Math 221 Notes

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Contents

1 Week 1

1.1 Day 1

1.1.1 Factoring:

$$(a + b)(a^2 - ab + b^2) = a^3 + b^3$$
$$(a - b)(a^2 + ab + b^2) = a^3 - b^3$$

1.1.2 Rules:

$$a < b \rightarrow -a > -b$$
$$-ax < b \rightarrow x > -\frac{b}{a}$$
$$|x| = b \rightarrow x = b \lor x = -b$$
$$|x| < b \rightarrow -b < x < b$$
$$|x| = b \rightarrow x > b \lor x < -b$$
$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$(a^m)^n = a^{mn}$$
$$\sqrt{a} = a^{\frac{1}{2}}$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$
$$\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$$

1.1.3 Functions:

Definition: Assigns each element x in a set D exactly one element, called $f(x)$, in set E.

In terms of a graph, a curve can only be a function if no vertical lines intersect the curve more than once (vertical line test).

- The set D is called the domain – possible x values.

- The set E is called the range – possible y values.

- If f is a function with domain D, then it graph is the set of ordered pairs $\{(x, f(x)|x)$ is a element of, $\in, D\}$

1.1.3.1 Finding domain:

1. $f(x) = \sqrt{x+2} \to x + 2 \geq 0 \to x \geq -2 \to [-2, +\infty)$

2. $f(x) = \frac{1}{x^2 - x} \to x^2 - x \neq 0 \to x(x-1) \neq 0 \to x \neq 0 \land x \neq 1 \to (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$

1.1.3.2 Picewise:

- Open circles, ∘, and circle brackets, ( ), are non-inclusive.

- Closed circles, •, and square brackets, [ ], are inclusive.

- Formatted as:

$$f(x) = \begin{cases} y = 5 \& x < 0 \\ y = x^2 \& x \geq 0 \end{cases}$$

1.1.3.3 Types

Even:

- A function is even if $f(x) = f(-x)$

- The graph is symmetric with respect to the y-axis.

- Examples: $x^4 - 2, x^{20} + x^6, \cos(x), |x|$

Odd:

- A function is even if $f(x) = -f(x)$

- The graph has rotational symmetry about origin.

- Examples: $x^3, x^7 + x, \sin(x), |x|x$

- Even times odd function is always odd.

- Even times even is always even.

- Odd times odd is always even.

1.1.3.4 Increasing & Decreasing:

Increasing : A function f is called increasing on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be positive.

Decreasing : A function f is called increasing on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be negative.

1.1.4 Limits

Definition: Supposing $f(x)$ is defined when x is near the number a, we write $\lim_{x \to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L".

- $\lim_{x \to a} f(x) = L$ if and only if (iff, $\iff$) $\lim_{x \to a^-} f(x) = L \wedge \lim_{x \to a^+} f(x) = L$

- That is, the limit does not exist, $\nexists$, if x approaches different values when from the left and right sides

- Approaching from left (from $-\infty \to \infty$) is notated as $x \to a^-$

- Approaching from right (from $\infty \to -\infty$)) is notated as $x \to a^+$

- iff, $\iff$:

$$A \iff B \to$$

$$A \text{ is necessary and sufficient for B } \to$$

$$B \text{ is necessary and sufficient for A } \to$$

$$A \text{ is equivalent to } B$$

- A vertical asymptote exists if the limit from left side is $+\infty$ or $-\infty$, and the limit from the right side is the opposite.

1.1.4.1 Infinite Limits

- If function limit is $\pm\infty$ (if the denominator is 0 at $f(a)$), you can find whether its + or − by solving the limit for each term. If the term is positive, then it's $+\infty$, and vice-versa, e.g.

$$\lim_{x \to -2^+} \frac{x - 1}{x^2(x + 2)}$$

$$\frac{\lim_{x \to -2^+}[x-1]}{\lim_{x \to -2^+}[x^2] \cdot \lim_{x \to -2^+}[(x+2)]}$$

$$\frac{\ominus}{\oplus \cdot \oplus}$$

$$\ominus \to -\infty$$

1.1.5 Lines

- Slope-Point form: $y - b = m(x + a) - (a, b)$ will be a point of the equation.

- Slope-Intercept form: $y = mx + b - (0, b)$ will be the y-intercept.

- Vertex form: $y = a(x - h) + k - (h, k)$ will be the vertex.

- Point-Point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - (x_1, y_1)$ and (x_2, y_2) will be points of the equation.

- Intercept form: $\frac{x}{a} + \frac{y}{b} = 1 - (a, b)$ will be a point of the equation.

1.2 Day 2

1.2.1 Limit Laws:

Supposing c is a constant and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists, then

$$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$

$$\lim_{x \to a}[cf(x)] = c \cdot \lim_{x \to a} f(x)$$

$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)} \text{ if } \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a}[f(x)]^n = [\lim_{x \to a} f(x)]^n$$

$$\lim_{x \to a} c = c$$

$$\lim_{x \to a} x = a$$

$$\lim_{x \to a} x^n = a^n \text{ where } n = \mathbb{Z}^+$$

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a} \text{ where } n = \mathbb{Z}^+$$

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)} \text{ if } f(a) \geq 0 \text{ and } n_{even}$$

Direct Substitution Property: If f is a polynomial or rational function, and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$

Theorem 1.6.1: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$

Theorem 1.5.1: If $f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$
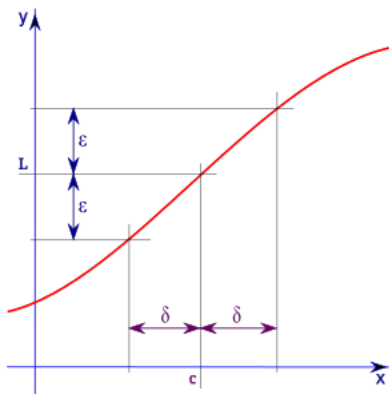
Theorem 1.6.2: If $f(x) \leq g(x)$ when x is near a and the limits of f and g both

exist as x approaches a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

1.2.2 Piecewise / (ϵ, δ) definition of limit

If for every small number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

1.3 Day 3

1.3.1 Continuous Function:

Definition: A function f is continuous at a number if $\lim_{x \to a} f(x) = f(a)$. Graphically, a function is continuous if you can draw it without having your pen leave paper. More formally, $f(x)$ is continuous at $x = a$ iff :

1. $f(a)$ is defined ($a \in D$: a is in the domain of f).

2. $\lim_{x \to a} f(x)$ exists, and equals $f(x) = f(a)$.

If one of the aforementioned statements is incorrect, then $f(x)$ is discontinuous at $x = a$

Theorem 1: A function is continuous on an interval if it's continuous at every number in the interval.

If f and g are continuous at $x = a$, then the following are also continuous at a:

$$f + g, f - g, cf, fg, \frac{f}{g} \text{ for } g(a) \neq 0$$

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Theorem 2: The following types of functions are continuous at every number in there domain:

1. Polynomials (always $(-\infty, \infty)$)

2. Rational functions

3. Root functions

4. Trig functions

Theorem 3: Intermediate Value Theorem (IVM): Suppose that f is continuous on the close interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists at least on number c in (a, b) such that $f(c) = N$
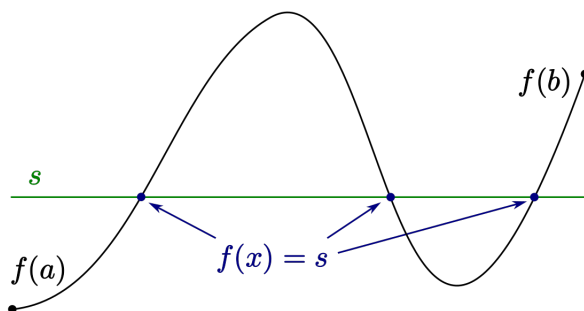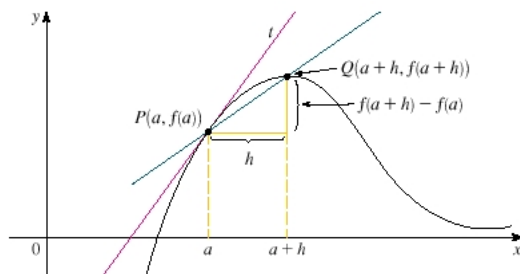

1.4   Day 4

1.4.1   Line:

1.4.1.1   Secant Line: A line that locally intersects two points on a curve.

$$\frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$

1.4.1.2   Tangent Line: The line through a pair of infinitely close points on the curve so that the line is "just touching". Slope equation (also known as "Difference Quotient"):

$$\lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$$

1.4.1.3   Derivatives: The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right]$$

and the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ can be written in point-slope form as

$$y - f(a) = f'(a)(x - a)$$