Tests of Significance

One-tailed & two-tailed tests

A test of any statistical hypothesis where the alternative hypothesis is one-tailed (right-tailed or left-tailed) is called a one-tailed test.

For eg, a test for testing the mean of a population

H_o : $\mu = \mu_o$ with H_1 : $\mu > \mu_o$ (right-tailed)

or $H_1 : \mu < \mu_o$ (left-tailed) is a single tailed test.

A test of statistical hypothesis where the alternative hypothesis is two-tailed such as :

H_o : $\mu = \mu_o$ against H_1 : $\mu \neq \mu_o$ is known as two-tailed test.

Now the critical region lies entirely on one tail (right or left tail) for one-tailed tests & lies in both tails of the probability curve of the test statistic for two-tailed tests.

Large Samples ($n \geq 30$)

1. Test of Significance (TOS) of the difference between sample mean and population mean:

Sample size n ($n \geq 30$)

Population mean: μ

Sample mean : $\bar{x}$

Population S. D. : σ

Null Hypothesis : $H_0 : \mu = \mu_0$

Alternate Hypothesis : $H_1 : \mu \neq \mu_0$ (Two tailed)

$H_1 : \mu < \mu_0$ (One tailed (left))

$H_1 : \mu > \mu_0$ (One tailed (right))

Test Statistic

If σ is known

$$Z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$$

If σ is unknown,

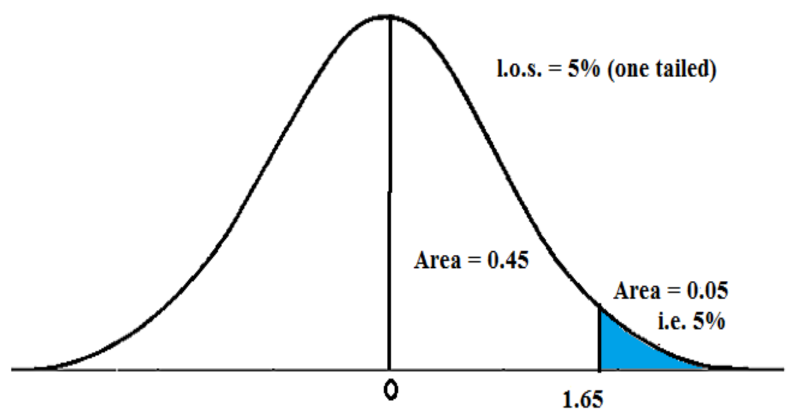
since sample size is large, we can take s, the sample s.d. to be an (unbiased) estimate of σ]

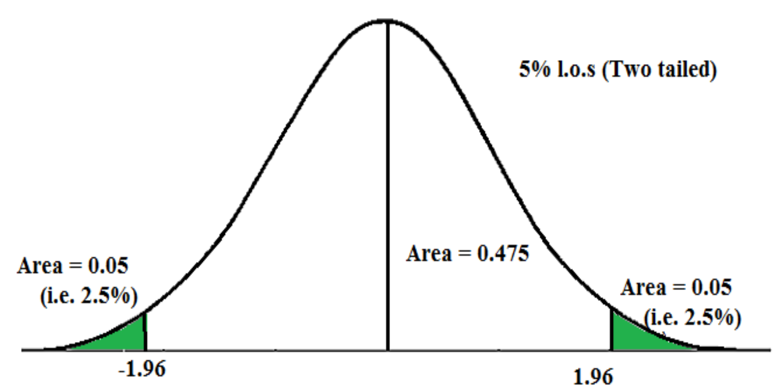
$$Z = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} \sim N(0,1)$$

Let Level of significance (LOS) = α

Conclusion of the Test:

If $|Z| > Z_\alpha$, Reject H_0 at α level of significance. Otherwise we have no reason to reject H_0.





Use the table for normal distribution to find $Z_\propto$. For example, if the tests is one tailed, and you want to test for 5% l.o.s., the critical region will be the 5% area on any one tail of the normal distribution graph. Say it's on the right side. Then, the critical value $Z_\propto$ separates the region of acceptance and rejection. Between $Z_\propto$ and 0 lies the area $0.5 - 0.05 = 0.45$. From the table, you will see that this area is given by $Z_\propto = 1.65$. On the other hand, if the test was two tailed, and we want to test at 5% l.o.s. then the 5% area gets distributed on both the tails, and thus, the area on any one side is 2.5% or 0.025. This is the area to the right of $Z_\propto$. This means that the area between 0 and $Z_\propto$ is $0.5 - 0.025 = 0.475$. From the table, we get $Z_\propto = 1.96$.

Examples:

1. A sample of 100 students is taken from a large population. The mean height of the students in this sample is 160 cm. Can it be reasonably regarded that in the population, the mean height is 165 cm and S.D is 10 cm?

   Solution: Given $n = 100$ (large sample)

$$\bar{x} = 160$$

Level of significance: $\alpha = 5\%$

$$H_0 : \mu = 165$$
$$H_1 : \mu \neq 165 \ (two-tailed\ test)$$

Test Statistic: $Z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$

$$Z = \frac{160 - 165}{10 / \sqrt{100}} = -5$$

From the tables, $Z_\alpha = 1.96$

Since $|Z| > Z_\alpha$ we reject H_0 at 5% level of significance. That is, **the population mean height cannot be regarded as 165**.

2. A random sample of 200 measurements from a large population gave a mean value as 50 and S.D of 9. Can the population mean be regarded as 52? Obtain 95% confidence interval for the mean of the population.

Solution: Given $n = 200$ (large sample)

$$\bar{x} = 50;\ s = 9;$$

Level of significance: $\alpha = 5\%$

$$H_0 : \mu = 52$$
$$H_1 : \mu \neq 52\ (two-tailed\ test)$$

Test Statistic: $Z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} = \dfrac{\bar{x} - \mu_0}{s / \sqrt{n}} \sim N(0,1)$

(Since the sample size is large, we can replace σ *by* s

$$Z = \frac{50 - 52}{9 / \sqrt{200}} = 3.1427$$

From the tables, $Z_\alpha = 1.96$

Since $|Z| > Z_\alpha$ we reject H_0 at 5% level of significance. That is, **the population mean cannot be regarded as 52**.

The 95% confidence limits (or interval) for the population mean is given by $\bar{x} \pm \dfrac{1.96\sigma}{\sqrt{n}}$,

(where $\bar{x}$ is the sample mean, σ, the population standard deviation & n, the size of the sample).

i.e. the 95% confidence limits for the population mean is given by

$$\bar{x} \pm \frac{1.96\sigma}{\sqrt{n}} = 50 \pm \frac{1.96(9)}{\sqrt{200}} = (48.75,\ 51.24)$$

That is, you can say with 95% certainty that the mean of the population lies between 48.75 and 51.24.

3. The mean breaking strength of cables supplied by a manufacturer is 1800 with a S.D of 100. By a new technique in the manufacturing process, it is claimed that the breaking strength of the cable has increased. To test his claim, a sample of 50 is tested and it is found that the mean breaking strength is 1850. Can we support the claim at 1% los?

Solution: Given $n = 50$ (large sample)

$$\bar{x} = 1850, \sigma = 100;$$

Level of significance: $\alpha = 1\%$

$H_0 : \mu = 1800$

$H_1 : \mu > 1800 \, (mean\, breaking\, strength\, has\, increased)(one - tailed\, test)$

Test Statistic: $Z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$

$Z = \dfrac{1850 - 1800}{100 / \sqrt{50}} = 3.5355$

From the tables, $Z_\alpha = 2.33$

Since $|Z| > Z_\alpha$ we reject H_0 and accept H_1 at 1% level of significance. Therefore we support the claim that the breaking strength of the cable has increased.

4. Can it be concluded that the average life span of an Indian is more than 70 years, if a random sample of 100 Indians has an average life span of 71.8 years with s.d of 7.8 years?

Solution: Given $n = 100$ (large sample)

$$\bar{x} = 71.8$$

Level of significance: $\alpha = 5\%$

$H_0 : \mu = 70$

$H_1 : \mu > 70 \, (one - tailed\, test)$

Test Statistic: $Z = \dfrac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \sim N(0,1)$

$Z = \dfrac{71.8 - 70}{7.8 / \sqrt{100}} = 2.3077$

From the tables, $Z_\alpha = 1.65$

Since $|Z| > Z_\alpha$ we reject H_0 at 5% level of significance. That is, **the average lifespan cannot be regarded as 70 years**.

Remark: If we take the level of significance: $\alpha = 1\%$, then

From the tables, $Z_\alpha = 2.33$

Since $|Z| < Z_\alpha$ we have no reason to reject H_0 at 1% level of significance. That is, we accept H_0 and conclude that **the average lifespan can be regarded as 70 years**.

TOS of difference between two sample means:

Sometimes, we are interested in comparing two populations on a particular aspect. For example, if we wanted to compare the average heights of 12 year old boys and girls, or if I want to compare the average salaries of men and women in an institution to check if there is any discrimination on the basis of sex. In such a case, we have two populations, and we will take a sample from each population. We obtain the test statistic from the means of the samples, and the S.D. and to support of reject out hypothesis about the population.

Sample Sizes : $n_1 \geq 30,\ n_2 \geq 30$

Population means : μ_1 , μ_2

Population s.d.'s : σ_1 , σ_2

Sample means : $\bar{x}_1$, $\bar{x}_2$

Null Hypothesis : $H_0 : \mu_1 = \mu_2$

Alternate Hypothesis : $H_1 : \mu_1 \neq \mu_2$ (Two tailed)

$H_1 : \mu_1 < \mu_2$ (One tailed (left))

$H_1 : \mu_1 > \mu_2$ (One tailed (right))

Test Statistic:

Let LOS be α

<u>Case 1</u> : If the population s.d.'s $\sigma_1\, and\, \sigma_2$ are *not known*, we can replace $\sigma_1\, by\, s_1$ and $\sigma_2\, by\, s_2$ and the test statistic is:

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim N(0,1)$$

Case 2 : If the population s.d.'s $\sigma_1 = \sigma_2 = \sigma (say)$ replace σ_1 by s_1 and σ_2 by s_2 and the test statistic is:

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$

Case 3 : If the population s.d.'s $\sigma_1 = \sigma_2 = \sigma (say)$ and σ is *unknown*, replace σ by its estimate based on the sample s.d.'s and the test statistic is:

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\frac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim N(0,1)$$ and after simplification we get,

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_2} + \frac{s_2^2}{n_1}}} \sim N(0,1)$$

Case 4 : If the population s.d.'s $\sigma_1 \neq \sigma_2$ and are *known*, the test statistic is:

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

Examples:

1. Test the significance of the difference between the means of the samples, drawn from two normal populations with the same SD, using the following data:

	size	Mean	SD
Sample 1	100	61	4
Sample 2	200	63	6

Solution: Given $n_1 = 100;\ n_2 = 200$ (large samples)

$$\bar{x}_1 = 61;\ \bar{x}_2 = 63;\ s_1 = 4;\ s_2 = 6$$

Level of significance: $\alpha = 5\%$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2\ (two-tailed\,test)$$

Test Statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_2} + \dfrac{s_2^2}{n_1}}} \sim N(0,1)$

$$Z = \frac{61-63}{\sqrt{\dfrac{4^2}{200} + \dfrac{6^2}{100}}} = -3.02$$

From the tables, $Z_\alpha = 1.96$

Since $|Z| > Z_\alpha$ we reject H_0 at 5% level of significance. That is, **the difference between the population means is significant.**

2. The average marks scored by 32 boys is 72 with an S.D of 8, while that for 36 girls is 70 with an S.D of 6. Test at 1% LOS whether the boys perform better than the girls.

Solution: Given $n_1 = 32;\ n_2 = 36$ (large samples)

$$\bar{x}_1 = 72;\ \bar{x}_2 = 70;\ s_1 = 8;\ s_2 = 6$$

Level of significance: $\alpha = 1\%$

$$H_0 : \mu_1 = \mu_2$$
$$H_1 : \mu_1 > \mu_2\ (one-tailed\ test)$$

Test Statistic: $Z = \dfrac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}} \sim N(0,1)$

$$Z = \frac{72-70}{\sqrt{\frac{64}{32} + \frac{36}{36}}} = 1.15$$

From the tables, $Z_\alpha = 2.33$

Since $|Z| < Z_\alpha$ we have no reason to reject H_0 at 1% level of significance. That is, we accept H_0 and conclude that **the difference is insignificant and boys do not perform better than girls.**