Modeling the Motion: A Comprehensive Study on the Dynamics of Pendulums

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Github Source Code

Contents

Abstract

In this research study, we attempt to model, observe, and study the dynamics governing the behavior of a pendulum. We begin by performing a dimensional analysis to determine the pendulum's period of oscillation (τ) and write the governing ordinary differential equation (ODE) using Newton's second law of motion. For small deviation angles (θ), we linearize the ODE and obtain the time scale of the model using the concept of dimensional homogeneity. Lastly, we implement Euler's method for differential equations to determine a numerical solution to the original nonlinear ODE. Next, accounting for the effect of drag force on the pendulum, we compare the results observed through this methodology for both the simple and the physical pendulum by graphically analyzing and discussing their behavior. We use the Python programming language to create our graphical models, along with the NumPy library for advanced mathematical computations and the Matplotlib plotting library for graph visualization. We present our source code along with this paper.

1 Introduction

Pendulums have fascinated researchers and enthusiasts for centuries due to their rich variety of motion, ranging from simple harmonic oscillations embodied by the simple pendulum to chaotic swings defining a double pendulum system. Thus, it is no surprise that the pendulum is a fundamental problem that has been modeled and studied by mathematicians, engineers, scientists, and researchers for centuries.

Acknowledging these previous attempts, our study aims to develop a deeper understanding of the underlying mathematics and physics behind their behavior by examining the factors that influence pendulum motion, such as length, mass, drag force, and initial conditions. We focus our research on studying the dynamics of the simple pendulum using mathematical concepts such as dimensional analysis. Next, we compare our observations with that of a physical pendulum and discuss how small differences, such as the distribution of mass across the pendulum, can cause significant changes in its behavior. Finally, we introduce the double pendulum and discuss its implications for our research.

2 Theory and Model Description

2.1 Simple Pendulum

We begin our study by considering a simple pendulum consisting of a spherical mass m anchored to a fixed point O via a mass-less rod of length l, depicted in Figure 1.

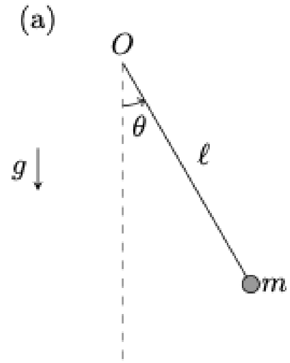


Figure 1: *A simple pendulum consists of a spherical mass m, also called a bob, anchored to a fixed point O via a mass-less rod of length l. When released, it swings back and forth in a plane under the influence of gravity, following a simple harmonic motion with the period determined by the length of the rod and acceleration due to gravity.*

2.1.1 Dimensional Analysis

We perform a dimensional analysis on a simple pendulum to determine the variables that will impact the pendulum's oscillation period. Specifically, the oscillation period of a simple pendulum relies on the length of the rod (l), the acceleration due to gravity (g), and the mass of the pendulum (m). The length of the rod is measured in meters, the mass of the pendulum is measured in kilograms, and the gravitational acceleration is measured in meters per second squared.

We express these quantities in the fundamental dimensions of Mass (M), Length (L), and Time (T).

$$\text{Length of the rod } (l)\text{: } M^0L^1T^0$$
$$\text{Mass of the pendulum } (m)\text{: } M^1L^0T^0$$
$$\text{Acceleration due to gravity } (g)\text{: } M^0L^1T^{-2}$$

Now, we write the period of oscillation as a function of the quantities it relies upon. Here, our function f relates the period of oscillation of the pendulum to the length of the rod of the pendulum (l), mass of the pendulum (m), and the acceleration due to gravity (g).

$$\tau = f(l, m, g)$$

In order to determine the dimensions of the period of oscillation, we apply the concept of dimensional homogeneity. Dimensional homogeneity states that the combination of dimensions assigned to each variable should be consistent with each side of the equation. Since the period of oscillation (τ) has dimensions of Time (T), we write our balanced equation:

$$\tau = l^a m^b g^c \tag{1}$$

where a, b, and c represent the unknown exponents to be determined.

Conducting the dimensional analysis for the balanced equation:

$$[T] = [L]^a [M]^b [LT^{-2}]^c \tag{2}$$

$$L^0 M^0 T^1 = L^a L^c M^b T^{-2c} \tag{3}$$

$$L^0 M^0 T^1 = L^{a+c} M^b T^{-2c} \tag{4}$$

$$\begin{gathered} -2c = 1 \implies c = -\tfrac{1}{2} \\ a + c = 0 \implies a = -c = \tfrac{1}{2} \\ b = 0 \end{gathered}$$

Thus, we have computed:

$$[T] = [L]^{\frac{1}{2}} [M]^0 [LT^{-2}]^{-\frac{1}{2}} \tag{5}$$

which means that our period of oscillation is directly proportional to the square root of the length of the rod, inversely proportional to the square root of the acceleration due to gravity, and independent of the mass of the pendulum. In terms of these specified quantities:

$$\tau \propto l^{\frac{1}{2}} m^0 g^{-\frac{1}{2}} \tag{6}$$

Finally,

$$\tau = \theta \sqrt{\frac{l}{g}} \tag{7}$$

where θ is a dimensionless constant.

So, for our simple pendulum, we have derived the equation for the period of oscillation,

$$\tau = 2\pi \sqrt{\frac{l}{g}} \tag{8}$$

2.1.2 ODE and Newton's Second Law

Newton's second law of motion states that the net force (F) acting on an object is equal to its mass (m) multiplied by its acceleration (a)($F = ma$). In the context of pendulums, Newton's second law of motion helps describe the relationship between the forces acting on the pendulum and its resulting angular acceleration. The law provides insights into the pendulum's oscillatory motion, allowing us to derive the governing equations and analyze the system's behavior.

We know that the period of oscillation of a simple pendulum is given by the formula:

$$T = 2\pi\sqrt{\frac{L}{g}} \tag{9}$$

where L is the length of the rod of the pendulum and g is the acceleration due to gravity.

Now, we use Newton's second law to derive an ODE describing the motion of the simple pendulum:

$$mL\theta'' = -mg\sin\theta \tag{10}$$

$$\theta'' = -\frac{g}{L}\sin\theta \tag{11}$$

where θ'' depicts the second-order derivative of $\theta(t)$ with respect to t. We use $\sin\theta \approx \theta$ as approximation for small θ, reducing the above differential equation to:

$$\theta'' = -\frac{g}{L}\theta \tag{12}$$

The above equation represents a linear homogeneous differential equation with constant coefficients. Now, we can assume that $\theta = A\cos(\omega t)$ where ω and A both are constants. Substituting this value into the differential equation, we are left with:

$$A\omega^2\cos(\omega t) = \frac{g}{L}A\cos(\omega t) \tag{13}$$

Simplifying,

$$\omega^2 = \frac{g}{L} \tag{14}$$

We have found the solution to the linearized equation:

$$\theta = A\cos((\frac{g}{L})t) \tag{15}$$

Now, assuming the initial conditions $\theta = \theta_0$ and $\dot{\theta} = 0$:

$$\theta_0 = A\cos((\frac{g}{L})t) \tag{16}$$

2.1.3 Euler's Method

We first rewrite the second-order differential equation as a system of two first-order differential equations.

Let $y1 = \theta$ and $y2 = \theta'$. We can write this as:

$$y1' = y2 \tag{17}$$

$$y2' = -\frac{g}{L}\sin(y1) \tag{18}$$

Now, we apply Euler's forward equation in order to approximate the solution of the differential equation. We do this by iteratively solving the following two equations:

$$y1(t + \Delta t) = y1(t) + \Delta t \cdot y2(t) \tag{19}$$

$$y2(t + \Delta t) = y2(t) - \Delta t(\frac{g}{L}) \sin(y1(t)) \tag{20}$$

2.1.4 Drag Force

Drag force is the force that opposes the motion of an object moving through a fluid due to the resistance of the fluid to the object's motion. This drag force is responsible for the loss of energy of the object and is necessary to accurately model a pendulum's behavior. When a simple pendulum swings through the air, it is subjected to a drag force due to air resistance.

This drag force on a simple pendulum is expressed as:

$$F_D = -kAv \tag{21}$$

where F_D is the drag force, k is a dimensionless constant, A is the cross-sectional area of the pendulum, and v is the velocity of the pendulum. This drag force acts opposite to the direction of motion of the pendulum and causes the body to lose energy. Due to this drag force, the oscillations of the pendulum should be damped, meaning that the amplitude of oscillations should decrease over time.

Now that we have established a way of measuring drag force, we can rewrite the equation of motion for a simple pendulum under the effect of this drag force:

$$mL\theta'' = -mg \sin\theta - F_D \tag{22}$$

$$mL\theta'' = -mg \sin\theta - kA\theta' \tag{23}$$

$$\theta'' = -\frac{g}{L} \sin\theta - kA\theta' \tag{24}$$

where m is the mass of the pendulum bob, L is the length of the pendulum, g is the acceleration due to gravity, θ is the angle of displacement of the pendulum, θ' is the angular velocity of the pendulum, k is the dimensionless constant related to drag force, and A is the cross-sectional area.

This equation shows that the natural frequency of the pendulum is reduced by the damping effect due to the drag force. As a result, the period of the pendulum's oscillation is longer than the period of an ideal pendulum with no drag force, and it decreases as the amplitude of the oscillations decreases.

Thus, we once again apply Euler's forward equations to approximate the solution of our new differential equation. This procedure is similar to the case where we did not account for drag force. We do this by iteratively solving the following equations:

$$y1(t + \Delta t) = y1(t) + \Delta t \cdot y2(t) \tag{25}$$

$$y2(t + \Delta t) = y2(t) - \Delta t(\frac{g}{L}) \sin(y1(t)) - kA\theta' \tag{26}$$

With these equations, we conclude our explanation of the theory governing the motion of a simple pendulum.

2.2 Physical Pendulum

We defined the simple pendulum as an ideal pendulum with all its mass concentrated at one point, the bob. Realistically, however, we cannot model most real pendulums as concentrated point masses attached to mass-less rods. We refer to such a real pendulum as a physical pendulum.

We define our physical pendulum as a circular rod of length l and uniform mass m anchored to a fixed point O, as depicted in Figure 2.

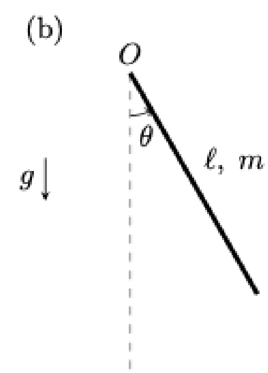


Figure 2: *A physical pendulum consists of a circular rod of length l and uniform mass m, anchored to a fixed point O and having thickness $2a$. When released, it rotates around an axis of rotation under the influence of gravity, following a harmonic motion with the period determined by its length, distribution of mass, and moment of inertia. The motion of a physical pendulum is more complex than that of a simple pendulum because it can oscillate in multiple directions and can have varying periods of motion depending on the location of its center of mass and the axis of rotation.*

2.2.1 Dimensional Analysis

We begin our study of the physical pendulum by performing dimensional analysis. For determining the period of oscillation, we consider the following parameters: the length of the rod (l), the acceleration due to gravity (g), the mass of the pendulum (m), moment of inertia (I), and the angular displacement (θ).

We express these quantities in the fundamental dimensions of Mass (M), Length (L), and Time (T).

Length of the rod (l): $M^0L^1T^0$
Mass of the pendulum (m): $M^1L^0T^0$
Acceleration due to gravity (g): $M^0L^1T^{-2}$
Moment of inertia (I): $M^1L^2T^0$
Angular displacement (θ): $M^0L^0T^0$

Now, once again, we describe the period of oscillation as a function of these quantities, relating it to the length of the rod of the pendulum (l), mass of the pendulum (m), the acceleration due to gravity (g), and the moment of inertia (I). We do not take the angular displacement (θ) into consideration since it is a dimensionless quantity.

$$\tau = f(l, m, g, I)$$

In order to determine the dimensions of the period of oscillation, we must apply the concept of dimensional homogeneity such that the combination of dimensions assigned to each variable should be consistent with each side of the equation. Since the period of oscillation (τ) has dimensions of Time (T), we write our balanced equation:

$$\tau = l^a m^b g^c I^d \tag{27}$$

where a, b, c, and d represent the unknown exponents to be determined.

Conducting the dimensional analysis for the balanced equation:

$$[T] = [L]^a[M]^b[LT^{-2}]^c[ML^2]^d \tag{28}$$

$$L^0M^0T^1 = L^{a+c+2d}M^{b+d}T^{-2c} \tag{29}$$

$$a + c + 2d = 0$$
$$b + d = 0$$
$$-2c = 1$$

Here we have four unknown variables with three linear equations, so our results are not as straightforward as they were in the case of the simple pendulum. Moreover, the physical pendulum has additional parameters, such as the moment of inertia (I), which is related to both the mass (m) and the geometry of the pendulum which makes the system of equations derived from the dimensional analysis underdetermined. Solving this system of equations yields an infinite number of solutions which implies that the period of oscillation of a physical pendulum cannot be accurately discerned using simply dimensional analysis. To fully understand and analyze the behavior of a physical pendulum, it is necessary to consider the governing equation of motion, derived from the torque, and solve it under specific conditions. This will provide a more accurate and detailed understanding of the relationship between the period of oscillation and the other parameters.

2.2.2 ODE and Newton's Second Law

We aim to derive the equation of motion for a physical pendulum by analyzing the torque acting on the pendulum due to the gravitational force. We consider the gravitational force acting at the center of mass of the pendulum and denote the distance between the fixed pivot point and the center of mass as $\frac{L}{2}$.

Now, we write the equation for torque (τ):

$$\tau = I\alpha \tag{30}$$

where I is the moment of inertia of the pendulum about the pivot point, and α is the angular acceleration. However, this torque is also equal to the cross product of the lever arm vector of the pendulum and the gravitational force vector. For small angles, we can approximate $\sin\theta \approx \theta$ so the torque due to gravity becomes,

$$\tau = -\frac{mgL\theta}{2} \tag{31}$$

$$I\alpha = -\frac{mgL\theta}{2} \tag{32}$$

Now, using the definition of angular acceleration and its relation with angular displacement, we can write this equation as,

$$I\theta'' = -\frac{mgL\theta}{2} \tag{33}$$

$$\theta'' = -\frac{mgL\theta}{2I} \tag{34}$$

This is a second-order linear differential equation and its solution describes the motion of the physical pendulum. This equation resembles the second-order linear differential equation for simple harmonic motion:

$$\theta'' = -\omega^2 x \tag{35}$$

where x is the displacement of the oscillating object from its equilibrium position, and ω is the angular frequency of the oscillation. This equation is derived from Newton's second law of motion. Comparing equations 34 and 35,

$$\omega^2 = \frac{mgL}{2I} \tag{36}$$

$$\omega = \sqrt{\frac{mgL}{2I}} \tag{37}$$

Now, we relate the period of oscillation, T, with the angular frequency ω,

$$T = \frac{2\pi}{\omega} \tag{38}$$

Substituting the expression we just derived for ω,

$$T = 2\pi(\sqrt{\frac{2I}{mgL}}) \tag{39}$$

For this pendulum, we assume that the mass m of the rod is uniform with a center of gravity towards its center. So, the moment of inertia (I) for this rod is given by,

$$I = \frac{mL^2}{3} \tag{40}$$

Substituting this value in our equation for the period of oscillation,

$$T = 2\pi(\sqrt{\frac{2}{mgL}(\frac{mL^2}{3})}) \tag{41}$$

$$T = 2\pi\sqrt{\frac{2L}{3g}} \tag{42}$$

2.2.3 Euler's Method

Now that we have the equation for the period of oscillation of the physical pendulum, we can begin adapting the Euler's forward method to solve the differential equations governing the behavior of the pendulum. Let $y1 = \theta$ and $y2 = \theta'$. We can write this as:

$$y1' = y2 \tag{43}$$

$$y2' = -\frac{3g}{2L}\sin(y1) \tag{44}$$

Applying Euler's forward equation, we can iteratively solve the following equations:

$$y1(t + \Delta t) = y1(t) + \Delta t \cdot y2(t) \tag{45}$$

$$y2(t + \Delta t) = y2(t) - \Delta t(\frac{3g}{2L})\sin(y1(t)) \tag{46}$$

2.2.4 Drag Force

We remarked that the drag force on a simple pendulum is a force due to air resistance that opposes the motion of the pendulum and causes the gradual loss of energy, leading to damped oscillations and a decrease in their amplitude. In terms of a physical pendulum, drag force works in a similar manner, causing loss of energy in the physical pendulum and damped oscillations. This drag force is almost identical to the one discussed earlier.

$$F_D = -kAv \tag{47}$$

Now, we can rewrite our equations for Euler's method for a physical pendulum experiencing drag force.

$$\theta'' = -\frac{mgL\theta}{2I} = -\frac{3g\theta}{2L} - kA\theta' \tag{48}$$

This drag force causing damping of the oscillations, leading to a decrease in the amplitude of the oscillations over time. Now, we consider the effects of the drag force while applying Euler's forward equations in computing the solution of the differential equation, giving us the following equations:

$$y1(t + \Delta t) = y1(t) + \Delta t \cdot y2(t) \tag{49}$$

$$y2(t + \Delta t) = y2(t) - \Delta t(\frac{3g}{2L}) \sin(y1(t)) - kA\theta' \tag{50}$$

With these equations, we conclude our explanation of the theory governing the motion of a physical pendulum.

3 Graphical Analysis: Results and Observations

We construct graphical models of the simple and physical pendulums and compare their behavior under different conditions. We focus on measuring and graphing a limited set of parameters, such as angular displacement and angular velocity, and investigate the effects of factors such as the presence or absence of drag force. By using graphical models, we seek to visualize the behavior of the pendulums and provide insights into the underlying dynamics of these systems.

3.1 Simple Pendulum

For all plots concerning the simple pendulum, we keep consistency in choosing our initial conditions so that our observations are precise and based off of correct and comparable information. We keep the initial angular displacement, $\theta = \frac{\pi}{2}$, timestep $dt = 0.001$, length of the pendulum $L = 1$, $\tau = 2\pi\sqrt{\frac{L}{g}}$, and otherwise noted. In Figure 3, we plot a graph of angular displacement (θ) versus dimensionless time ($\frac{t}{\tau}$) for different timesteps.

From the graph, we can observe that as the time step values increase, our solution becomes increasingly more inaccurate. When the value of $dt = 0.1$, our numerical solution diverges off the analytical extremely and almost immediately, in contrast to when our $dt = 0.001$, where the numerical solution very closely follows the analytical solution and shows little deviation as time goes on. So, as the time step changes, the accuracy of the numerical solution obtained using the Euler method is affected. Smaller time steps lead to more accurate solutions, while larger time steps result in less accurate solutions.

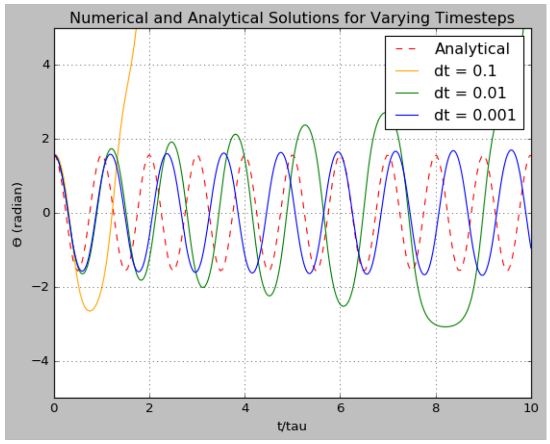


Figure 3: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for Varying Time Steps*

Next, we model the graph of angular displacement (θ) versus the dimensionless time ($\frac{t}{\tau}$) for different initial angular displacements (θ_0). We expect the amplitude of the oscillations to decrease with decreasing initial angular displacements. Our results are depicted in Figure 4 and Figure 5.

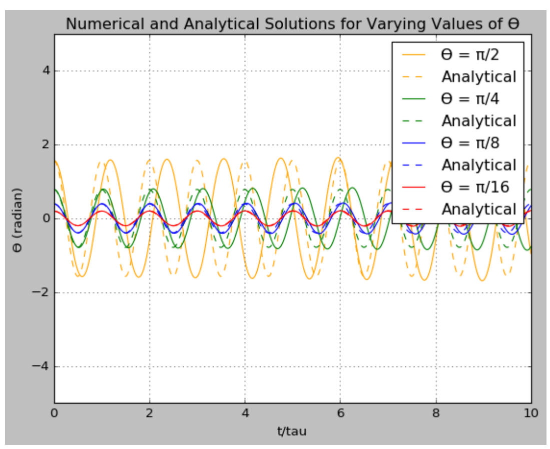
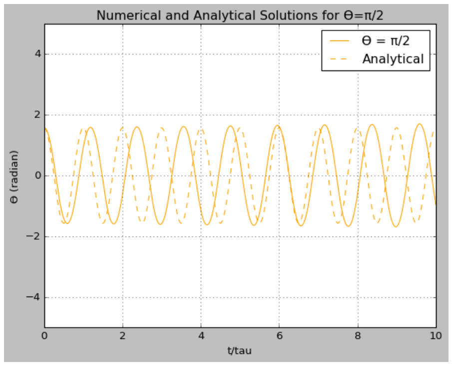
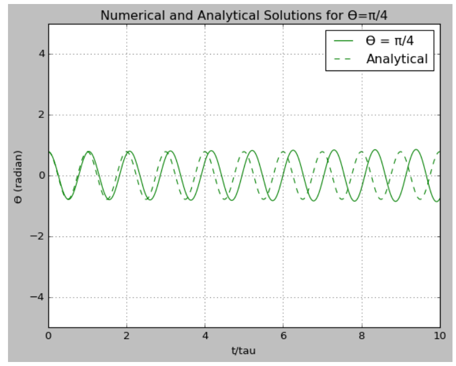


Figure 4: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for Varying Initial Angular Displacements Combined*

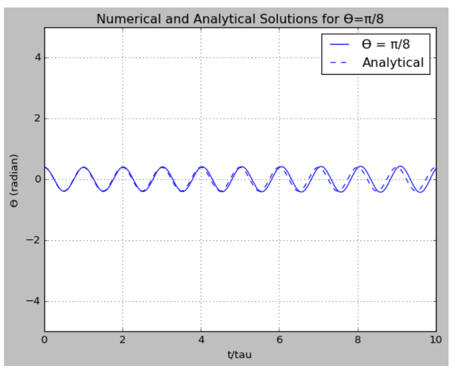
From these figures, we notice that as the initial angular displacement decreases, the amplitude of the oscillations also decreases, as expected. Moreover, the motion of the pendulum becomes closer to the conditions of the small angle approximation and we observe improving agreement between the analytical and



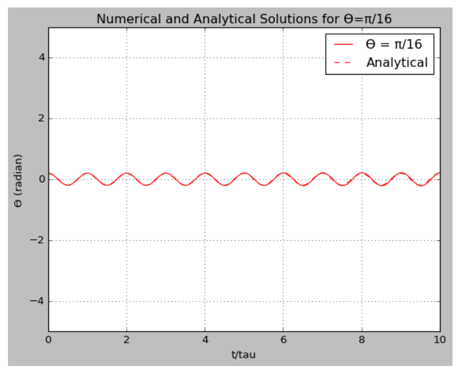
(a) θ *versus* $\frac{t}{\tau}$ *for condition* $\theta_0 = \frac{\pi}{2}$



(b) θ *versus* $\frac{t}{\tau}$ *for condition* $\theta_0 = \frac{\pi}{4}$



(c) θ *versus* $\frac{t}{\tau}$ *for condition* $\theta_0 = \frac{\pi}{8}$



(d) θ *versus* $\frac{t}{\tau}$ *for condition* $\theta_0 = \frac{\pi}{16}$

Figure 5: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for Varying Initial Angular Displacements ($\theta_0 = \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}$)*

numerical solutions. We see a noticeable difference between Figure $5(a)$, representing an initial displacement of $\frac{\pi}{2}$, and Figure $5(d)$, representing an initial displacement of $\frac{\pi}{16}$. In Figure $5(d)$, the numerical and analytical solutions agree with each other to such a great extent that they appear to be the same, whereas there is a significant difference in the curves for the numerical and analytical solution in Figure $5(a)$.

Next, we graphically visualize the effect of drag force on the simple pendulum by graphing angular displacement (θ) versus time ($\frac{t}{\tau}$) for two instances of the simple pendulums: with the influence of drag force and without the influence of drag force. Our model can be observed in Figure 6, where the blue curve is the one experiencing the effect of drag force. As we theorized, the drag force is resulting in a loss of energy depicted by the damped oscillations and the convergence of the curve to an equilibrium, stable level ($\theta = 0$). The amplitude of the oscillations appears to be decreasing over time in the case of the simple

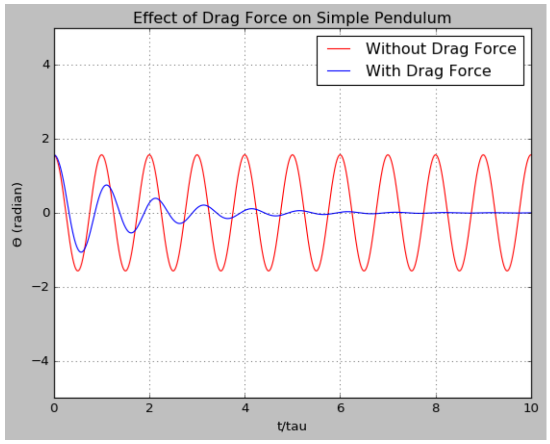


Figure 6: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for a Simple Pendulum With and Without Drag Force*

pendulum experiencing drag force while the other one seems unaffected. These results are in agreement with our knowledge of the effect of drag force on a simple pendulum and confirm our expectations for their behavior.

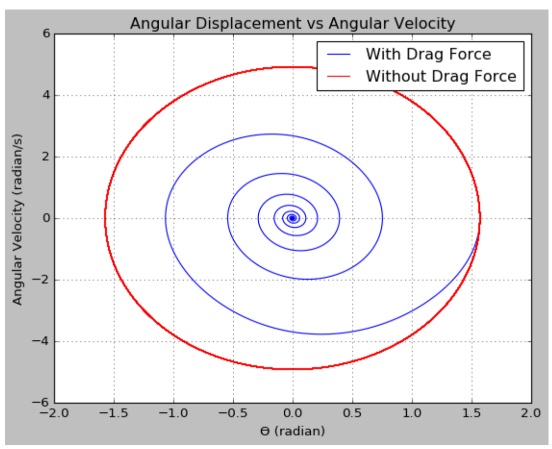


Figure 7: *Angular Velocity ($\dot{\theta}$) versus Angular Displacement (θ) for a Simple Pendulum With and Without Drag Force*

Finally, to better evaluate the effects of drag force on the simple pendulum, we also visualize the graph plotting angular velocity ($\dot{\theta}$) versus angular displacement (θ) for a simple pendulum with and without the effects of drag force. Straight away, we observe that the pendulum experiencing the drag force (blue curve) spirals inward whereas the one not experiencing drag (red curve) mimics the shape of a closed loop. Due to the drag force and subsequent amplitude of oscillation decreasing over time, the motion of the pendulum eventually slows down till it comes to a complete stop, as can be observed in Figure 7. So, in the absence of damping forces, the pendulum exhibits periodic and harmonic oscillations with constant amplitude and period. However, when a drag force is introduced, the system loses energy over time, leading to a decrease in amplitude and eventually bringing the pendulum to a stop at its equilibrium position. These phase plots visually demonstrate these differences in model behavior.

3.2 Physical Pendulum

Similar to the simple pendulum, we maintain consistency in choosing the initial conditions for all models depicting the physical pendulum, unless noted otherwise. This will help us generate accurate plots and make it easier to compare our results. We begin by plotting a graph depicting angular displacement (θ) versus time ($\frac{t}{\tau}$) for a simple and physical pendulum (depicted Figure 8).

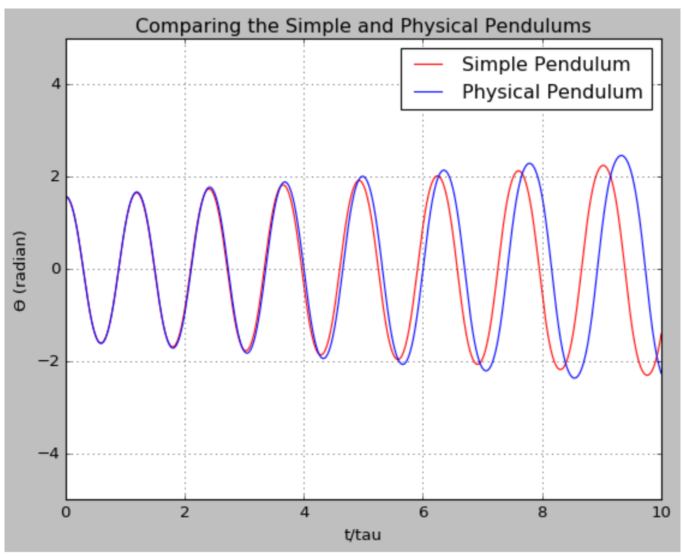


Figure 8: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for a simple (red curve) and physical (blue curve) Pendulum*

The fundamental dissimilarity between the physical and simple pendulums is in their moment of inertia. The physical pendulum's moment of inertia is dependent on the distribution of mass around its pivot point, while the simple pendulum's moment of inertia is proportional to the mass at the end of its rod. As a consequence, the physical pendulum exhibits more intricate motion patterns than the simple pendulum, which is evident from the graph whereby the blue curve (physical pendulum) displays a less regular pattern of oscillations in comparison to the red curve (simple pendulum).

Moreover, the simple and physical pendulums have different time periods (τ), as indicated by the different slopes of the normalized time axis. The time period of the simple pendulum is given by $\tau_1 = 2\pi\sqrt{\frac{L}{g}}$, while the time period of the physical pendulum is given by $\tau_1 = 2\pi\sqrt{\frac{2L}{3g}}$. As expected, the physical pendulum has a longer time period than the simple pendulum which may also contribute to the irregularities in their relative motion.

Next, we observe the motion of the physical pendulum by plotting angular displacement (θ) versus time ($\frac{t}{\tau}$) for different values of initial angular displacement (θ_0). Our observations, both visually and analytically, seem to be similar to those of the simple pendulum. For this reason, we do not include individual plots and only the combined plot. This plot, therefore, demonstrates the consistency between the numerical and analytical solutions for a physical pendulum with varying initial angular displacements. It also correlates the relationship between initial angular displacement on the amplitude of oscillation with a direct proportionality defining their relationship. More significantly, the overall oscillatory behavior and sinusoidal nature of the curve/wave remains consistent across different initial conditions and the type of pendulum (pictured Figure 9).

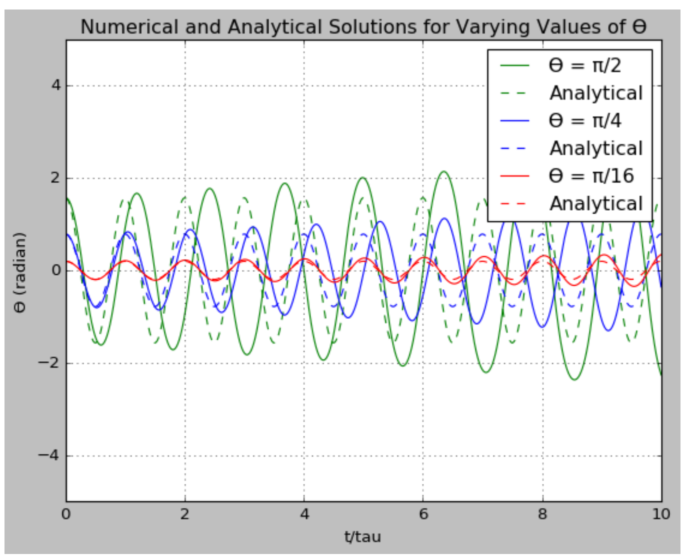


Figure 9: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for varying values of initial angular displacement (θ_0)*

We examine how drag force affects the physical pendulum in relation to the simple pendulum by plotting a graph of angular displacement (θ) versus time

($\frac{t}{\tau}$) for both instances of the physical pendulum: with the influence of drag force and without the influence of drag force. This is depicted in Figure 10. Our observations and results are once again nearly identical to those of the simple pendulum, with little to no differences in how drag force relatively affects the movement of the two pendulums where compared against each other. While both the simple and physical pendulums with drag exhibit damped oscillations and decay in amplitude, the specifics of their oscillation behavior, time period, and equilibrium convergence can differ due to their respective equations of motion and model parameters.

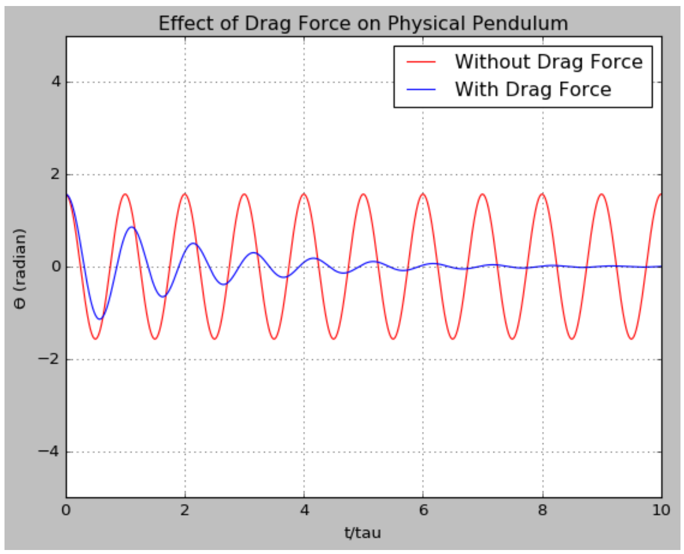


Figure 10: *Angular Displacement (θ) versus dimensionless Time ($\frac{t}{\tau}$) for varying values of initial angular displacement (θ_0)*

Finally, we plot the angular velocity ($\dot{\theta}$) versus angular displacement (θ) for a physical pendulum with and without drag force and discuss its observations (visualised in Figure 11).

Like the simple pendulum, both curves show a loop-like shape, indicating periodic behavior in the motion of the physical pendulum. The curve with drag (blue) appears more stretched and elongated compared to the curve without drag (green). This is due to the damping effect of the drag force on the pendulum's oscillations. As compared to the simple pendulum, the shape of the curves is slightly different, the reason being the differences in their equation of motions which tends to give the physical pendulum's curves a more rounded figure as compared to the more elongated figures we observed for the simple pendulum. Also, due to the loss of energy and subsequent damping of the oscillations, the blue curve spirals inward, and stops once it reaches the equilibrium point of

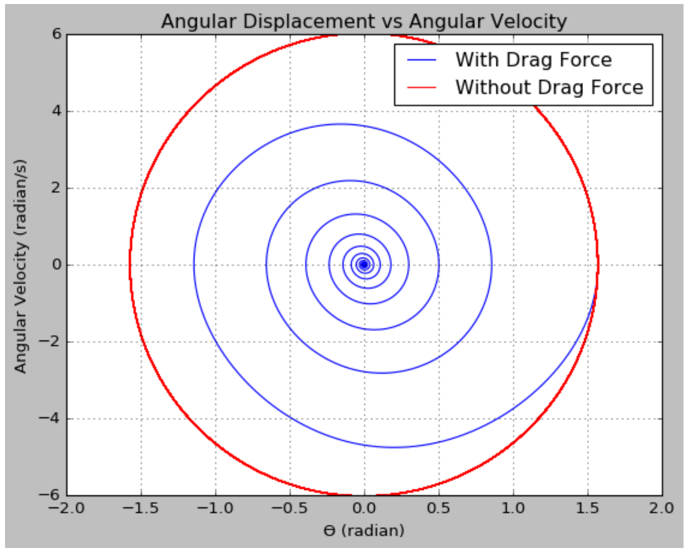


Figure 11: *Angular Velocity ($\dot{\theta}$) versus Angular Displacement (θ) for a physical pendulum with and without drag force*

zero. So our model is accurate and predictive of the behavior expected from a physical pendulum.

Therefore, our understanding of the physical pendulum as well as the simple pendulum has been strengthened through graphical analysis and this concludes our results discussion.

4 Conclusion

In conclusion, our graphical analysis of the simple and physical pendulums has provided valuable insights into the behavior of these systems under different conditions. By examining various parameters, such as angular displacement and angular velocity, we have been able to identify and compare the effects of factors like the presence or absence of drag force on both pendulum types.

Our findings indicate that smaller time steps result in more accurate numerical solutions, while larger time steps lead to less accurate solutions. We have also observed that the amplitude of oscillations decreases with decreasing initial angular displacements and that the motion of the pendulum approaches the small angle approximation conditions as the initial displacement decreases. The introduction of drag force leads to damped oscillations and energy loss, eventually bringing the pendulum to a stop at its equilibrium position.Comparing the simple and physical pendulums, we noted that the physical pendulum exhibits more intricate motion patterns due to its moment of inertia dependence on the mass distribution around its pivot point. The physical pendulum also has a longer time period than the simple pendulum. Despite these differences, the effects of drag force and the relationship between initial angular displacement and amplitude of oscillation are similar in both pendulums.

Our study has successfully visualized the behavior of simple and physical pendulums using graphical models, thereby enhancing our understanding of the underlying dynamics of these systems.

5 Future Work

A scientist's work is never finished! Future research could conduct a nonlinear analysis and study the effect of nonlinear damping forces or nonuniform mass distribution systems. Graphing and visualising such a task could prove to be a difficult issue, which would naturally lean into the requirement of advanced modeling techniques to develop more reliable and predictive models of the pendulums. These might help identify new insights into their behavior.

We can also explore the behavior of more complex pendulum systems, such as the double pendulum, which consists of two pendulums attached to each other. The double pendulum exhibits chaotic and unpredictable motion, making it a more unique and more challenging system to study. Graphical models can be used to visualize the motion of the double pendulum and identify the parameters that affect its behavior. Additionally, we can investigate the effects of various internal and external factors on the motion of the double pendulum, such as the length and mass of each pendulum, the angle between them, and the presence of damping forces.

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