Season 1: Mathematical Foundations for Modern Control

Prerequisites: Basic linear algebra

MATLAB Version: R2025b

Toolboxes Required: Symbolic Math Toolbox

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```
close all; clear; clc;
rng(0);  % For reproducibility
```

Section 0:

Basic Matlab Functions:

Basic arithmetic operators

```
Addition = 2 + 3
```

```
Addition =
5
```

```
Subtraction = 2 - 3
```

```
Subtraction =
-1
```

```
Multiplication = 2 * 3
```

```
Multiplication =
6
```

```
Division = 2 / 3
```

```
Division =
0.6667
```

```
syms x y;

x = 2 + 3 * 4
```

```
x =
14
```

```
x = (2 + 3) * 4
```

```
x =
20
```

```
y = 1 / (2 + 3^2) + 4/5 * 6/7
```

```
y =
0.7766
```

```
y = 1 / 2 + 3^2 + 4/5 * 6/7
```

```
y =
10.1857
```

Mathematical functions

```
rad2deg(1)       % Radians to Degrees
```

```
ans =
57.2958
```

```
deg2rad(90)      % Degrees to Radians
```

```
ans =
1.5708
```

```
cosine_val = cos(x)   % Cosine
```

```
cosine_val =
0.4081
```

```
sine_val = sin(x)     % Sine
```

```
sine_val =
0.9129
```

```
tan_val = tan(x)      % Tangent
```

```
tan_val =
2.2372
```

```
arcc_val = acos(x)   % Arc cosine
```

```
arcc_val =
0.0000 + 3.6883i
```

```
arcs_val = asin(x)   % Arc sine
```

```
arcs_val =
1.5708 - 3.6883i
```

```
arct_val = atan(x)   % Arc tangent
```

```
arct_val =
1.5208
```

```
exp_val = exp(x)     % Exponential
```

```
exp_val =
4.8517e+08
```

```
sqrt_val = sqrt(x)   % Square root
```

```
sqrt_val =
4.4721
```

```
log_val = log(x)     % Natural
```

```
log_val =
2.9957
```

```
log10_val = log10(x) % Common logarithm
```

```
log10_val =
1.3010
```

```
x = -20;
abs_val = abs(x)     % Absolute value
```

```
abs_val =
20
```

```
sign_val = sign(x)   % Signum function
```

```
sign_val =
-1
```

```
x = [1 2 3 7 465 -2 7.6];
max_val = max(x)     % Maximum value
```

```
max_val =
465
```

```
min_val = min(x)     % Minimum value
```

```
min_val =
-2
```

```
x = 45.3;
ceil_val = ceil(x)   % Round towards +∞
```

```
ceil_val =
46
```

```
floor_val = floor(x) % Round towards -∞
```

```
floor_val =
45
```

```
round_val = round(x) % Round to nearest integer
```

```
round_val =
45
```

```
x = 2 + 7i;
real(x)
```

```
ans =
2
```

```
imag(x)
```

```
ans =
7
```

```
angle_val = angle(x) % Phase angle
```

```
angle_val =
1.2925
```

```
conj_val = conj(x)   % Complex conjugate
```

```
conj_val =
2.0000 - 7.0000i
```

Matrix

```
A = [1 2 3; 4 5 6; 7 8 9]
```

```
A = 3×3
     1     2     3
     4     5     6
     7     8     9
```

```
A(2,1)
```

```
ans =
4
```

```
A(3,3) = 0
```

```
A = 3×3
     1     2     3
     4     5     6
     7     8     0
```

```
m = 2;
n = 3;
eye1 = eye(m,n)    % Returns an m-by-n matrix with 1 on the main diagonal
```

```
eye1 = 2×3
     1     0     0
     0     1     0
```

```
eye2 = eye(n)      % Returns an n-by-n square identity matrix
```

```
eye2 = 3×3
     1     0     0
     0     1     0
     0     0     1
```

```
zeros_example = zeros(m,n) % Returns an m-by-n matrix of zeros
```

```
zeros_example = 2×3
     0     0     0
     0     0     0
```

```
ones_example = ones(m,n)   % Returns an m-by-n matrix of ones
```

```
ones_example = 2×3
     1     1     1
     1     1     1
```

```
diag_example = diag(A)     % Extracts the diagonal of matrix A
```

```
diag_example = 3×1
     1
     5
     0
```

```
rand_example = rand(m,n)   % Returns an m-by-n matrix of random numbers
```

```
rand_example = 2×3
    0.8147    0.1270    0.6324
    0.9058    0.9134    0.0975
```

Section 1.1: Vector and Matrix Operations

Concept Overview

Matrix operations form the foundation of state-space control theory. Understanding element-wise operations, matrix multiplication, and basic matrix properties is essential for analyzing linear systems.

```
% Create sample matrices
A = [1 2 3; 4 5 6; 7 8 10];

fprintf('Matrix A:\n'); % Use fprintf to write data to the screen or a text file,
refer to the documentation for writing to a text file
```

```
Matrix A:
```

```
disp(A); % display the output without the variable name
```

```
     1     2     3
     4     5     6
     7     8    10
```

```
B = [2 1 0; 1 3 1; 0 1 2]
```

```
B = 3×3
     2     1     0
     1     3     1
     0     1     2
```

```
disp(B);
```

```
     2     1     0
     1     3     1
     0     1     2
```

```
v = [1; 2; 3];
```

Basic matrix operations

A + B (addition)

```
C_add = A + B
```

```
C_add = 3×3
     3     3     3
     5     8     7
     7     9    12
```

A * B (matrix multiplication):

```
C_mult = A * B
```

```
C_mult = 3×3
     4    10     8
    13    25    17
    22    41    28
```

A .* B (element-wise multiplication):

```
C_elem = A .* B
```

```
C_elem = 3×3
     2     2     0
     4    15     6
     0     8    20
```

A.^2 (Element-wise power)

```
C_power = A.^2
```

```
C_power = 3×3
     1     4     9
    16    25    36
    49    64   100
```

Transpose operations

```
A_transpose = A.'       % Non-conjugate transpose
```

```
A_transpose = 3×3
     1     4     7
     2     5     8
     3     6    10
```

```
A_hermitian = A'        % Conjugate transpose (same for real matrices)
```

```
A_hermitian = 3×3
     1     4     7
     2     5     8
     3     6    10
```

Matrix concatenation

```
H_concat = [A, B]       % Horizontal concatenation
```

```
H_concat = 3×6
     1     2     3     2     1     0
     4     5     6     1     3     1
     7     8    10     0     1     2
```

```
V_concat = [A; B]       % Vertical concatenation
```

```
V_concat = 6×3
     1     2     3
     4     5     6
     7     8    10
     2     1     0
     1     3     1
     0     1     2
```

Section 1.2: Matrix Properties and Functions

Mathematical Background

Key matrix properties:

- Determinant: det(A) - nonzero for invertible matrices
- Rank: number of linearly independent rows/columns
- Trace: sum of diagonal elements, tr(A) = sum of eigenvalues
- Inverse: A^(-1) exists if det(A) ≠ 0

Determinant

```
det_A = det(A);
```

```
fprintf('Determinant of A: %.4f\n', det_A);
```

```
Determinant of A: -3.0000
```

Rank

```
rank_A = rank(A);
fprintf('Rank of A: %d (size %dx%d)\n', rank_A, size(A,1), size(A,2));
```

```
Rank of A: 3 (size 3x3)
```

Trace (*Sum of diagonal elements)*

```
trace_A = trace(A);
fprintf('Trace of A: %.4f\n', trace_A);
```

```
Trace of A: 16.0000
```

Inverse (if exists)

```
if det_A ~= 0
    A_inv = inv(A);
    fprintf('Inverse of A:\n');
    disp(A_inv);

    % Verify A * inv(A) = I
    identity_check = A * A_inv;
    fprintf('A * inv(A) (should be identity):\n');
    disp(identity_check);
else
    fprintf('Matrix A is singular (not invertible)\n');
end
```

```
Inverse of A:
   -0.6667   -1.3333    1.0000
   -0.6667    3.6667   -2.0000
    1.0000   -2.0000    1.0000
A * inv(A) (should be identity):
    1.0000         0   -0.0000
         0    1.0000         0
         0         0    1.0000
```

Condition number (measures sensitivity to perturbations)

A *condition number* for a matrix and computational task measures how sensitive the answer is to changes in the input data and roundoff errors in the solution process.

The *condition number for inversion* of a matrix measures the sensitivity of the solution of a system of linear equations to errors in the data. It gives an indication of the accuracy of the results from matrix inversion and the linear equation solution. For example, the 2-norm condition number of a square matrix is

$$\kappa(A) = \|A\| \|A - 1\| \ .$$

In this context, a large condition number indicates that a small change in the coefficient matrix A can lead to larger changes in the output b in the linear equations $Ax = b$ and $xA = b$. The extreme case is when A is so poorly conditioned that it is singular (an infinite condition number), in which case it has no inverse and the linear equation has no unique solution.

```
cond_A = cond(A);
fprintf('Condition number of A: %.4f\n', cond_A);
```

```
Condition number of A: 88.4483
```

```
fprintf('(Higher values indicate ill-conditioning)\n\n');
```

```
(Higher values indicate ill-conditioning)
```

Section 1.3: Matrix Norms

Concept Overview

Matrix norms measure the "size" of a matrix and are crucial for:

- Stability analysis
- Error bounds in numerical computations
- Convergence analysis

Common Norms:

- 1-norm: maximum absolute column sum
- 2-norm (spectral): largest singular value
- ∞-norm: maximum absolute row sum
- Frobenius norm: sqrt(sum of squared elements)

Create a test matrix with complex entries

```
A_complex = [1-2i, 2-1i, 5; 7, 5+4i, 3; 3, 8, 9+1i];
```

Different norm types

1-norm (max column sum)

```
norm_1 = norm(A_complex, 1)
```

```
norm_1 =
17.0554
```

2-norm (spectral/largest singular value)

```
norm_2 = norm(A_complex, 2)
```

```
norm_2 =
15.8431
```

inf-norm (max row sum)

```
norm_inf = norm(A_complex, inf)
```

```
norm_inf =
20.0554
```

Frobenius norm

```
norm_fro = norm(A_complex, 'fro')
```

```
norm_fro =
17
```

Vector norms

```
v_test = [3; 4; 0];
v_norm1 = norm(v_test, 1);      % Sum of absolute values
v_norm2 = norm(v_test, 2);      % Euclidean norm
v_norminf = norm(v_test, inf);  % Maximum absolute value
disp(v_test);
```

```
     3
     4
     0
```

```
fprintf('1-norm: %.4f\n', v_norm1);
```

```
1-norm: 7.0000
```

```
fprintf('2-norm (Euclidean): %.4f\n', v_norm2);
```

```
2-norm (Euclidean): 5.0000
```

```
fprintf('inf-norm: %.4f\n\n', v_norminf);
```

```
inf-norm: 4.0000
```

Section 1.4: Triangular Matrices

Concept Overview

Upper and lower triangular matrices are important for:

- Efficient solving of linear systems
- QR and LU decompositions
- Analyzing stability (eigenvalues on diagonal)

```
M = [3.1, -1.6, 11.1; -8.6, 6.2, -8; -0.3, 11, 0.7]
```

```
M = 3×3
    3.1000   -1.6000   11.1000
   -8.6000    6.2000   -8.0000
   -0.3000   11.0000    0.7000
```

Extract upper and lower triangular parts

Upper triangular

```
U = triu(M)  % Upper triangular
```

```
U = 3×3
    3.1000   -1.6000   11.1000
         0    6.2000   -8.0000
         0         0    0.7000
```

Lower triangular

```
L = tril(M)  % Lower triangular
```

```
L = 3×3
    3.1000         0         0
   -8.6000    6.2000         0
   -0.3000   11.0000    0.7000
```

Diagonal extraction

```
D = diag(M) % Extract diagonal elements
```

```
D = 3×1
    3.1000
    6.2000
    0.7000
```

Section 1.5: Linear Equations, Null Space, and Orthogonality

Mathematical Background

For system Ax = b:

- Null space N(A): all vectors x where Ax = 0
- Range/Column space R(A): all possible outputs Ax
- Orthogonal complement: vectors perpendicular to a subspace

Key Property: rank(A) + dim(null(A)) = n (number of columns)

Create a rank-deficient matrix

```
A_rank_def = [1 2 3; 2 3 4; 4 5 6; 25 34.5 44];

fprintf('Rank-deficient matrix A:\n');
```

```
Rank-deficient matrix A:
```

```
disp(A_rank_def);
```

```
    1.0000    2.0000    3.0000
    2.0000    3.0000    4.0000
    4.0000    5.0000    6.0000
   25.0000   34.5000   44.0000
```

```
fprintf('Rank: %d, Columns: %d\n', rank(A_rank_def), size(A_rank_def, 2));
```

```
Rank: 2, Columns: 3
```

Null space

Null space basis (should satisfy A*Z ≈ 0):

```
Z = null(A_rank_def)
```

```
Z = 3×1
    0.4082
   -0.8165
    0.4082
```

Verify (A * null(A) should be ≈ 0)

```
verification = A_rank_def * Z
```

```
verification = 4×1
10^-14 ×
    0.1332
         0
   -0.2220
   -0.3553
```

```
fprintf('Norm of A*null(A): %.2e\n\n', norm(verification));
```

```
Norm of A*null(A): 4.40e-15
```

Orthonormal basis for column space

```
Q = orth(A_rank_def)
```

```
Q = 4×2
   -0.0593    0.7358
   -0.0865    0.2860
   -0.1408   -0.6136
   -0.9845    0.0183
```

Verify orthonormality: Q'*Q should be identity

```
orthogonality_check = Q' * Q;
fprintf('Q^T * Q (should be identity of size %dx%d):\n', size(Q,2), size(Q,2));
```

```
Q^T * Q (should be identity of size 2x2):
```

```
disp(orthogonality_check);
```

```
    1.0000    0.0000
    0.0000    1.0000
```

Dot product and orthogonality check

```
u = randi(10, [3,1]); %what's the difference between randi and rand?
v = randi(10, [3,1]);
dot_uv = dot(u, v);
fprintf('\nVectors u and v:\n');
```

```
Vectors u and v:
```

```
fprintf('u = '); disp(u');
```

```
u =      3     6    10
```

```
fprintf('v = '); disp(v');
```

```
v =     10     2    10
```

```
fprintf('Dot product u·v = %.4f\n', dot_uv);
```

```
Dot product u·v = 142.0000
```

```
if abs(dot_uv) < 1e-10
    fprintf('Vectors are orthogonal\n');
else
    fprintf('Vectors are not orthogonal\n');
end
```

```
Vectors are not orthogonal
```

Solving linear equations Ax = b

```
A_solve = [2 1 -1; -3 -1 2; -2 1 2];
b_solve = [8; -11; -3];
```

Using backslash operator (most efficient)

```
x_solution = A_solve \ b_solve;

fprintf('\nSolving Ax = b:\n');
```

```
Solving Ax = b:
```

```
fprintf('A:\n'); disp(A_solve);
```

```
A:
     2     1    -1
    -3    -1     2
    -2     1     2
```

```
fprintf('b:\n'); disp(b_solve');
```

```
b:
     8   -11    -3
```

```
fprintf('Solution x:\n'); disp(x_solution');
```

```
Solution x:
    2.0000    3.0000   -1.0000
```

Verify solution

```
residual = norm(A_solve * x_solution - b_solve);
fprintf('Residual ||Ax - b||: %.2e\n', residual);
```

```
Residual ||Ax - b||: 8.88e-16
```

Alternative: using linsolve for more control

```
[x_linsolve, R] = linsolve(A_solve, b_solve);
fprintf('Linsolve solution is:\n'); disp(x_linsolve);
```

```
Linsolve solution is:
    2.0000
    3.0000
   -1.0000
```

```
fprintf('Reciprocal condition estimate: %.2e\n', R);
```

```
Reciprocal condition estimate: 1.30e-02
```

Section 1.6: Eigenvalues and Eigenvectors

Mathematical Background

For matrix A, eigenvalue λ and eigenvector v satisfy:

$$Av = \lambda v$$

Key Properties:

- Characteristic polynomial: $\det(A - \lambda I) = 0$
- Trace = sum of eigenvalues
- Determinant = product of eigenvalues
- Eigenvalues determine stability of dynamic systems

```
A_eig = [5 11 4; 12 8 5; 1 7 3];

fprintf('Matrix A:\n'); disp(A_eig);
```

```
Matrix A:
     5    11     4
    12     8     5
     1     7     3
```

Compute eigenvalues and eigenvectors

```
[V, D] = eig(A_eig);

fprintf('Eigenvalues (diagonal of D):\n'); disp(diag(D));
```

```
Eigenvalues (diagonal of D):
   20.3073
   -5.1816
    0.8743
```

```


fprintf('Eigenvector matrix V:\n'); disp(V);
```

```
Eigenvector matrix V:
   -0.6063   -0.5284   -0.2421
```

```
   -0.7244    0.6759   -0.2501
   -0.3280   -0.5137    0.9375
```

Verify: A*V should equal V*D

```
verification_eig = A_eig * V
```

```
verification_eig = 3×3
  -12.3123    2.7381   -0.2117
  -14.7112   -3.5023   -0.2187
   -6.6614    2.6619    0.8196
```

```
expected_eig = V * D
```

```
expected_eig = 3×3
  -12.3123    2.7381   -0.2117
  -14.7112   -3.5023   -0.2187
   -6.6614    2.6619    0.8196
```

```

fprintf('Verification: max|A*V - V*D| = %.2e\n', max(max(abs(verification_eig -
expected_eig))));
```

```
Verification: max|A*V - V*D| = 5.33e-15
```

Characteristic polynomial (using poly)

```
char_poly = poly(A_eig);
fprintf('\nCharacteristic polynomial coefficients:\n');
```

```
Characteristic polynomial coefficients:
```

```
fprintf('p(λ) = ');
```

```
p(λ) =
```

```
for i = 1:length(char_poly)
    if i == 1
        fprintf('%.4fλ^%d', char_poly(i), length(char_poly)-i);
    else
        if char_poly(i) >= 0
            fprintf(' + %.4fλ^%d', char_poly(i), length(char_poly)-i);
        else
            fprintf(' - %.4fλ^%d', abs(char_poly(i)), length(char_poly)-i);
        end
    end
end
```

```
1.0000λ^3
 - 16.0000λ^2 - 92.0000λ^1
 + 92.0000λ^0
```

Alternative: use charpoly

```
syms lambda;
charpoly(A_eig,lambda)
```

ans = $\lambda^3 - 16\,\lambda^2 - 92\,\lambda + 92$

Properties

```
trace_sum = sum(diag(D));
det_prod = prod(diag(D)); % product of the array elements

fprintf('Trace of A: %.4f\n', trace(A_eig));
```

```
Trace of A: 16.0000
```

```
fprintf('Sum of eigenvalues: %.4f\n', trace_sum);
```

```
Sum of eigenvalues: 16.0000
```

```
fprintf('Determinant of A: %.4f\n', det(A_eig));
```

```
Determinant of A: -92.0000
```

```
fprintf('Product of eigenvalues: %.4f\n\n', det_prod);
```

```
Product of eigenvalues: -92.0000
```

Section 1.7: Similarity Transformations

Concept Overview

Two matrices A and B are similar if: B = T^(-1) * A * T

Similar matrices have:

- Same eigenvalues
- Same determinant, trace, rank
- Same characteristic polynomial

Application: Transform systems to diagonal or canonical forms

Using eigenvector matrix as transformation

```
A_original = A_eig;
T = V;  % Eigenvector matrix

if abs(det(T)) > 1e-10
    A_transformed = inv(T) * A_original * T;

    fprintf('Original matrix A:\n');
    disp(A_original);

    fprintf('Transformation matrix T (eigenvectors):\n');
    disp(T);

    fprintf('Transformed matrix T^(-1)*A*T (should be diagonal):\n');
    disp(A_transformed);
```

```
    % Check eigenvalues are preserved
    eig_original = sort(eig(A_original));
    eig_transformed = sort(eig(A_transformed));

    fprintf('Eigenvalues of original: ');
    fprintf('%.4f ', eig_original);
    fprintf('\n');
    fprintf('Eigenvalues of transformed: ');
    fprintf('%.4f ', eig_transformed);
    fprintf('\n\n');
end
```

```
Original matrix A:
     5    11     4
    12     8     5
     1     7     3
Transformation matrix T (eigenvectors):
   -0.6063   -0.5284   -0.2421
   -0.7244    0.6759   -0.2501
   -0.3280   -0.5137    0.9375
Transformed matrix T^(-1)*A*T (should be diagonal):
   20.3073    0.0000   -0.0000
   -0.0000   -5.1816   -0.0000
   -0.0000   -0.0000    0.8743
Eigenvalues of original:
-5.1816 0.8743 20.3073
Eigenvalues of transformed:
-5.1816 0.8743 20.3073
```

Section 1.8: Jordan Normal Form

Mathematical Background

Every square matrix is similar to its Jordan normal form:

$$A = TJT^{-1}$$

where J is block-diagonal with Jordan blocks.

Jordan Block: Upper triangular matrix with eigenvalue on diagonal

Matrix with repeated eigenvalues

```
A_jordan = [5 11 4; 12 8 5; 1 7 3];

fprintf('Matrix A:\n');
```

```
Matrix A:
```

```
disp(A_jordan);
```

```
     5    11     4
    12     8     5
     1     7     3
```

```

```

Compute Jordan form

```
[T_jordan, J] = jordan(A_jordan);

fprintf('Jordan form J:\n');
```

```
Jordan form J:
```

```
disp(J);
```

```
  20.3073 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -5.1816 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.8743 - 0.0000i
```

```

fprintf('Transformation matrix T:\n');
```

```
Transformation matrix T:
```

```
disp(T_jordan);
```

```
   1.8483 + 0.0000i   1.0286 - 0.0000i  -0.2583 + 0.0000i
   2.2084 + 0.0000i  -1.3158 + 0.0000i  -0.2668 - 0.0000i
   1.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
```

Verify: A = T*J*inv(T)

```
A_reconstructed = T_jordan * J * inv(T_jordan);
fprintf('Reconstructed A from T*J*T^(-1):\n');
```

```
Reconstructed A from T*J*T^(-1):
```

```
disp(real(A_reconstructed));
```

```
    5.0000   11.0000    4.0000
   12.0000    8.0000    5.0000
    1.0000    7.0000    3.0000
```

```

reconstruction_error = norm(A_jordan - A_reconstructed);
fprintf('Reconstruction error: %.2e\n\n', reconstruction_error);
```

```
Reconstruction error: 4.66e-15
```

Example with defective matrix (non-diagonalizable)

```
A_defective = [2 1 0; 0 2 0; 0 0 3];
fprintf('Defective matrix (non-diagonalizable):\n');
```

```
Defective matrix (non-diagonalizable):
```

```
disp(A_defective);
```

```
     2     1     0
     0     2     0
     0     0     3
```

```
[T_def, J_def] = jordan(A_defective);
fprintf('Jordan form (note the 1 above diagonal for repeated eigenvalue):\n');
```

```
Jordan form (note the 1 above diagonal for repeated eigenvalue):
```

```
disp(J_def);
```

```
     2     1     0
     0     2     0
     0     0     3
```

Section 1.9: Matrix Exponential

Mathematical Background

Matrix exponential is defined as:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots$$

Key Application: Solution to linear differential equations

For $\dot{x} = Ax$, solution is $x(t) = e^{At}x_0$

State matrix for a stable system

```
A_exp = [0 1; -2 -3];

fprintf('Matrix A (typical state matrix):\n');
```

```
Matrix A (typical state matrix):
```

```
disp(A_exp);
```

```
     0     1
    -2    -3
```

Compute matrix exponential at t = 0

```
exp_A0 = expm(A_exp * 0);
fprintf('exp(A*0) (should be identity):\n');
```

```
exp(A*0) (should be identity):
```

```
disp(exp_A0);
```

```
     1     0
     0     1
```

Compute at t = 1

```
t = 1;
exp_At = expm(A_exp * t);
fprintf('exp(A*%.1d):\n', t);
```

```
exp(A*1):
```

```
disp(exp_At);
```

```
    0.6004    0.2325
   -0.4651   -0.0972
```

State transition: if x(0) = [1; 0], what is x(1)?

```
x0 = [1; 0];
x_t = exp_At * x0;
fprintf('If x(0) = [1; 0], then x(%.1d) = exp(A*%.1d)*x(0) =\n', t, t);
```

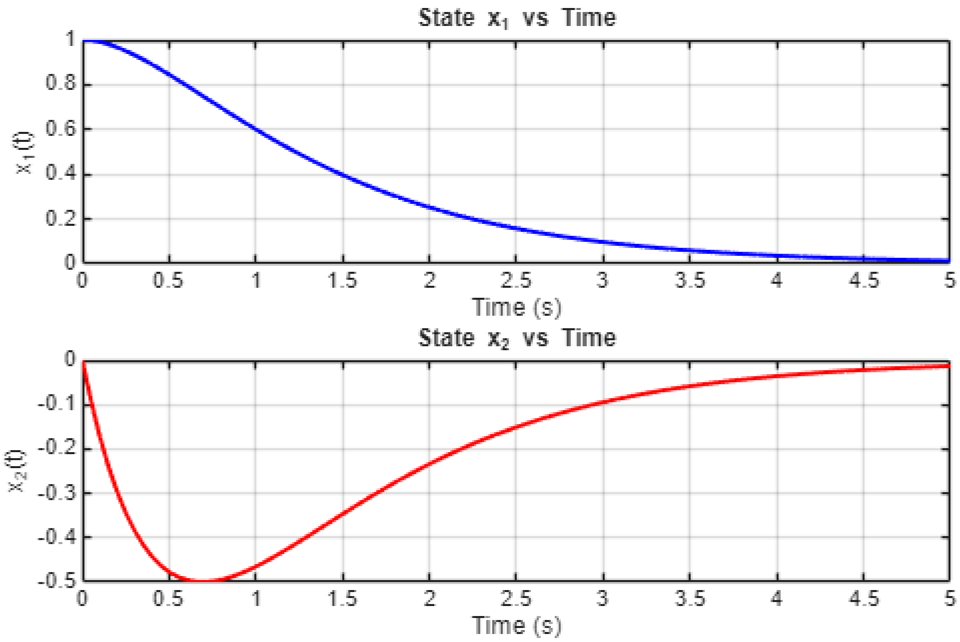
```
If x(0) = [1; 0], then x(1) = exp(A*1)*x(0) =
```

```
disp(x_t');
```

```
    0.6004   -0.4651
```

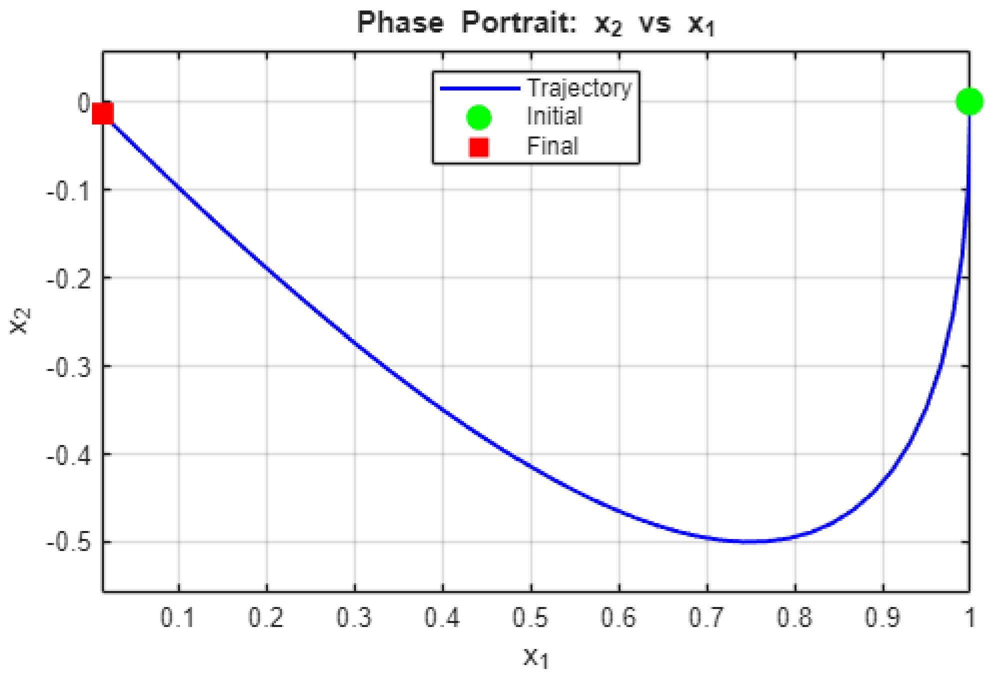
Visualize state trajectory

```
t_span = linspace(0, 5, 100); %from 0 to 5, 100 points inbetween
x_trajectory = zeros(2, length(t_span));

for i = 1:length(t_span)
    x_trajectory(:,i) = expm(A_exp * t_span(i)) * x0;
end

figure('Name', 'State Trajectory using Matrix Exponential');
subplot(2,1,1);
plot(t_span, x_trajectory(1,:), 'b-', 'LineWidth', 1.5);
grid on;
xlabel('Time (s)');
ylabel('x_1(t)');
title('State x_1 vs Time');

subplot(2,1,2);
plot(t_span, x_trajectory(2,:), 'r-', 'LineWidth', 1.5);
grid on;
xlabel('Time (s)');
ylabel('x_2(t)');
title('State x_2 vs Time');
```



Phase portrait

```
figure('Name', 'Phase Portrait');
plot(x_trajectory(1,:), x_trajectory(2,:), 'b-', 'LineWidth', 1.5);
hold on;
plot(x0(1), x0(2), 'go', 'MarkerSize', 10, 'MarkerFaceColor', 'g');
plot(x_trajectory(1,end), x_trajectory(2,end), 'rs', 'MarkerSize', 10,
'MarkerFaceColor', 'r');
grid on;
xlabel('x_1');
ylabel('x_2');
title('Phase Portrait: x_2 vs x_1');
legend('Trajectory', 'Initial', 'Final', 'Location', 'best');
axis equal;
```




```
fprintf('\nSystem decays to zero (stable) as eigenvalues have negative real parts\n');
```

```
System decays to zero (stable) as eigenvalues have negative real parts
```

```
fprintf('Eigenvalues of A: ');
```

```
Eigenvalues of A:
```

```
fprintf('%.4f ', eig(A_exp));
```

```
-1.0000 -2.0000
```

```
fprintf('\n');
```

Section 1.10: Summary and Key Takeaways

Key Concepts Covered:

1. Matrix operations: addition, multiplication, transpose, concatenation
2. Matrix properties: determinant, rank, trace, inverse, condition number
3. Matrix norms: 1-norm, 2-norm, infinity-norm, Frobenius norm
4. Triangular matrices: upper, lower, diagonal
5. Linear equations: solving Ax=b, null space, orthogonality
6. Eigenanalysis: eigenvalues, eigenvectors, characteristic polynomial
7. Similarity transformations and invariants
8. Jordan normal form for matrices with repeated eigenvalues
9. Matrix exponential for solving linear differential equations

MATLAB Functions Mastered:

`eig`, `inv`, `det`, `rank`, `trace`, `norm`, `triu`, `tril`, `null`, `orth`, `dot`, `linsolve`, `poly`, `jordan`, `expm`

Next Steps:

These mathematical tools will be applied to:

- State-space modeling (Season 2)
- Controllability and observability analysis (Season 3)
- Stability analysis using Lyapunov theory (Season 5)
- Observer and controller design (Season 6)