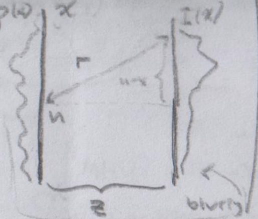
General LSI imaging system

$p(x)$ = unknown source image

$$I(x) = \int_{-\infty}^{\infty} p(u)\cdot \frac{1}{r(u,x;z)^{P}}\,du$$



$$w/\ r(u,x;z) = \left((u-x)^2+z^2\right)^{1/2}$$

$$I(x) = (p * h)(x) \quad w/\ h(x) = \left(x^2+z^2\right)^{-P/2}$$

FWHM: $\frac{1}{2}(z^2)^{-P/2} = (\hat{x}^2+z^2)^{-P/2}$

$$\hat{x} = \pm\sqrt{z^2(2^{2/P}-1)}$$

$$\Rightarrow \text{FWHM} = 2|\hat{x}| = 2\times z\sqrt{2^{2/P}-1}$$

Sampling:

$$F_n = \int_{[n-\frac{1}{2}]\Delta}^{[n+\frac{1}{2}]\Delta} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} f(x)\,\Pi\left(\frac{x-n\Delta}{\Delta}\right)dx$$

$u \quad \frac{n\Delta-u}{\Delta} \longleftarrow$ sub ; $\Pi(x)=\Pi(-x)$

$$= f(n\Delta) * \Pi\left(\frac{n\Delta}{\Delta}\right)$$

$$= \left[f(x) * \Pi\left(\frac{x}{\Delta}\right)\right]_{x=n\Delta} = \left(f(x) * \Pi\left(\frac{x}{\Delta}\right)\right)\cdot \delta(x-n\Delta)$$

$$f_s[x] = \sum_{n=-\infty}^{\infty} F_n$$

$$= \Big[\underbrace{f(x) * \Pi\left(\tfrac{x}{\Delta}\right)}_{\text{sampling avg area}}\Big]\times \underbrace{\sum \delta\left(\Delta\cdot\frac{x-n\Delta}{\Delta}\right)}_{\frac{1}{\Delta}\sum\delta\left(\frac{x-n\Delta}{\Delta}\right) \;=\; \text{Ш}(x/\Delta)}$$

$$F_s[k] = \left[F(k)\cdot\Delta\,\mathrm{sinc}(k\Delta)\right] * \frac{1}{\Delta}\,\text{Ш}(k\Delta)\,\Delta$$

$$= \Delta\sum_m F\left(k-\frac{m}{\Delta}\right)\mathrm{sinc}\left(\frac{k-m}{\Delta}\Delta\right)$$

↳ Copies every $1/\Delta$

No overlap if $2k_0 < 1/\Delta$

⟺ $F(k)$ bandlimit to $\pm k_0$

Recovery:

$$F(k) = F_s[k]\cdot\Pi(k\Delta)\,/\,\Delta\,\mathrm{sinc}(k\Delta)$$

Cascade N-LSI sys.

Single PSF: $g_n(x) = e^{-\pi(x/w_n)^2}$

overall spatial response is then

$$h(x) = g_1(x) * g_2(x) \cdots * g_N(x)$$

$$\Updownarrow$$

$$H(k) = \prod_{i=1}^{N}|w_i|\,e^{-\pi(kw_i)^2}$$

$$\cdots = \left[\prod^{N}|w_i|\right]\times\exp\left\{-\pi k^2\sum w_i^2\right\}$$

$\therefore$ FWHM $= \sqrt{\sum w_i^2}$

$\vdots \quad h(x) = \frac{w_1\cdots w_N}{\sqrt{w_1^2+\cdots w_N^2}}\,N\left(x/\sqrt{\sum w_i^2}\right)$

Constraints:

- Measured: Using phantom, we can approximate $\delta(x)$ and thus $H(k)$.
  - ↳ This informs us of the system-level contributions to blur (hardware, geometry, radiation, etc.) and thus how could improve our sys ⟹ images
  - ↳ Can also measure FOV, letting us determine spatial sampling rate
- Imposed: Depend on judgement
  - l_1/l_2 are a priori math constraints, rather than calibration or physically motivated constraints (e.g. Maxwell's eqn holds, $\mu \geq 0$ for x-ray, img $\in \mathbb{R}$)
  - Inherit real-world effects, i.e. magnetic field imperfections, may make these physical assumptions invalid
  - Leads to false credibility: MD may assume img determined 100% by data, when $\exists\,\infty$ solns and we chose one based on (arbitrary) weights chosen

FWHM Heuristic: $\Delta_x \leq \frac{1}{2}\text{FWHM}(h)$

Deconvolution

Additive noise: $m(x) = (f * h)(x) + n(x)$

$$\Updownarrow$$

$$M(k) = F(k)H(k) + N(k)$$

- ↳ Dividing both by $H(k)$ will lead to $N(k)/H(k)$ term which blows up for big $k \Rightarrow H(k)\to 0$
  - ⟹ salt & pepper noise
- ↳ Could crop when noise dominates, but this'll marginally improve SNR ($\approx 50\%$) but not resolution;
  - FWHM improved by high-freq., which'll narrow transfer fctn
  - But noise dominates high-freq, and ~100 SNR limits how wide we can crop
- ↳ Fundamentally ill-posed problem; Could infer high-k values using constraints, but this means the output is no long determined by just data measured (possibly introducing artifacts)

Estimating sampling rate for $h(x) = N\left(\frac{x}{w}\right)$

Want SNR ≈ 100 ⟹ For what $\hat{x}$ do we go below $\frac{1}{100}$ of peak?

$$H(\hat{k}) = w\exp[-\pi(w\hat{k})^2] = 1/100$$

$$\Rightarrow \hat{k} = (\ln 100/\pi w^2)^{1/2}$$

$$\Updownarrow$$

$\hat{x} = w\,(\pi/\ln 100)^{1/2}$ } BW 'til noise

$\therefore$ Nyquist $= 2\hat{x} \approx 0.4\,w$

WTF # samples ⟺ what n s.t. $h(n\Delta_x) \geq \frac{1}{100}h(0)$

$$\exp[-\pi(n\cdot 0.4w/w)^2] \geq 1/100$$

$$\Longrightarrow n \leq \left(\ln\tfrac{1}{100}/-\pi\cdot 0.16\right)^{1/2} \approx 2.95$$

i.e. $n=2$ is last visible pt

for $N = \{\pm 2, \pm 1, 0\}$ five pts in all

WTS $\delta\left(\frac{x}{a}\right) = |a|\,\delta(x)$

Intuit: scale width effectively scales height

Sampling $f(\cdot)$ with $\delta\left(\frac{x}{a}\right)$:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x/a)\,dx \longleftarrow y = x/a$$

$$= \int_{-\infty}^{\infty} f(ay)\,\delta(y)\times a\,dy \qquad dx = a\,dy$$

$$= af(ay)\big|_{y=0} = aF(0)$$

$$= a\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

IF $a<0$ then the integration limits are reversed, so $\delta\left(\frac{x}{a}\right) = -a\,\delta(x)$ □

Deblurring: IF transfer fctn has wide tail, we can high-pass filter which has no gain at high freqs

Ex $h(x) = N\left(\frac{x}{1}\right) + \frac{1}{4}N\left(\frac{x}{4}\right)$

$$\Updownarrow$$

$$H(k) = N(k) + N(4k)$$

Desire $\hat{H}(k) = N(k) \therefore H_{eq}(k) = \frac{N(k)}{N(k)+N(4k)}$

No noise increase:

$$\lim_{k\to\infty} H_{eq} = \lim \frac{e^{-\pi k^2}}{e^{-\pi k^2}+e^{-\pi(4k)^2}} = 1\cdot\frac{1}{1+0} = 1$$

↳ Only suppress low-freq; don't risk amplifying noise, degrading SNR

$$\frac{e^{-\pi k^2}}{e^{-\pi k^2}}\cdot\frac{1}{(1+e^{-\pi 15k^2})}$$

↳ Wispy gaus. leads to blurring, so we attenuate only low freqs to reshape, concentrating PSF towards middle where relevant info lies

Undersample: k-space bw = FOV_x

Sampling in k-space at interval Δk

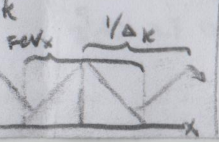
Alias region $= FOV_A = FOV_x - \frac{1}{\Delta k}$

Non-a " $= FOV_{NA} = \frac{1}{\Delta k} - FOV_A$

Nyq: $FOV_x = 1/\Delta k \quad = \frac{2}{\Delta k} - FOV_x$

Nyquist Sampling: $\Delta_{kx} = \frac{1}{FOV_x}$

$N_x = \frac{k_0}{\Delta_{kx}}$ = # samples along x



where k_0 = resolution

FOV_x = bw of signal

Ex 1x1mm res over 10x10cm FOV

$$\Delta_{kx} = \frac{1}{10\,cm}\cdot\frac{1\,cm}{10\,mm} = \frac{1}{100}\,mm^{-1}$$

$$N_x = \frac{k_0}{\Delta_{kx}} = \frac{1\,mm^{-1}}{1/100\,mm^{-1}} = 100$$

$\therefore$ total samples $= 10{,}000$ } inside $k_0\times k_0$ box in k-space

Ex Sampling $M(k) = H\cdot F(k) + N(k)$ in k-space

$F(x)$ bw $L \Longrightarrow \Delta_k = 1/L$ (really $\Delta_k \leq 1/L$)

$F(k)$ bw K_1

$H(k)$ bw $K_0 \Longrightarrow N = \frac{k_0}{\Delta_k} = L\,k_0$

LSI | IF sys LSI, they can only blur input image

1] Lin.: $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \alpha g_1 + \beta g_2$

2] SI: $\mathcal{L}\{f(x)\} = g(x) \Leftrightarrow \mathcal{L}\{f(x-\tau)\} = g(x-\tau)$

Ex | $\mathcal{L}\{\cdot\} = f(x)^2$

L: $\mathcal{L}\{\alpha f_1 + \beta f_2\} = (\alpha f_1 + \beta f_2)^2$

SI: trivial $= \alpha^2 f_1^2 + \beta^2 f_2^2 + 2\alpha\beta f_1 f_2 \neq \alpha^2 f_1^2 + \beta^2 f_2^2$

Ex | $\mathcal{L}\{\cdot\} = F(2x)$ - shrink by 2

SI: $\mathcal{L}\{F(x-\tau)\} = F(2(x-\tau))$

$g(x-\tau) = F(2x-\tau)$ $\neq$

Ex | $\mathcal{L}\{\cdot\} = f'(x)$

LI: $\mathcal{L}\{\alpha f_1 + \beta f_2\} = \frac{d}{dx}[\alpha f_1(x) + \beta f_2(x)] = \alpha f_1'(x) + \beta f_2'(x)$ □

SI: $\mathcal{L}\{f(x-\tau)\} = \frac{d}{dx} f(x-\tau)$, $\tilde{x} = x - \tau$, $d\tilde{x} = dx$

$= \frac{d}{d\tilde{x}} F(\tilde{x})$

$g(x-\tau) = F'(x-\tau)$

Ex | Let input $= u(x)$, output $= F(x)$

IF LSI sys. is invertible, it's inverse must also be LSI (swap in & out)

Ex | $\frac{dF(x)}{dx} + bf(x) = u(x)$

L: $\mathcal{L}\{\alpha_1 f_1 + \alpha_2 f_2\}$ w/ $u_i = \frac{df_i}{dx} + bf_i$

$= \frac{d}{dx}[\alpha_1 f_1 + \alpha_2 f_2] + b[\;''\;]$

$= \underbrace{\alpha_1\left(\frac{d}{dx}f_1 + bf_1\right)}_{\mathcal{L}\{f_1\} = u_1} + \underbrace{\alpha_2\left(\frac{d}{dx}f_2 + bf_2\right)}_{\mathcal{L}\{f_2\} = u_2}$ □

SI: $\mathcal{L}\{h(x)\} = \frac{d}{dx}h(x) + bh(x)$ for $h(x) = f(x-\tau)$

$= \frac{d}{dx}f(x-\tau) + bf(x-\tau)$

$= u(x-\tau) := \mathcal{L}\{f(x)\}$

Ex | $\frac{dF}{dx} = -b\,u(x)\,F(x) \equiv u(x) = \frac{dF}{dx} / -bF(x)$

L: $\mathcal{L}\{\alpha_1 f_1 + \alpha_2 f_2\} = \frac{d}{dx}[\alpha_1 f_1 + \alpha_2 f_2] / -b(\alpha_1 f_1 + \alpha_2 f_2)$

$\neq \alpha_1 \mathcal{L}\{F_1(x)\} + \alpha_2 \mathcal{L}\{F_2\}$

SI: $\mathcal{L}\{\overbrace{F(x+\tau)}^{\text{let } h(x)}\} = \frac{d}{dx}[h(x)] / -bh(x)$

$= \frac{d}{dx}F(x-\tau) / -bf(x-\tau)$

$= u(x-\tau)$ □

$\mathcal{L}_{LSI}\{F(x)\} = \int F(u)\,h(x-u)\,du =: (f * h)(x)$

$= \lim_{\Delta \to 0} \sum_n f(n\Delta)\, \underbrace{\mathcal{L}_{LSI}\left\{\frac{1}{\Delta}\Pi\left(\frac{x-n\Delta}{\Delta}\right)\right\}}_{=:h(x)} \Delta$

const. area

Shift: IF $g(x) = (F_1 * F_2)(x)$

then $g(x-\tau-\gamma) = F_1(x-\tau) * F_2(x-\gamma)$

as $'' = \int F_1(\underbrace{u-\tau}) F_2(\underbrace{(x-\gamma)-u})\,du$

Let $\alpha \Rightarrow x-\gamma-(\alpha+\tau)$

$'' = \int F_1(\alpha)\, F_2((x-\tau-\gamma)-\alpha)\,d\alpha$ □

Scale: $|b|\,K\left(\frac{x}{b}\right) = (F * g)\left(\frac{x}{b}\right)$

as $'' = \int F(u/b)\, g\left(\frac{1}{b}(x-u)\right) du$

Let $l \Rightarrow \frac{x}{b} - l$; $dl = \begin{cases} +du/b : b > 0 \\ -\;'' : o/w \end{cases}$

$'' = \underbrace{\int F(l)\, g\left(\frac{x}{b} - l\right)|b|\,dl}_{K(x/b)}$ □

Area: $\int (F * g)(x)\,dx = \int\left[\int F(u)\,g(x-u)\,du\right]dx$

$\xrightarrow{\text{swap}} = \int\left[F(u)\int g(x-u)\,dx\right]du$

$\cdots = \int F(u) \int g(l)\,dl\,du = \int F(u)\,du \times \int g(l)\,dl$

Fourier Properties:

Defn | $F(x) = \int_{-\infty}^{\infty} F(k)\, e^{j2\pi kx}\,dk$

$F(k) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi kx}\,dx$

Shift: $f(x-\tau) \leftrightarrow e^{-j2\pi k\tau} F(k)$

Scale: $f(x/a) \leftrightarrow |a|\,F(k)$

$\frac{1}{w}\Pi\left(\frac{x}{w}\right) \longleftrightarrow \mathrm{sinc}(wk)$ | $Ш(x) \leftrightarrow Ш(k)$

$\exp\left(-\pi\left(\frac{x}{w}\right)^2\right) \longleftrightarrow |w|\exp\left(-\pi(kw)^2\right)$ $\equiv \frac{1}{w}N\left(\frac{x}{w}\right) \updownarrow N(kw)$

$\delta(x-\tau) \longleftrightarrow 1 \cdot \exp(-j2\pi\tau k)$

$\cos 2\pi\gamma x \longleftrightarrow \frac{1}{2}[\delta(k-\gamma) + \delta(k+\gamma)]$

$\sin 2\pi\gamma x \longleftrightarrow \frac{1}{2}[\delta(k-\gamma) - \delta(k+\gamma)]$

Alt | Let $y = f(x)$, for $\frac{dy}{dx} = -b\,u(x)\,y$

LI: $\mathcal{L}\{h(x)\}$

$= \exp\left(-b\int[\alpha_1 u_1 + \alpha_2 u_2]\,dx + c\right)$

$= \exp(\alpha_1)\,\mathcal{L}\{u_1\} + \exp(\alpha_2)\,\mathcal{L}\{u_2\}$

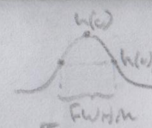
$\neq \alpha_1 F_1(x) + \alpha_2 F_2(x)$

$\int y'\,dy = \int -b\,u(x)\,dx$

$\ln(y) + c = \;''$

$\therefore\; y = \exp\left(-b\int u(x)\,dx + c\right)$

Rayleigh/Houston Criteria:



Two sources are resolvable iff they're separated by more than FWHM; o/w they blur together, are indistinguishable

↳ Heuristic of "real-world" delta Fctn should be at-most half width of FWHM

Ex | Expected resolution 200 μm → ≥100 μm-wide phantom

IF we detect $I(x) = \exp[-\mu x]$ then desire $1/\mu \approx 200\,\mu m$

Oversampling (beyond Nyquist) gains no novel information; blurring occurs before sensors - cannot recover what's already lost due to falloff

Gaus. FWHM: $\frac{1}{2} = \exp\left(-\pi\left(\frac{\tilde{x}}{w}\right)^2\right) =: N\left(\frac{x}{w}\right)$

$\Rightarrow -\ln 2 = -\pi\left(\frac{\tilde{x}}{w}\right)^2 \Rightarrow \tilde{x} = W\sqrt{\ln 2/\pi}$

$\therefore$ FWHM $= 2|\tilde{x}| \approx 0.94\,W \approx W$

Delta Properties: $\delta(x) * h(x) = h(x)$

$\delta(x-\tau) * h(x) = h(x-\tau)$

$\int \delta(x-\tau)\,f(x)\,dx = f(\tau)$

$\delta(x) = \lim_{w\to 0} \frac{1}{w}\Pi\left(\frac{x}{w}\right) = \lim_{w \to 0} \frac{1}{w}N\left(\frac{x}{w}\right)$

$\mathcal{F}\{\delta(x)\} = \int \delta(x)\, e^{-j2\pi kx}\,dx = e^{-j2\pi k\cdot 0} = 1$

$Ш\left(\frac{x-\tau}{\Delta}\right) = |\Delta| \sum_n \delta(x-\tau-n\Delta) = \sum_n \delta((x-\tau-n\Delta)/\Delta)$

Let $s(x) = Ш\left(\frac{x}{\Delta}\right) \leftrightarrow S(k) = \Delta\, Ш(k\Delta) = \sum_m \delta\left(k - \frac{m}{\Delta}\right)$

PF) $s(x) := \sum_n \delta\Big(\underbrace{\frac{x}{\Delta} - n}_{\frac{1}{\Delta}(x-n\Delta)}\Big)$

$= \sum \Delta\,\delta(x-n\Delta)$

$\updownarrow$

$S(k) = \Delta \sum_m \underbrace{\int \delta(x-m\Delta)\, e^{-j2\pi kx}\,dx}_{\exp(-j2\pi k(m\Delta))}$

↳ $S(k)$ periodic w/ $1/\Delta$

↳ $S(0) \to \infty$, $S(k) \to 0$ for $k \neq \frac{p}{\Delta}$

- Argument to Fctns must be dimensionless; ex. rect(x) is actually…

$\Pi(x/1\text{mm}) = \begin{cases} 1; & |x/1\text{mm}| < 1/2 \\ 0, & o/w \end{cases}$

Discretization:

$(f * h)(x) = \sum_n \underbrace{f(n\Delta x)\,\Delta x}_{\text{mass of } n^{th}}\; \underbrace{h(x - n\Delta x)}_{\text{mold}}$

Ctns int	disc. sum
u	$n\Delta x$
du	Δx
$\int_a^b$	$\sum_{n=a/\Delta x}^{b/\Delta x}$

Units: $H(k)$ is ratio, unitless (V_{in}/V_{out})

$h(x) := 1/\text{mm} =: \delta(x)$; $V(k) = V \cdot \text{mm}$

Solving FWHM: $\frac{1}{2}h(x_{max}) = h(x_{FWHM}/2)$

Sum of Gaus: Let $h_i = \frac{1}{w_i}N\left(\frac{x}{w_i}\right)$

$H(k) = H_1 \times H_2 = \exp\left(-\pi k^2(w_1^2 + w_2^2)\right)$

$\updownarrow$

$h(x) = \frac{1}{\sqrt{w_1^2 + w_2^2}} N\left(\frac{x}{\sqrt{w_1^2 + w_2^2}}\right)$

$\therefore$ FWHM $= \sqrt{w_1^2 + w_2^2}$

↳ Dominated by blurriest sys. (narrowest in k-space)

Averaging improves SNR $\propto \sqrt{N}$

Ex | Want to downscale $1 \times 1\,\text{mm} \to \frac{1}{4} \times \frac{1}{4}\,\text{mm}$ while maintaining SNR

SNR $\propto$ Area, so decreases by 16

Boost SNR by avging $16^2 = 256$ samples

IF we can reduce # samples read, then we could increase the sampling rate; this would tackle aliasing in frequency

↳ This is done in X-ray CT, ignoring higher freqs and reducing radiation exposure

↳ Broadest structures in low freqs; aliasing of noise in x-space will be mostly high freqs