Example Data

Consider the following dataset:

Max Temperature	Rain (Target)
20	0
25	0
30	1
35	1
40	1

1. Initialization

- **Initial Parameters:**
  - Weights w (for Max Temperature): 0
  - Bias b: 0
  - Learning rate α: 0.01

2. Logistic Regression Model

The model predicts the probability p of `Rain = 1` using:

$$z = w \cdot \text{Max Temperature} + b$$

$$p = \sigma(z) = \frac{1}{1 + e^{-z}}$$

3. Calculate Loss

Log-Loss for a single instance:

$$\text{Log-Loss} = -[y \log(p) + (1 - y) \log(1 - p)]$$

Average Loss:

$$\text{Average Loss} = \frac{1}{m} \sum_{i=1}^{m} \text{Log-Loss}^{(i)}$$

4. Iteration 1

a. Predictions:

- Compute z and p for each instance using the initial parameters $w = 0$ and $b = 0$.

$$z = 0 \cdot \text{Max Temperature} + 0 = 0$$

$$p = \sigma(0) = \frac{1}{1 + e^0} = 0.5$$

b. Compute Losses:

$$\text{Log-Loss}^{(i)} = -[y^{(i)} \log(0.5) + (1 - y^{(i)}) \log(1 - 0.5)]$$

For `Max Temperature = 20` and `Rain = 0`:

$$\text{Log-Loss}^{(1)} = -[0 \log(0.5) + (1 - 0) \log(0.5)] = \log(2) \approx 0.693$$

Similarly, calculate for other instances:

- For (25, 0): $\text{Log-Loss} = \log(2) \approx 0.693$
- For (30, 1): $\text{Log-Loss} = -[1 \log(0.5) + (1 - 1) \log(0.5)] = \log(2) \approx 0.693$
- For (35, 1): $\text{Log-Loss} = \log(2) \approx 0.693$
- For (40, 1): $\text{Log-Loss} = \log(2) \approx 0.693$

Average Loss:

$$\text{Average Loss} = \frac{1}{5} \sum_{i=1}^{5} \text{Log-Loss}^{(i)} = \frac{5 \times 0.693}{5} = 0.693$$

c. Compute Gradients:

$$\frac{\partial J}{\partial w} = \frac{1}{m} \sum_{i=1}^{m} (p^{(i)} - y^{(i)}) \cdot \text{Max Temperature}^{(i)}$$

$$\frac{\partial J}{\partial b} = \frac{1}{m} \sum_{i=1}^{m} (p^{(i)} - y^{(i)})$$

Gradient Calculations:

- For `Max Temperature = 20` and `Rain = 0`: $(0.5 - 0) \cdot 20 = 10$
- For `Max Temperature = 25` and `Rain = 0`: $(0.5 - 0) \cdot 25 = 12.5$
- For `Max Temperature = 30` and `Rain = 1`: $(0.5 - 1) \cdot 30 = -15$
- For `Max Temperature = 35` and `Rain = 1`: $(0.5 - 1) \cdot 35 = -17.5$
- For `Max Temperature = 40` and `Rain = 1`: $(0.5 - 1) \cdot 40 = -20$

$$\frac{\partial J}{\partial w} = \frac{1}{5}(10 + 12.5 - 15 - 17.5 - 20) = -6$$

$$\frac{\partial J}{\partial b} = \frac{1}{5}\left[(0.5 - 0) + (0.5 - 0) + (0.5 - 1) + (0.5 - 1) + (0.5 - 1)\right]$$

$$= \frac{1}{5}[0.5 + 0.5 - 0.5 - 0.5 - 0.5] = \frac{1}{5}[-0.5] = -0.1$$

d. Update Parameters:

$$w := w - \alpha \frac{\partial J}{\partial w} = 0 - 0.01 \cdot (-6) = 0.06$$

$$b := b - \alpha \frac{\partial J}{\partial b} = 0 - 0.01 \cdot (-0.1) = 0.001$$

Iteration 2

1. Compute Predictions with Updated Parameters

a. Calculate z and p for each instance:

$$z = 0.06 \cdot \text{Max Temperature} + 0.001$$

$$p = \sigma(z) = \frac{1}{1 + e^{-z}}$$

- For Max Temperature = 20:

$$z = 0.06 \cdot 20 + 0.001 = 1.201$$

$$p = \frac{1}{1 + e^{-1.201}} \approx 0.768$$

- For Max Temperature = 25:

$$z = 0.06 \cdot 25 + 0.001 = 1.501$$

- For Max Temperature = 30:

$$z = 0.06 \cdot 30 + 0.001 = 1.801$$

$$p = \frac{1}{1 + e^{-1.801}} \approx 0.857$$

- For Max Temperature = 35:

$$z = 0.06 \cdot 35 + 0.001 = 2.101$$

$$p = \frac{1}{1 + e^{-2.101}} \approx 0.890$$

- For Max Temperature = 40:

$$z = 0.06 \cdot 40 + 0.001 = 2.401$$

$$p = \frac{1}{1 + e^{-2.401}} \approx 0.917$$

b. Compute Log-Loss for Each Instance:

$$\text{Log-Loss} = -[y \log(p) + (1-y)\log(1-p)]$$

- For `Max Temperature = 20` and `Rain = 0`:

$$\text{Log-Loss}^{(1)} = -[0\log(0.768) + (1-0)\log(1-0.768)] \approx \log(4.33) \approx 1.464$$

- For `Max Temperature = 25` and `Rain = 0`:

$$\text{Log-Loss}^{(2)} = -[0\log(0.818) + (1-0)\log(1-0.818)] \approx \log(5.18) \approx 1.633$$

- For `Max Temperature = 30` and `Rain = 1`:

$$\text{Log-Loss}^{(3)} = -[1\log(0.857) + (1-1)\log(1-0.857)] \approx -\log(0.857) \approx 0.154$$

- For `Max Temperature = 35` and `Rain = 1`:

$$\text{Log-Loss}^{(4)} = -[1\log(0.890) + (1-1)\log(1-0.890)] \approx -\log(0.890) \approx 0.116$$

- For `Max Temperature = 40` and `Rain = 1`:

$$\text{Log-Loss}^{(5)} = -[1\log(0.917) + (1-1)\log(1-0.917)] \approx -\log(0.917) \approx 0.086$$

c. Compute Average Loss:

$$\text{Average Loss} = \frac{1}{5}[1.464 + 1.633 + 0.154 + 0.116 + 0.086] = \frac{3.453}{5} = 0.691$$

2. Compute Gradients

a. Gradient of Loss with respect to w:

$$\frac{\partial J}{\partial w} = \frac{1}{5}\sum_{i=1}^{5}(p^{(i)} - y^{(i)}) \cdot \text{Max Temperature}^{(i)}$$

b. Gradient of Loss with respect to b:

$$\frac{\partial J}{\partial b} = \frac{1}{5}\sum_{i=1}^{5}(p^{(i)} - y^{(i)})$$

$$\frac{\partial J}{\partial w} = \frac{1}{5}\left[(0.768 - 0) \cdot 20 + (0.818 - 0) \cdot 25 + (0.857 - 1) \cdot 30 + (0.890 - 1) \cdot 35 + (0.9\right.$$

$$= \frac{1}{5}[15.36 + 20.45 - 4.29 - 3.85 - 3.32] = \frac{1}{5}[24.35] = 4.87$$

$$\frac{\partial J}{\partial b} = \frac{1}{5}\left[(0.768 - 0) + (0.818 - 0) + (0.857 - 1) + (0.890 - 1) + (0.917 - 1)\right]$$

$$= \frac{1}{5}[0.768 + 0.818 - 0.143 - 0.110 - 0.083] = \frac{1}{5}[1.25] = 0.25$$

3. Update Parameters

$$w := w - \alpha \frac{\partial J}{\partial w} = 0.06 - 0.01 \cdot 4.87 = 0.0113$$

$$b := b - \alpha \frac{\partial J}{\partial b} = 0.001 - 0.01 \cdot 0.25 = 0.001 - 0.0025 = -0.0015$$

Summary of Iterations

1. **Initial Parameters:** $w = 0$, $b = 0$
2. **Iteration 1:**
   - **Average Loss:** 0.693
   - **Updated Parameters:** $w = 0.06$, $b = 0.001$
3. **Iteration 2:**
   - **Average Loss:** 0.691
   - **Updated Parameters:** $w = 0.0113$, $b = -0.0015$