Calculus 1-2 Notes

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Contents

1 Factoring:

$$(a+b)(a^2-ab+b^2) = a^3+b^3$$
$$(a-b)(a^2+ab+b^2) = a^3-b^3$$

2 Rules:

$$a < b \rightarrow \; -a > -b$$
$$-ax < b \rightarrow x > -\frac{b}{a}$$
$$|x| = b \rightarrow x = b \vee x = -b$$
$$|x| < b \rightarrow \; -b < x < b$$
$$|x| = b \rightarrow x > b \vee x < -b$$
$$a^m a^n = a^{m+n}$$
$$\frac{a^m}{a^n} = a^{m-n}$$
$$(ab)^m = a^m b^m$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$
$$(a^m)^n = a^{mn}$$
$$\sqrt{a} = a^{\frac{1}{2}}$$
$$\sqrt[n]{a} = a^{\frac{1}{n}}$$
$$\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$$
$$\sqrt{a}\sqrt{b} = \sqrt{ab}$$
$$\sqrt[n]{a}\,\sqrt[n]{b} = \sqrt[n]{ab}$$

3 Functions:

Definition: Assigns each element x in a set D exactly one element, called $f(x)$, in set E.
In terms of a graph, a curve can only be a function if no vertical lines intersect the curve more than once (vertical line test).

- The set D is called the domain – possible x values.
- The set E is called the range – possible y values.
- If f is a function with domain D, then it graph is the set of ordered pairs $\{(x, f(x)|x)$ is a element of, $\in$, $D\}$

3.1 Finding domain:

1. $f(x) = \sqrt{x+2} \rightarrow x+2 \geq 0 \rightarrow x \geq \; -2 \rightarrow [-2, +\infty)$

2. $f(x) = \frac{1}{x^2-x} \to x^2 - x \neq 0 \to x(x-1) \neq 0 \to x \neq 0 \land x \neq 1 \to (-\infty, 0) \cup (0,1) \cup (1, +\infty)$

3.2 Picewise:

- Open circles, $\circ$, and circle brackets, ( ), are non-inclusive.
- Closed circles, $\bullet$, and square brackets, [ ], are inclusive.
- Formatted as:

$$f(x) = \begin{cases} y = 5 : x < 0 \\ y = x^2 : x \geq 0 \end{cases}$$

3.3 Types

Even:

- A function is even if $f(x) = f(-x)$
- The graph is symmetric with respect to the y-axis.
- Examples: $x^4 - 2, x^{20} + x^6, \cos(x), |x|$

Odd:

- A function is even if $f(x) = -f(x)$
- The graph has rotational symmetry about origin.
- Examples: $x^3, x^7 + x, \sin(x), |x|x$

- Even times odd function is always odd.
- Even times even is always even.
- Odd times odd is always even.

3.4 Increasing & Decreasing:

- **Increasing :** A function f is called increasing on an interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be positive.
- **Decreasing :** A function f is called increasing on an interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$. In other words, the slope must always be negative.

4 Limits

Definition: Supposing $f(x)$ is defined when x is near the number a, we write $\lim_{x \to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L".

- $\lim_{x\to a} f(x) = L$ if and only if (iff, $\iff$) $\lim_{x\to a^-} f(x) = L \wedge \lim_{x\to a^+} f(x) = L$
- That is, the limit does not exist, $\nexists$, if x approaches different values when from the left and right sides
- Approaching from left (from $-\infty \to \infty$) is notated as $x \to a^-$
- Approaching from right (from $\infty \to -\infty$)) is notated as $x \to a^+$
- iff, $\iff$:

$$A \iff B \to$$

$$A \text{ is necessary and sufficient for B } \to$$

$$B \text{ is necessary and sufficient for A } \to$$

$$A \text{ is equivalent to } B$$

- A vertical asymptote exists if the limit from left side is $+\infty$ or $-\infty$, and the limit from the right side is the opposite.

4.1 Infinite Limits

- If function limit is $\pm\infty$ (if the denominator is 0 at $f(a)$), you can find whether its + or - by solving the limit for each term. If the term is positive, then it's $+\infty$, and vice-versa, e.g.

$$\lim_{x\to -2^+} \frac{x-1}{x^2(x+2)}$$

$$\frac{\lim_{x\to -2^+}[x-1]}{(\lim_{x\to -2^+}[x])^2 \cdot \lim_{x\to -2^+}[(x+2)]}$$

$$\frac{\ominus}{\oplus \cdot \oplus}$$

$$\ominus \to -\infty$$

5 Lines

- Slope-Point form: $y - b = m(x + a)$ – (a, b) will be a point of the equation.
- Slope-Intercept form: $y = mx + b$ – $(0, b)$ will be the y-intercept.
- Vertex form: $y = a(x - h) + k$ – (h, k) will be the vertex.
- Point-Point form: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ – (x_1, y_1) and (x_2, y_2) will be points of the equation.
- Intercept form: $\frac{x}{a} + \frac{y}{b} = 1$ – (a, b) will be a point of the equation.

6 Limit Laws:

Supposing c is a constant and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[cf(x)] = c \cdot \lim_{x\to a} f(x)$$

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$$

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim f(x)}{\lim g(x)} \text{ if } \lim_{x\to a} g(x) \neq 0$$

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$$

$$\lim_{x\to a} c = c$$

$$\lim_{x\to a} x = a$$

$$\lim_{x\to a} x^n = a^n \text{ where } n = \mathbb{Z}^+$$

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a} \text{ where } n = \mathbb{Z}^+$$

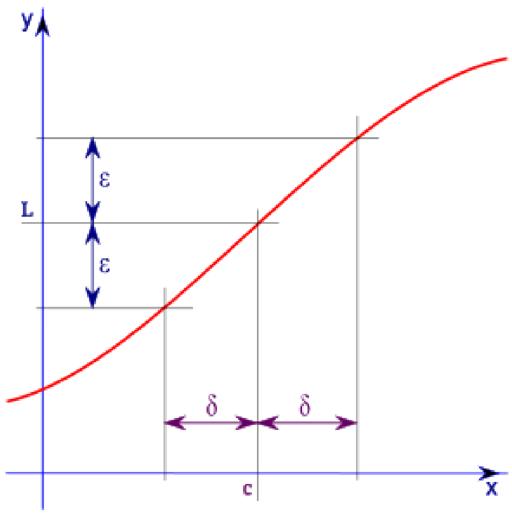
$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)} \text{ if } f(a) \geq 0 \text{ and } n_{even}$$

Direct Substitution Property: If f is a polynomial or rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$
Theorem 1.6.1: If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$
Theorem 1.5.1: If $f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$
Theorem 1.6.2: If $f(x) \leq g(x)$ when x is near a and the limits of f and g both

exist as x approaches a, then $\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$
Squeeze Theorem: If $f(x) \leq g(x) \leq h(x)$ when x is near a and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

7 Piecewise / (ϵ, δ) definition of limit

If for every small number $\epsilon > 0$ there is a number $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|f(x) - L| < \epsilon$.



Solving for δ, rewrite the term defining ϵ to be equal to the term defining δ. E.g., solving for δ if $|x - 2| < \delta$, then $|4x - 8| < \epsilon$, where $\epsilon = 0.1$:

$$|4x - 8| = 4|x - 2| < 0.1$$
$$|x - 2| < \frac{0.1}{4}, \text{ therefore } \delta = \frac{0.1}{4}$$

For non-linear equations, find the lesser and greater δ, and choose the one that results in the smaller ϵ.

8 Continuous Function:

Definition: A function f is continuous at a number if $\lim_{x \to a} f(x) = f(a)$. Graphically, a function is continuous if you can draw it without having your pen leave paper. More formally, $f(x)$ is continuous at $x = a \iff$:

1. $f(a)$ is defined ($a \in D$: a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists, and equals $f(x) = f(a)$.

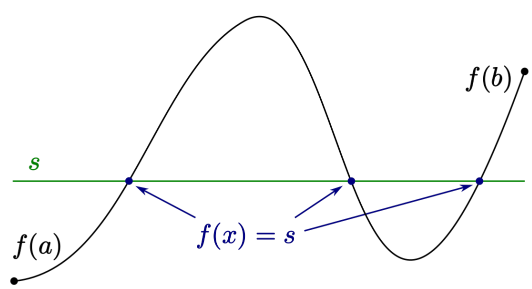
If one of the aforementioned statements is incorrect, then $f(x)$ is discontinuous at $x = a$

Theorem 1: A function is continuous on an interval if it's continuous at every number in the interval. That is, if f and g are continuous at $x = a$, then the following are also continuous at a:

$$f + g, f - g, cf, fg, \frac{f}{g} \text{ for } g(a) \neq 0$$

Theorem 2: The following types of functions are continuous at every number in there domain: Polynomials, Rational functions, Root functions, & Trig functions.

Theorem 3: Intermediate Value Theorem (IVM): Suppose that f is continuous on the close interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists at least on number c in (a, b) such that $f(c) = N$.

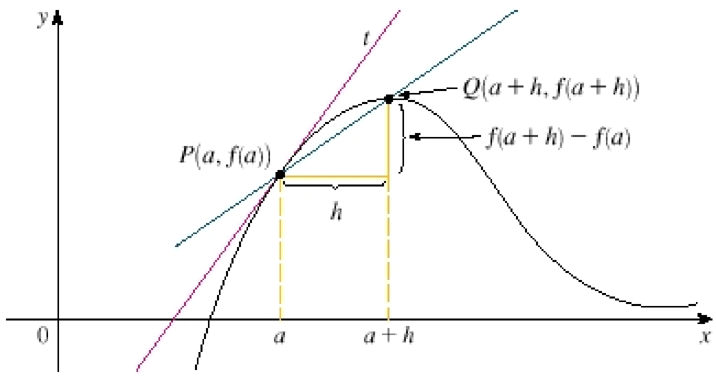


9 Lines:

9.1 Secant Line:

A line that locally intersects two points on a curve.

$$\frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$



9.2 Tangent Line:

The line through a pair of infinitely close points on the curve so that the line is "just touching". Slope equation (also known as "Difference Quotient"):

$$\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

10 Derivatives:

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a)=\lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right]$$

and the equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ can be written in point-slope form as

$$y - f(a) = f'(a)(x - a)$$

11 Derivatives (cont.):

- Other notation is $\frac{dy}{dx}|_{x=a}$ (Leibniz Notation), $\frac{df}{dx}$, $\frac{d}{dx}$ $f(x)$, $f(x)$, $Df(x)$, & $D_x f(x)$
- Function $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists (same as Theorem 1.5.1).
- Therefore, Not Continuous $\implies$ Not Differentiable.

- If $f'(a)$ exists, then $\lim_{x \to a} f(x) = f(a)$
- The derivative is a function, not a constant.
- Because f' is also a function, f' may have a derivative of its own, denoted by $(f')' = f''$ and called the **second derivative** of f. This can also be written as $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

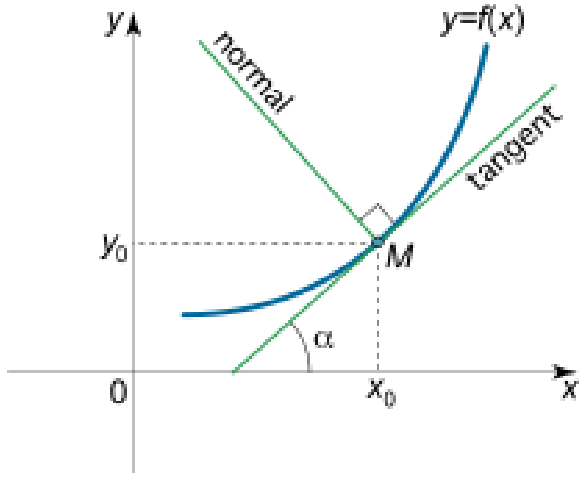
12 Derivatives Rules:

- Constant: $\frac{d}{dx}(c) = 0$
- Linear: $\frac{d}{dx}(x) = 1$
- Linear + Constant: $\frac{d}{dx}(ax) = a$
- Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

- Constant Multiple: $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx}(f(x))$
- Sum/Difference: $\frac{d}{dx}[f(x) \pm g(x)] = f' \pm g'$
- Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] = f' \cdot g + f \cdot g'$
- Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g \cdot f' - f \cdot g'}{g^2} = \frac{\text{lo·d hi - hi·d lo}}{\text{lo·lo}}$

13 Lines:

Theorem: If the graph of $y = m_1x + b_1$ is perpendicular to the graph of $y = m_2x + b_2$, then $m_1m_2 = -1$.
Normal Line: The normal line to a curve at point M is the line through M that is perpendicular to the tangent line at M.



14 Trig Review:

$$\csc = \frac{1}{\sin}, \quad \sec = \frac{1}{\cos}, \quad \cot = \frac{1}{\tan} = \frac{\cos}{\sin}$$

15 Trig Identities:

$$\sin^2 + \cos^2 = 1$$

$$\text{Dividing by } \sin^2 : 1 + \frac{\cos^2}{\sin^2} = \frac{1}{\sin^2} \to 1 + \cot^2 = \csc^2$$

$$\text{Dividing by } \cos^2 : \frac{\sin^2}{\cos^2} + 1 = \frac{1}{\cos^2} \to \tan^2 + 1 = \sec^2$$

16 Derivative of Trig Functions:

$$(\sin)' = \cos$$
$$(\cos)' = -\sin$$
$$(\tan)' = \sec^2$$

$$(\csc)' = -\cot \cdot \csc$$
$$(\sec)' = \sec \cdot \tan$$
$$(\cot)' = -\csc^2$$

17 Limit of Trig Functions:

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

18 Composite Function:

A new function can be composed to two old functions, such as $f \circ g$. Formally, given two functions f and g, the composite function $f \circ g$ and $g \circ f$ (also called the composition of f and g) is defined by $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$

19 Decomposite Function:

Many functions can be decomposed into small functions, $h(x) = f(g(x))$

20 Chain Rule

If the composite function $F(x) = f \cdot g$ is defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)) \cdot g'(x) \text{ or } OUT'(in) \cdot IN'$$

The power rule can be combined with the chain rule as

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

21 Explicit and Implicit:

Explicit: A function given in terms of the independent variable, e.g. $y = \sqrt{x^3 + 1}$.
Implicit: A function given in terms of both dependent and independent variables, e.g. $x^3 - y^2 + 1 = 0,\ y \geq 0$.
Implicit Differentiation: If the function is stuck in implicit form, then differentiate both sides of the equation with respect to x and solve the resulting equation for y'.
Note:

$$\frac{d}{dx}(y^n) \neq ny^{n-1} \qquad \frac{d}{dx}(y^n) = ny^{n-1} \cdot y'$$

22 Related Rates:

A related rates problem is to compute the rate of change (derivative) of one quantity in terms of the rate of change of another quantity, which is more easily measured.

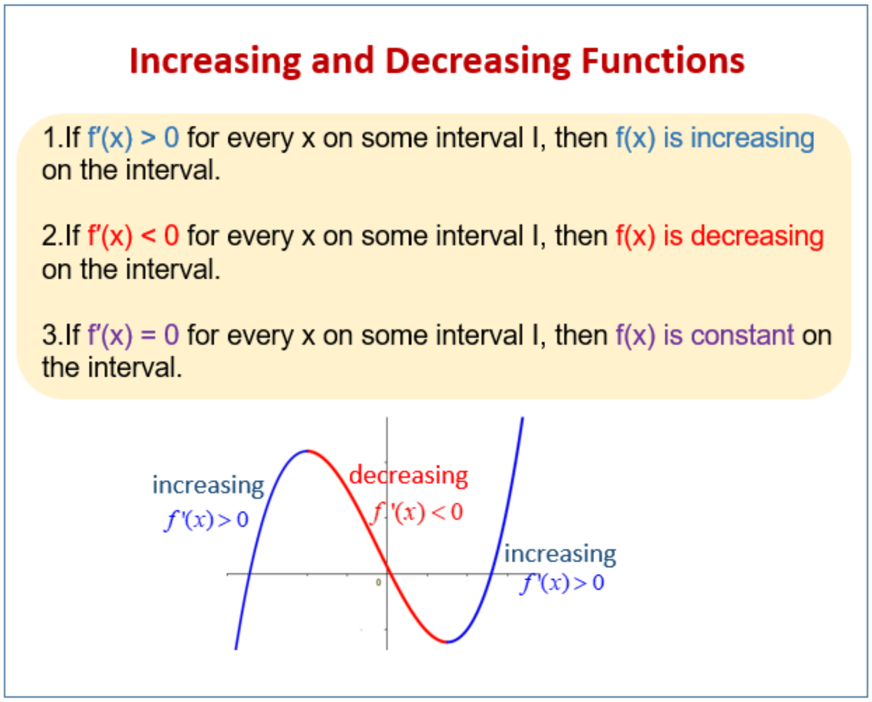
How to approach a related rates problem:

1. Read the problem, and draw a diagram if possible.
2. Assign symbols to all variables that are a function of time.
3. Find an equation that relates the two variables.
4. Using the Chain Rule to differentiate both sides with respect to time.
5. Substitute the given information into resulting equation and solve for the unknown rate.

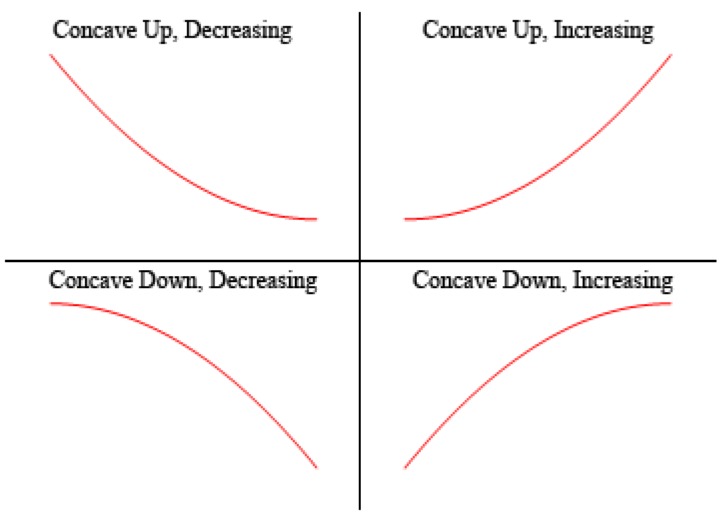
23 Maximum and Minimum Values:

Absolute or Global [Max/Min]: The [max/min] y-value in the domain.
Relative or Local [Max/Min]: The [max/min] y-value in a given range around a x-value. Can only occur when the graph goes from [increasing to decreasing / vice-versa]
Note: The endpoint of a graph can never be a relative [max/min] because one side of the point is always undefined.

24 Increasing/Decreasing Functions:



25 Concavity:



26 Extreme Value Theorem:

Suppose that $f(x)$ is continuous on the interval $[a, b]$ then there are two numbers, $a \leq c, d \leq d$ so that $f(c)$ is an absolute maximum for the function and $f(d)$ is an absolute minimum for the function.

27 Fermat's Theorem:

If f has a local max or min at a, and $f'(a)$ exists, then $f'(a) = 0$.

28 Critical Number:

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c) = \text{DNE}$

29 Absolute Extreme:

To find the absolute max and min values of a continuous function f on a closed interval $[a, b]$:

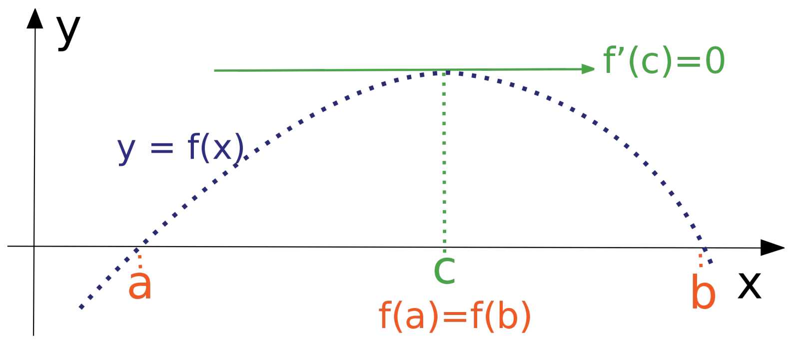
1. Find the critical numbers of the function on (a, b)
2. Find the values of f at the critical numbers of f in (a, b)
3. Find the values of f at the end points of the interval, in other words, find $f(a)$ and $f(b)$.
4. The largest of the values from steps 2 and 3 is the absolute max, the smallest is the absolute min.

30 Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$
2. f is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Then there is at least one number c in (a, b) such that $f'(c) = 0$



30.1 Proof

- Case 1: $f(x) = k$, a constant. Then, $f'(x) = 0$ so the number c can be taken to be any number in (a, b).
- Case 2: $f(x) > f(a)$, concave down, for some x in (a, b). By the extreme value theorem, f has a max value in $[a, b]$. Since $f(a) = f(b)$, it must attain it's max value at c; $c \in (a, b)$. Then, f has a local max at c, f is differentiable at c, therefore $f'(c) = 0$ by Fermat's theorem.
- Case 3: $f(x) < f(a)$, concave up, for some x in (a, b). By the extreme value theorem, f has a min value in $[a, b]$. Since $f(a) = f(b)$, it must attain it's min value at c; $c \in (a, b)$. Then, f has a local min at c, f is differentiable at c, therefore $f'(c) = 0$ by Fermat's theorem.

31 Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

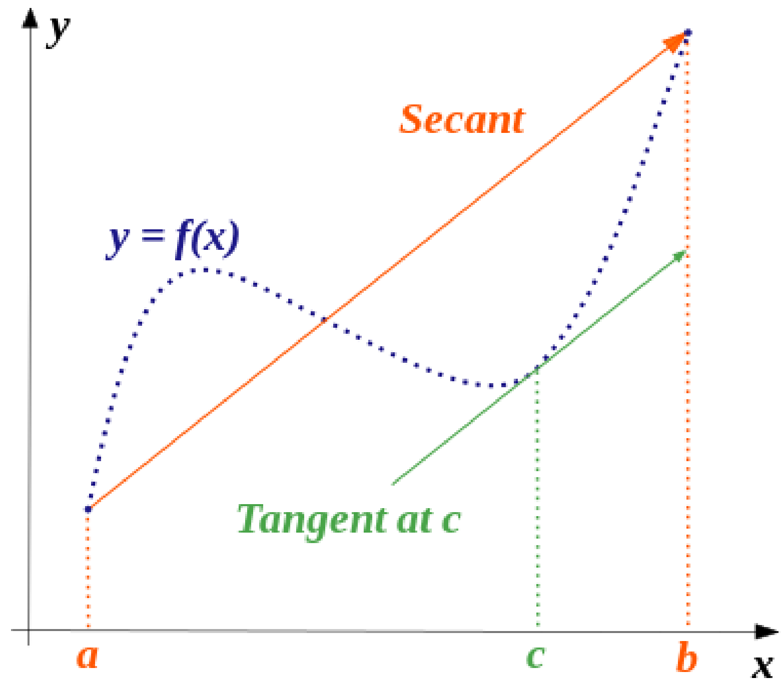
- f is continuous on the closed interval $[a, b]$
- f is differentiable on the open interval (a, b)

Then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\text{Slope of tangent line at } c = \text{Slope of Secant Line}$$

$$f(b) - f(a) = f'(c)(b - a)$$



32 Determining Increasing/Decreasing:

1. Find the critical numbers
2. Mark the critical numbers on a real line
3. Pick a test number, c, in each interval:

· If $f'(c) > 0$, f is increasing at that interval

· If $f'(c) < 0$, f is decreasing at that interval

33 Local max and min:

1. Find the critical numbers

2. Mark the critical numbers on a real line

3. Pick a test number, c, in each interval:

   · If f' changes from $\oplus$ to $\ominus$, then f has a local max at $f(c)$

   · If f' changes from $\ominus$ to $\oplus$, then f has a local min at $f(c)$

   · If f' doesn't change sign, then f has no local min or min at $f(c)$

34 Second Derivative Test

Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local max at c.

2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local min at c.

3. If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test says nothing about the point c, a possible inflection point

35 Finding Concavity

1. $f(x)$ increasing implies $f'(x) > 0$

2. $f'(x)$ increasing implies $f''(x) > 0$

3. $f''(x) > 0$ for all x in I, then the graph of f is concave upwards on I

4. $f''(x) < 0$ for all x in I, then the graph of f is concave downwards on I

35.1 Inflection Point:

A point p on a curve $f(x)$ is called an inflection point if f is continuous there and the curve changes at p from concave up to concave down or vice versa.

35.2 Theorem:

At a point of inflection, $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ DNE.

36 Limit at ∞

Let f be a function defined on some interval $(a, +\infty)$. Then $\lim_{x\to\infty} f(x) = L$ means that the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently large. In other words, for every $\epsilon > 0$, there is a number N such that if $x < N$ then $|f(x) - L| < \epsilon$. "N": Large negative number.

37 Horizontal Asymptote

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

37.1 Theorem

If $r > 0$ is a rational number, then $\lim_{x\to\infty} \frac{1}{x^r} = 0$
If $r > 0$ is a rational number such that x^r is defined for all x, then $\lim_{x\to-\infty} \frac{1}{x^r} = 0$

38 Optimization

1. Understand the problem. What is the unknown? What are the given quantities? What are the given conditions?

2. Draw a diagram. Show variables.

3. Introduce notation. Assign Q to the quantity that is to be maximized or minimized.

4. Express Q in terms of other variables.

5. If Q is expressed with more than one variable, use the given info to find a relationship among these variables.

6. Find the absolute max or min value of the Q (the function).

39 Area Under Curve:

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} (f(x_1)\Delta x) + f(x_2)\Delta x) + ... + f(x_n)\Delta x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A = \lim_{n\to\infty} r_n = \lim_{n\to\infty} (f(x_0)\Delta x) + f(x_1)\Delta x) + ... + f(x_{n-1})\Delta x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

40 Absolute Extreme Values

Suppose that c is a critical number of a continuous function f defined on an interval.

- If $f'(x) > 0$ for all $x < c$, and $f'(x) < 0$ for all $x > c$, then $f(c)$ is the absolute max for values of f.
- If $f'(x) < 0$ for all $x < c$, and $f'(x) > 0$ for all $x > c$, then $f(c)$ is the absolute min for values of f.

41 Anti-derivative

A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.
For example, $F(x) = x^3$ and $f(x) = 3x^2$:

- $f(x)$ is derivative of function $F(x)$
- $F(x)$ is antiderivative of function $f(x)$

41.1 Theorem

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x) + C$ where C is an arbitrary constant.

41.2 Initial Condition Steps

1. Find the anti-derivative of $f(x)$
2. Plug in the initial condition to evaluate constant C
3. Find the solution

42 Differential Equations

An equation that involves the derivatives of a function is called a differential equation. The solution of a differential equation is a function that satisfies the equation.

43 Definite Integral

If f is continuous on $[a, b]$, or if f has only a finite number of jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral exists.

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

43.1 Properties

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b c \cdot dx = c(b-a)$$

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \text{ where } a < b < c$$

43.2 Comparison Properties

$$f(x) > 0 \text{ for } a < x < b, \text{ then } \int_a^b f(x)dx > 0$$

$$f(x) > g(x) \text{ for } a < x < b, \text{ then } \int_a^b f(x)dx > \int_a^b g(x)dx$$

$$m < f(x) < M \text{ for } a < x < b, \text{ then } m(b-a) < \int_a^b f(x)dx < M(b-a)$$

44 Fundamental Theorem of Calculus (1)

If $f(x)$ is continuous on $[a, b]$, then the function g defined by $g(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$, is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x)=f(x)$. In other words,

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

45 Fundamental Theorem of Calculus (2)

If $f(x)$ is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$. where $F(x)$ is any antiderivative of f, that is, a function such that $F' = f$.

45.1 Notation:

$$F(b) - F(a) = F(x)\Big|_a^b = F(x)\Big]_a^b$$

46 Properties of Definite Integral

$$\int_{-a}^{a} f(x)dx = 0 \text{ if } f(x) \text{ is odd}$$

$$\int_{-a}^{a} f(x)dx = 2 \cdot \int_0^a f(x)dx \text{ if } f(x) \text{ is even}$$

47 Indefinite Integral

If $F(x)$ is any anti-derivative of $f(x)$, then the most general anti-derivative of $f(x)$ is called an indefinite integral and denoted as

$$\int f(x)dx = F(x) + C$$

47.1 Indefinite vs Definite Integrals

Definite is a number, where as indefinite is a family of functions

48 Integral Rules

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

49 Net Change Theorem

The integral of a rate of change is the net change, $\int_a^b F'(x)dx = F(b) - F(a)$

50 Integration by Substitution

A function F is called an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

1. Let $u = g(x)$, where $g(x)$ is part of the integrand.
2. Calculate $\frac{du}{dx}$ (or $g'(x)$)
3. Use the substitution $u = g(x)$ and $du = g'(x)dx$ to convert the entire integral into one involving only u
4. If indefinite, evaluate the resulting integral and plug in u by $g(x)$.

5. If definite, plug in the lower limit of the integral and the upper limit of the integral into the expression $u = g(x)$ and get the new lower and upper limits. Then, evaluate the new definite integral (involving only u) by finding the anti-derivative of f and use the Fundamental Theorem.

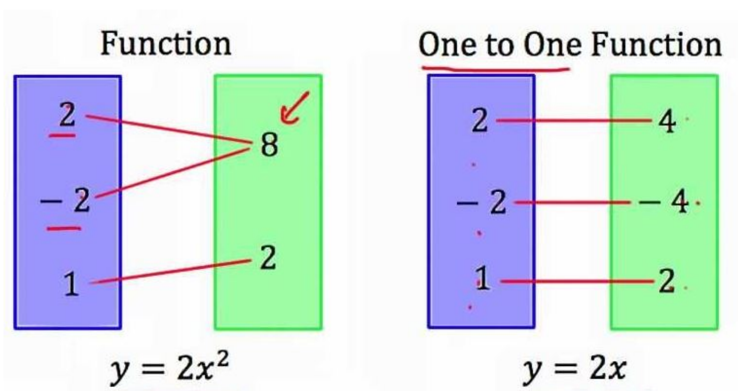
51 Substitution Rule for Definite Integral

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x)) \cdot g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

52 One-to-One Function

A function f that never takes one a same value twice; that is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.



52.1 Horizontal Line Test

If you draw a horizontal line through the graph, it should only intersect once if it's a 1-to-1 function.

52.2 Remark

All strictly increasing/decreasing functions are 1-to-1

- $f'(a) > 0$ means strictly increasing
- $f'(a) < 0$ means strictly decreasing

53 Inverse Function

Let function f be a 1-to-1 function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by $f^{-1}(y) = x \iff f(x) = y$ for any y in B.

53.1 Remark

- The domain of f^{-1} is range of f
- The range of f^{-1} is the domain of f
- $f^{-1}(f(x)) = x$
- $f(f^{-1}(x)) = x$

53.2 Steps to find inverse

1. Write $y = f(x)$
2. Solve this equation for x in terms of y
3. To express f^{-1} as a function of x, interchange x and y.

54 Derivative of Inverse Function

If f is a 1-to-1 differentiable function with inverse function f^{-1}, then the inverse function is differentiable and the following is true:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

55 Exponential Functions

A function f is exponential if it is in the form $f(x) = a^x, a > 0, a \neq 1$.

- If $a > 1$, the range is $(0, \infty)$ and greater the a, greater the slope
- If $1 > a > 0$, the range is $(\infty, 0)$ and lower the a, greater the slope

55.1 Derivative of Exponential Function

$$a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \cdot f'(0)$$

56 e

The function $f(x) = e^x$ is called the natural exponential function because the derivative is itself.

57 Log Functions:

Exponential functions are 1-to-1, hence it has an inverse function f^{-1}:

$$f^{-1}(x) = log_a x, x > 0$$
$$y = log_a x \iff a^y = x$$

Therefore, if $a > 1$ the function $f(x) = log_a x$ is a 1-to-1, continuous, increasing function with the domain $(0, \infty)$ and range $(-\infty, \infty)$. If x, $y > 0$ and r is any real number, then:

$$log_a(xy) = log_a x + log_a y$$
$$log_a(x/y) = log_a x - log_a y$$
$$log_a(x^r) = r \cdot log_a x$$

$$lim_{x \to \infty}[log_a x] = \infty$$
$$lim_{x \to 0^+}[log_a x] = -\infty, a > 1$$

58 Inverse Function Rules:

$$(f \circ f^{-1})(x) = a^{log_a x} = x, x > 0$$
$$(f^{-1} \circ f)(x) = log_a(a^x) = x$$

59 Theorem

If f is a 1-to-1 differentiable function with the inverse function f^{-1}, then the inverse function is differentiable and we can use 54:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}, (-1 < x < 1) \text{ between } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\frac{d}{dx}\left(\cos^{-1} x\right) = -\frac{1}{\sqrt{1-x^2}}, (-1 < x < 1) \text{ between } 0 \leq y \leq \pi$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}, x \in R \text{ between } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

60 Indeterminate Forms:

$\frac{0}{0}$	$0 \cdot \infty (\neq 0)$
$\frac{\infty}{\infty}$	$\infty - \infty (\neq 0)$
0^0	∞^0

61 L'Hospital's Rule:

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

$$\text{or that } \lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \infty$$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$)

62 Area Between Lines

The area A of the region bounded by the curves $y = f(x),\ y = g(x)$, and the lines $x = a, x = b$, where f and g are continuous and $f(x) > g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Or, if we just want to find the area inclsed we can use

$$A = \int_a^b |f(x) - g(x)|\, dx$$

63 Average value of f

We define the average value of f on the interval $[a, b]$ as

$$f_{avg} = \frac{1}{b-a} \cdot \int_a^b f(x)dx$$

64 Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f(x) = f_{avg} = f(c) \cdot (b - a)$$

65 Area of Shape

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the volume of S is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)dx$$

65.1 Volume

$$V = \int_a^b A(x)dx$$

65.2 Disk

$$A(x) = \pi \cdot r^2 \text{ if rotating about } x\text{-axis}$$

65.3 Cross-Section (washer)

$$A(x) = \pi \cdot (R^2 - r^2) \text{ if rotating about } x\text{-axis}$$

65.4 Cylinder

$$A(x) = 2\pi x \cdot f(x) \cdot dx \text{ if rotating about } y\text{-axis}$$

66 Integration by Parts

$$\int f \cdot g' \, dx = f \cdot g - \int g \cdot f' \, dx$$

$$\int f \, dg = f \cdot g - \int g \, df$$

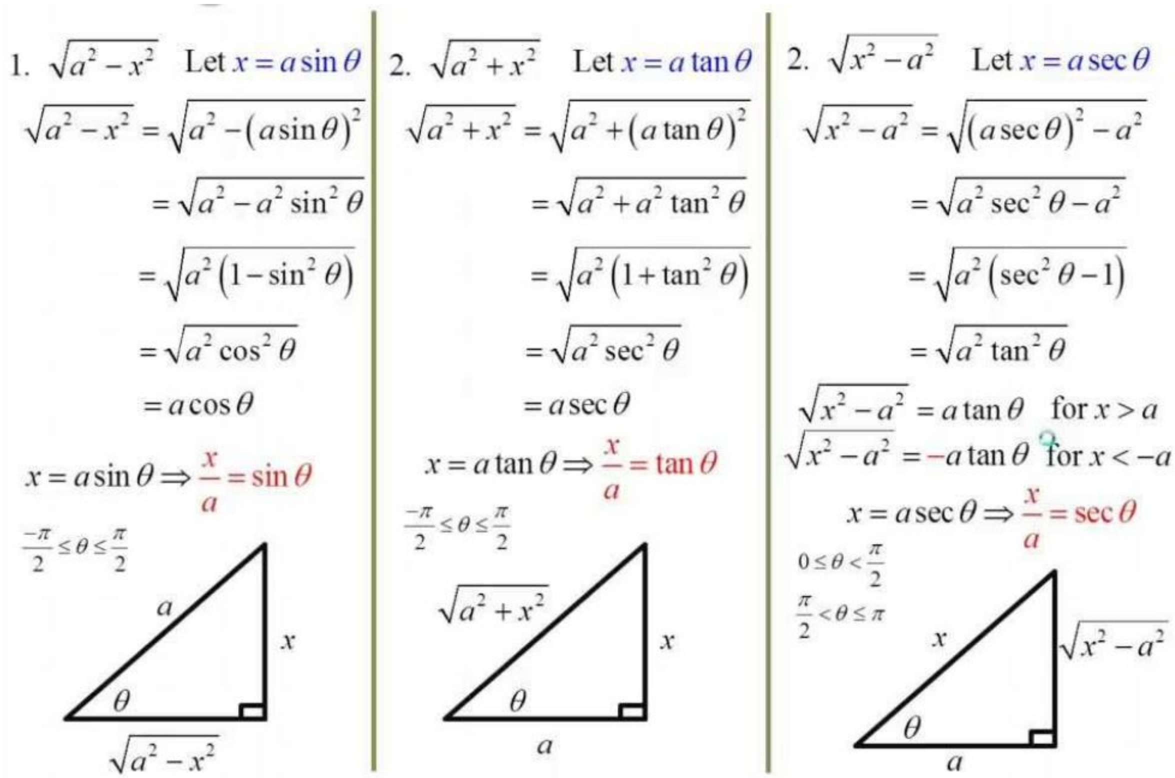
- Make f something that results in the most simple equation when derived.
- DI Method is very quick and easy

67 Partial Fraction Decomposition (PFD)

Table 8-1 The Four Cases for Setting Up Partial Fractions

Case	*Example*	*As Partial Fractions*
Case #1: Distinct linear factors	$\frac{x}{(x+4)(x-7)}$	$\frac{A}{x+4} + \frac{B}{x-7}$
Case #2: Distinct irreducible quadratic factors	$\frac{8}{(x^2+3)(x^2+9)}$	$\frac{A+Bx}{x^2+3} + \frac{C+Dx}{x^2+9}$
Case #3: Repeated linear factors	$\frac{2x+2}{(x+5)^2}$	$\frac{A}{x+5} + \frac{B}{(x+5)^2}$
Case #4: Repeated quadratic factors	$\frac{x^2-2}{(x^2+6)^2}$	$\frac{A+Bx}{x^2+6} + \frac{C+Dx}{(x^2+6)^2}$

68 Trig Sub



69 Improper Integrals

$$\int_a^{\infty} f(x)\, dx = \lim_{B\to\infty} \int_a^B f(x)\, dx$$

- If the limit exists, it converges
- If the limit doesn't exist, it diverges

e.g.

$$\int \frac{1}{x^P}\, dx$$

if $P \le 1$, the integral diverges

70 Separable Differential Equations

1. Put all y's on one side and all x's on the other.
2. Integrate both sides (don't forget $+c$!)
3. If possible, solve for y.
4. Apply initial conditions (if given.)
5. Verify the solution.

Note: Make sure dy and dx are in numerators!

71 First-Order Linear Differential Equations

$$\frac{dy}{dx} + P(x) \cdot y = Q(x)$$

1. Find the integrating factor:
$$e^{\int P(x)\ dx}$$
2. Multiply both sides by the integrating factor
3. Recognize LHS as product rule
4. Integrate both sides
5. Solve for y
6. Verify Solution

72 Taylor Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + ...$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + ...$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + ...$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

$$\tan^{-1}(x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$$

72.1 Maclaurin Series

Where a is the given center, by default 0

$$f(x) = \frac{f(a)(x-a)^0}{0!} + \frac{f^1(a)(x-a)^1}{1!} + \frac{f^2(a)(x-a)^2}{2!} + \frac{f^3(a)(x-a)^3}{3!} + ...$$

72.2 Linear Approximation

$$P_1(x) = f(a) + f^1(a)(x-a)$$

72.3 Quadratic Approximation:

$$P_2(x) = f(a) + f^1(a)(x-a) + f^2(a) \cdot \frac{1}{2}(x-a)^2$$

73 Integrating using Series

1. Find series
2. Take integral
3. Evaluate at limits
4. Find the first term less than the error magnitude. Stop at one term before

74 Error

74.1 Alternating

For a Taylor polynomial of order n that alternates, the error will always be smaller than the first unused term. This error is given by the first unused term;

$$R_n(x) = P_{n+1}(x)$$

74.2 Converges

For a Taylor polynomial of order n that converges, the worst-case scenario is given by the following where $c \in (a, x)$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

75 Sequences & Series

- a_n represents the nth term of a **sequence**; $a_1, a_2, a_3, \ldots\ a_n$
- The **series** is a the summation of a sequence; $a_1 + a_2 + a_3 + \ldots\ + a_n$
- If the $\lim_{n\to\infty}$ of a sequence is ∞, the sequence diverges.
- If the $\lim_{n\to\infty}$ of a sequence is a constant, the sequence converges.

76 Monotonic Sequence Theorem

If a sequence $\{a_n\}$ is both bounded and monotonic (meaning it's either nonincreasing or nondecreasing), then the sequence converges.

$$\lim_{r\to\infty} r^n = 0 \text{ if } |r| < 1$$

77 Geometric Series

A geometric series is a series that has the first term of a and a common ration of r written as

$$\sum_{n=0}^{\infty} a \cdot r^n$$

- If $|r| < 1$ it converges to the sum of $\frac{a}{1-r}$
- If $|r| \geq 1$ it diverges

The nth partial sum for $\sum_{n=1}^{\infty} ar^{n-1}$ is

$$a \cdot \frac{1 - r^n}{1 - r}$$

and the interval of convergence is about a and

$$-1 < r < 1 \text{ solve for x}$$

When the endpoints are plugged in to the series equation and it diverges, then it isn't included using parenthesis. If it converges, it is included using square brackets. If it equals 0, the interval is $(-\infty, \infty)$. If it equals ∞, the interval is a single point.

77.1 Starting at $n \neq 0$

$$\sum_{n=b}^{\infty} a \cdot r^n = \sum_{n=0}^{\infty} (a \cdot r^b) \cdot r^n$$

78 p-Series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^p = \sum_{n=1}^{\infty} \frac{1}{n^p} = \sum_{n=1}^{\infty} n^{-p}$$

- Converges if $p > 1$
- Diverges if $p \leq 1$

79 n-th term test

If the terms of a series (the sequence) don't converge to 0, then the series diverges; however, the converse is not true! (e.g. $\sum \frac{1}{n}$ diverges even though its terms $\to 0$)

80 Ratio Test

For $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$ where r is the ratio

- $r < 1$, series converges
- $r > 1$, series diverges
- $r = 1$, test inconclusive
  — For Power Series
- $r = 0$ the radius is ∞ so interval is $(-\infty, \infty)$
- $r = \infty$ the radius 0 because the series converges at the center value.
- $r = \frac{|x-c|}{R}$ the radius is R so interval is $(-R+c, R+c)$ but we don't know if the endpoints $|c|$ are included, so we should plug them into a_n for x

81 Root Test

- $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, series converges
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, series diverges
- $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, test inconclusive

82 Integral Test

If $\sum_{n=1}^{\infty} a_n$ has positive, decreasing terms and $f(n) = a_n$ is continuous, then $\int_1^{\infty} f(x)dx$ and $\sum_{n=1}^{\infty} a_n$ either both converge or both diverge. Our resulting value is not what the series converges to!

83 Limit Comparison Test

For some $b_n > a_n$, $\lim_{n\to\infty} \frac{a_n}{b_n} = c$,

- $c > 0$ and finite $\implies \sum a_n$ and $\sum b_n$ either both converge or diverge.
- $c = 0 \implies \sum a_n$ and $\sum b_n$ converge
- $c = \infty \implies \sum a_n$ and $\sum b_n$ diverge

84 Alternating Series Test

If $\sum a_n$ is alternating, and the terms to go 0 continually, then the series converges.

85 Absolute vs Conditional

- If $\sum |a_n|$ converges, then it converges absolutely.
- If $\sum |a_n|$ diverges but $\sum a_n$ converges, it conditionally converges

86 Factorial Rule

$$(ax+n)! = (ax+n) \cdot (ax+(n-1)) \cdot (ax+(n-2)) \cdot ... \cdot (ax)!$$

87 Feedback

Negative Feedback

- $y'' = -y$
- $y = \cos(x) \to y'' = -\cos(x)$
- $y = \sin(x) \to y'' = -\sin(x)$

Positive Feedback

- $y'' = y$
- $y = e^x \to y'' = e^x$
- $y = e^{-x} \to y'' = e^{-x}$

88 Euler's Formula

$$e^{ix} = \cos(x) + i \cdot \sin(x)$$

89 2nd-order Linear DE

89.1 Factoring

1. Two real distinct roots; r_1, r_2; $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
2. Two complex roots; $\alpha \pm \beta x$; $y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$
3. One repeated real root; r; $y = c_1 e^{rx} + c_2 x e^{rx}$

89.2 Undetermined Coefficients

1. Constant; $y_p = A$
2. $e^{\alpha x}$; $y_p = Ae^{\alpha x}$
3. $\sin(\alpha x)$ or $\cos(\alpha x)$; $y_p = A\cos(\alpha x) + B\sin(\alpha x)$

89.3 Solving

89.3.1 If given two possible equations, y_1/y_2

1. Plug in the y_1/y_2 into the given equation to verify that the given functions work
2. Write the general solution
3. Plug in initial values, solve for constants
4. Plug in constants to find particular solution

89.3.2 Initial Value

1. Turn the equation into it's auxiliary form and find it's roots
2. Find it's respective case and write the general solution
3. **If non-homogeneous (RHS $\neq$ 0)**, find RHS's case and add it to the end of the general solution
4. Non-homo: Plug y_p into diff eq as y and solve for constants A (and sometimes B)
5. Set up the initial values equations and find c_1 and c_2
6. Rewrite general solution with constants to get the particular solution

90 Parabolic Foci

$$4py = x^2$$

- Focus at $(0, p)$
- Directrix at $y = -p$
- Opens along y

$$4px = y^2$$

- Focus at $(p, 0)$
- Directrix at $x = -p$
- Opens along x

- p is the focal length; distance from vertex to focus

91 Elliptical Foci

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- Wider than it is tall
- a is the larger horizontal radius
- b is the smaller vertical radius
- Focus at $(c, 0)$
- Vertices at $(a, 0), (-a, 0)$
- Transverse axis about x

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

- Taller than it is wide
- b is the smaller horizontal radius
- a is the larger vertical radius
- Focus at $(0, c)$
- Vertices at $(0, a), (0, -a)$
- Transverse axis about y

- $c = \pm\sqrt{a^2 - b^2}$

92 Hyperbolic Foci

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

- Intersects with x axis
- Asymptote/slope along $\pm\frac{b}{a}$
- Transverse axis about x
- Focus at $(c, 0)$

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

- Intersects with y axis
- Asymptote/slope along $\pm\frac{a}{b}$
- Transverse axis about y
- Focus at $(0, c)$

- a always associated with positive
- $c = \sqrt{a^2 + b^2}$

93 3D

93.1 Distance in 3D

$$d = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

93.2 Shapes

1. Ellipsoid – 3 ellipses; $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$
2. Paraboloid – 2 parabolas, 1 ellipse[2]; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$
3. Hyperboloid of one sheet[1] – 1 ellipse, 2 hyperbolas; $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
4. Hyerpboloid of two sheets[1] – 1 ellipse[2], 2 hyperbolas; $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
5. Elliptical Cone – 1 ellipse[2], 2 lines that cross; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$
6. Cylinder – Either 1 parabola, ellipse, or hyperbola.

[1] One sheet has differing hyperbolic transverse axis while two sheets has the same
[2] non-zero value will need to be plugged in

93.3 Finding Intercepts

For each of the three variables, plug in 0 for two and solve for the remaining.

94 Parametric Equations

$$x = f(t), y = g(t)$$

- Each t value gives a point on the curve: $(f(t), g(t))$.
- Circle goes clockwise if x is with sin or y with cos.

94.1 Solving Methods

1. Solve for t, plug in
2. Plot points

94.2 Parametrizing the Curve

Given two points – $(x_0, y_0), (x, y)$ – and a time interval, Δt

- $x(t) = x_0 + at$ where $a = \frac{\Delta x}{\Delta t}$
- $y(t) = y_0 + bt$ where $b = \frac{\Delta y}{\Delta t}$

94.3 Size Constraints

- $x = t$: Right half: [x-shift] $\leq t < \infty$
- $x = t$: Left half: $-\infty < t \leq$ [x-shift]
- $y = t$: Top half: [y-shift] $\leq t < \infty$
- $y = t$: Bottom half: $-\infty < t \leq$ [y-shift]

94.4 Slope of Parametric Curve

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

94.5 Tangent Lines to Parametric

Given point (x_0, y_0) and two parametric equations, $x(t), y(t)$

1. Find t by setting x or y equal to 0 in their respective equation
2. Find the slope equation by solving $\frac{dy}{dx}$
3. Plug in t-values into step 2 equation to find slope m
4. Plug in everything to the formula $y - y_0 = m(x - x_0)$

94.6 Second Derivative of Parametric Curve

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

94.7 Parametric Area

$$A = \int_a^b f(x)\ dx = \int_a^b y(t) \cdot \frac{dx}{dt} \cdot dx$$

94.8 Parametric Arc Length

$$s = \int_a^b \sqrt{x' + y'}\ dt$$

94.9 Parametric Surface Area

$$SA = \int_a^b 2\pi \cdot y(t)\sqrt{x' + y'}\ dt$$

95 Polar Coordinates

95.1 Converting

Polar to Cartesian

- $x = r \cdot \cos\theta$
- $y = r \cdot \sin\theta$

Cartesian to Polar

- $r^2 = x^2 + y^2$
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

95.2 Conics in Polar

$$r = \frac{l}{1 \pm e\cos\theta}$$

where l is the size factor and e is the eccentricity.

- $e = 0$: Circle
- $0 < e < 1$: Ellipse
- $e = 1$: Parabola
- $e > 1$: Hyperbola

95.3 Area

$$\int_0^{2\pi} \frac{1}{2} \cdot (r(\theta))^2 \; d\theta$$

96 Vectors

96.1 Scalar Multiple

Give $\vec{v}$ and scalar r;

$$r\vec{v} \; = < rx, ry > = \; r \cdot ||\vec{v}||$$

96.2 Unit Vector

Vector of magnitude 1, such as $\hat{i}, \hat{j}, \hat{k}$

96.3 Direction Vector

A unit vector in some given direction found by dividing a vector $\vec{v}$ by it's magnitude

96.4 Dot Product

$$< a, b, c > \bullet < d, e, f >= ad + be + cf$$

$$\vec{a} \bullet \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cdot \cos\theta$$

- Tells us how similar two vectors are

96.5 Cross Product

$$< a_1, a_2, a_3 > \times < b_1, b_2, b_3 > = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & b_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}_{\text{determinant}}$$

$$< a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 >$$

Note that the cross product is not commutative

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$

96.6 Projection

$$\vec{a} \text{ on } \vec{b}\ = \text{projection}_{\vec{b}}\vec{a} = \frac{\vec{a} \bullet \vec{b}}{||\vec{b}||} \cdot \frac{\vec{b}}{||\vec{b}||}$$