Math 221 Notes

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Contents

1 Week 1

1.1 Day 1

1.1.1 Factoring:

$$(a+b)(a^2-ab+b^2) = a^3+b^3$$
$$(a-b)(a^2+ab+b^2) = a^3-b^3$$

1.1.2 Rules:

$$a<b \to -a>-b$$
$$-ax<b \to x>-\frac{b}{a}$$
$$|x|=b \to x=b \vee x=-b$$
$$|x|<b \to -b<x<b$$
$$|x|=b \to x>b \vee x<-b$$
$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$(a^m)^n = a^{mn}$$
$$\sqrt{a} = a^{\frac{1}{2}}$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$
$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$$

1.1.3 Functions:

Definition: Assigns each element x in a set D exactly one element, called $f(x)$, in set E.
In terms of a graph, a curve can only be a function if no vertical lines intersect the curve more than once (vertical line test).

- The set D is called the domain – possible x values.
- The set E is called the range – possible y values.
- If f is a function with domain D, then it graph is the set of ordered pairs $\{(x, f(x)|x)$ is a element of, $\in$, $D\}$

1.1.3.1 Finding domain:

1. $f(x) = \sqrt{x+2} \rightarrow x+2 \geq 0 \rightarrow x \geq \ -2 \rightarrow [-2, +\infty)$
2. $f(x) = \frac{1}{x^2-x} \rightarrow x^2 - x \neq 0 \rightarrow x(x-1) \neq 0 \rightarrow x \neq 0 \wedge x \neq 1 \rightarrow (-\infty, 0) \cup (0,1) \cup (1, +\infty)$

1.1.3.2 Picewise:

- Open circles, $\circ$, and circle brackets, ( ), are non-inclusive.
- Closed circles, $\bullet$, and square brackets, [ ], are inclusive.
- Formatted as:

  $$f(x) = \begin{cases} y = 5 : x < 0 \\ y = x^2 : x \geq 0 \end{cases}$$

1.1.3.3 Types

Even:

- A function is even if $f(x) = f(-x)$
- The graph is symmetric with respect to the y-axis.
- Examples: $x^4 - 2, x^{20} + x^6, \cos(x), |x|$

Odd:

- A function is even if $f(x) = -f(x)$
- The graph has rotational symmetry about origin.
- Examples: $x^3, x^7 + x, \sin(x), |x|x$

- Even times odd function is always odd.
- Even times even is always even.
- Odd times odd is always even.

1.1.3.4 Increasing & Decreasing:

Increasing : A function f is called increasing on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be positive.

Decreasing : A function f is called increasing on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be negative.

1.1.4 Limits

Definition: Supposing $f(x)$ is defined when x is near the number a, we write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L".

- $\lim_{x\to a} f(x) = L$ if and only if (iff, $\iff$) $\lim_{x\to a^-} f(x) = L \wedge \lim_{x\to a^+} f(x) = L$
- That is, the limit does not exist, $\nexists$, if x approaches different values when from the left and right sides
- Approaching from left (from $-\infty \to \infty$) is notated as $x \to a^-$
- Approaching from right (from $\infty \to -\infty$)) is notated as $x \to a^+$
- iff, $\iff$:

$$A \iff B \to$$

$$A \text{ is necessary and sufficient for B } \to$$

$$B \text{ is necessary and sufficient for A } \to$$

$$A \text{ is equivalent to } B$$

- A vertical asymptote exists if the limit from left side is $+\infty$ or $-\infty$, and the limit from the right side is the opposite.

1.1.4.1 Infinite Limits

- If function limit is $\pm\infty$ (if the denominator is 0 at $f(a)$), you can find whether its + or - by solving the limit for each term. If the term is positive, then it's $+\infty$, and vice-versa, e.g.

$$\lim_{x\to -2^+} \frac{x-1}{x^2(x+2)}$$

$$\frac{\lim_{x\to -2^+}[x-1]}{(\lim_{x\to -2^+}[x])^2 \cdot \lim_{x\to -2^+}[(x+2)]}$$

$$\frac{\ominus}{\oplus \cdot \oplus}$$

$$\ominus \to -\infty$$

1.1.5 Lines

- Slope-Point form: $y - b = m(x + a)$ – (a, b) will be a point of the equation.
- Slope-Intercept form: $y = mx + b$ – $(0, b)$ will be the y-intercept.
- Vertex form: $y = a(x - h) + k$ – (h, k) will be the vertex.
- Point-Point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ – (x_1, y_1) and (x_2, y_2) will be points of the equation.
- Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$ – (a, b) will be a point of the equation.

1.2 Day 2

1.2.1 Limit Laws:

Supposing c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x)$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x = a$$

$$\lim_{x\to a} x^n = a^n \text{ where } n = \mathbb{Z}^+$$

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \text{ where } n = \mathbb{Z}^+$$

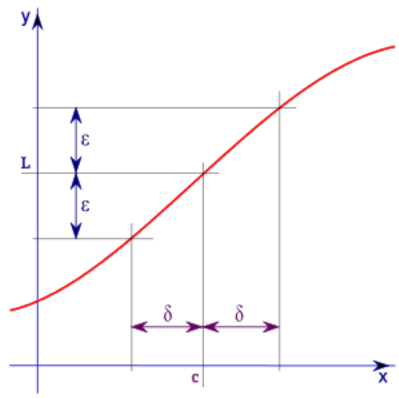
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)} \text{ if } f(a) \geq 0 \text{ and } n_{even}$$

Direct Substitution Property: If f is a polynomial or rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$
Theorem 1.6.1: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$
Theorem 1.5.1: If $f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$
Theorem 1.6.2: If $f(x) \leq g(x)$ when x is near a and the limits of f and g both

exist as x approaches a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$
Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$

1.2.2 Piecewise / (ϵ, δ) definition of limit

If for every small number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.



Solving for δ, rewrite the term defining ϵ to be equal to the term defining δ. E.g., solving for δ if $|x - 2| < \delta$, then $|4x - 8| < \epsilon$, where $\epsilon = 0.1$:

$$|4x - 8| = 4|x - 2| < 0.1$$
$$|x - 2| < \frac{0.1}{4}, \text{ therefore } \delta = \frac{0.1}{4}$$

For non-linear equations, find the lesser and greater δ, and choose the one that results in the smaller ϵ.

1.3 Day 3

1.3.1 Continuous Function:

Definition: A function f is continuous at a number if $\lim_{x\to a} f(x) = f(a)$. Graphically, a function is continuous if you can draw it without having your pen leave paper. More formally, $f(x)$ is continuous at $x = a \iff$:

1. $f(a)$ is defined ($a \in D$: a is in the domain of f).

2. $\lim_{x \to a} f(x)$ exists, and equals $f(x) = f(a)$.

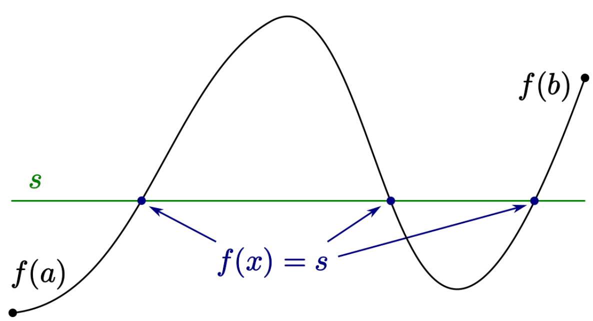
If one of the aforementioned statements is incorrect, then $f(x)$ is discontinuous at $x = a$

Theorem 1: A function is continuous on an interval if it's continuous at every number in the interval. That is, if f and g are continuous at $x = a$, then the following are also continuous at a:

$$f + g, f - g, cf, fg, \frac{f}{g} \text{ for } g(a) \neq 0$$

Theorem 2: The following types of functions are continuous at every number in there domain: Polynomials, Rational functions, Root functions, & Trig functions.

Theorem 3: Intermediate Value Theorem (IVM): Suppose that f is continuous on the close interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists at least on number c in (a, b) such that $f(c) = N$.



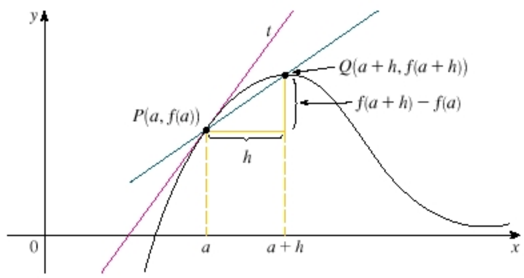
1.4 Day 4

1.4.1 Lines:

1.4.1.1 Secant Line:

A line that locally intersects two points on a curve.

$$\frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$



1.4.1.2 Tangent Line: The line through a pair of infinitely close points on the curve so that the line is "just touching". Slope equation (also known as "Difference Quotient"):

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

1.4.2 Derivatives:

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

and the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ can be written in point-slope form as

$$y - f(a) = f'(a)(x - a)$$

1.5 Day 5

1.5.1 Derivatives (cont.):

- Other notation is $\frac{dy}{dx}|_{x=a}$ (Leibniz Notation), $\frac{df}{dx}$, $\frac{d}{dx}$ $f(x)$, $f(x)$, $Df(x)$, & $D_x f(x)$
- Function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists (same as Theorem 1.5.1).
- Therefore, Not Continuous $\implies$ Not Differentiable.
- If $f'(a)$ exists, then $\lim_{x\to a} f(x) = f(a)$

- The derivative is a function, not a constant.
- Because f' is also a function, f' may have a derivative of its own, denoted by $(f')' = f''$ and called the **second derivative** of f. This can also be written as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

1.6 Day 6

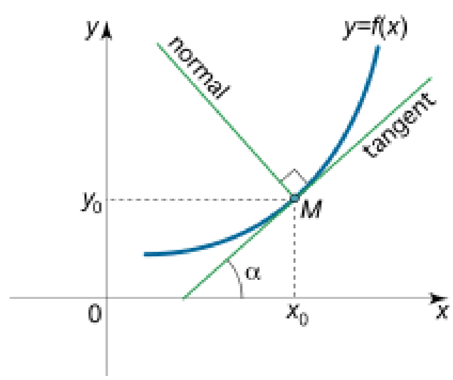
1.6.1 Derivatives Rules:

- Constant: $\frac{d}{dx}(c) = 0$
- Linear: $\frac{d}{dx}(x) = 1$
- Linear + Constant: $\frac{d}{dx}(ax) = a$
- Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

- Constant Multiple: $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}(f(x))$
- Sum/Difference: $\frac{d}{dx}[f(x) \pm g(x)] = f' \pm g'$
- Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f' \cdot g + f \cdot g'$
- Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g \cdot f' - f \cdot g'}{g^2} = \frac{\text{lo·d hi - hi·d lo}}{\text{lo·lo}}$

1.6.2 Lines:

Theorem: If the graph of $y = m_1x+b_1$ is perpendicular to the graph of $y = m_2x+b_2$, then $m_1m_2 = -1$.
Normal Line: The normal line to a curve at point M is the line through M that is perpendicular to the tangent line at M.



1.7 Day 7

1.7.1 Trig Review:

$$\csc = \frac{1}{\sin},\ \sec = \frac{1}{\cos},\ \cot = \frac{1}{\tan} = \frac{\cos}{\sin}$$

1.7.2 Trig Identities:

$$\sin^2 + \cos^2 = 1$$
$$\text{Dividing by } \sin^2 : 1 + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2} \to 1 + \cot^2 = \csc^2$$
$$\text{Dividing by } \cos^2 : \frac{\sin^2}{\cos^2} + 1 = \frac{1}{\cos^2} \to \tan^2 + 1 = \sec^2$$

1.7.3 Derivative of Trig Functions:

$$(\sin)' = \cos$$
$$(\cos)' = -\sin$$
$$(\tan)' = \sec^2$$

$$(\csc)' = -\cot \cdot \csc$$
$$(\sec)' = \sec \cdot \tan$$
$$(\cot)' = -\csc^2$$

1.7.4 Limit of Trig Functions:

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$