Solution of Laplace's equation: Uniqueness theorem

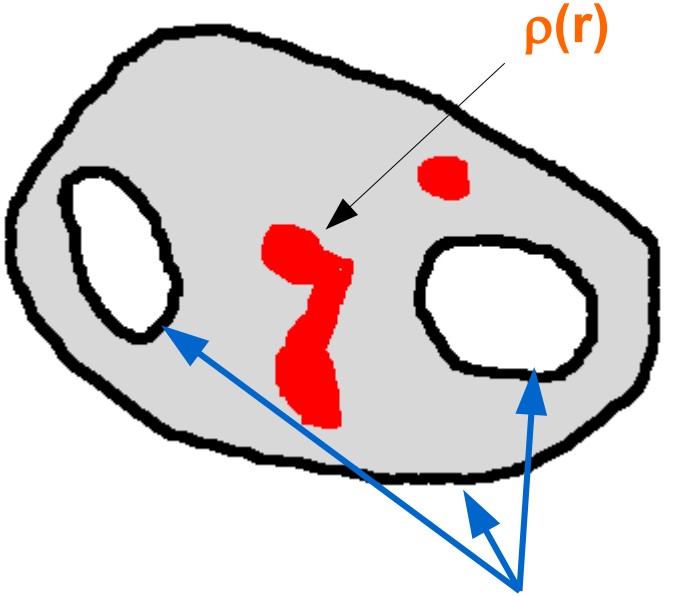
Two functions V_1 and V_2 satisfy $\nabla^2 V = -\frac{\rho}{\epsilon_0}$

With the same boundary conditions

Let $\psi = V_1 - V_2$ Then

$\nabla^2 \psi = 0$ and $\psi = 0$ on ALL boundaries

Implies $\psi = 0$ everywhere



To prove this:

consider the vector function: $\psi \vec{\nabla} \psi$

$$\int_{vol} \nabla . (\psi \vec{\nabla} \psi) d\tau = \int_{surface} \psi \vec{\nabla} \psi . d\vec{S}$$

$$\int_{vol} \left[\psi \nabla^2 \psi + \left| \vec{\nabla} \psi \right|^2 \right] d\tau = 0$$

$$\int_{vol} \left| \vec{\nabla} \psi \right|^2 d\tau = 0$$

If a "guess" satisfies the boundary condition then that MUST be the solution

Possible only if ψ =constant=0 everywhere

Why is a metal cavity a "shield" ?

Arbitrary charges are outside the cavity (Q1...Qn)

Charges will be induced in the wall of the cavity.

But the wall remains an equipotential.

Inside the cavity V=0 is one possible solution satisfying the boundary conditions.

THAT IS THE UNIQUE SOLUTION.

What if the wall is not fixed at V=0 (i.e. floating)?

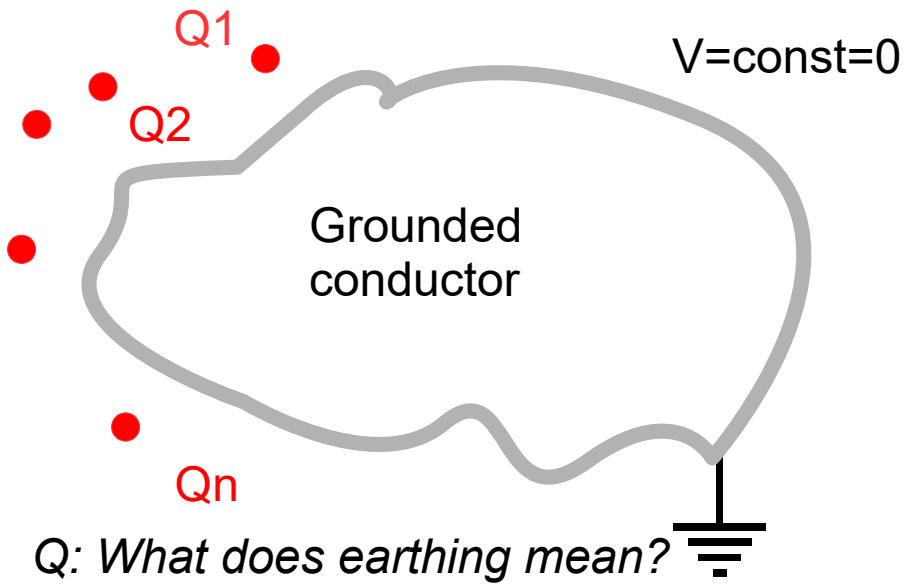
V=constant is still correct, but the constant will depend on the charge distribution outside.

If charges are placed inside?

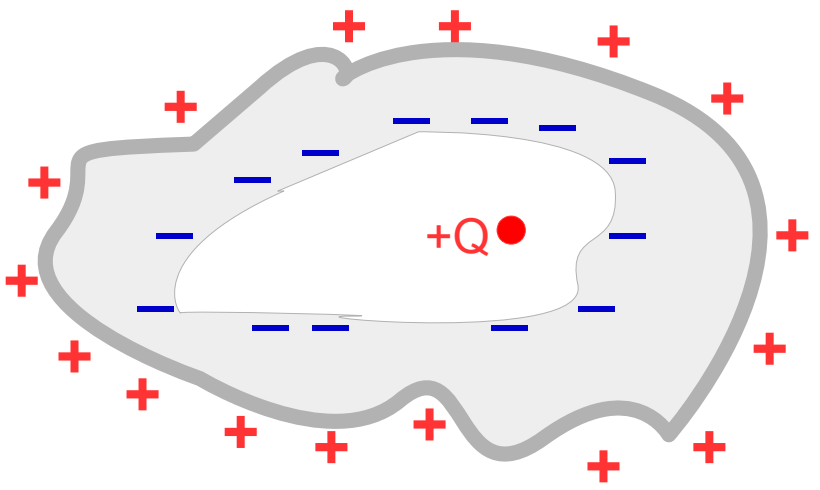
$$\nabla^2 V = -\frac{\rho_{in}}{\epsilon_0}$$

$V = 0$ (boundary condition)

irrespective of ρ_{out}

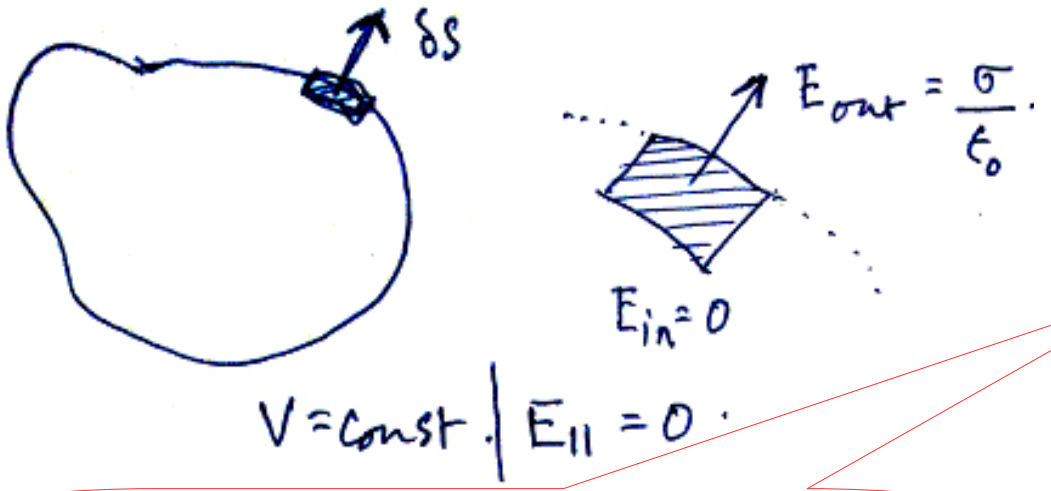


Q: What does earthing mean? Why is the potential of the earth fixed?

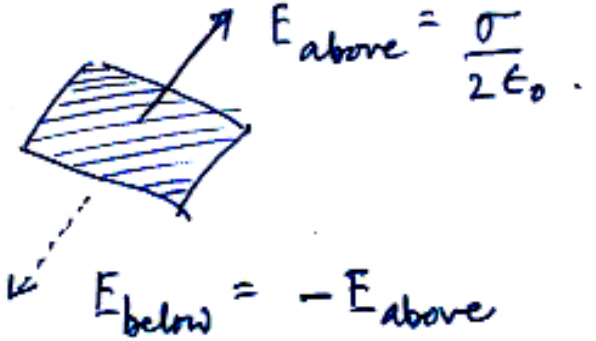


Floating conductor
Equal amounts +Q and -Q on inner and outer surfaces.

Electrostatic pressure on a conducting shell/ charged bubble



If δS was treated in isolation



Force on the element dS is due to E field created by all the other charges.

$$\delta Q = \sigma \delta S \text{ creates}$$

$$E_{\perp} = \frac{\sigma}{2\epsilon_0} \quad \text{above and below}$$

$$\text{But } E_{\perp} = \frac{\sigma}{\epsilon_0} \quad \text{(above)}$$

$$E_{\perp} = 0 \quad \text{(below/inside)}$$

The difference must be due to the field created by all the other charges.

The force on dS is
Charge in dS x Field due to all charges NOT in dS

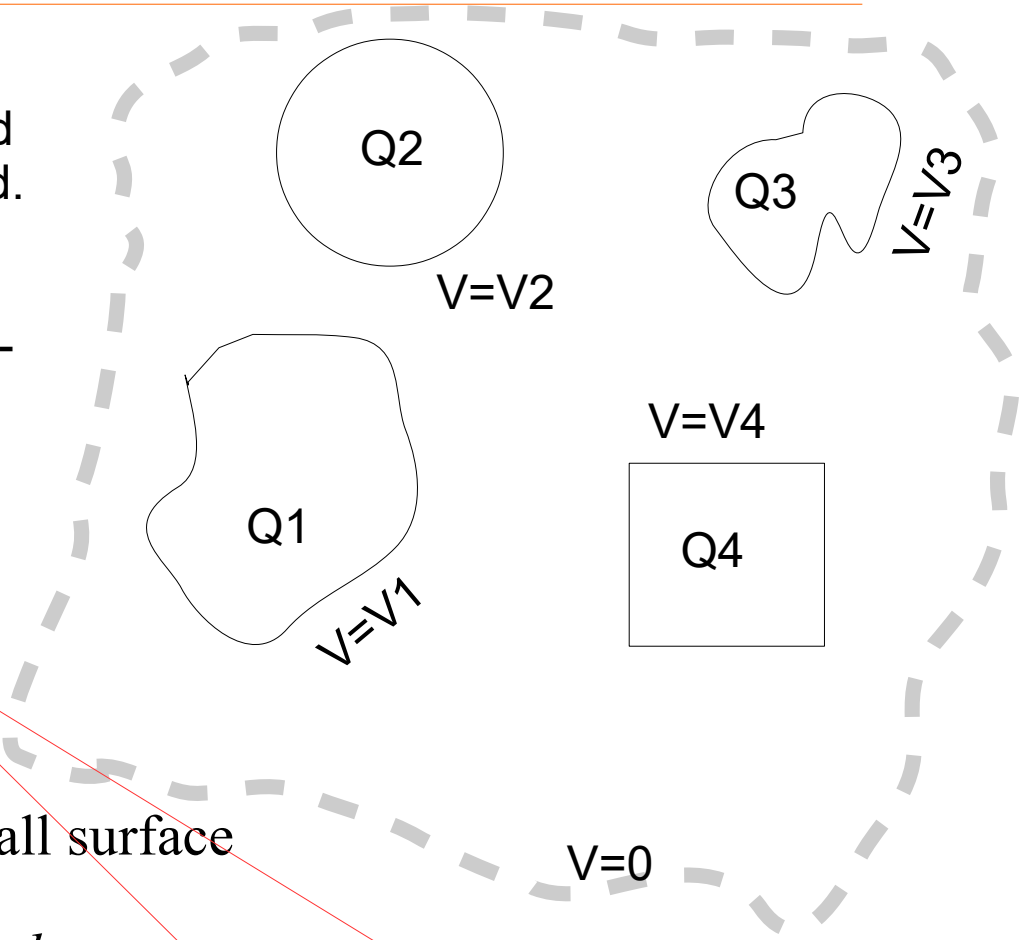
$$\delta F = (\sigma \delta S)\frac{\sigma}{2\epsilon_0} = \left(\frac{\sigma^2}{2\epsilon_0}\right)\delta S$$

Outward Pressure=Force/area
The conducting surface simplifies the calculation. For an arbitrary surface it is more complex.....

Electrostatics of conductors : uniqueness theorem 2 & capacitance

If the charge on ALL the **conductors** is specified then the potential V(x,y,z) is uniquely determined.

Notice that we are not specifying the charge distribution, only the total charge. That's the non-trivial content.



Suppose two distinct solutions exist

$U(x,y,z), V(x,y,z)$: *Both must satisfy*

$$\int_{surf\ i} \left(-\vec{\nabla} U\right).d\vec{S} = \int_{surf\ i} \left(-\vec{\nabla} V\right).d\vec{S} = Q_i$$

define $\psi = U - V$, & *integrate* $\psi\vec{\nabla}\psi$ over all surface

$$\sum_i \int_{surf\ i} \left(\psi\vec{\nabla}\psi\right).d\vec{S} = \int_{all\ vol} \left[\psi\nabla^2\psi + |\nabla\psi|^2\right] d\tau$$

$LHS = 0$: *why ?*

= 0

So $\int_{all\ vol} |\vec{\nabla}\psi|^2 d\tau = 0$

Hence $U - V = 0$

U and V must give equipotentials On each conducting surface, but we do not claim that they are the same constant to start with.

Where all do we come across Laplace's equation?

1. Fluid flow : Incompressible, "inviscid", "irrotational"

flow of "DRY water", quite far from reality, still useful as a starting point

$$(\rho = const. \quad \eta = 0) \Rightarrow \nabla . v = 0$$
$$If \ \nabla \times \boldsymbol{v} = 0 \quad then \quad \boldsymbol{v} = \nabla \phi \quad (velocity\ potential)$$
$$\nabla^2 \phi = 0$$

2. Heat conduction (Fourier), Diffusion equation (in steady state, time derivative =0)

$$D \nabla^2 \theta = \frac{\partial \theta}{\partial t}$$

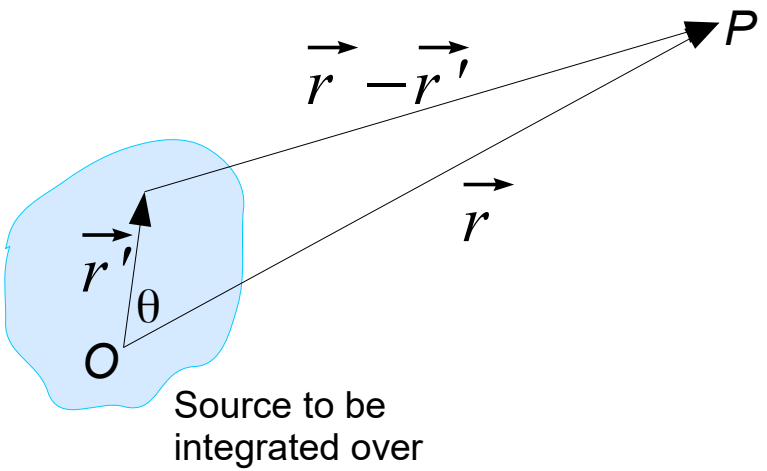
3. Electrostatic lensing :
   Electron microscope, Ion trap, particle acceleration/beam steering, mass spectrometer

   Interesting differences from optical lensing:
   - Charged nature of particles,
   - Not possible to have focussing from all sides

How does a charge distribution look from far away?

Quantitatively this means : With what power law does it fall off ….inverse square, cube, fourth ?

Often the charge is limited to a small area.



Answer:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\,\frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

In many cases $r \gg r'$

So expand in a power series in $\frac{r'}{r}$

From the figure :

$$\frac{1}{|\vec{r}-\vec{r}'|} = [r^2 + r'^2 - 2rr'\cos\theta]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\left[1 - \left\{2\frac{r'}{r}\cos\theta - \left(\frac{r'}{r}\right)^2\right\}\right]^{-\frac{1}{2}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta)$$

Legendre polynomials again!

Binomial expansion .....

$$(1-x)^{-\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 \ldots$$

Multipole expansion of the electrostatic potential

$$V(P) = \frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int d^3\vec{r}\,'\left[\rho(\vec{r}\,')\,r'^{l}\,P_l(\cos\theta)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int d^3\vec{r}\,'\rho(\vec{r}\,') \; + \right.$$

monopole

$$\frac{1}{r^2}\int d^3\vec{r}\,' r'\cos\theta\,\rho(\vec{r}\,') \; +$$

dipole

$$\left.\frac{1}{r^3}\int d^3\vec{r}\,'(r')^2\frac{1}{2}(3\cos^2\theta-1)\rho(\vec{r}\,') \; + \; \ldots\right]$$

quadrupole

If the total charge is zero:
Dipole term dominates.

If that is also zero
Quadrupole dominates

suppose

$$\rho(\vec{r}\,') = q\,\delta(\vec{r}\,'-\vec{a})-q\,\delta(\vec{r}\,'+\vec{a})$$

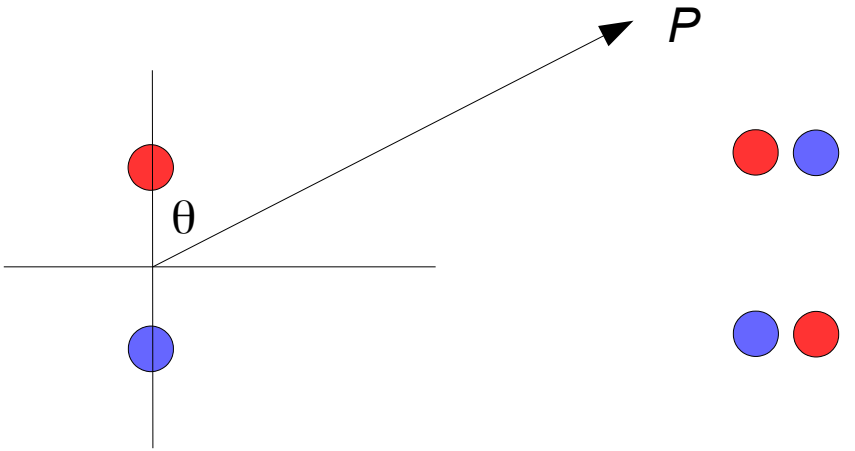
how will the dipole integral look?

can write this as

$$V_{dipole} = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\,.\,\hat{\vec{r}}}{r^2}$$

with $\vec{p} = \sum q_i \vec{r}_i\,'$

and some other equivalent forms...



Stick two monopoles
to get a dipole.

Stick two dipoles to
get a quadrupole.

Choice of the co-ordinate system and origin in multipole expansion

We could have done the expansion in a more cartesian way...

$$\frac{1}{|\vec{r}-\vec{r}'|} = [r^2 + r'^2 - 2\vec{r}.\vec{r}']^{-\frac{1}{2}}$$

This would have given successive terms like....

$$V_{mono} = \frac{1}{4\pi\epsilon_0}\frac{Q_{total}}{r}$$

$$V_{dip} = \frac{1}{4\pi\epsilon_0}\frac{\sum \hat{r}_i p_i}{r^2}$$

$$V_{quad} = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\frac{\sum_{ij} \hat{r}_i \hat{r}_j Q_{ij}}{r^3}$$

$$p_i = \int d^3\vec{r}\, r_i' \rho(r')$$

$$Q_{ij} = \int d^3\vec{r}\,' \left(3r_i' r_j' - r'^2 \delta_{ij}\right)\rho(r')$$

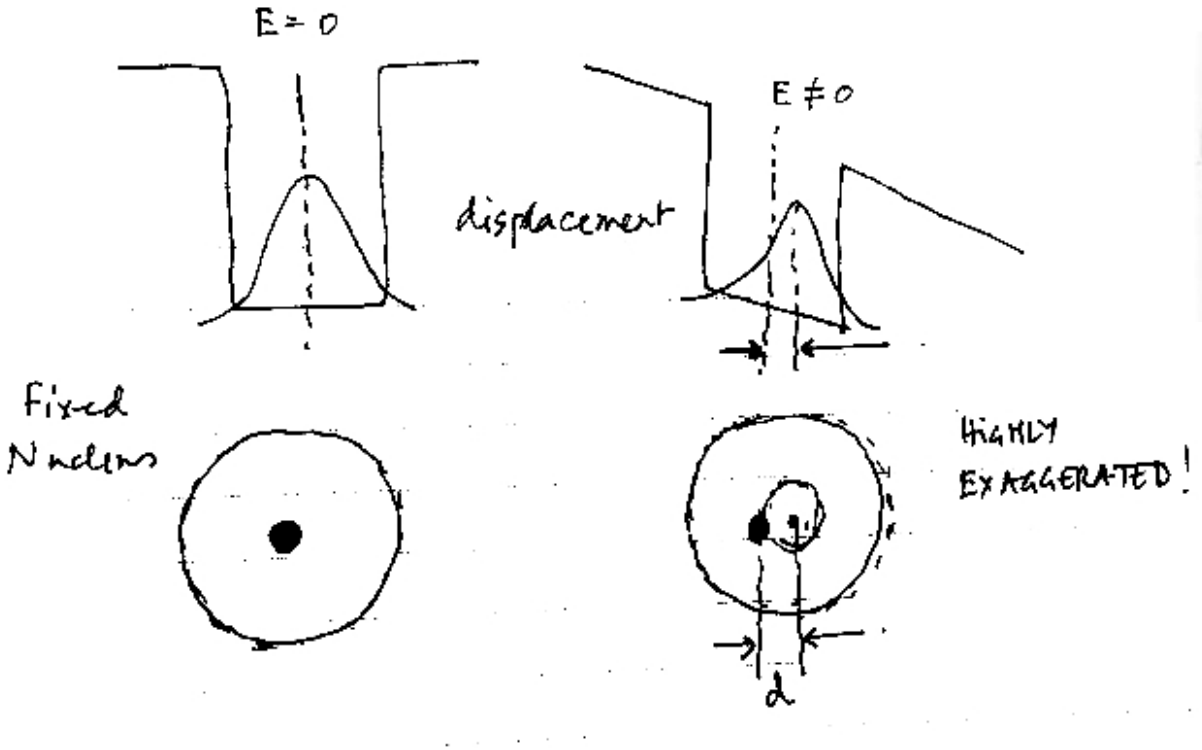
Dipole moment is a vector
Quadrupole moment is a tensor

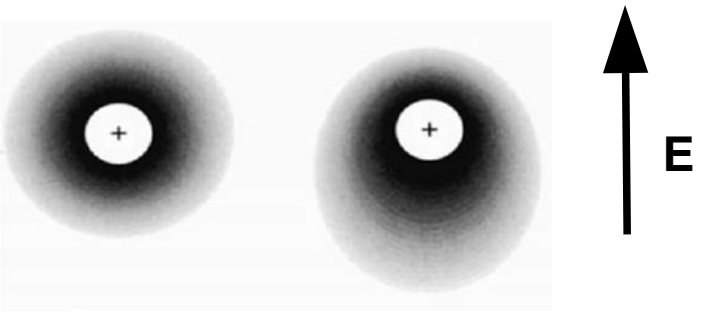
The lowest non-vanishing moment is independent of the choice of the origin.
The higher moments are NOT necessarily so.

So if the total charge (monopole) is zero then dipole term is origin-independent.
If the dipole also vanishes then quadrupole is origin independent. (Prove it!)

Dipole is more common in electronic charge distributions.
Nucleii often have quadrupole moments.
Earth's gravitational potential has a significant quadrupole component.

Response of atoms and molecules to an electric field





Q is the total charge
A is the atomic radius
p = Qd (induced moment)

Electron cloud is an uniformly charged sphere..(assume)
Force on the nucleus due to displaced electron cloud = External force on nucleus

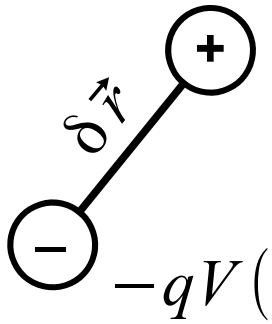
$$E = \frac{1}{4\pi\epsilon_0}\frac{Qd}{a^3} \qquad \textit{hence} \qquad \vec{p} = 4\pi\epsilon_0 a^3 \vec{E}$$

$$\sim 4\pi\epsilon_0 \times 10^{-30} \quad \text{in SI}$$

Atomic polarizability
Small for inert gases
Large for atoms with partially filled outer shell
Estimated values and observed value agree (order of magnitude)

Force and torque on a dipole

Potential Energy and force

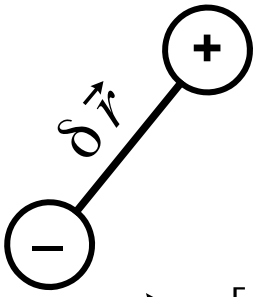


$$qV(\vec{r}+\delta\vec{r}) = q(V+\delta\vec{r}.\vec{\nabla}V)$$

Add the two contributions

$$\vec{p} = q\delta\vec{r}$$
$$U_{dip} = -\vec{p}.\vec{E}$$
$$\vec{F}_{dip} = (\vec{p}.\vec{\nabla})\vec{E}$$

Torque



$$(\vec{r}+\delta\vec{r})\times[q\vec{E}(\vec{r}+\delta\vec{r})]$$

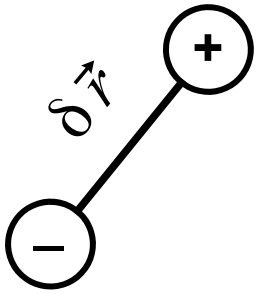
$$\vec{r}\times[-q\vec{E}(\vec{r})]$$

$$\vec{\tau}_{dip} = \vec{p}\times\vec{E}$$

Although the E field is different at two sites, the difference in the final expression would be second order.....

Now we can calculate the interaction force between two dipoles....easily!
If we have two dipoles...the E field of the first will act on the second and vice versa,

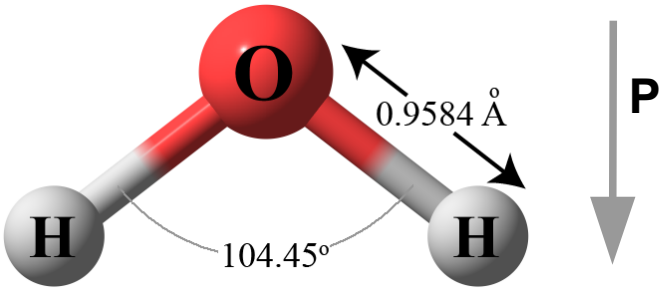
Correct expression for the force on a dipole



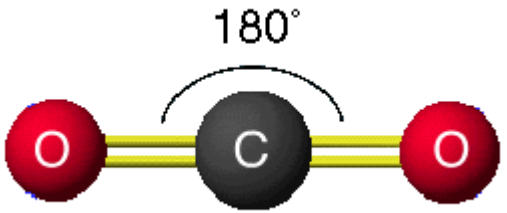
$$\vec{F} = -q\vec{E}(\vec{r}) + q\vec{E}(\vec{r}+\delta\vec{r})$$
$$F_x = -qE_x(\vec{r}) + qE_x(\vec{r}+\delta\vec{r})$$

$$\left.\begin{aligned} F_x &= q\,\delta\vec{r}.\nabla E_x \\ F_y &= q\,\delta\vec{r}.\nabla E_y \\ F_z &= q\,\delta\vec{r}.\nabla E_z \end{aligned}\right\} \quad \vec{F} = (\vec{p}.\nabla)\vec{E}$$

Atoms and molecules in an electric field: frozen moment of molecules



$6.2\times10^{-30}\,Coul.mt$
$(1.85\,D)$



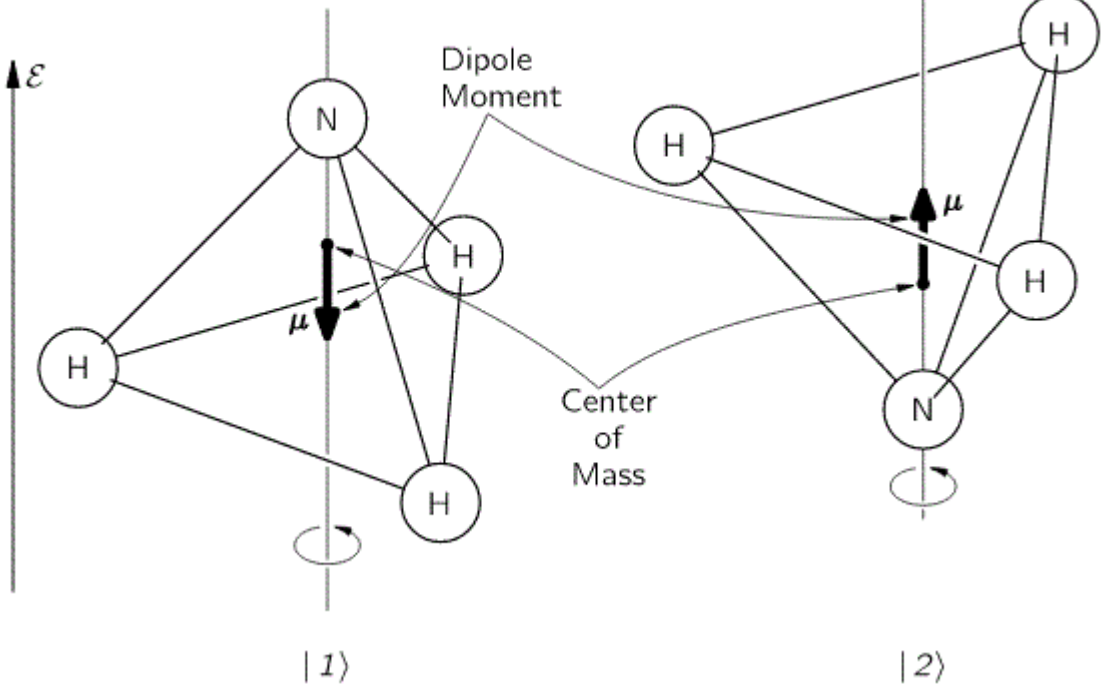
No dipole moment

SI unit = Coul-mt.

1 Debye unit (historical but useful)
Dipole moment of 10^{-10} esu of charge separated by 1 angstrom
Useful for molecular scale since
Electron charge is 4.8 x10^{-10} esu

Electron distribution in the bonds can give rise to built in dipole moment

$4.8\times10^{-30}\,Coul.mt$
$(1.42\,D)$



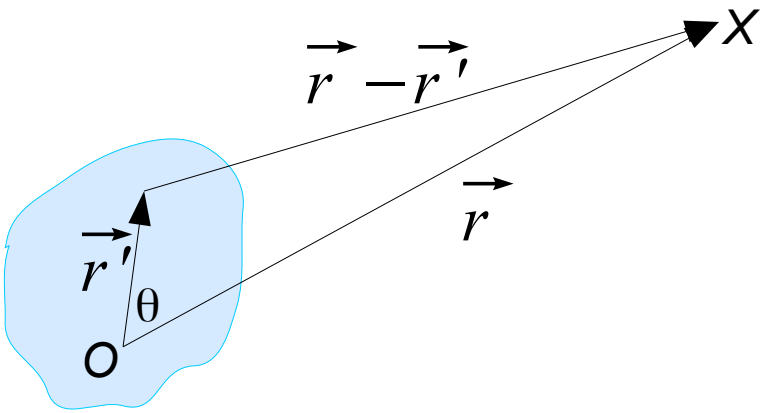
Induced dipole moment and electric field are not necessarily in the same direction for a molecule. Since the bonds do not shift uniformly in all directions...."easy" and "hard" directions...
P and E are related by a matrix/tensor

Potential of an extended distribution of dipoles

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\,\vec{P}.\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

$$\nabla_{r'}\frac{1}{|\vec{r}-\vec{r}\,'|} = \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$

Prove this by writing out in (x-x').....



Hence

$$V(X) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left[\nabla\frac{\vec{P}}{|\vec{r}-\vec{r}\,'|} - \frac{1}{|\vec{r}-\vec{r}\,'|}\nabla.\vec{P}\right]$$

dipole distribution to be integrated over

$$= \frac{1}{4\pi\epsilon_0}\left[\int\frac{d\vec{S}\,'.\vec{P}}{|\vec{r}-\vec{r}\,'|} - \int d^3\vec{r}\,'\frac{\nabla.\vec{P}}{|\vec{r}-\vec{r}\,'|}\right]$$

Here integration and differentiation are w.r.t. primed co-ordinates

Surface charge

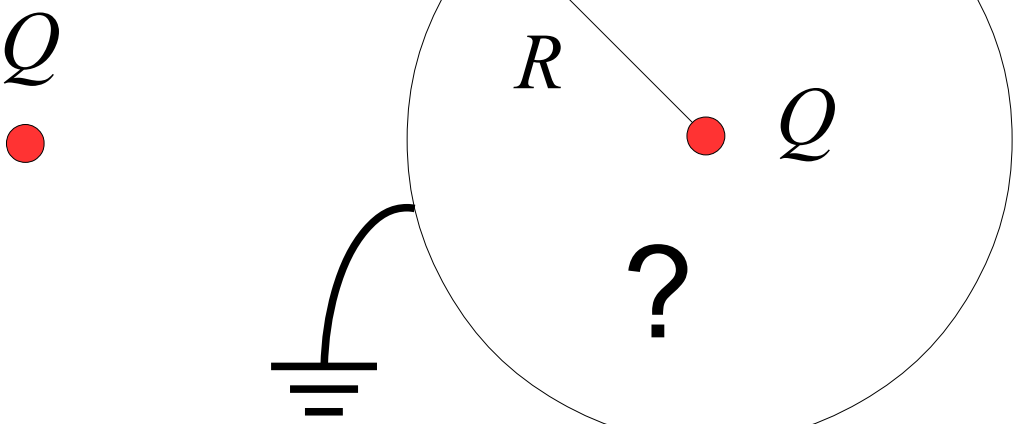
$$\sigma = \vec{P}.\hat{n}$$

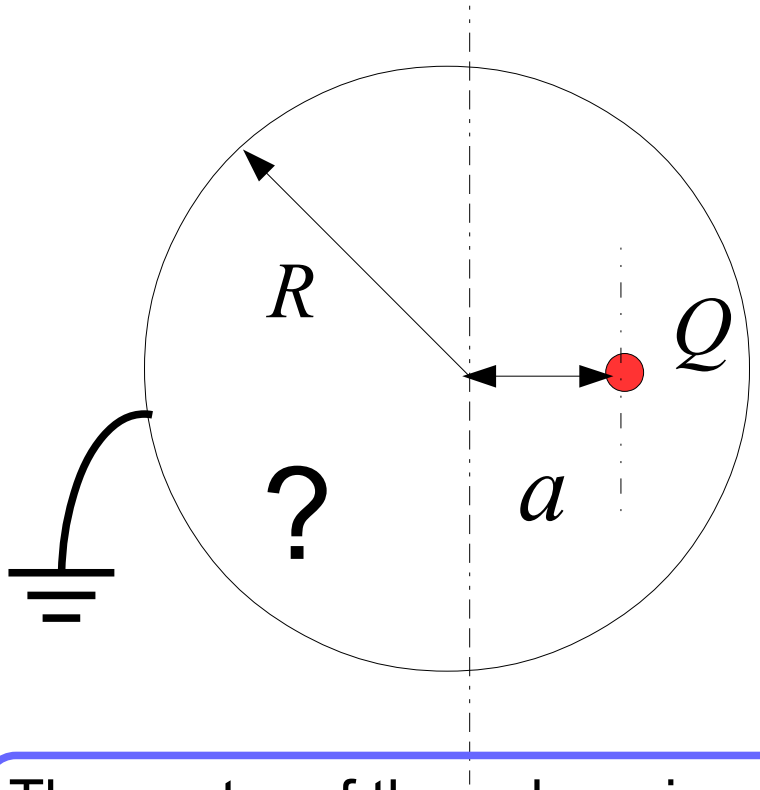
Volume charge

$$\rho = -\nabla.\vec{P}$$

The Image charge "trick"

Charge distribution and boundary condition





$$V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$$

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{R}\right)$$

The center of the sphere is taken as the origin.
The xy plane only considered

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-a)^2+y^2}} - \frac{R}{a}\frac{1}{\sqrt{\left(x-\frac{R}{a}\right)^2+y^2}}\right)$$

Boundary condition changes the form of the solution in non-trivial ways.

Solving the Laplace equation : Image charge method

Problem: A charge distribution and some boundary conditions are given. The usual boundary conditions are fixed potentials over some surfaces.
Solve for V(r) in a certain region.

A "trick“ works for some (!! not all !!) problems.

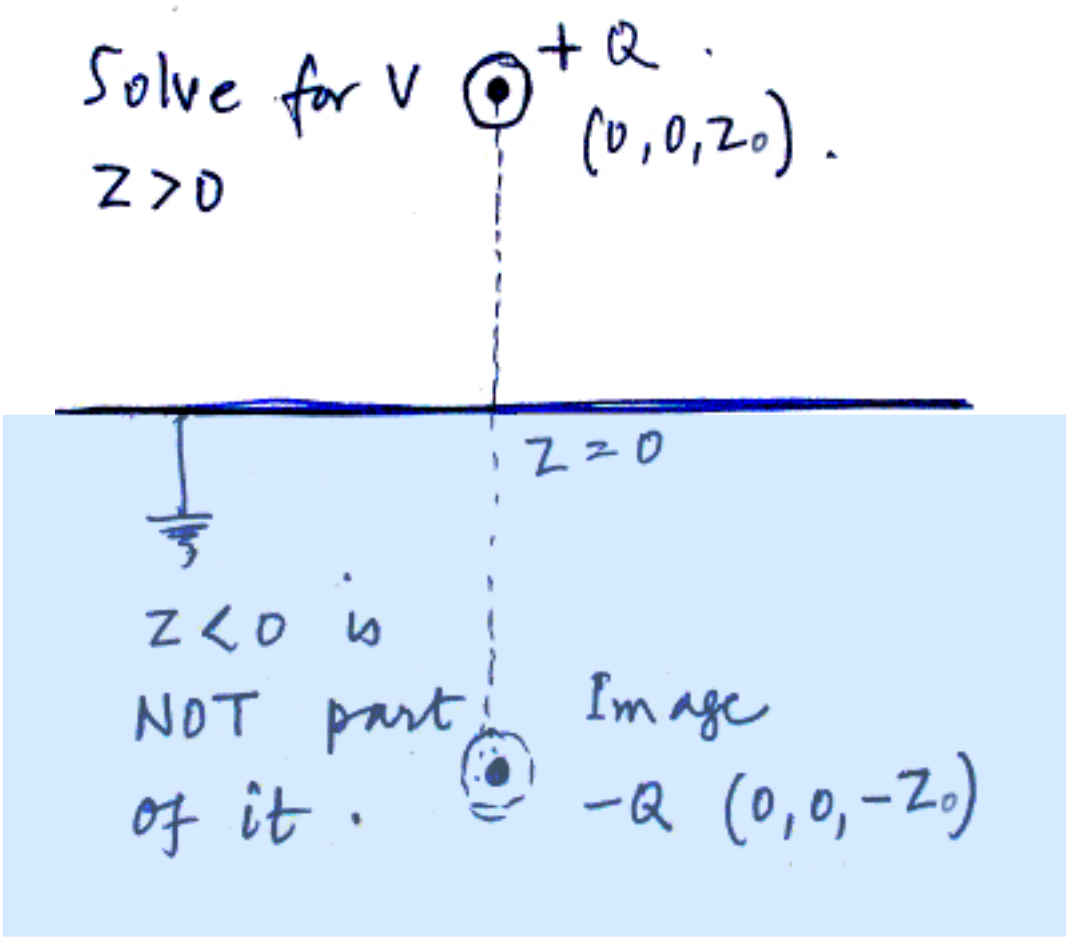
STEP 1: put some point charges in the regions NOT part of the region where you need to solve for the potential.

STEP 2: Try to arrange these external charges, so that the external + given charges together produce the desired potential at the boundaries. Forget all else!

STEP 3: Calculate the total potential in the certain (given) region using all the charges in the problem + external charges.

STEP 4: The total field/potential produced by the ALL the charges is the solution to the problem. The extra charges are called Image charges.

Image charge method : point charge near a conducting grounded plane



PROBLEM:
A point charge +Q is kept at $(0,0,z_0)$

z=0 is a grounded conducting sheet.

What is V(x,y,z) for z >0
Subject to boundary conditions:
V=0 at z=0
$V \to 0$ as $x,y,z \to \infty$

SOLUTION:

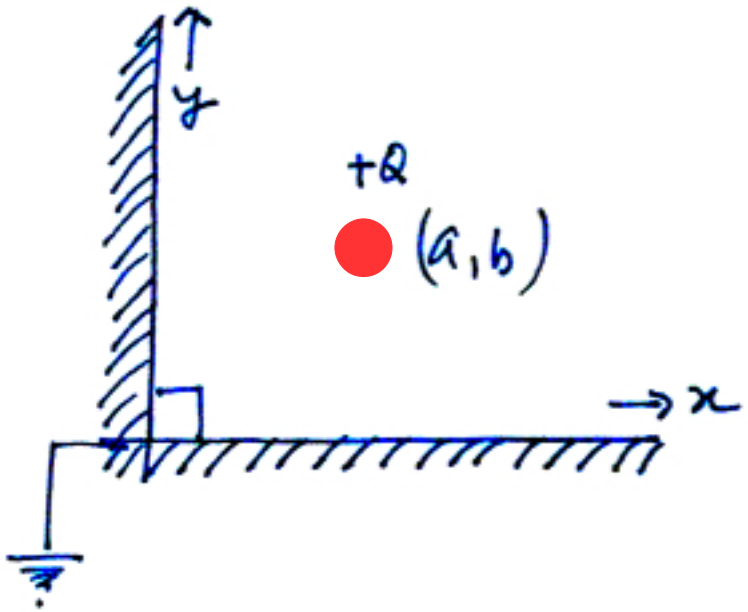
Put an extra charge -Q at $(0,0,-z_0)$

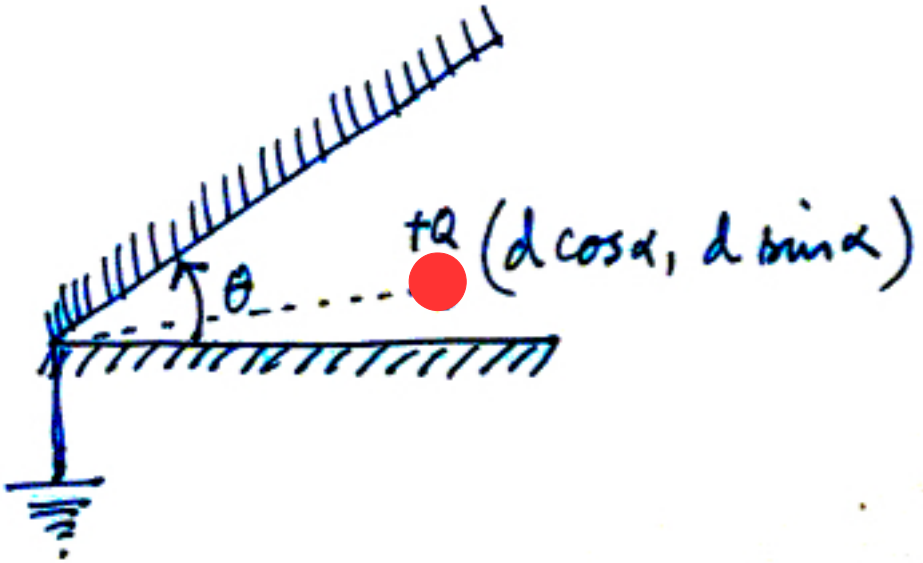
The potential due to both is

$$V(x,y,z) = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}}\right]$$

For z=0, the two terms cancel giving V=0. This must be the solution (uniqueness).
We can now calculate the force between the charge +Q, induced surface charge at every point etc.

Image charge method : point charge near a conducting grounded plane





To solve this we need three image charges:

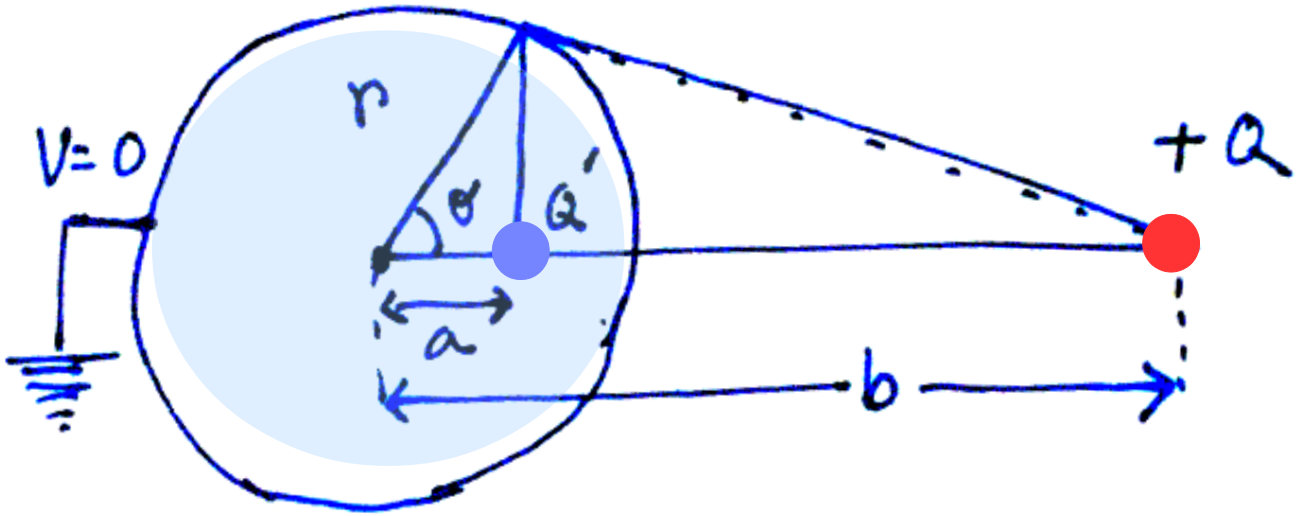
- Q at (a, -b)
+Q at (-a,-b)
- Q at (-a,b)

This problem can be solved with a finite number of images if

$$\theta \times integer = \pi$$

There is no generic method! It is a combination of guess and some calculation.....

Image charge method : point charge near a conducting grounded sphere



Solution wanted outside the sphere only

The distance of an arbitrary point on the surface from the charges Q and Q'

$$d'^2 = r^2 + a^2 - 2ar\cos\theta$$
$$d^2 = r^2 + b^2 - 2br\cos\theta$$

can we adjust a and Q' such that $\frac{Q}{d} - \frac{Q'}{d'} = 0$ for all θ?

$$\frac{r^2 + a^2 - 2ar\cos\theta}{Q'^2} = \frac{r^2 + b^2 - 2br\cos\theta}{Q^2}$$

Equate the terms independent of cosθ and coefficient of cosθ on both sides

$$a = \frac{r^2}{b}$$
$$\frac{Q'}{Q} = \frac{r}{b}$$

How would you solve it if the sphere is NOT grounded?

The Green's function and boundary value

What does a Green's function do ?

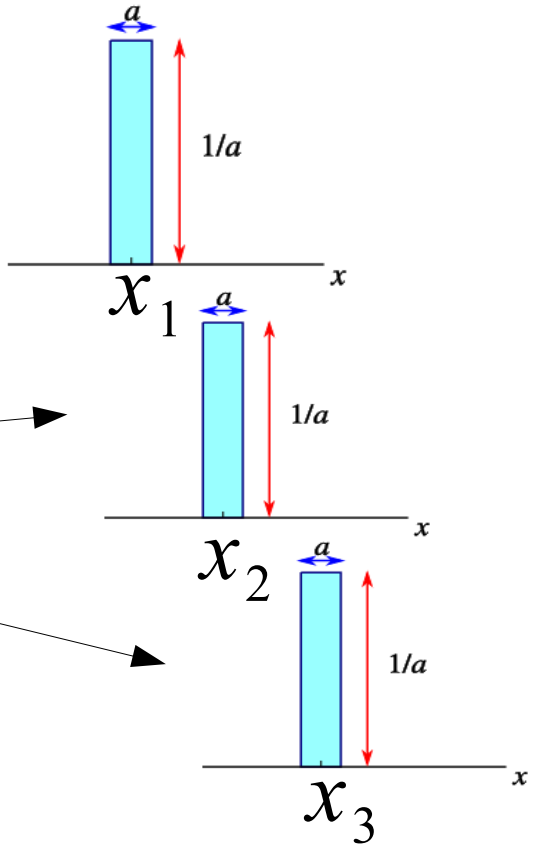
$$[\text{Operator}]\,G(x-x') = -\delta(x-x')$$

Some operator

Some func F1

Some func F2

Some func F3



We assume that superposition works...

Add F1, F2, F3.....to build up LHS

Build up RHS by assembling spikes of different heights.
Like breaking up an integral into rectangles

Can add to G any function that gives RHS = 0

Delta function is a "simple" thing in k-space.

How do we put these together ?

Begin with two arbitrary functions $\psi(r), \phi(r)$

$$\left.\begin{aligned} \nabla.(\phi\nabla\psi) &= \phi\nabla^2\psi + \nabla\psi.\nabla\phi \\ \nabla.(\psi\nabla\phi) &= \psi\nabla^2\phi + \nabla\phi.\nabla\psi \end{aligned}\right\}$$

$$\int_{vol}\left(\phi\nabla^2\psi-\psi\nabla^2\phi\right)d\tau = \oint_{surf}\left(\phi\nabla\psi-\psi\nabla\phi\right).d\vec{S}$$

Now make the choice

$$\begin{cases} \psi = G & \text{where} \quad \nabla^2 G(\vec{r}-\vec{r}') = -\delta(\vec{r}-\vec{r}') \\ \phi = \Phi & \text{where} \quad \nabla^2\Phi = -\dfrac{\rho}{\epsilon_0} \end{cases}$$

$$\int_{vol}\left[\Phi\left(-\delta(\vec{r}-\vec{r}')\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau = \oint_{surf}\left[\Phi\frac{\partial G}{\partial n}-G\frac{\partial\Phi}{\partial n}\right]dS$$

Formal solution in terms of $G(r-r')$

Make a choice $G=0$ on the surface S (Dirichlet)

$$\int_{vol}\left(\Phi\left[-\delta\left(\vec{r}-\vec{r}'\right)\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau=\oint_{surf}\left(\Phi\frac{\partial G}{\partial n}-G\frac{\partial \Phi}{\partial n}\right)dS$$

interchange the role of r and r'

$$\Phi(\vec{r})=\frac{1}{\epsilon_0}\int_{vol}G(\vec{r}-\vec{r}')\rho(r')d\tau' \;-\; \oint_{surf}\Phi(r')\frac{\partial G}{\partial n}dS'$$

The formal solution for potential when the charge distribution is given and the potential is specified on the surface S.

But we need to start solving for G in various geometries.

The form of G depends crucially on the boundary conditions!

Interpretation of the terms

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0} \int_{vol} G(\vec{r} - \vec{r}\,') \rho(r') d\tau' - \oint_{surf} \Phi(r') \frac{\partial G}{\partial n} d S' ¿$$

The first term gives the contribution of the volume charge.

But imposing a boundary condition (potential) on S requires a (surface) charge distribution to be "pasted" on S. The second term results from that.

If there is no "volume charge", then the potential is entirely determined by the "surface" term. It can be calculated if we know the function G.

Any other possibility ? (von Nuemann...)

Can we make $\frac{\partial G}{\partial n}=0$ on the surface S ? !!NO !!

$$\int_{vol}\left(\Phi\left[-\delta\left(\vec{r}-\vec{r}'\right)\right)-G\left(-\frac{\rho}{\epsilon_0}\right)\right]d\tau=\oint_{surf}\left(\Phi\frac{\partial G}{\partial n}-G\frac{\partial \Phi}{\partial n}\right)dS$$

$$\int_{vol}\nabla.\left(\nabla G\right)d\tau=\int_{vol}\nabla^2 G\,d\tau \;=\; \oint_{surf}\left(\nabla G\right).d\vec{S}=\oint_{surf}\frac{\partial G}{\partial n}dS$$

Here S is the area of the bounding surface.

$$\frac{\partial G}{\partial n} = -\frac{1}{S} \;:\; \text{simplest choice}$$

This choice is used in heat flow related problems. However "mixed boundary value" problems do occur in electostatics. An example is an aperture in a metallic sheet.

$G(\vec{r}-\vec{r}\,')$ for a plane

PROBLEM : The potential is given everywhere on a plane. It is not necessarily constant. How to solve for the potential everywhere?

$$\nabla^2 G = -\delta(x-x')\delta(y-y')\delta(z-z') \;:\; G=0 \text{ if } z=0$$

Dimension of G (for Laplacian) is $[L]$ in 1D, dimensionless in 2D, $[L]^{-1}$ in 3D. Why?

The simplest image charge problem in disguise !
Point charge above a "grounded" conducting plane.

$$G(\vec{r}-\vec{r}\,') = \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right)$$

$G(\vec{r}-\vec{r}\,')$ for a plane

$$\frac{\partial G}{\partial n} = -\frac{\partial G}{\partial z'}\bigg|_{z'=0} = -\frac{1}{2\pi}\frac{z}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

Why is the direction of $\hat{n}$ along $-z'$?

$$\Phi(\vec{r}) = \frac{1}{\epsilon_0}\int_{vol} G(\vec{r}-\vec{r}\,')\rho(r')d\tau' - \oint_{surf}\Phi(r')\frac{\partial G}{\partial n}dS'$$

$$\Phi(\vec{r}) = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dx'dy'\frac{\Phi(x',y')}{\left[(x-x')^2+(y-y')^2+z^2\right]^{3/2}}$$

The potential must be specified everywhere...no holes or slits!
The divergence theorem that we used as our starting point, holds only if the surface is closed.....

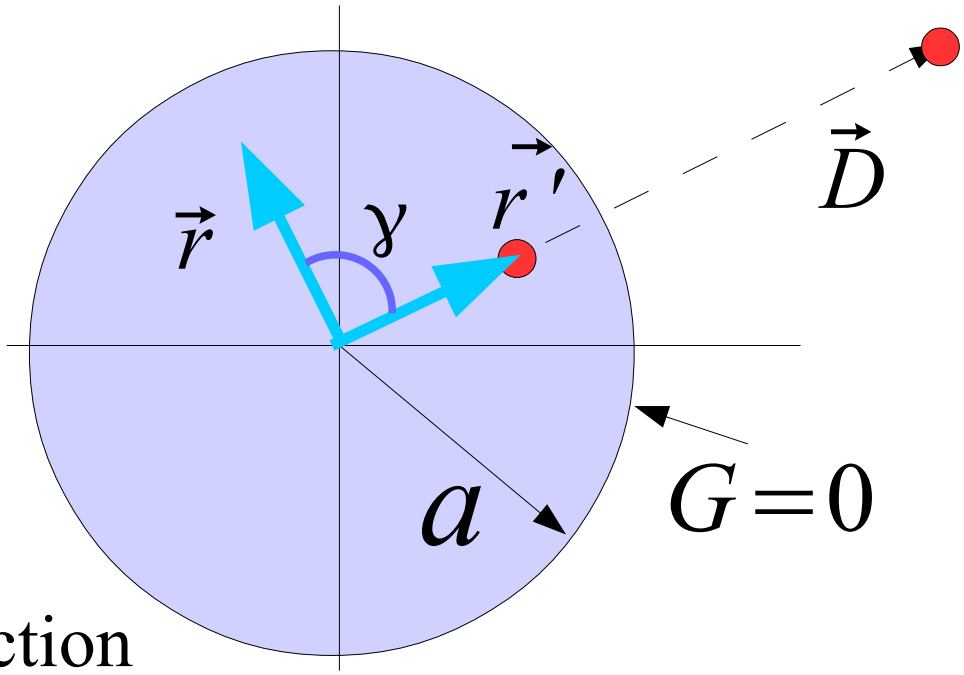
$G(\vec{r}-\vec{r}\,')$ for a sphere

An image charge problem, really...

$$Dr' = a^2 \qquad Q' = -\frac{a}{r'}Q$$

$$\nabla^2 G = \delta(\vec{r}-\vec{r}\,')$$

$$G = \frac{1}{4\pi}\left(\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{a/r'}{|\vec{r}-\vec{D}|} \right)$$



since $\vec{r}\,'$ and $\vec{D}$ are in same direction

$$G = \frac{1}{4\pi}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\gamma}} - \frac{1}{\sqrt{(rr'/a^2)+a^2-2rr'\cos\gamma}} \right)$$

Normal derivative $\dfrac{\partial G}{\partial n} = \begin{cases} \left.\dfrac{\partial G}{\partial r'}\right|_{r'=a} & \text{for} \quad r<a \\ -\left.\dfrac{\partial G}{\partial r'}\right|_{r'=a} & \text{for} \quad r>a \end{cases}$

r and r' are interchange-able. Why?

Eignefunction expansion of a δ function

You would have noticed that the functions appearing in the Green's functions are the same functions frequently seen in eignefunction problems. What is the connection ?

Basic fact: We know that any function can be expanded using the "complete" and "orthonormal" set of eignefuctions → So we should be able to expand a delta-fn also in a similar way.

Where does this lead to?

Consider an operator eigenfunction : $L u_n(x) = \lambda_n u_n(x)$

$$\sum_n A_n u_n(x) = f(x) = \delta(x - x')$$

Solve for A_n

Eignefunction expansion of a δ function

$$\int_a^b \sum_n A_n u_n(x) u_m(x)\, dx = \int_a^b \delta(x-x') u_m(x)\, dx$$

$$\sum_n A_n \int_a^b u_n(x) u_m(x)\, dx = u_m(x')$$

Correct normalisation assumed

$$\sum_n A_n \delta_{mn} = u_m(x')$$

$$\delta(x-x') = \sum_n u_n(x) u_n(x')$$

So the RHS of a Green's function can be expanded in eigenfuctions for each delta function. The LHS can also be written in terms of eigenfunctions. The solution is guaranteed but not the most handy expression in many cases.

What should a δ function look like in (ρ, θ, z)?

$$\int_{vol} \delta(\vec{r}-\vec{r}')\, d\tau = 1 \qquad \text{must hold}$$

$$\int_{vol} \frac{\delta(\rho-\rho')}{\rho}\delta(\theta-\theta')\delta(z-z')\, d\rho\,\rho\, d\theta\, dz$$

In general, for u_1, u_2, u_3

$$d\tau = h_1 h_2 h_3 d u_1 d u_2 d u_3$$

$$\delta(\vec{r}-\vec{r}') = \frac{\delta(u_1-u_1')}{h_1}\frac{\delta(u_2-u_2')}{h_2}\frac{\delta(u_3-u_3')}{h_3}$$

It is possible to integrate out angular co-ordiantes if there is no angle depndence of the functions that are being dealt with. For example

$$\delta(\vec{r}-\vec{r}') = \frac{1}{4\pi r^2}\delta(r-r') \quad \text{in } (r, \theta, \phi) \text{ with only } r \text{ dependence}$$