalDelay: [0×1 double]
       InputDelay: 0
      OutputDelay: 0
        InputName: {''}
        InputUnit: {''}
       InputGroup: [1×1 struct]
       OutputName: {''}
       OutputUnit: {''}
      OutputGroup: [1×1 struct]
            Notes: [0×1 string]
         UserData: []
             Name: ''
               Ts: 0
         TimeUnit: 'seconds'
     SamplingGrid: [1×1 struct]
```

```
% Compare step responses
figure('Name', 'Balanced Realization - Model Reduction');
step(sys_unbal, 'b-', sys_reduced, 'r--', 5);
grid on;
legend('Original (4 states)', sprintf('Reduced (%d states)', n_keep), ...
    'Location', 'best');
title('Step Response: Original vs Reduced Model');
```

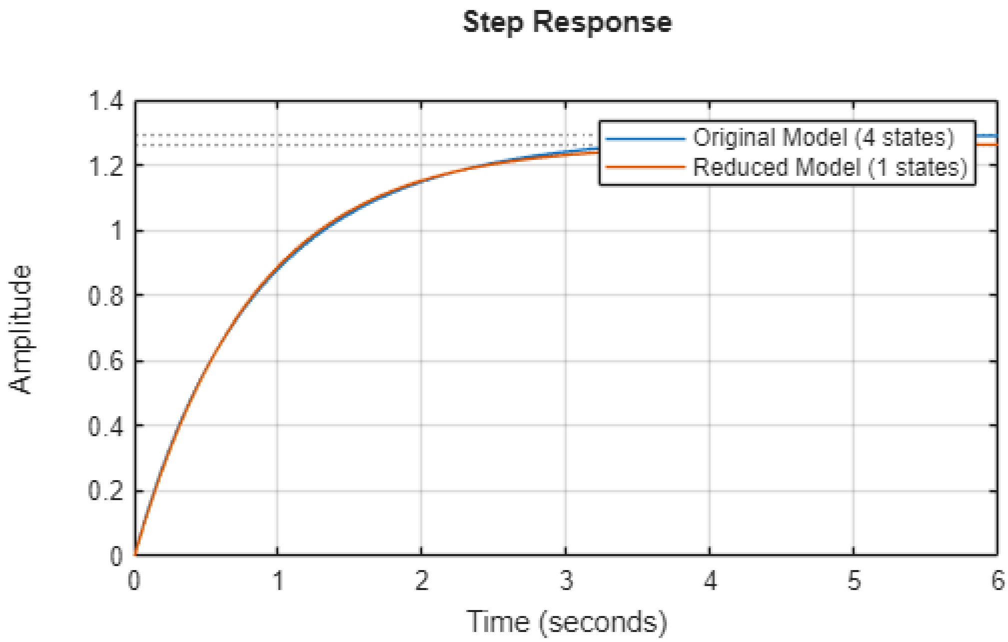


using a task:

```
%% Reduce LTI model order using balanced truncation

% Compute reduced-order approximation
R = reducespec(sys_unbal,'balanced');
% Compute MOR data once
R = process(R);
% Get reduced-order model
sysReduced = getrom(R,Order=1,Method='truncate');

% Create comparison plot
f = figure();
h = stepplot(f,sys_unbal,sysReduced);
legend(h,['Original Model (',mat2str(order(sys_unbal)),' states)'],...
         ['Reduced Model (',mat2str(order(sysReduced)),' states)']);
h.AxesStyle.GridVisible = true;
```




```
% Remove temporary variables from Workspace
clear R f h
```

Section 3.9: Minimal Realization from Transfer Function

Concept Overview

Given a transfer function, find the minimal state-space realization:

1. Convert to state-space: `tf2ss`
2. Remove uncontrollable/unobservable modes: `minreal in which` eliminates uncontrollable or unobservable state in state-space models, or cancels pole-zero pairs in transfer functions or zero-pole-gain models.

```
% Transfer function with pole-zero cancellation
```

Goal transfer function: $\frac{(s+1)(s+2)}{(s+1)(s+3)(s+4)} \longrightarrow \frac{(s+2)}{(s+3)(s+4)}$

```
num_cancel = conv([1 1], [1 2]);  % (s+1)(s+2)
den_cancel = conv([1 1], conv([1 3], [1 4]));  % (s+1)(s+3)(s+4)

fprintf('Transfer function with cancellation:\n');
```

```
Transfer function with cancellation:
```

```
fprintf('Numerator (zeros at s=-1,-2): ');
```

```
Numerator (zeros at s=-1,-2):
```

```
fprintf('%.0f ', num_cancel);
```

```
1 3 2
```

```
fprintf('\n');
fprintf('Denominator (poles at s=-1,-3,-4): ');
```

```
Denominator (poles at s=-1,-3,-4):
```

```
fprintf('%.0f ', den_cancel);
```

```
1 8 19 12
```

```
fprintf('\n\n');

fprintf('G(s) =\n');
```

```
G(s) =
```

```
tf_cancel = tf(num_cancel, den_cancel)
```

```
tf_cancel =

       s^2 + 3 s + 2
  -----------------------
  s^3 + 8 s^2 + 19 s + 12

Continuous-time transfer function.
Model Properties
```

```

% Convert to state-space (may include cancelling mode)
sys_ss_cancel = ss(tf_cancel);

fprintf('State-space realization (before minreal): %d states\n',
size(sys_ss_cancel.A, 1));
```

```
State-space realization (before minreal): 3 states
```

Minimal realization

```
sys_min_cancel = minreal(sys_ss_cancel);
```

```
1 state removed.
```

```

fprintf('Minimal realization: %d states\n', size(sys_min_cancel.A, 1));
```

```
Minimal realization: 2 states
```

```

fprintf('\nMinimal state-space:\n');
```

```
Minimal state-space:
```

```matlab
fprintf('A =\n'); disp(sys_min_cancel.A);
```

```
A =
   -1.5067   -0.4533
    8.2133   -5.4933
```

```matlab
fprintf('B =\n'); disp(sys_min_cancel.B);
```

```
B =
    0.8889
   -1.7778
```

```matlab
fprintf('Poles (eigenvalues of A): ');
```

```
Poles (eigenvalues of A):
```

```matlab
fprintf('%.4f ', eig(sys_min_cancel.A));
```

```
-3.0000 -4.0000
```

```matlab
fprintf('\n');
fprintf('(Cancelled pole at s=-1 removed)\n\n');
```

```
(Cancelled pole at s=-1 removed)
```

```matlab
fprintf('Simplified transfer function after minreal:\n');
```

```
Simplified transfer function after minreal:
```

```matlab
tf_min_cancel = tf(sys_min_cancel)
```

```
tf_min_cancel =

      s + 2
  --------------
  s^2 + 7 s + 12

Continuous-time transfer function.
Model Properties
```

Section 3.10: Summary and Key Takeaways

Key Concepts Covered:

1. Controllability: rank test, PBH test, gramians
2. Observability: rank test, PBH test, gramians
3. Kalman decomposition and minimal realization
4. Canonical forms: controllable, observable, modal
5. State-space transformations and similarity
6. Balanced realization for model reduction
7. Minimal realization from transfer functions

MATLAB Functions Mastered:

`ctrb`, `obsv`, `rank`, `gram`, `canon`, `ss2ss`, `minreal`, `balreal`, `ss2tf`, `tf2ss`, `ssdata`

Structural Property Tests:

- Rank test: $rank(\mathscr{C}) = n$ or $rank(\mathscr{O}) = n$
- PBH test: $rank([\lambda I - A, B]) = n$ for all eigenvalues
- Gramian test: $W_c > 0$ or $W_o > 0$

Next Steps:

These structural properties enable:

- Stability analysis and state feedback design (Season 4)
- Observer design (requires observability) (Season 5)
- System simplification and model reduction