Small sample tests: t-distribution

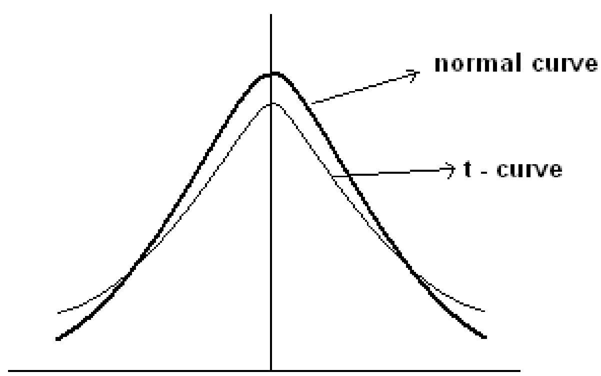
If the sample sizes are small (<30), the above tests (for large samples) do not hold good. Therefore, for estimation of the parameters as well as for testing a hypothesis, we cannot use the above methods.

Student's t-distribution

If we take a large number of samples of small (<30) sizes, calculate the mean of each sample, obtain the frequencies and obtain the frequency curve, we will find that the resulting sampling distribution of the mean is the **student's t-distribution**.

Properties of t distribution

1. The curve extends from – ∞ to + ∞
2. Like the normal distribution the t distribution is also symmetrical and has mean zero.
3. The variance of t distribution is greater than unity and approaches unity when the degrees of freedom (d.f.) i.e the size of the sample becomes large. The degress of freedom (d.f.) is given by $n-1$



t distribution curve

Uses of t distribution

Suppose it is not possible to take a sample of large size (due to lack of time or prohibitive costs), we can use the t distribution to do the 't-test' for testing hypothesis about the population mean/s and for estimating the population mean using the sample mean.

1. Test of Significance (TOS) of the difference between sample mean and population mean:

Sample size n $(n < 30)$

Population mean: μ

Sample mean : $\bar{x}$

Population S. D. : σ (always unknown)

Sample S. D. : s

Null Hypothesis : $H_0 : \mu = \mu_0$

Alternate Hypothesis : $H_1 : \mu \neq \mu_0$ (Two tailed)

$H_1 : \mu < \mu_0$ (One tailed (left))

$H_1 : \mu > \mu_0$ (One tailed (right))

Test Statistic

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n-1}} \sim t_{(n-1)} \; d.f$$

This t statistic is said to follow a t distribution with ν = $n - 1$ degrees of freedom. The t distribution has been derived under the hypothesis that the parent population is distributed normally.

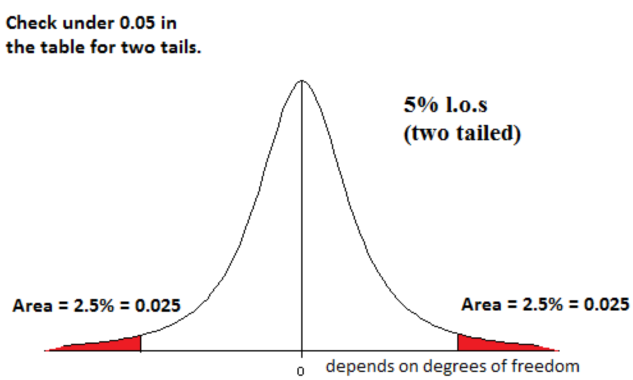
Conclusion of the test

Let LOS = α

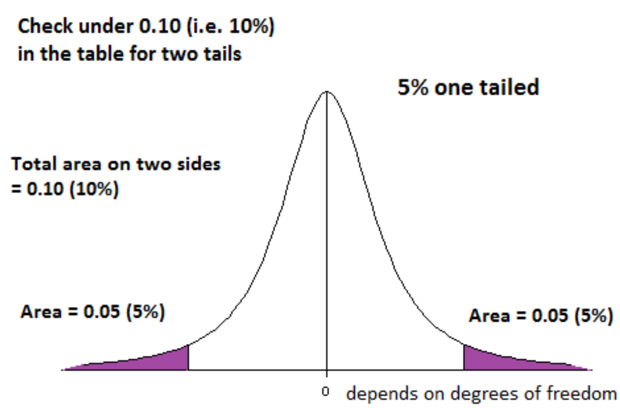
If $| t_{cal} | > t_\alpha$, Reject H_0 at α level of significance. Otherwise we have no reason to reject H_0.

How to read the table:

There is a different table given for t-distribution, since the curve is different, the areas will also differ from a normal distribution. The rows in the t-distribution table indicate degrees of freedom, and the columns indicate the most commonly used levels of significance. So if you wanted to find the critical value at 5% l.os.(two tailed) and with 9 degrees of freedom (i.e. the size of the sample is 10), then the value from the table is 2.26. This means that when the area in the critical region is 5% (2.5% on each side, since it is two-tailed, the critical value $Z_{\propto} = 2.26$.



Note that the table given to you indicates two-tail areas. In the event of a two-tailed test (depending on the alternative hypothesis), reading the critical values from the table is easy. However, if the test is one tailed, with 5% l.o.s. then you cannot look up the value under 5% in the table because it is the two tailed area, meaning the corresponding one-tailed area is 2.5%. However, if you look up the critical value for the two tailed area 10%, then the area is distributed evenly on both the tails (5% on each side), and the critical value corresponding to this area will be the same as the 5% area on a one-tailed test.



<u>Solved Examples:</u>

1. A machinist is expected to make engine parts with axle diameter of 1.75 cm. A random sample of 10 parts shows a mean diameter of 1.85 cm, with a S.D of 0.1 cm. On the basis of this sample, would you say that the work of the machinist is inferior?

   Solution: Given $n = 10$ (small sample)

   $$\bar{x} = 1.85$$

   Level of significance: $\alpha = 5\%$

   $$H_0 : \mu = 1.75$$
   $$H_1 : \mu \neq 1.75 \ (two-tailed\ test)$$

   Under H_0, the test statistic is given by

   $$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n-1}} \sim t_{(n-1)} = t_9 \ d.f$$

   $$i.e\ t = \frac{1.85 - 1.75}{0.1 / \sqrt{10-1}} = 3$$

   From the tables, $t_{9,0.05} = 2.262$

   Since $|t| > 2.262$ we reject H_0 at 5% level of significance. That is, **the work is inferior at 5% LOS.**

   Remark: But from the tables, $t_{9,0.01} = 3.250$. Since $|t| < 2.262$ we have no reason to reject H_0 at 1% level of significance. That is, **we cannot consider the work to be inferior at 1% LOS.**

2. A certain injection administered to each of the 12 patients resulted in the following increases of blood pressure: $5,\ 2,\ 8,\ -1,\ 3,\ 6,\ 0,-2,-1, 5, 0, 4$

Can it be concluded that the injection will be, in general,

accompanied by an increase in BP?

Solution: We have n=12 (small sample).

The mean of the sample is the average increase in B.P.

$$\bar{x} = \frac{1}{n}\sum x = \frac{1}{12}(31) = 2.58$$

The S.D of the sample is given by

$$s^2 = \frac{1}{n}\sum x^2 - \left(\frac{1}{n}\sum x\right)^2 = \frac{1}{12}(185) - (2.58)^2 = 8.76$$

$$\Rightarrow s = 2.96$$

Level of significance: $\alpha = 5\%$

$H_0 : \mu = 0$ (there is no increase in B.P)
$H_1 : \mu > 0$ (*one-tailed test*)

Under H_0, the test statistic is given by

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n-1}} \sim t_{(n-1)} = t_{11} d.f$$

$$i.e\ t = \frac{2.58 - 0}{2.96/\sqrt{11}} = 2.89$$

From the tables, $t_{11, 0.05}\ (one-tailed) = 1.796$

Since $|t| > 1.796$ we reject H_0 at 5% level of significance. And conclude that, **the injection will, in general, result in increase in B.P**

Exercise: The annual rainfall at a certain place is normally distributed with mean 30mm. If the rainfalls during the past 8 years (in mm) are $31.1,\ 30.7,\ 24.3,\ 28.1,\ 27.9,\ 32.2,\ 25.4$ and 29.1, can we conclude that the average rainfall during the last 8 years is less than the normal rainfall?

TOS of difference between two sample means (Small samples) and Paired t-tests

TOS of difference between two sample means (Small samples)

Sample Sizes : $n_1 < 30,\ n_2 < 30$

Population means : $\mu_1\ ,\ \mu_2$

Population s.d.'s : $\sigma_1\ ,\ \sigma_2$ (unknown)

Sample means : $\bar{x}_1\ ,\ \bar{x}_2$

Sample s.d.'s : $s_1\ ,\ s_2$

Null Hypothesis : $H_0 : \mu_1 = \mu_2$

Alternate Hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two tailed)

$H_1 : \mu_1 < \mu_2$ (One tailed (left))

$H_1 : \mu_1 > \mu_2$ (One tailed (right))

Test Statistic: (for when the two samples are independent)

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{(n_1+n_2-2)}\ d.f$$

Let LOS be α

If $|t| > t_{\alpha,\,(n_1+n_2-2)}$, Reject H_0 at α level of significance. Otherwise we have no reason to reject H_0.

Examples

1. The mean height and S.D of 8 randomly chosen soldiers are 166.9 and 8.29 respectively. The corresponding values of 6 randomly chosen sailors are 170.3 and 8.50 cm respectively. Based on these data, can we conclude that soldiers are, in general, shorter than sailors?

   Solution: Given $n_1 = 8;\ n_2 = 6$ (small samples)

   $$\overline{x_1} = 166.9;\ \overline{x_2} = 170.3;\ s_1 = 8.29;\ s_2 = 8.50$$

   Level of significance: $\alpha = 5\%$

   $H_0 : \mu_1 = \mu_2$

   $H_1 : \mu_1 < \mu_2$ $(one-tailed\, test)$

   Test Statistic: $t = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} \sim t_{(n_1 + n_2 - 2)}\ d.f$

   We have

   $t = \dfrac{166.9 - 170.3}{\sqrt{\dfrac{983.29}{12}\left(\dfrac{1}{8} + \dfrac{1}{6}\right)}} = -0.695$

   From the tables, $t_{12,0.05}\ (one - tailed) = 1.782$

   Since $|t| < 1.782$ we have no reason to reject H_0 at 5% level of significance. That is, we accept H_0 and conclude that **soldiers are, in general, of the same average height as sailors.**

2. Samples of two types of electric bulbs were tested for length of life and the following data were obtained:

	size	Mean	SD
Sample 1	8	1234 h	36 h
Sample 2	7	1036 h	40 h

Is the difference in the means sufficient to warrant that Type 1 bulbs are superior to type 2 bulbs?

Solution: Given $n_1 = 8;\ n_2 = 7$ (small samples)

$$\overline{x}_1 = 1234;\ \overline{x}_2 = 1036;\ s_1 = 36;\ s_2 = 40$$

Level of significance: $\alpha = 5\%$

$H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 > \mu_2$ $(one-tailed\ test)$

Test Statistic: $t = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} \sim t_{(n_1+n_2-2)}\ d.f$

We have $t = \dfrac{1234 - 1036}{21.0807} = 9.39$

From the tables, $t_{13,0.05}$ $(one-tailed) = 1.771$

Since $|t| > 1.782$ we reject H_0 at 5% level of significance and conclude that **type 1 bulbs are superior to type 2 bulbs.**

<u>Small sample tests</u> (for dependent samples)

<u>Paired *t* tests</u>

The paired t test is generally used when measurements are taken from the same subject before and after some manipulation such as injection of a drug, extra coaching, adding water treatment etc. For example, you can use a paired t test to determine the significance of a difference in blood pressure before and after administration of an experimental pressure substance or the effectiveness of a water additive in reducing bacterial numbers by sampling water from different sources and comparing bacterial counts in the treated versus untreated water sample. Each different water source would give a different pair of data points.

Let x_1 and x_2 be the variables denoting the first and second series respectively

$$Let\ d = x_1 - x_2$$

$$and\ \bar{d} = \frac{1}{n}\sum\left(x_1 - x_2\right) \text{ (the mean (average) of the differences)}$$

Let $s^2 = \mathrm{var}(d) = \frac{1}{n}\sum d^2 - \left(\bar{d}\right)^2$

Then the test statistic is given by

$$t = \frac{\bar{d}}{\frac{s}{\sqrt{n-1}}} \sim t_{(n-1)}\ d.f$$

Example:

1. The following data relate to marks obtained by 11 students in 2 tests, one held at the beginning of the year and the other at the end of the year after intensive coaching:

Test 1	19	23	16	24	17	18	20	18	21	19	20
Test 2	17	24	20	24	20	22	20	20	18	22	19

Do the data indicate that the students have benefited from coaching?

Solution: We have,

Test 1	19	23	16	24	17	18	20	18	21	19	20
Test 2	17	24	20	24	20	22	20	20	18	22	19
$d = x_1 - x_2$	2	-1	-4	0	-3	-4	0	-2	3	-3	1
d^2	4	1	16	0	9	16	0	4	9	9	1

$$\bar{d} = \frac{1}{n}\sum\left(x_1 - x_2\right)$$

$$\Rightarrow \bar{d} = \frac{1}{11}(-11) = -1$$

$$s^2 = \text{var}(d) = \frac{1}{n}\sum d^2 - \left(\bar{d}\right)^2$$

$$= \frac{1}{11}(69) - \left(-1\right)^2 \quad = 5.27$$

$$\Rightarrow s = 2.296$$

Let the Level of significance: $\alpha = 5\%$

$$H_0 : \bar{d} = 0 \text{ (no benefit from coaching)}$$
$$H_1 : \bar{d} < 0 \ (one - tailed\ test)$$

We have the test statistic to be given by

$$t = \frac{\bar{d}}{\frac{s}{\sqrt{n-1}}} \quad t_{(n-1)}\ d.f$$

$$i.e\ t = \frac{-1}{\frac{2.296}{\sqrt{10}}} = -1.38$$

From the tables, $t_{10,0.05}\ (one - tailed) = 1.812$

Since $|t| < 1.812$ we have no reason to reject H_0 at 5% level of significance and conclude that **the students have not benefited from coaching.**