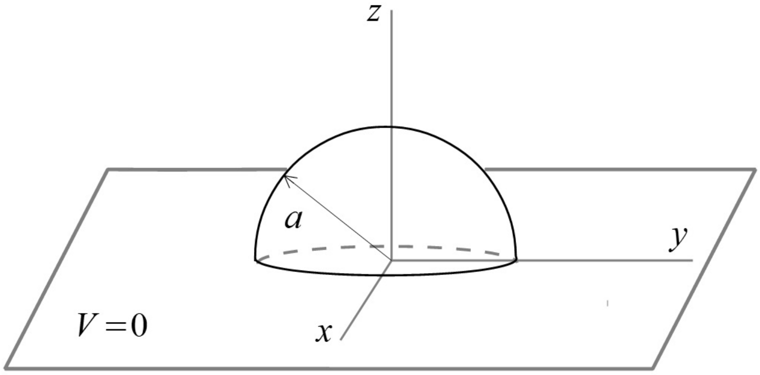
PH312 (July - Dec 2024): Electricity & Magnetism : Tutorial 1

22 Aug 2024

1. Consider a conducting surface formed by joining the xy plane and a conducting hemispherical shell of radius a, centered at the origin as shown. A point charge q is placed at the location $(0,0,L)$ where $L > a$ and the conducting surface is held at a potential $V = 0$.



   (a) What is the force experienced by the point charge? Your answer should be in closed form.

   (b) Utilise this and generalise your solution to get the Green's function for this geometry. This means solving for

$$\nabla^2 G = -\delta^3(\vec{r} - \vec{r'}) \qquad \begin{cases} G = 0 & r \to \infty \\ G = 0 & \text{plane} + \text{hemisphere} \end{cases}$$

   Note that $\vec{r'} \equiv (x', y', z')$ is not necessarily on the z axis. Your answer should be in closed form. Use (x, y, z, x', y', z') co-ordinates to write your answer.

2. Consider the Laplacian operator ∇^2 in two dimensions.

   (a) Find the Green's function for the entire plane using r, θ co-ordinates. The operator acts on the r variables not the r' variables.

$$\nabla^2 G(\vec{r}, \vec{r'}) = -\delta^2(\vec{r} - \vec{r'})$$

   (b) Convert this to cartesian co-ordinates where the delta function is at (x', y')

   (c) Modify your solution if the solution is to be restricted to the upper half plane only.

   (d) Now suppose you are given that the potential on the x-axis is

$$V(x, 0) = \begin{cases} -V_0 & (-\infty < x < 0) \\ V_0 & (0 < x < \infty) \end{cases}$$

   Solve for the potential $V(x, y)$ at all points.

3. A conducting sphere of radius R is grounded (held at $V = 0$) and kept in an initially uniform electric field $\vec{E} = E_0\hat{z}$. Find the induced charge as a function of θ. What is the total charge induced ?

4. A spherical shell of radius R is kept at a potential $V(\theta) = V_0 \sin^2\frac{\theta}{2}$. Where θ denotes the angle made with the z axis in standard spherical polar co-ordinates.

   (a) Show that the potential inside $(r < R)$ is : $V(r, \theta) = \frac{V_0}{2}\left(1 - \frac{r}{R}\cos\theta\right)$

(b) Show that the potential outside ($r > R$) is : $V(r,\theta) = \dfrac{V_0 R}{2r}\left(1 - \dfrac{R}{r}\cos\theta\right)$

5. If the total amount of charge (monopole) contained in a distribution is zero, show that the dipole moment is independent of the choice of the origin. Then show if the monopole and dipole moments of a charge distribution are both zero, the quadrupole moment is independent of the choice of the origin.

6. A circular disc of radius R lies in $z = 0$ plane, centred at the origin. The following charge density is frozen in it.

$$\sigma(r',\phi) = \sigma_0 r' \cos\phi$$

(a) What is the monopole moment of this distribution?

(b) Calculate the dipole contribution at a point $(0,0,z)$, using the expression for the dipole contribution in polar form.

(c) Then calculate all the Cartesian components of the dipole moment vector $\vec{p}$ for this charge distribution. Using this result calculate the dipole contribution at $(0,0,z)$, using the cartesian expression for dipole contribution. Do you get the same result in both cases?

(d) Now calculate the quadrupole contribution to the potential at $(0,0,z)$ using the polar expression.

$$V_{quad} = \frac{1}{4\pi\varepsilon_0 r}\int_{disc} d^2\vec{r'}\left(\frac{r'}{r}\right)^2 P_2(\cos\theta)\sigma(\vec{r'}) \tag{1}$$

(e) Then consider the definition of the quadrupole moment matrix (or tensor) in cartesian co-ordinates. The elements are defined as

$$Q_{ij} = \int_{disc} d^2\vec{r'}\left(3r'_i r'_j - r'^2\delta_{ij}\right)\sigma(\vec{r'}) \tag{2}$$

Where $r'_1 = x, r'_2 = y, r'_3 = z$. So some of the elements will be like:

$$Q_{xx} = \int_{disc} dxdy\left(2x^2 - y^2 - z^2\right)\sigma(x,y)$$

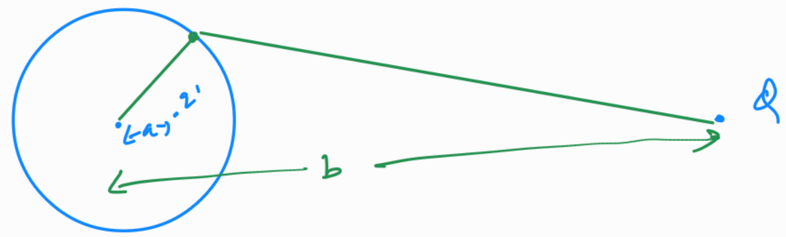
$$Q_{xy} = \int_{disc} dxdy\ 3xy\ \sigma(x,y)$$

How many independent elements will be there in this 3×3 matrix? Calculate them.

(f) Using the result calculate the quadrupole contribution at $(0,0,z)$ again using cartesian expression.

7. Consider a charge distribution given by $-2q$ at $(0,0,0)$, $+q$ at $(0,0,a)$ and $+q$ at $(0,0,-a)$. Calculate all elements of the quadrupole moment tensor for this distribution.

Q1



conducting sphere

$$\frac{q^2}{b^2+R^2-2bR\cos\theta} = \frac{-q'^2}{a^2+R^2-2aR\cos\theta}$$

$$\frac{b^2+R^2}{q^2} = \frac{a^2+R^2}{q'^2} \qquad \frac{2bR}{q} = \frac{2aR}{q'}$$

$$q'^2 = \frac{aq^2}{b}$$

$$\frac{a}{b}(b^2+R^2) = a^2+R^2$$

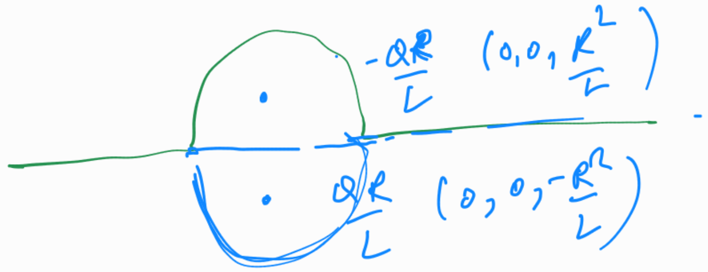
$$ab^2+aR^2 = a^2b+bR^2$$

$$ab^2-a^2b = R^2(b-a)$$

$$\boxed{ab = R^2}$$

$$b = \frac{R^2}{a} \qquad q'^2 = \frac{a^2q^2}{R^2} \Rightarrow q = -\frac{aq}{R} = -\frac{Rq}{b}$$

$\cdot\, Q(0,0,L)$



$\cdot\, -Q\ (0,0,-L)$

Force on Q :-

$$\frac{1}{4\pi\epsilon_0}\left[\frac{-Q^2}{4L^2} + \frac{Q^2 R}{L\left(\frac{R^2}{L}+L\right)^2} - \frac{Q^2 R}{L\left(L-\frac{R^2}{L}\right)^2}\right]$$

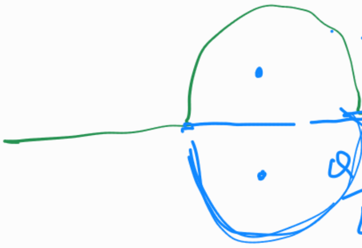
$$= \frac{1}{4\pi\epsilon_0}\left[\frac{-Q}{4L^2} + \frac{Q^2 R L}{(R^2+L^2)^2} - \frac{Q^2 R L}{(L^2-R^2)^2}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{-Q}{4L^2} + Q^2 R L\left[\frac{-4L^2R^2}{(L^4-R^4)^2}\right]\right.$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{-Q}{4L^2} - \frac{4Q^2L^3R^3}{(L^4-R^4)^2}\right]$$

$\cdot\, Q(0,0,L)$

(b)



$-\frac{QR}{L}\ \left(0,0,\frac{R^2}{L}\right)$

$\frac{QR}{L}\ \left(0,0,-\frac{R^2}{L}\right)$

$\cdot\, -Q\ (0,0,-L)$

$$\varphi = \frac{Q}{\sqrt{x^2+y^2+(L-z)^2}} - \frac{QR}{L\left(x^2+y^2+\left(z-\frac{R^2}{L}\right)^2\right)^{1/2}} + \frac{QR}{L\left(x^2+y^2+\left(z+\frac{R^2}{L}\right)^2\right)^{1/2}}$$

$$- \frac{Q}{\sqrt{x^2+y^2+(z+L)^2}}$$

Q2

$$\nabla^2 G(\vec{r}, \vec{r}') = -\delta^2(\vec{r} - \vec{r}')$$

$$\frac{\partial^2 G}{\partial r^2} + \frac{1}{r}\frac{\partial G}{\partial r} + \frac{1}{r^2}\frac{\partial G}{\partial \theta^2} = -\delta(r - r')$$

$$f_2 \frac{\partial^2 f_1}{\partial r^2} + \frac{f_2}{r}\frac{\partial f_1}{\partial r} + \frac{f_1}{r^2}\frac{\partial^2 f_2}{\partial \theta^2} = -\frac{\delta(r-r')}{r}\delta(\theta - \theta')$$

We expand $f_2 = A_m \sin(m\theta)$

Expansion of delta function,

$$\int_0^{2\pi} N^2 \sin^2(m\theta)\, d\theta$$

$$= \frac{N^2}{2}\int_0^{2\pi} 1 - \cos(2m\theta)\, d\theta$$

$$= \frac{N^2}{2}(2\pi) = N^2\pi \quad \therefore \quad N = \frac{1}{\sqrt{\pi}}$$

we have :-

$$A_m \sin(m\theta)\frac{\partial^2 f_1}{\partial r^2} + \frac{A_m \sin(m\theta)}{r}\frac{\partial f_1}{\partial r} - \frac{f_1 A_m m^2 \sin(m\theta)}{r^2}$$

$$= \frac{-\delta(r - r')}{\pi r}\sum_m \sin(m\theta) \cdot \sin(m\theta')$$

$$\sum_m A_m \sin(m\theta)\left[\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - \frac{f_1 m^2}{r^2} + \frac{\delta(r-r')}{\pi r}\sin(m\theta')\right] = 0$$

Solving.

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - \frac{f_1 m^2}{r^2} + \frac{\delta(r-r')}{\pi r}\sin(m\theta') = 0$$

Solving :- $r < r'$

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - \frac{m^2 f_1}{r^2} = 0$$

$$f_1 = B_k r^k$$

$$B_k k(k-1) r^{k-2} + B_k k r^{k-2} - m^2 B_k r^{k-2} = 0$$

$$k(k-1) + k - m^2 = 0$$

$$k^2 - m^2 = 0$$

$$k = \pm m$$

$$f_1 = B_m r^m$$

For $r > r'$

$$f_1 = C_m \frac{1}{r^m}$$

f_1 must be continuous at $r = r'$

$$\therefore \quad B r'^m = C \frac{1}{r'^m}$$

$$\boxed{B r'^{2m} = C}$$

$$-mC \frac{1}{r^{m+1}} - mBr^{m-1} = -\frac{\sin(m\theta')}{\pi r}$$

$$-mC \frac{1}{r'^{m+1}} - mBr'^{m-1} = -\frac{\sin(m\theta')}{\pi r'}$$

$$-mC \frac{1}{r'^m} - mBr'^m = -\frac{\sin(m\theta')}{\pi}$$

$$+2mBr'^m = \frac{\sin(m\theta')}{\pi}$$

$f_1 = -\ln r$

$\frac{\partial f_1}{\partial r} = -\frac{1}{r}$

$\frac{\partial^2 f}{\partial r^2} = \frac{1}{r^2}$

$\left(-\frac{1}{r}\right)\left(-\frac{1}{r}\right)$

$B = + \frac{\sin(m\theta')}{2m\pi} \frac{1}{r'^m}$

$C = + \frac{\sin(m\theta')}{2m\pi} r'^m$

$$G = \sum_m \frac{\sin(m\theta')}{2m\pi} \left(\frac{r}{r'}\right)^m \sin(m\theta) \, , \quad r < r'$$

$$\sum_m \frac{\sin(m\theta')}{2m\pi} \left(\frac{r'}{r}\right)^m \sin(m\theta) \, , \quad r > r'$$

$$\sum_m \frac{e^{-im\theta'}}{2m\pi} \left(\frac{r}{r'}\right)^m e^{im\theta}$$

$$C\left(\cos 2\theta - \left(\frac{J}{K}\right)^{*} \sin 2\theta\right) = \alpha J$$

$$D\left(\sin 2\theta + \left(\frac{J}{K}\right)^{*} \cos 2\theta\right) = -\alpha K$$

$$\cos 2\theta - \left(\frac{J}{K}\right)^{*} \sin 2\theta = \alpha \frac{J}{C}$$

$$\sin 2\theta + \left(\frac{J}{K}\right)^{*} \cos 2\theta = -\alpha \frac{K}{C}$$

$$0 - \left(\frac{J}{K}\right)^{*} = \alpha \frac{J}{C}$$

$$1 = -\alpha \frac{K}{C}$$

$$\boxed{\alpha = -\frac{C}{K}} \quad , \quad \left(\frac{J}{K}\right)^{*} = \frac{C}{K} \frac{J}{C}$$

$$\left(\frac{J}{K}\right)^{*} = \frac{J}{K}$$

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} = 0 \qquad (r < r') \quad (r > r')$$

$$\Rightarrow \quad \frac{\partial^2 f_1}{\partial r^2} = -\frac{1}{r}\frac{\partial f_1}{\partial r}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial f_1}{\partial r}\right) = -\frac{1}{r}\frac{\partial f_1}{\partial r}$$

$$\partial y = -\frac{y}{r}\,\partial r$$

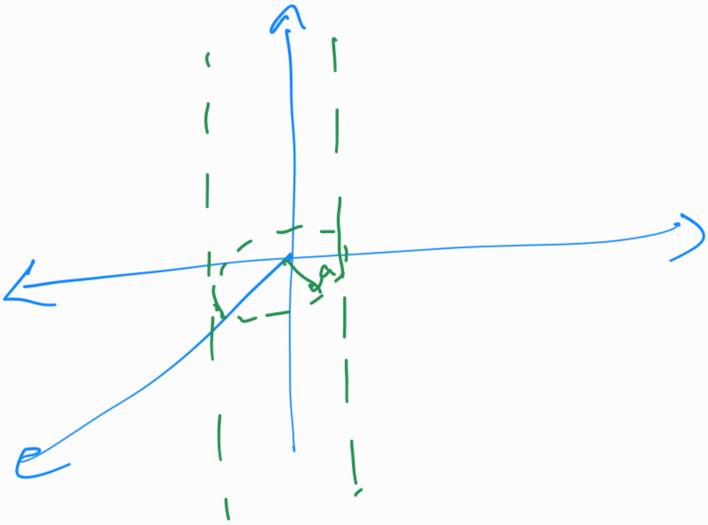
$$\ln y = -\ln r + c$$

$$\boxed{y = A\ln r + c}$$

Jackson:-

(a) $$G = \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$$

(b)



$G(\rho, \theta, z) =$

$$\Rightarrow \frac{\rho}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')+(z-z')^2}} + \frac{\rho}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')+(z+z')^2}}$$

$r > r'$:

$$\phi = \int_V \rho\, G\, d^3r - \frac{1}{4\pi} \oint \phi(x') \frac{\partial G}{\partial z}\, da$$

$$= \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho'^2\, d\rho'\, d\phi}{(\rho^2 + \rho'^2 - 2\rho\rho' \cos(\theta - \theta') + z^2)^{3/2}}$$

$$= \frac{Vz}{2\pi} \int_0^a \int_0^{2\pi} \frac{\rho'\, d\rho'\, d\phi}{(\rho'^2 + z^2)^{3/2}}$$

$\rho = z \tan\theta$

$d\rho = z \sec^2\theta\, d\theta$

$$= Vz \left[\int_0^{\tan^{-1}(a/z)} \frac{z^2 \tan\theta \sec^2\theta\, d\theta}{z^3 \sec^3\theta} \right]$$

$$= V \left[\int_0^{\tan^{-1}(a/z)} \sin\theta\, d\theta \right]$$

$$= V \left[1 - \frac{a/z}{\sqrt{\left(\frac{a}{z}\right)^2 + 1}} \right]$$

(d)

$$\frac{Vz}{2\pi}\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi}{(\rho^2+\rho'^2-2\rho\rho'\cos(\theta-\theta')+z^2)^{3/2}}$$

$$\frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^a\int_0^{2\pi}\frac{\rho' d\rho' d\phi}{\left(1+\frac{\rho'^2-2\rho\rho'\cos(\theta-\theta')}{\rho^2+z^2}\right)^{3/2}}$$

$$\frac{Vz}{2\pi}\frac{1}{(\rho^2+z^2)^{3/2}}\int_0^a\int_0^{2\pi}(\rho')^2\left[1-\frac{3}{2}\left(\frac{\rho'^2-2\rho\rho'\cos(\theta-\theta')}{\rho^2+z^2}\right)\right]$$

$$-\frac{3}{2}\int_0^a\int_0^{2\pi}\frac{\rho_1^2-2\rho\rho'\cos(\theta-\theta')}{\rho^2+z^2)}$$

$$\int_0^a\int_0^{2\pi}\rho'[\rho_1^{3}-2\rho\rho'\cos(\theta-\theta')]\,d\rho$$

$$\frac{1}{\quad}$$

(2.12) $f_2 = A_m e^{im\theta}$

$$A_m e^{im\theta} \frac{\partial^2 f_1}{\partial r^2} + \frac{A_m e^{im\theta}}{r} \frac{\partial f_1}{\partial r} - \frac{A_m m^2 f_1}{r^2} e^{im\theta} =$$

$$-\frac{\delta(r-r')}{2\pi r} \sum_m e^{im\theta} e^{-im\theta'}$$

$$\Rightarrow \sum_m e^{im\theta} \left[A_m \frac{\partial^2 f_1}{\partial r^2} + \frac{A_m}{r} \frac{\partial f_1}{\partial r} - \frac{A_m m^2 f_1}{r^2} + \frac{\delta(r-r')}{2\pi r} e^{-im\theta'} \right]$$

$$= 0.$$

Solve

$$A_m \frac{\partial^2 f_1}{\partial r^2} + \frac{A_m}{r} \frac{\partial f_1}{\partial r} - \frac{A_m m^2 f_1}{r^2} = -\frac{\delta(r-r')e^{-im\theta'}}{2\pi r}$$

For $r < r'$

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r} \frac{\partial f_1}{\partial r} - \frac{m^2 f_1}{r^2} = 0$$

$$f_1 = \alpha_m r^m$$

$$= \beta_m \cdot \frac{1}{r^m}$$

Special case