Complex | Phasors

$e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$z := x + jy = |z| e^{j\theta}$

$\to |z| = \sqrt{x^2+y^2} = z\cdot\bar{z}$

$\to \theta = atan2(y, x)$

$\bar{z} = x - jy = |z| e^{-j\theta}$

$\overline{\alpha e^{j\theta}} = \bar{\alpha} e^{-j\theta}$

$z^{\frac{1}{2}} = \pm|z|^{\frac{1}{2}} e^{j\theta/2}$

$z^n = |z|^n e^{jn\theta}$

$\measuredangle z^k = k\measuredangle z \, ; \, \measuredangle kz = \measuredangle z$

$\measuredangle z_1/z_2 = \measuredangle z_1 - \measuredangle z_2$

$j = -\frac{1}{j}$

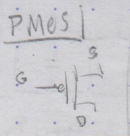
MOSFETS |

V_{in}	V_{out}	NM-	PMOS
V_{DD}	0	on	off
0	V_{DD}	off	on

Ⓖ Ⓓ — On = closed/short

$z + \bar{z} = Re\{z\}$

PMOS |

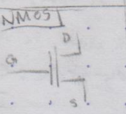


G • S

D

$\underbrace{V_{GS} \le -|V_{tp}|}_{\text{closed/short condition}}$

NMOS |

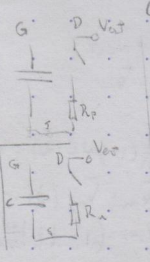


G • D

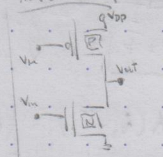
S

$V_{GS} \ge V_{tn}$

$V_c = V_{GS}$

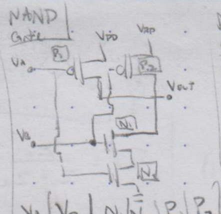


CMOS Inverter |



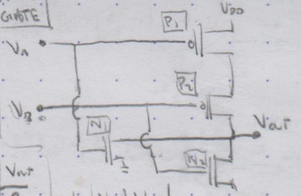
V_{in}	V_{out}	NM	PM
V_{DD}	0	on	off
0	V_{DD}	off	on

NAND Gate |



V_A	V_B	N_1	N_2	P_1	P_2	V_{out}
V_{DD}	V_{DD}			off	off	0
0	V_{DD}					V_{DD}
V_{DD}	0					V_{DD}
0	0					V_{DD}

NOR GATE |



V_A	V_B	V_{out}
0	0	V_{DD}
V_{DD}	0	0
0	V_{DD}	0

$m = 10^{-3}$	K
$\mu = 10^{-6}$	M
$n = 10^{-9}$	G
$p = 10^{-12}$	

DEs | Homo |

$\frac{d}{dt}x(t) = \lambda x(t)$

$x(t) = x_0 e^{\lambda t}$

Non-Homo |

$\frac{d}{dt}x(t) = \lambda x(t) + \alpha \quad \leftarrow z := x + \frac{\alpha}{\lambda}$

$x(t) = x_0 e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$

NHomo + Nonconst Input |

$\frac{d}{dt}x = \lambda x(t) + bu(t)$

$x = x_0 e^{\lambda(t-t_0)} + be^{\lambda t}\int_{t_0}^{t} e^{-\lambda\theta} u(\theta)\, d\theta$

Vector |

$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t)$

$\vec{x}(t) = \underbrace{e^{At}}_{diag(e^{a_{ii}t})}\vec{x}_0 + \int_0^t e^{A(t-\theta)} B\vec{u}(\theta)\, d\theta$

$\vec{x}(t) = V\underbrace{e^{\Lambda t}}_{diag(e^{\lambda_{ii}t})}V^{-1}\vec{x}_0 + \int_0^t Ve^{\Lambda(t-\theta)}V^{-1}B\vec{u}(\theta)\, d\theta$

Discrete |

$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$

$\vec{x}[i] = A^i\vec{x}[0] + \left(\sum_{j=0}^{i-1} u[i]A^{i-1-j}\right)\vec{b}$

Discrete Matrix |

$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$

$\vec{x}[i] = A^i\vec{x}_0 + \sum_{k=0}^{i-1} A^{i-k-1} B\vec{u}[k]$

Steps to Solve

a.) set up diff eq: $\frac{d}{dt}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$

1.) Find eigens: $\det(\lambda I - A) \Rightarrow (\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$

$V := [\vec{v}_1, \vec{v}_2]; \; \Lambda := diag(\lambda_1, \lambda_2)$

2) COB: $\tilde{\vec{x}} := V^{-1}\vec{x} \Rightarrow \frac{d}{dt}\tilde{\vec{x}} = \underbrace{V^{-1}AV}_{\Lambda}\tilde{\vec{x}}(t) \; ; \; \tilde{\vec{x}}_0 = V^{-1}\vec{x}_0$

$+ \underbrace{V^{-1}\vec{b}\, u(t)}$ $\quad \tilde{\vec{u}}_0 = V^{-1}\vec{b}\, u(0)$

3) Decompose & Solve: $\frac{d}{dt}\tilde{x}_i = \lambda_i \tilde{x}_i(t)$

$\Longrightarrow \tilde{x}_i(t) = \left(\tilde{x}_{0i} + \frac{\tilde{u}_{0i}}{\lambda_i}\right)e^{\lambda_i t} - \frac{\tilde{u}_{0i}}{\lambda_i}$

4) Original Basis: $\vec{x} = V\tilde{\vec{x}}$

Inverse | 2x2:

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Mean-Product |

$\left.\begin{matrix} m = \sum^n diag/n \\ p = \det(M) \end{matrix}\right\} m \pm \sqrt{m^2 - p}$

$$\begin{tikzcd} \frac{d}{dt}\vec{x}(t) = A\vec{x}(t) \arrow[d, "\tilde{\vec{x}} = V^{-1}\vec{x}"] & \vec{x}(t) = \ldots \\ \frac{d}{dt}\tilde{\vec{x}} = \tilde{A}\tilde{\vec{x}} \arrow[r] & \tilde{\vec{x}} = \ldots \arrow[u, "\vec{x} = V\tilde{\vec{x}}"'] \end{tikzcd}$$

$\tilde{A} = V^{-1}AV = \Lambda = diag(\lambda)$

Capacitors | Series: $(C_1^{-1} + C_2^{-1})$

$I(t) = C\cdot\frac{dV(t)}{dt}$

$Z_C = 1/j\omega C = (\omega C)^{-1} e^{-j\frac{\pi}{2}}$

Inductor | Series: $L_1 + L_2$

$= I_0 L e^{\omega t}$

$V_L = L\cdot\frac{dI(t)}{dt}$

$Z_L = j\omega L = \omega L e^{j\frac{\pi}{2}}$

Phasors |

$\omega = 2\pi f \left[\frac{rad}{s}\right]$

$f = T^{-1}$ [periods]

$\sin\theta = \cos(\theta - \frac{\pi}{2})$

$\cos\theta = \sin(\theta + \frac{\pi}{2})$

Second-Order DE |

$\frac{d^2}{dt^2}x(t) + a\cdot\frac{d}{dt}x(t) + bx(t) = 0$

$\lambda^2 + a\lambda + b = 0 \Rightarrow -\frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 - 4b}$

Generally, define state vector:

$A\begin{bmatrix} x(t) \\ \frac{d}{dt}x(t) \end{bmatrix} + \ldots$ and solve DE $= \begin{bmatrix} \frac{d^2}{dt^2} \\ \frac{d}{dt} \end{bmatrix} x(t)$

Complex Eigs |

1.) All real: Time response is a decaying exp, no osc.

2) All complex: Response is sinusoidal osc. w/ no decay

3) Both: Osc. + decays exp.

1.) Converting to Phasor Domain

$v(t) = V_0\cos(\omega t + \phi) = Re(V_0 e^{j\phi_v} e^{j\omega t}) \Rightarrow \tilde{V} = \underbrace{\frac{V_0}{2}}_{|\tilde{V}|} \underbrace{e^{j\phi_v}}_{\measuredangle\tilde{V}}$

$i(t) = I_0\cos(\omega t + \phi) = Re(I_0 e^{j\phi_i} e^{j\omega t}) \Rightarrow \tilde{I} = \frac{I_0}{2} e^{j\phi_i}$

2.) Compute impedance: $Z = \frac{V}{I}$

3.) Solve: Use KCL/KVL/NVA (+ convert back w/ 1)

Stability |

- Discrete: $|\lambda| < 1$
- Continuous: $Re\{\lambda\} < 0$

$|a+b| \le |a| + |b|$

$|\int f_x dx| \le \int |f_x| dx$

$a + ar + ar^2 \to \frac{a}{1-r}$ iff $|r| < 1$

Feedback |

$u := \vec{f}^T\vec{x}(t) \Longrightarrow \vec{x}[i+1] = \underbrace{(A + \vec{b}\vec{f}^T)}_{A_{CL}}\vec{x}[i]$

Controllability |

Let $A := diag(\lambda_n), \; B = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix};$

$A_d := diag(e^{\lambda_n t}), \; B_d = diag(\int_0^t e^{\lambda s})$

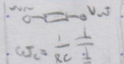
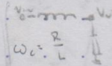
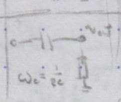
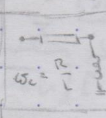
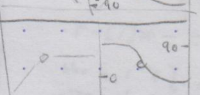
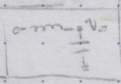
$\vec{z} = V^{-1}\vec{x}$

$\vec{z}_d[i+1] = A_d\vec{z}_d[i] + B_dV^{-1}Bu_d[i]$

$\vec{x}_d[i+1] = VA_dV^{-1}\vec{x}[i] + VD_dV^{-1}Bu_d[i]$

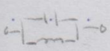
Transfer Functions |

$H(j\omega) = \frac{V_{out}}{V_{in}} = |H(j\omega)| e^{j\measuredangle H(j\omega)}$

LP	 $\omega_c = \frac{1}{RC}$	 $\omega_c = \frac{R}{L}$	$H = \frac{1}{1 + j\omega/\omega_c}$	
HP	 $\omega_c = \frac{1}{RC}$	 $\omega_c = \frac{R}{L}$	$H = \frac{j\omega/\omega_c}{1 + j\omega/\omega_c}$	
LC			$H(j\omega) = \frac{1}{1-\omega^2LC}$	$Z = \frac{Z_{bottom}}{Z_{tot}}$; $\lvert H(j\omega)\rvert = \frac{1}{\sqrt{2}}\lvert H(\omega_c)_{max}\rvert$

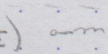
Type	$\omega < \frac{\omega_c}{10}$	$\omega = \frac{\omega}{10}$	$\omega = \omega_c$	$\omega = 10\omega_c$	$\omega > 10\omega_c$	
LP	0	-5.7	-45	-84.3	-90	$\left(1 + \frac{\omega^2}{\omega_c^2}\right)^{-\frac{1}{2}}$; $-atan2(\frac{\omega}{\omega_c}, 1)$
HP	90	84.3	45	5.7	0	$\left(1 + \frac{\omega_c^2}{\omega^2}\right)^{-\frac{1}{2}}$; $-atan2(\frac{-\omega_c}{\omega}, 1)$

$Z_L \parallel Z_C = \frac{j\omega L \cdot \frac{1}{j\omega C}}{j\omega L + (j\omega C)^{-1}} = \frac{j\omega L}{(j\omega L \cdot j\omega C) + 1} = j\frac{\omega L}{1 - \omega^2 LC}$



@ $\omega_s = \frac{1}{\sqrt{LC}}$; $H = \infty$

$Z_L + Z_C = j\left(\omega L - \frac{1}{\omega C}\right)$



@ ω_s : $H = 0$

Controllability | $\mathcal{C} \in \mathbb{R}^{n\times(i\cdot m)}$

$C_i = [A^{i-1}B \cdots AB \; B]$

Must be rank n

$x[i] = A^i x[0] + C\begin{bmatrix} \vec{u}[0] \\ \vdots \\ \vec{u}[i-1] \end{bmatrix}$ unpacks

$\vec{x}^*$ reachable in i^* from $\vec{x}_0$

$\Longleftrightarrow \vec{x}^* - A^{i^*}\vec{x}_0 \in Col(C_{i^*}) = C_{i^*}\vec{\omega}$

$A := \begin{bmatrix} \vec{0}_{n-1} & I_{n-1} \\ & -\vec{\alpha}^T & \end{bmatrix} \in \mathbb{R}^{n\times n}$

$\vec{b} := \begin{bmatrix} \vec{0}_{n-1} \\ 1 \end{bmatrix} \in \mathbb{R}^n \quad P_A = \lambda^n + \sum_{i=1}^{n} a_i\lambda^{i-1}$

Discretization |

$\frac{d}{dt}x(t) = \lambda x(t) + bu(t)$

$\Rightarrow x_d[i+1] = \underbrace{e^{\lambda\Delta}}_{\lambda_d} x_d[i] + \underbrace{b\left(\frac{e^{\lambda\Delta} - 1}{\lambda}\right)}_{b_d} u_d[i]$

System ID |

$\vec{y} \approx D\vec{p}\,(+\vec{e})$

$\vec{y} \in \mathbb{R}^l$; measurements

$D \in \mathbb{R}^{l\times n}$

$\vec{p}$; unknown coefs $\to [a_1, a_2 \cdots b_1 \cdots b_n]$

Scalar:

$x[i+1] = \lambda x[i] + bu[i] + e[i]$

$\underbrace{\begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[l-1] & u[l-1] \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} e[0] \\ \vdots \\ e[l-1] \end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[l] \end{bmatrix}}_{\vec{y}}$

$\hat{\vec{p}} := (D^TD)^{-1}D^T\vec{y}$

Vector |

$\vec{x}[i]^T = \vec{x}[i-1]^TA^T + \vec{u}[i-1]^TB^T + e[l-1]^T$

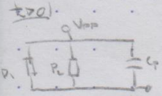
$\begin{bmatrix} \vec{x}[0]^T & u[0]^T \\ \vdots & \vdots \\ \vec{x}[l-1]^T & u[l-1]^T \end{bmatrix}\underbrace{\begin{bmatrix} A^T \\ B^T \end{bmatrix}}_{[\vec{p}_1 \cdots \vec{p}_n]} + \underbrace{\begin{bmatrix} \vec{e}[0]^T \\ \vdots \\ e[l-1]^T \end{bmatrix}}_{[\vec{e}_1 \cdots \vec{e}_n]} = \underbrace{\begin{bmatrix} \vec{x}[1] \\ \vdots \\ \vec{x}[l] \end{bmatrix}}_{[\vec{y}_1 \cdots \vec{y}_n]}$

NAND |

t < 0 : Hi

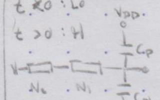
t > 0 : Lo

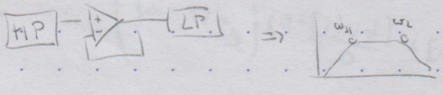
t > 0 |



t < 0 : Lo

t > 0 : H





$H(j\omega) = \frac{R}{(j\omega C)^{-1} + R + j\omega L}$

$R_p = R_{p1} = R_{p2}$

$C_{N,inv}\frac{dV_{out}}{dt} + C_{P,inv}\frac{d(V_{out} - V_{DD})}{dt} + \frac{2}{R_p}(V_{out} - V_{DD}) = 0$

$\Rightarrow \frac{dV_{out}(t)}{dt} = \underbrace{-2(R_p(C_p + C_N))^{-1}}_{\lambda}V_{out} + \underbrace{2(R_p(C_p + C_N))^{-1}}_{b}V_{DD}$

$V_{out} = V_{out}(0)e^{\lambda t} + b\int_0^t V_{DD}e^{\lambda(t-\tau)}d\tau$

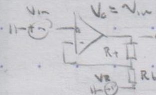
$= V_{out}(0)e^{\lambda t} + \frac{b}{\lambda}V_{DD}(e^{\lambda t} - 1)$

$ax^2 + bx + c = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

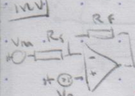
Non-inv

$V_o = V_{in}\left(1+\frac{R_f}{R_b}\right) - V_R\left(\frac{R_f}{R_b}\right)$



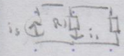
Inv

$V_o = V_{in}\left(-\frac{R_f}{R_s}\right) + V_R\left(\frac{R_f}{R_s}+1\right)$



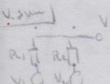
cur. div

$i_1 = i_s \cdot \frac{R_2}{R_1+R_2}$



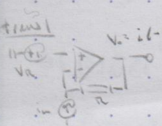
V. sum

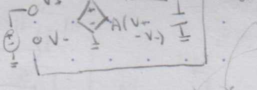
$V_{out} = V_1\left(\frac{R_2}{R_1+R_2}\right) + V_2\left(\frac{R_1}{R_1+R_2}\right)$



trans

$V_o = i(-R) + V_R$





$\tilde{V}_- = \tilde{V}_{out}$

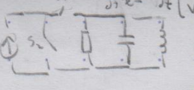
$\tilde{V}_+ = \tilde{V}_{in}$

$\tilde{V}_{out} = \frac{Z_c}{Z_c + Z_R} A(\tilde{V}_+ + \tilde{V}_-) = \frac{1}{A}(1 + j\omega C \cdot R) = \tilde{V}_{in} - \tilde{V}_o$

$\Rightarrow \tilde{V}_{in} = \tilde{V}_{out}\left(\frac{1}{A}(1 + j\omega RC + 1)\right)$

$\therefore H(j\omega) = \left(1 + \frac{1}{A} + \frac{j\omega}{\omega_c A}\right)^{-1};\ \omega_c = \frac{1}{RC}$

RLC



$\frac{d}{dt}\vec{x} = \frac{d}{dt}\begin{bmatrix} I_L(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} 0 & L^{-1} \\ -C^{-1} & (RC)^{-1} \end{bmatrix}\begin{bmatrix} I_L \\ V_c \end{bmatrix}$

$V_c = V_L = L\frac{dI}{dt} \Rightarrow \frac{dI}{dt} = \frac{1}{L}V_c$

$t = 0$: S_1 open, S_2 closed

KCL: $i_R + i_C + i_L = 0$

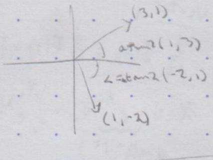
$\frac{V}{R} + C\frac{dV}{dt} + I_L = 0 \Rightarrow \frac{dV}{dt} = \frac{-V}{RC} - \frac{1}{C}I_L$

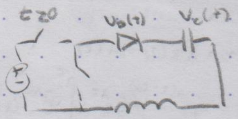
$t < 0$: S_1 closed, S_2 open

Steady-state: Capacitors: Open $\rightarrow V_c(0) = 0V$

Inductors: Short $\rightarrow I_L(0) = 1A = I_s$

Example Complex





Let $\vec{x} = \begin{bmatrix} v_c \\ i_L \end{bmatrix}(t)$

$\frac{d}{dt}\vec{x}(t) = \vec{f}(v_c, i_L, u)$

$i_D \approx e^{V_D} - 1$

KCL: $C\frac{dV_c}{dt} = i_L \Rightarrow \frac{d}{dt}v_c = \frac{1}{C}i_L(t) = f_1$

KVL: $u(t) = V_D + V_c + L\frac{dI_L}{dt}$

$\Rightarrow u(t) = \ln(i_L(t)+1) + v_c(t) + L\frac{di_L}{dt}$

$\Rightarrow \frac{d}{dt}i_L = -\frac{1}{L}v_c(t) - \frac{1}{L}\ln(i_L+1) + \frac{1}{L}u(t) = f_2$

$\frac{d}{dt}y(t) = \frac{1}{L}\ln(y(t)+\alpha) + \frac{1}{L}u(t)$

$y^* = 0,\ u^* = 1V$

$f(*) + \frac{\partial f}{\partial y}\Big|_*(y - y^*) + \frac{\partial f}{\partial u}\Big|_*(u - u^*)$

$= \frac{1}{L}\ln(0+\alpha) + \frac{1}{L}\cdot 1 + \left(\frac{1}{y(t)+\alpha}\right)\Big|_*(y(t) - 0)$

$+ \frac{1}{L}(u(t) - 1)$

$\Rightarrow \frac{dy(t)}{dt} \approx \frac{1}{\alpha L}y(t) + \frac{1}{L}u(t) + \frac{1}{L}\ln(\alpha)$

Upper Diag

$A = VTV^T$

$\frac{d}{dt}x(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}}\vec{x}(t) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}u(t)$

Feedback

$\lambda = -2, -2,\ v_\lambda = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

or e_2

$\vec{q}_1 = \vec{e}_1 - \langle \vec{v}_1, \vec{e}_1 \rangle \vec{v}_1$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} - [1\ 0]\begin{bmatrix} \ \end{bmatrix}\begin{bmatrix} \ \end{bmatrix}$

$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{5}}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 4/5 \\ 2/5 \end{bmatrix}$

$\vec{p}_1 = \frac{\vec{q}_1}{\|\vec{q}_1\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore V = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$

$T = V^TAV$

$= \begin{bmatrix} v_1^TAv_1 & v_1^TAp_1 \\ p_1^TAv_1 & p_1^TAp_1 \end{bmatrix}$

$= \begin{bmatrix} \lambda_1 v_1^Tv_1 & v_1^TAp_1 \\ \lambda_1 p_1^Tv_1 & p_1^TAp_1 \end{bmatrix}$

$= \begin{bmatrix} -2 & -5 \\ 0 & -2 \end{bmatrix}$ □

Set $\{\lambda\} = \{-4, -5\} \Rightarrow (\lambda+4)(\lambda+5) = \lambda^2 + \lambda 9 + 20$

w/ $\vec{u} = \vec{K}^T\vec{x}$:

$\frac{d}{dt}\delta x(t) = (A + b\vec{K}^T)\delta\vec{x}(t)$

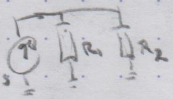
$= \begin{bmatrix} 0 & 1 \\ -4+4K_1 & -4+4K_2 \end{bmatrix}\delta x(t)$

$\lambda^2 + \overbrace{(4-4K_2)}^{9}\lambda + \overbrace{(4-4K_1)}^{20}$ $\therefore K_1 = -4$

$\therefore K_2 = -\frac{5}{4}$

$\det(A) > 0 = 4 - 4K_1 \Rightarrow 1 > K_1$

$\mathrm{tr}(A) < 0 = -4 - 4K_2\quad -1 < K_2$



$P = i_1^2R_1 + i_2^2R_2 = i_1^2R_1 + (i_s - i_1)^2R_2$

$\Rightarrow \frac{\partial P}{\partial i_1} = 2i_1R_1 - 2(i_s - i_1)R_2 = 0$

$\therefore i_1 = \frac{R_2}{R_2+R_1}i_s$

$i_2 = i_s - i_1 = i_s -$

$\vec{x} = \begin{bmatrix} i_1\sqrt{R_1} \\ i_2\sqrt{R_2} \end{bmatrix} \therefore P = x_1^2 + x_2^2 = \|\vec{x}\|^2$

$i_s = i_1 + i_2 = \frac{x_1}{\sqrt{R_1}} + \frac{x_2}{\sqrt{R_2}}$

$\therefore \underbrace{\begin{bmatrix} \frac{1}{\sqrt{R_1}} & \frac{1}{\sqrt{R_2}} \end{bmatrix}}_{A}\vec{x} = i_s$

Alt:

$A = [a\ b] = \overbrace{[1]}^{U}\overbrace{\left[\sqrt{a^2+b^2}\right]}^{\Sigma}\overbrace{\left[\frac{a}{\sqrt{\cdots}}\ \frac{b}{\sqrt{\cdots}}\right]}^{V^T}$

$A^+ = V\Sigma^{-1}U^T$

w/ $\vec{x} = A^+ i_s$

$\Rightarrow \vec{i} = \left[\frac{1}{\sqrt{R_1}}\ \frac{1}{\sqrt{R_2}}\right]\vec{x}$

$x[i+1] = \alpha x[i] + \beta u[i] + \gamma$

a) Let $\alpha = 1,\ \beta = 0,\ \gamma \neq 0$

st $x[i+1] = x[i] + \gamma$

$\lim_{i\to\infty}|x[i]| = \lim_{i\to\infty}|x[0] + i\gamma| = \infty$

not BIBO!

b) $\alpha \neq 0,\ \beta > 0,\ \gamma \neq 0,\ \lambda_0 = 0$

$\&\ u[i] = f \cdot x[i]$

$f = \frac{-1-\alpha}{\beta} \Rightarrow \alpha + \beta f = -1$

$\Rightarrow x[i+1] = (\alpha + \beta f)x[i] + \gamma$

$= -x[i] + \gamma$

$= \begin{cases} \gamma : i \text{ odd} \\ 0 : i \text{ even} \end{cases}$

$\therefore$ bounded by $|\gamma|$

$\Rightarrow$ Generally bounded by

$f \in \left[\frac{-1-\alpha}{\beta}, \frac{1+\alpha}{\beta}\right)$

Since $|(\alpha + \beta f)| < 1$

$\hookrightarrow$ coef. to $x[i]$

$D\vec{p} \approx \vec{s}$

$D := \begin{bmatrix} x_0 & u_0 & 1 \\ \vdots & \vdots & \vdots \\ x_{l-1} & u_{l-1} & 1 \end{bmatrix},\ \vec{p} := \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

$\vec{s} = \begin{bmatrix} x_1 \\ \vdots \\ x_l \end{bmatrix}$

$\frac{d}{dt}x = ax(t) + v$

$x[i+1] = e^{a(t-t_0)}x(t_0) + \frac{v}{a}\left(e^{a(t-t_0)} - 1\right)$

$\frac{d}{dt}x(t) = ax(t) + bu(t) + c;\ t \in [i\Delta, (i+1)\Delta]$

$\Rightarrow x(t) = e^{a(t-i\Delta)}x[i] + \frac{(bu[i]+c)}{a}\left(e^{a(t-i\Delta)}\right)$

$\alpha = e^{a}$

$\beta = \frac{(e^{a}-1)b}{a}$

$\gamma = \frac{e^{a}-1}{a}c$

$a = \log\alpha$

$b = \beta\frac{\log(\alpha)}{\alpha - 1} = \frac{a}{e^{a}-1}\beta$

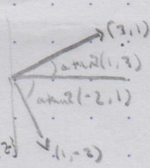
$c = \gamma$ ↓

Complex

$e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$ ↖ Im{z}

$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$ ↖ Re{z}



$j = -\frac{1}{j}$

$z := x + jy = |z|e^{j\theta}$

$\bar{z} = x - jy = |z|e^{-j\theta}$

$|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$

$x = |z|\cos\theta$; $y = |z|\sin\theta$

$\theta = \text{atan2}(y, x)$

$i^0 = 1$
$i^1 = i$
$i^2 = -1$
$i^3 = -i$

$-z = |z|e^{j(\theta\pm\pi)}$

$\sqrt{z} = \pm|z|^{\frac{1}{2}}e^{j\frac{\theta}{2}}$

$z^n = |z|^n e^{jn\theta}$

$\angle z^n = n\cdot\angle z$

$|z_1/z_2| = |z_1|/|z_2|$

$\angle z_1\cdot z_2 = \angle z_1 + \angle z_2$

$\angle z_1/z_2 = \angle z_1 - \angle z_2$

$\angle kz = \angle z$

Conjugate distributes over +, −, ·, /

$z^{-1} = \bar{z}/|z|^2$

$\overline{z^n} = (\bar{z})^n$

$\overline{\exp(z)} = \exp(\bar{z})$

$\overline{\ln(z)} = \ln(\bar{z})$

$A^* = (\bar{A})^T = \overline{A^T}$ | $(A^*)^* = A$

$(A+B)^* = A^* + B^*$ | $\det(A^*) = \overline{\det(A)}$

$(z\cdot A)^* = \bar{z}A^*$ | $\text{tr}(A^*) = \overline{\text{tr}(A)}$

$(AB)^* = B^*A^*$ | Eigens of A^* = conj. eigens of A

$\langle A\vec{x}, \vec{y}\rangle = \langle\vec{x}, A^*\vec{y}\rangle$

$\langle\vec{x},\vec{y}\rangle = \vec{y}^*\vec{x} = \overline{\langle\vec{y},\vec{x}\rangle} = \sum_i x_i\bar{y}_i$

$\langle\vec{u},\vec{v}\rangle \le \langle\vec{u},\vec{u}\rangle\cdot\langle\vec{v},\vec{v}\rangle$

$\Rightarrow |\langle\vec{u},\vec{v}\rangle| \le \|\vec{u}\|\|\vec{v}\|$

Inverse of $A\in\mathbb{R}^{2\times 2}$ $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a\end{bmatrix}$

Mean-Product $m = \text{tr}(A)/n$, $p = \det(A)$ | $m \pm \sqrt{m^2 - p}$

Units	p	n	μ	m	K	M	G
	$1e^{-12}$	$1e^{-9}$	10^{-6}	10^{-3}	10^{3}	10^{6}	10^{9}

CCB

$$\begin{tikzcd}
\frac{d}{dt}\vec{x} = A\vec{x} \arrow[d, "\tilde{\vec{x}} = V^{-1}\vec{x}"'] & \vec{x} = \ldots \\
\frac{d}{dt}\tilde{\vec{x}} = \tilde{A}\tilde{\vec{x}} \arrow[r] & \tilde{\vec{x}} = \ldots \arrow[u, "\vec{x} = V\tilde{\vec{x}}"']
\end{tikzcd}$$

↳ $\tilde{A} = V^{-1}AV = \Lambda = \text{diag}(\lambda_i)$

Eigens

1) All real: No oscillations, decays
2) All imaginary: Oscillates + no decays
3) Complex: Oscillates + decays

Stability

Continuous $|x(t)| < \infty$

In the 2x2 case:

$\det(I\lambda - A) = \lambda^2 - \underbrace{(a+d)}_{\text{tr}(A)}\lambda + \underbrace{(ad-bc)}_{\det(A)}$

$\Rightarrow \{\lambda\} = \frac{1}{2}\left(\text{tr}(A) \mp \sqrt{\text{tr}(A)^2 - 4\det(A)}\right)$

∴ Stable iff $\det(A) > 0$ & $\text{tr}(A) < 0$

(1) $\det(A) \le 0 \Rightarrow \sqrt{\ldots} \ge \sqrt{\text{tr}(A)^2} = |\text{tr}(A)|$
$\therefore \lambda_2 = \frac{\text{tr}(A) + |\text{tr}(A)|}{2} \ge 0$

(2) $\text{tr}(A) \ge 0 \Rightarrow \ldots < 0 \Rightarrow \sqrt{\ldots} \in \mathbb{C} \therefore \text{Re}\{\lambda\} = \frac{\text{tr}(A)}{2} < 0$
$\ldots > 0 \Rightarrow \sqrt{\ldots} \ge 0 \therefore \lambda_2 \ge \text{tr}(A) \ge 0$

$\therefore \lambda_1 \le \frac{\text{tr}(A)}{2} < 0$, $\lambda_2 < \frac{\text{tr}(A) + |\text{tr}(A)|}{2} \le 0$

Discrete:	Continuous:	
$\lvert\lambda\rvert < 1$	$\text{Re}\{\lambda\} < 0$	$\lvert a+b\rvert \le \lvert a\rvert + \lvert b\rvert$ $\lvert\int f(x)dx\rvert \le \int\lvert f(x)\rvert dx$ $a + ar + ar^2 \Rightarrow \frac{a}{1-r}$ iff $\lvert r\rvert < 1$

Phasors

$\omega = 2\pi f\ [\text{rad}/s]$, $f = T^{-1}$; $T = \frac{2\pi}{\omega}$

$\sin\theta = \cos(\theta - \frac{\pi}{2})$; $\cos\theta = \sin(\theta + \frac{\pi}{2})$

① Convert to Phasor Domain:

$v(t) = \underbrace{V_0}_{\text{peak}}\cos(\underbrace{\omega t}_{\text{freq.}} + \underbrace{\phi}_{\text{offset/delay}}) = \text{Re}\{V_0e^{j\theta}e^{j\omega t}\}$

$\Rightarrow \tilde{V} = \frac{V_0}{2}e^{j\theta} = |\tilde{V}|e^{\angle\tilde{V}}$

② Compute Impedance: $Z = \frac{V}{I} = \frac{Z_{\text{bottom}}}{Z_{\text{top}}}$

$Z_R = R$, $Z_L = j\omega L$, $Z_C = (j\omega C)^{-1}$

③ Solve + convert back

KCL/KVL/NVA

Capacitor | series: $(C_1^{-1} + C_2^{-1})^{-1}$

$I(t) = C\frac{dV}{dt}$

$Z_C = \frac{1}{j\omega C} = \frac{1}{\omega C}e^{-j\frac{\pi}{2}}$

Inductor

$V(t) = L\frac{dI}{dt}$

$Z_L = j\omega L = \omega L e^{j\frac{\pi}{2}}$

DEs

Homog. $\frac{d}{dt}x(t) = \lambda x(t)$
$\Rightarrow x(t) = x_0e^{\lambda t}$

$\frac{d}{dt}x = \lambda x + \alpha \Rightarrow \frac{d}{dt}z = \lambda z$
↳ $z := x + \alpha/\lambda$
$\Rightarrow z = z_0e^{\lambda t} \Rightarrow x(t) = x_0e^{\lambda t} + \frac{\alpha}{\lambda}(e^{\lambda t} - 1)$

$\frac{d}{dt}x(t) = \lambda x(t) + b u(t)$
$\Rightarrow x(t) = x_0e^{\lambda(t-t_0)} + be^{\lambda t}\int_{t_0}^{t}e^{-\lambda\theta}u(\theta)\,d\theta$

$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}u[i]$
$\Rightarrow \vec{x}[i] = A^i\vec{x}_0 + \left(\sum_{j=0}^{i-1}u[j]A^{i-1-j}\right)\vec{b}$

$\vec{x}[i+1] = A\vec{x}[i] + B\vec{u}[i]$
$\Rightarrow \vec{x}[i] = A^i\vec{x}_0 + \sum_{j=0}^{i-1}A^{i-j-1}B\vec{u}[j]$

$\frac{d}{dt}\vec{x}(t) = A\vec{x} + \vec{b}\vec{u}(t) \Longrightarrow \vec{x}(t) = \underbrace{e^{At}}_{\text{diag}(e^{a_{ii}})}\vec{x}_0 + \int_0^t e^{A(t-\theta)}V^{-1}B\vec{u}(\theta)\,d\theta = Ve^{\Lambda t}V^{-1}x_0 + \int_0^t Ve^{\Lambda(t-\theta)}V^{-1}\tilde{B}\vec{u}(\theta)\,d\theta$

① Set up diffeq: $\frac{d}{dt}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}$

② Find Eigens: $\det(\lambda I - A) \Rightarrow \{\lambda_1, \lambda_2\}$
$V := [\vec{v}_{\lambda_1}\ \vec{v}_{\lambda_2}]$, $\Lambda := \text{diag}(\lambda)$

③ CCB: $\tilde{\vec{x}} = V^{-1}\vec{x}$ | $\tilde{\vec{x}}_0 = V^{-1}\vec{x}_0$
$\frac{d}{dt}\tilde{\vec{x}} = \underbrace{V^{-1}AV}_{\Lambda}\tilde{\vec{x}} + \underbrace{V^{-1}\vec{B}}_{\tilde{B}}u(t)$

④ Decomp: $\frac{d}{dt}\tilde{x}_i = \lambda_i\tilde{x}_i(t) + \tilde{b}_iu(t)$
$\Rightarrow \tilde{x}(t)_i = \left(\tilde{x}_{0,i} + \frac{\tilde{u}_{0,i}}{\lambda_i}\right)e^{\lambda_it} - \frac{\tilde{u}_{0,i}}{\lambda_i}$

⑤ CG Basis: $\vec{x} = V\tilde{\vec{x}}$

2nd ODE

$\frac{d^2}{dt^2}x(t) + a\frac{d}{dt}x(t) + bx(t) = 0$
$\Rightarrow \lambda^2 + a\lambda + b = 0 \Longrightarrow \frac{1}{2}\left(-a \mp \sqrt{a^2 - 4b}\right)$

General Define state vector $[x\ \frac{d}{dt}x]^T$ st $\begin{bmatrix}\frac{d}{dt}x\\ \frac{d^2}{dt^2}x\end{bmatrix} = A\begin{bmatrix}x\\ \frac{d}{dt}x\end{bmatrix}$

$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix}2 & 3\\1 & 4\end{bmatrix}\vec{x}(t) \Rightarrow \{\lambda\} = \{1, 5\}$ ∴ $[15e^t + \frac{1}{4}e^{5t} \mid -\frac{3}{4}e^t + \frac{1}{4}e^{5t}]^T$

$= \begin{bmatrix}2 & -1\\1 & 4\end{bmatrix}$ 3 ∴ $[4e^{3t} + 3\sqrt{2}te^{3t} \mid -e^{3t} + 3\sqrt{2}te^{3t}]^T$

$= \begin{bmatrix}1 & -1\\2 & 4\end{bmatrix}$ 2,3 ∴ $[7e^{2t} - 3e^{3t} \mid -7e^{2t} + 6e^{3t}]^T$

$= \begin{bmatrix}0 & -1\\1 & 2\end{bmatrix}$ 1 ∴ $[4e^{t} + 3\sqrt{2}te^{t} \mid -e^{t} - 3\sqrt{2}te^{t}]^T$

$= \vec{x}$

↖ Exponential power = eigens

Controllability

$\vec{x}[i+1] = A\vec{x}[i] + \vec{b}\vec{u}[i]$

$C := [A^{n-1}\vec{b}\ \cdots\ A\vec{b}\ \ \vec{b}]$ must be rank n

$\vec{x}[i] = A^i\vec{x}_0 + C\underbrace{\begin{bmatrix}u[0]\\ \vdots\\ u[i-1]\end{bmatrix}}_{\vec{\omega}}$

$\vec{x}^*$ reachable in i^* from $\vec{x}_0$
iff $x^* - A^{i^*}\vec{x}_0 \in \text{Col}(C_{i^*})$:
$C_{i^*}\vec{\omega} = \vec{x}^* - A^{i^*}\vec{x}_0 \Rightarrow \vec{\omega} = C_{i^*}^{\dagger}(\vec{x}^* - A^{i^*}\vec{x}_0)$

CCF $\vec{x}[i+1] = A\vec{x}[i] + \vec{b}\vec{u}[i]$

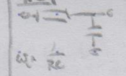
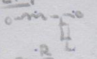
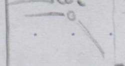
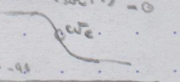
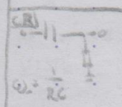
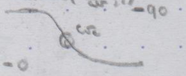
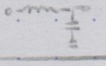
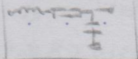
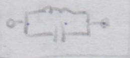
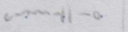
$A = \begin{bmatrix}\vec{0}_{n-1} & I_{n-1}\\ a_1 & \cdots\ a_n\end{bmatrix} \in \mathbb{R}^{n\times n}$, $\vec{b} = \begin{bmatrix}\vec{0}_{n-1}\\ 1\end{bmatrix} \in \mathbb{R}^n$

$\Rightarrow p_A(\lambda) = \lambda^n - \sum_{i=1}^{n}a_i\lambda^{i-1}$

Feedback: $u := \vec{f}^T\vec{x}$
$\Rightarrow x[i+1] = \overbrace{(A + \vec{b}\vec{f}^T)}^{A_{cl}}\vec{x}[i]$

Transfer Funcs + Filters

$H(j\omega) = \frac{V_{out}}{V_{in}} = |H(j\omega)|e^{j\angle H(j\omega)}$

LP	RC  $\omega_c = \frac{1}{RC}$	LR  $\omega_c = \frac{R}{L}$	H(jω) $\left(1 + j\frac{\omega}{\omega_c}\right)^{-1}$	$\lvert H\rvert$ 	$-\text{atan2}(\frac{\omega}{\omega_c}, 1)$ 
HP	CR  $\omega_c = \frac{1}{RC}$	RL  $\omega_c = \frac{R}{L}$	$\frac{j\frac{\omega}{\omega_c}}{1 + j\frac{\omega}{\omega_c}}$		$-\text{atan2}(-\frac{\omega_c}{\omega}, 1)$ 
LC		$H = (1 - \omega^2LC)^{-1}$			
LRC		$H = \frac{Z_C}{Z_C + Z_L + Z_R} = \frac{\frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}}$			
L‖C		$\frac{Z_L\cdot Z_C}{Z_L + Z_C} = \frac{j\omega L}{(j\omega L + j\omega C) + 1} = j\frac{\omega L}{1 - \omega^2LC}$			
L+C		$j\omega L + \frac{1}{j\omega C} = j\left(\omega L - \frac{1}{\omega C}\right)$			

↖ @ $\omega = \frac{1}{\sqrt{LC}}$: $H = \infty$

ω	LP	HP
$< \omega_c/10$	0	90
$= \omega_c/10$	−5.7	84.3
$= \omega_c$	−45	45
$= 10\omega_c$	−84.3	5.7
$> 10\omega_c$	−90	0

$U_R = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \Sigma_r = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, V_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Find bound α st $\|A\vec{x}\| \leq \alpha$ for $\|\vec{x}\| \leq 1$

$\rightarrow$ Bound by largest singular value: $\alpha = 2$

Full SVD:

$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \Sigma = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix}, V = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$

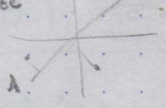
U, V orthon, Σ descending

- Singular Values = sqrt of eigens of $A^T A$ or AA^T

  $\hookrightarrow \{\|a_1\| \cdots \|a_n\|\}$ for orthon. A

$A = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & 1 \end{bmatrix}$ w/ $\{\sigma\} = \{\sqrt{21}, \sqrt{7}\}$

Find $y = \alpha x$ to minimize perp lines



$\hookrightarrow$ Left singular vector of A $\overset{\sqrt{21}}{\wedge}$

$AA^T \vec{v} = \sqrt{21}\,\vec{v}$

$\begin{bmatrix} 14 & 7 \\ 7 & 14 \end{bmatrix}\vec{v} = 21\vec{v}$

$\Rightarrow \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \therefore \frac{v_2}{v_1} = \frac{1}{1} = 1 = \alpha$

Orthon. preserve Θ between vectors

$\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\|\|\vec{y}\| \cos\theta$

R = ortho st $R^T R = RR^T = I$

$\tilde{x}, \tilde{y} = R\vec{x}, R\vec{y}$ w/ $\langle \tilde{x}, \tilde{y} \rangle = (R\vec{y})^T R\vec{x} = y^T R^T R \vec{x}$

$= \vec{y}^T \vec{x}$

Hermitian iff $A^* = \bar{A}^T = A$

$\rightarrow \langle A\vec{x}, \vec{y} \rangle = y^* A\vec{x} = y^* A^* x = (Ay)^* x = \langle x, Ay \rangle$

$\rightarrow \langle Ax, y \rangle = \langle x, Ay \rangle \Rightarrow A = A^*; a_{ij} = \tilde{a}_{ij}$

$\vec{x} = \vec{e}_i, \vec{y} = \vec{e}_j$ st:

$y^* A x = y^* A^* x \equiv e_j^* A e_i = e_j^* A^* e_i$

$\Rightarrow a_{ij} = \tilde{a}_{ij}$ □

$\overbrace{0 \rightarrow v(t) \quad 0}^{h(t)} \rightarrow v_L$

$h'(t) = v_L - v(t)$

$v'(t) = a - bv^2(t) + cu(t)$; Want $h^* > 0$

$\rightarrow$ Find operating point (v^*, u^*)

$h' = 0 = v_L - v^* \Rightarrow v^* = v_L$

$v' = 0 = a - bv^{*2} + cu^* \Rightarrow u^* = \frac{1}{c}(-a + bv_L^2)$

$\frac{d}{dt}\begin{bmatrix} \delta h(t) \\ \delta v(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} \delta h(t) \\ \delta v(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\delta u(t)$

w/ $\delta h(t) = h(t) - h^*$; Find A, b

Let $\vec{f} = \begin{bmatrix} h' \\ v' \end{bmatrix} = \begin{bmatrix} v_L - v(t) \\ a - bv(t)^2 - cu(t) \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} h \\ v \end{bmatrix}$

Thus $J_{\vec{x}}\vec{f}\big|_* = \begin{bmatrix} 0 & -1 \\ 0 & -2bv^* \end{bmatrix} = A$, $J_u \vec{f}\big|_* = \begin{bmatrix} 0 \\ c \end{bmatrix} = \vec{b}$

since $\frac{d}{dt}\begin{bmatrix} \delta h(t) \\ \delta v(t) \end{bmatrix} = J_{\vec{x}}\vec{f}(*)\begin{bmatrix} \delta h \\ \delta v \end{bmatrix} + J_u \vec{f}(*)\,\delta u(t)$

$\frac{d}{dt}\begin{bmatrix} \delta h \\ \delta v \end{bmatrix}(t) = \begin{bmatrix} 0 & -1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} \delta h \\ \delta v \end{bmatrix}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\delta u(t)$

w/ $u(t) = u^* + K_1 \delta h(t) + K_2 \delta v(t)$: $\underbrace{\text{Stabalize}}$

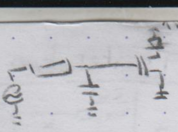
$= \begin{bmatrix} 0 & -1 \\ K_1 & -4 + K_2 \end{bmatrix}\vec{x}$

det(A) > 0

tr(A) < 0

det(A) = $K_1 > 0$

tr(A) = $K_2 < 4$ □



LED on if

$V_c \geq 0.5V$

$i_R = \frac{V_{in}(t) - V_c(t)}{R}$, $i_c = \frac{dV_c(t)}{dt}C$

KCL: $i_R = i_c$

$\Rightarrow \frac{dV_c(t)}{dt} = \frac{1}{RC}(V_{in} - V_c)(t)$

$\therefore \frac{d}{dt}V_c(t) = \underbrace{-\frac{1}{RC}}_{A} V_c(t) + \underbrace{\frac{1}{RC}}_{B} V_{in}(t)$

---

let $\lambda = -\frac{1}{RC} = A = -B$

$\frac{d}{dt}x(t) = \lambda x_t + bu_t$

$\Rightarrow x[i+1] = A_d x[i] + B_d u[i]$

$A_d := e^{\lambda\Delta}, B := \begin{cases} \frac{b}{\lambda}(e^{\lambda\Delta} - 1) : \lambda \neq 0 \\ b\Delta \quad : \lambda = 0 \end{cases}$

$\therefore x[i+1] = e^{\lambda\Delta}x + (-\lambda)\left(\frac{1}{\lambda}(e^{\lambda\Delta} - 1)\right)u$

$= \underbrace{e^{\lambda\Delta}}_{A_d} x + \underbrace{(1 - e^{\lambda\Delta})}_{B_d} u[i]$

$x^* = 0.5V = \underbrace{\begin{bmatrix} A_d^{l-1} B_d \cdots A_d b_d & b_d \end{bmatrix}}_{C_l} \begin{bmatrix} u_d[0] \\ \vdots \\ l-1 \end{bmatrix}$

w/ $C_l C_l^T = \frac{1 - e^{-.2l}}{.2\theta}$

$\rightarrow$ Minimize $\|\vec{u}_d\|^2 = \sum_{i=0}^{l-1} |u_d[i]|^2$

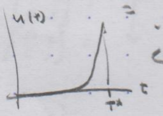
w/ $A_d = e^{-.1}$, $B_d = 0.1$

Min when $\vec{u}_d[l] = C_l^+ x^*$

$= C_l^T (C_l C^T)^{-1} x^*$

$= \frac{1}{2}\begin{bmatrix} c_{l-1} \\ \vdots \\ c_0 \end{bmatrix} \frac{.2\theta}{1 - e^{-.2l}} = \frac{.1\theta}{1 - e^{-.2l}}\begin{bmatrix} \lambda e^{-.1(l-1)} \\ \lambda e^{-.1} \\ \lambda \end{bmatrix}$

$\therefore U_d[i] = \underbrace{(1 - e^{-.2l}) e^{-.1(l-i-1)}}$



$= \ldots e^{.1i}$

$\underbrace{\qquad}$

Quad Approx $f_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

$f(x + \Delta x, y + \Delta y) =$

$f(*) + f_x(*)\Delta x + f_y(*)\Delta y$

$+ \frac{1}{2}(f_{xx}(*)\Delta x^2 + f_{yy}(*)\Delta y^2 +$

$\Delta x\Delta y(f_{xy}(*) + f_{yx}(*)))$

$f(\vec{x}^* + \Delta\vec{x}) \approx f(\vec{x}^*)$

$+ [D_{\vec{x}} f|_{\vec{x}^*}](\Delta x)$

$+ \frac{1}{2}(\Delta x)^T [H_{\vec{x}} f|_{x^*}](\Delta x)$

$H_{\vec{x}} f = \begin{bmatrix} f_{x_1x_1} & \cdots & f_{x_nx_1} \\ \vdots & & \vdots \\ f_{x_1x_n} & \cdots & f_{x_nx_n} \end{bmatrix}(\vec{x})$

System ID | $D\vec{p} \approx \vec{y}$

$\vec{y} \in \mathbb{R}^{\ell}$, measurements; $[x_1(1)\ x_n(1) \cdots x_1(m)\ x_n(m)]^T$

$D \in \mathbb{R}^{\ell \times m}$

$\vec{p} = [a_1 \cdots a_n\ b_1 \cdots b_n]^T$

Scalar | $x[i+1] = \lambda x[i] + b u[i] + e[i]$

$$\underbrace{\begin{bmatrix} x[0] & u[0] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \lambda \\ b \end{bmatrix}}_{\vec{p}} + \underbrace{\begin{bmatrix} \vec{e}[0] \\ \vdots \\ \vec{e}[\ell-1] \end{bmatrix}}_{\vec{e}} = \underbrace{\begin{bmatrix} x[1] \\ \vdots \\ x[\ell] \end{bmatrix}}_{\vec{y}}$$

$\hookrightarrow \hat{\vec{p}}_i = (D^T D)^{-1} D^T \vec{y}_i$

Vector | $\vec{x}[i+1] = A\vec{x}[i] + Bu[i] + \vec{e}[i]$

$$\underbrace{\begin{bmatrix} \vec{x}[0] & u[0] \\ \vdots & \vdots \\ x[\ell-1] & u[\ell-1] \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} A \\ B \end{bmatrix}}_{[\vec{p}_1 \cdots \vec{p}_n]^T} + \vec{e} = \underbrace{\begin{bmatrix} \vec{x}[0] \\ \vdots \\ \vec{x}[\ell-1] \end{bmatrix}}_{\vec{y}_1 \cdots \vec{y}_n}$$

Discretization | $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}[i] \Rightarrow A_d = Ve^{\Lambda\Delta}V^{-1} = e^{A\Delta}$

$t \in [i\Delta, (i+1)\Delta]$

$B_d = V(e^{\Lambda\Delta} - I_n)\Lambda^{-1}V^{-1}B$

$= (e^{A\Delta} - I_n)A^{-1}B$

s is orthon: $\sum_{i=1}^{\ell} \langle x, \vec{s}_i \rangle \vec{s}_i$

$\uparrow$

$\text{proj}_S \vec{x} = \sum_{i=1}^{\ell} \frac{\langle \vec{x}, \vec{s}_i \rangle}{\langle \vec{s}_i, \vec{s}_i \rangle} \vec{s}_i \leftarrow \|s_i\|^2$

$\langle A, B \rangle_F = tr(B^* A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \bar{b}_{ij}$

$A^* := (\bar{A})^T$; $\langle \vec{x}, \vec{y} \rangle = \vec{y}^* \vec{x} = \sum x_i \bar{y}_i$

$\text{rank}(A) + \dim(\text{Null}(A)) = n$

$\text{Null}(A) = \{\vec{x} \in \mathbb{C}^n \mid A\vec{x} = \vec{0}\} \subseteq \mathbb{C}^n$

$\text{Col}(A) = \text{Span}(\vec{a}_1 \cdots \vec{a}_n) \subseteq \mathbb{C}^m$

$AB = [A\vec{b}_1\ A\vec{b}_2 \cdots]; A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{n \times p}$

Schur Decomp |

$A \in \mathbb{C}^{n \times n}$; $A = UTU^*$

- U: unitary
- T: upper triang.

Func. $SD(A)$

$A \in \mathbb{C}^{1 \times 1}$: rt $[1], A$

$(\vec{q}_1, \lambda_1) = \text{Eigen}(A)$ ↙ extend basis

$Q = [\vec{q}_1\ \tilde{Q}] = GS(\{\vec{q}_1\}, \mathbb{C}^n)$

Compute $Q^* A Q = \begin{bmatrix} \lambda_1 & \tilde{a}_{12} \\ 0_{n-1} & \tilde{A}_{22} \end{bmatrix}$

$(P, \tilde{T}) = SD(\tilde{A}_{22})$

$U = [\vec{q}_1, \tilde{Q}P]$

$T = \begin{bmatrix} \lambda_1 & \tilde{a}_{12}P \\ 0_{n-1} & \tilde{T} \end{bmatrix}$

rt (U, T)

Orthon. | For $Q \in \mathbb{R}^{m \times n}$ ...

$m \geq n$	Tall	$Q^T Q = I_n$	orthon. cols	$\text{proj}_Q \vec{y}$ $QQ^T\vec{y}$
$m < n$	wide	$QQ^T = I_m$	orthon. rows	
$m = n$	square	$Q^T = Q^{-1}$	$\downarrow$	

Gram-Schmidt | Given $A = [\vec{a}_1, \vec{a}_2, \vec{a}_3] \in \mathbb{R}^{3 \times 3}$

1) $\vec{z}_1 = \vec{a}_1$

$\vec{p}_1 = \vec{z}_1 / \|\vec{z}_1\|$

$\vec{z}_2 = \vec{a}_2 - \langle \vec{a}_2, \vec{p}_1 \rangle \vec{p}_1$

$\vec{p}_2 = \vec{z}_2 / \|\vec{z}_2\|$

$\vec{z}_3 = \vec{a}_3 - \langle \vec{a}_3, \vec{p}_1 \rangle \vec{p}_1 - \langle \vec{a}_3, \vec{p}_2 \rangle \vec{p}_2$

$\vec{p}_3 = \vec{z}_3 / \|\vec{z}_3\|$

P is an orthon. basis for A (same span).

if we want to extend $A = [\vec{a}_1, \vec{a}_2] \in \mathbb{R}^{2 \times 2}$ to $\mathbb{R}^{3 \times 3}$, we computed GS with $\{\vec{a}_1, \vec{a}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, dropping $\vec{p}_i = \vec{0}$

Eigen Multiplicity |

Given $A \in \mathbb{R}^{n \times n}$ with d eigens $\lambda_1 \cdots \lambda_d$.

→ Algebraic Multiplicity: m_A^a of λ_i is the multiplicity of λ_i as a root of $p_A(\lambda)$

$p_A(\lambda) = \prod_{i=1}^{d} (\lambda - \lambda_i)^{m_i}$ ∴ $m_A^a(\lambda_i) = m_i$

→ Geometric Multiplicity: $m_A^g(\lambda_i)$ of λ_i is the # of lin indep. eigenvectors of A with eigenvalue λ_i; $m_A^g(\lambda_i) = \dim(\text{Null}(A - \lambda_i I))$

- $\sum_{i=1}^{d} m_A^a(\lambda_i) = n$
- $m_A^a(\lambda_i) \geq m_A^g(\lambda_i)\ \forall i$

→ A is diagonalizable iff $\forall i\ m_A^a(\lambda_i) = m_A^g(\lambda_i)$ st. $\sum_{i=1}^{d} m_A^g(\lambda_i) = n$

Triangular

$T = \begin{bmatrix} t_{11} & * \\ 0 & t_{nn} \end{bmatrix} \Rightarrow p_T(\lambda) = \prod_{i=1}^{n} (\lambda - t_{ii})$

$T\vec{x} = \vec{y}$: $y_k = \sum_{j=k}^{n} t_{kj} x_j$

SVD | $A \in \mathbb{R}^{m \times n}$, $\text{rank}(A) = r \leq \min(m, n)$

Compact: $A = \sum_{i=1}^{r} \sigma_i \vec{u}_i \vec{v}_i^T$

- $\{\vec{u}_i\} \subseteq \mathbb{R}^m$, ON basis of $\text{Col}(A)$
- $\{\vec{v}_i\} \subseteq \mathbb{R}^n$, ON $\downarrow$ $\text{Col}(A^T)$
- $\sigma_1 \geq \sigma_2 \geq \cdots > 0$

→ $(\sigma_i^2, \vec{u}_i)$ eigenpair of AA^T

→ $(\sigma_i^2, \vec{v}_i)$ $\downarrow$ $A^T A$

Full: $A = U\Sigma V^T$;

- $U = [U_r\ U_{m-r}] \in \mathbb{R}^{m \times m}$

→ $\text{Col}(U_r) = \text{Col}(A)$

→ $AA^T U_r = U_r \Sigma_r^2$

- $V = [V_r\ V_{n-r}] \in \mathbb{R}^{n \times n}$

→ $\text{Col}(V_r) = \text{Col}(A^T)$

→ $A^T A V_r = V_r \Sigma_r^2$

- $\Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{m \times n}$

→ $0 = 0_{(m-r) \times (n-r)}$

→ $AV_r = U_r \Sigma_r$

→ $\text{Col}(U_{m-r}) = \text{Null}(A^T)$

→ $\text{Col}(V_{n-r}) = \text{Null}(A)$

Steps |

① Choose smaller dim of $\{A^T A,\ AA^T\} = B$

② Find + Order Eigens of A

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$

$\lambda_{r+1} = \cdots = \lambda_n = 0$

$\sigma_i = \sqrt{\lambda_i}\ \forall i \in [\min(m, n)]$

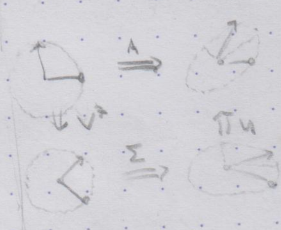
$\vec{v}_1 \cdots \vec{v}_n$ st. ..

$B\vec{v}_i = \lambda_i \vec{v}_i\ \forall i \in [n]$

③ For $i \in [r]$: $\vec{u}_i = A\vec{v}_i / \sigma_i$: $A^T A$

$= A^T\vec{v}_i / \sigma_i$

$\vec{v}_{r+1} \cdots v_n$ by Gram-Schmidt



$\text{Col}(AA^T) = \text{Col}(A)$

∴ U_r = basis of A

V_r = basis of A^T

$\text{Null}(AA^T) = \text{Null}(A)$

∴ V_{m-r} = basis of $N(A^T)$

V_{n-r} $\downarrow$ $N(A)$

Pseudo-Inverse |

Given $A = U\Sigma V^T \in \mathbb{R}^{m \times n}$

$A^+ \in \mathbb{R}^{n \times m}$ is given by $A^+ = V\Sigma^+ U^T$ w/.

$\Sigma^+ = \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0_{(m-r) \times (n-r)} \end{bmatrix}$

- $A^+ A = V_r V_r^T$ ($A^+ A$ proj $\text{Col}(A^T)$)
- $AA^+ = U_r U_r^T$ (proj onto $\text{Col}(A)$)
- $AA^+A = A$; $A^+ A A^+ = A^+$

Least-Squares | $m \geq n$

$\min_{\vec{x} \in \mathbb{R}^n} \|A\vec{x} - \vec{b}\|^2$

$\Rightarrow \vec{x}^*_{LS} = A^+ \vec{b}$

w/ $A^+ = (A^T A)^{-1} A^T$

$= V_r \Sigma_r^{-1} U_r^T$

Spectral | Let $A \in \mathbb{R}^{n \times n}$ be real + symmetric (Hermitian)

Then A is diagonalizable, has real orthog. eigens st

$A = V\Lambda V^T$

PCA | For n-rank A, Find k<n dim subspace to approx. col(A) w/ $A = \underbrace{[\ |\ |\ |\ |\ ]}_{\text{samples}}$ } features

→ k orthogonal directions to maximize information/spread/variation

$\tilde{A} = \text{argmin}_{D} \|D - A\|_F = \sum_{i=1}^{k} \sigma_i \vec{u}_i \vec{v}_i^T$

1) Find $\tilde{A}$ = k-rank approx using SVD

2) Principal comp $\vec{u}_i$ for $i \in [1, k]$ span k-dim subspace of features (row = sample ⇒ $\vec{v}_i$)

3) New data $\vec{x}$ represent by proj onto principal comp:

$\hat{x} = \begin{bmatrix} \vec{u}_1^T \vec{x} \\ \vdots \\ \vec{u}_k^T \vec{x} \end{bmatrix}$

Linearization | $f(\star) = \vec{0}_n$ for cont.

$f(x) \approx f(x^*) + f'(x^*)(x - x^*)$

w/ $x - x^* = \delta x$: $x(t) = x^* + \delta_x(t)$

- General: $\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(x^*)(x - x^*)^n$

$\star = (x^*, y^*)$

$f(x, y) \approx f(\star) + \underbrace{f_x(\star)}_{\left.\frac{\partial f(x,y)}{\partial x}\right|_{\star}}(x - x^*) + \underbrace{f_y(\star)}_{\left.\frac{\partial f(x,y)}{\partial y}\right|_{\star}}(y - y^*)$

$f(\vec{x}, \vec{y}) \approx f(\star) + \underbrace{(D_{\vec{x}} f)|_{\star}}_{\left.\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}\right|_{\star} = \nabla f(\star)}(\vec{x} - \vec{x}^*) + \cdots$

$J_{\vec{x}} = \begin{bmatrix} D_x f_1 \\ \vdots \\ D_x f_m \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}, J_u = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}$

$\vec{f}(\vec{x}, \vec{y}) \approx \vec{f}(\star) + J_x \vec{f}(\star)(x - x^*) + \cdots$

Least-Norm | $m \leq n$

$\min_{\vec{x} \in \mathbb{R}^n} \|\vec{x}\|^2$ st $A\vec{x} = \vec{b}$

$\vec{x}^*_{LN} = A^+ \vec{b}$

w/ $A^+ = A^T(AA^T)^{-1}$