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Remarks on Parallel Analysis

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Remarks on Parallel Analysis

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We investigate parallel analysis (PA), a selection rule for the number-of-factors problem, from the point of view of permutation assessment. The idea of applying permutation test ideas to PA leads to a quasi-inferential, non-parametric version of PA which accounts not only for finite-sample bias but sampling variability as well. We give evidence, however, that quasi-inferential PA based on normal random variates (as opposed to data permutations) is surprisingly independent of distributional assumptions, and enjoys therefore certain non-parametric properties as well. This is a justification for providing tables for quasi-inferential PA. Based on permutation theory, we compare PA of principal components with PA of principal factor analysis and show that PA of principal factors may tend to select too many factors. We also apply parallel analysis to so-called resistant correlations and give evidence that this yields a slightly more conservative factor selection method. Finally, we apply PA to loadings and show how this provides benchmark values for loadings which are sensitive to the number of variables, number of subjects, and order of factors. These values therefore improve on conventional fixed thresholds such as 0.5 or 0.8 which are used irrespective of the size of the data or the order of a factor.

Introduction

Parallel analysis (PA) is a method for deciding on the number of factors in principal components and principal factor analysis (Horn, 1965; Humphreys & Ilgen, 1969; Montanelli & Humphreys, 1976; Zwick & Velicer, 1986). The idea can be described as a modification of the root one criterion which retains components with eigenvalues greater than one. In PA, one replaces the threshold +1 with the mean eigenvalues generated from independent normal variates. The goal is to account for chance capitalization in sample eigenvalues under the null hypothesis of independent variables. The rationale is that even

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if all population eigenvalues of a correlation matrix are exactly one and no large components exist (as is the case for independent variates), any finite sample will still produce a number of empirical eigenvalues greater than one due to sampling variability.

To illustrate the importance of taking chance capitalization into account, a simple Monte Carlo simulation shows that for sample size $N = 20$ and number of variables $P = 10$, the largest eigenvalue of a correlation matrix of normal pseudo-random data will have a mean of 2.34. In PA, observed eigenvalues will be selected only if they are greater than their "null counterparts;" for example, the largest eigenvalue would have to be greater than 2.34 (versus 1.00).

In the present note we introduce several modifications and extensions of PA:

1. *Replacement of means by quantiles* leads to a quasi-inferential version of PA. In this form, PA can be fine-tuned through the use of quantiles other than the median (mean). Alternately, quantile-based p-values can be calculated to measure the strength of evidence for selection of a factor.

2. *Application of PA to resistant estimates of correlation matrices* (Gnanadesikan & Kettenring, 1972; Gnanadesikan, 1977) provides a version of PA which is resistant to the influence of small subsets of cases which may otherwise give rise to minor components.

3. *Application of PA to loadings* aids in the interpretation of selected factors. This may complement conventional benchmark values such as 0.5 and 0.8 by taking into account the sample size, the number of variables and the order of the eigenvalue in assessing the size of its loadings.

4. *Replacement of normal pseudorandom deviates with random permutations of the observed data* results in a nonparametric version of PA. In this case, the null reference set is conditional on the observed data values as opposed to relying on an assumed marginal distribution (e.g., the Gaussian) for the variables.

A couple of comments on terminology are in order. We adopt some language from hypothesis testing such as "inference," "null distribution," "significance," and "p-values," referring implicitly to the baseline assumption of independent variables as a null hypothesis. We prefer the term "quasi-inference" because the interpretation of these procedures as inference is not quite appropriate for all but the largest eigenvalue. Full independence of variables is no longer an interesting null hypothesis for testing the second largest eigenvalue once the largest eigenvalue has been established as significant. Yet this most accessible null assumption leads to useful baselines which one may want to exploit to a greater degree than has been done to date by just extracting null means.

Parallel analysis in its original form proposed by Horn (1965; i.e., noninferential, parametric, and applied to raw correlation matrices) has received considerable attention recently as it was found to be far superior as a method for factor selection compared to most commonly used rules. An extensive simulation study by Zwick and Velicer (1986) revealed PA as one of the two winners in a comparison which included the sample root one criterion (Kaiser, 1960), Bartlett's (1950, 1954) test at several significance levels, the scree test (Cattell, 1966; Cattell & Vogelmann, 1977), and minimum-average-partial criterion (Zwick and Velicer (1986). In an empirical comparative study of 30 published correlation matrices, Hubbard and Allen (1987) concurred with Zwick and Velicer's findings, stating that "Horn's test acquitted itself with distinction." At the same time, the commonly used sample root one criterion and Bartlett's test were shown to be too liberal and to result in solutions which were unlikely to replicate.

Another sign of the rising importance of PA has been the publication of empirical formulas for approximating the PA cutoff values, which would otherwise have to be obtained by simulations. This was done for principal factor analysis by Montanelli and Humphreys (1976) and for principal components analysis by Bobko and Schemmer (1984), Allen and Hubbard (1986), and Lautenschlager, Lance and Flaherty (1989). However, Lautenschlager (1989) casts serious doubt on the precision of the earlier formulas. He publishes tables for mean eigenvalues and shows that linear interpolation based on his tables results in more precise approximations than any of the previously published formulas.

The inferential feature discussed in this note increases the flexibility of PA. It was noted by Zwick and Velicer (1986) that PA has tendencies to include what they call "minor components." This can be ameliorated in the quasi-inferential version of PA by lowering the significance levels (i.e., using a higher quantile). The use of null averages as thresholds implies that the classical version of PA performs approximately at the 50% significance level – a liberal procedure from the point of view of conventional hypothesis testing. Whereas classical PA based on null means adjusts for finite sample bias, in the quasi-inferential form based on null quantiles it also adjusts for sampling variability.

The benefit of basing PA on data permutations rather than normal variates turns out to be of secondary importance. The reason is that PA has a certain degree of robustness under deviations from normal assumptions. This means that the null distributions of the eigenvalues are relatively independent of the marginal distributions of the variables. Therefore, the use of data permutations does not offer much advantage, and tabulation of threshold values derived from normal assumptions may prove useful and justifiable. The main use for permutation methods is in more complex situations where tabulation is impossible. If simulations are performed, one may as well use the more exact nonparametric method because random permutations are as fast to generate as normal random deviates.

On a different note, permutation distributions are useful for deriving theoretical results. In particular, it is possible to make an explicit comparison of PA for principal components (raw correlation matrices) and PA for principal factor analysis (R^2-adjusted diagonals). We will show that PA is somewhat more liberal when applied to adjusted correlation matrices, and the degree to which this is the case can be summarized in a useful rule of thumb. This result may lend some evidence against the use of PA in principal factor analysis.

Permutation null computations have previously been applied by de Leeuw and van der Burg (1986) to generalized canonical analysis or Alternating Least Squares (ALS) methods. These authors used the permutation approach less as a formal method for factor selection than a rough method for generating null baselines. Gnanadesikan (1977, p. 205) proposed a method for eigenvalue assessment which is related to parallel analysis. He uses so-called quantile-quantile plots of observed eigenvalues versus null eigenvalues. This is potentially a more sensitive graphical method than the more common eigenvalue profile plots overlaid with null profiles (e.g., Figure 1 below).

In the next section, we introduce the quasi-inferential version of PA in the context of data permutations. In subsequent sections, we show its application to raw, R^2-adjusted, and resistant correlation matrices. In the appendix, we provide selected tables of null quantiles for eigenvalues as well as loadings. The tables are based on raw correlation matrices of independent normal deviates.

Multivariate Data Permutation

The permutation principle is a method of generating "conditional null distributions" for arbitrary statistics. These distributions can be used to calculate conditional significance levels and p-values (Lehmann, 1986; Edgington, 1980; Noreen, 1989).

For clarity, we introduce some formal notation. The observations $x_{i,j}$ ($i = 1, 2, ..., N, j = 1, 2, ..., P$) are assumed to be obtained from a random sample of N subjects drawn from a large population: that is, they are realizations of random variables $X_{i,j}$ where the random vectors $(X_{i,1}, X_{i,2}, ..., X_{i,P})$ are independently and identically distributed for $i = 1, 2, ..., N$. No additional assumptions such as normality are made. The common marginal distribution F_j of the independent variables $X_{1,j}, X_{2,j}, ..., X_{N,j}$ is arbitrary, with the minor restriction that the variance be nonzero to insure the existence of correlations.

The null assumption is that there exists no stochastic dependence between the variables (i.e., $X_{i,1}, X_{i,2}, ..., X_{i,P}$ are independently distributed). This implies that the population eigenvalues from principal component analysis are all identical to +1, whereas those from principal factor analysis are all identical to 0. (We always assume that correlations are used rather than covariances.)

Under this null hypothesis, the variables in the N by P array

$$\mathbf{X} = \begin{bmatrix} X_{1,1} & X_{1,2} & X_{1,3} & \cdots & X_{1,P} \\ X_{2,1} & X_{2,2} & X_{2,3} & \cdots & X_{2,P} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{N,1} & X_{N,2} & X_{N,3} & \cdots & X_{N,P} \end{bmatrix}$$

are independently distributed, and the variables within column j share a common marginal distribution F_j, which may be different from column to column. The joint null distribution of these variables is invariant under permutation within the columns. Denoting by $\pi_j = [\pi_j(1), \pi_j(2), ..., \pi_j(N)]$ an arbitrary permutation of the N indices 1, 2, ..., N (for each j = 2, 3, ..., P), the permuted array of variables

$$\mathbf{X}_\pi = \begin{bmatrix} X_{1,1} & X_{\pi_2(1),2} & X_{\pi_3(1),3} & \cdots & X_{\pi_P(1),P} \\ X_{2,1} & X_{\pi_2(2),2} & X_{\pi_3(2),3} & \cdots & X_{\pi_P(2),P} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{N,1} & X_{\pi_2(N),2} & X_{\pi_3(N),3} & \cdots & X_{\pi_P(N),P} \end{bmatrix}$$

has the same joint distribution as the raw array **X**. (It is obvious that one column, e.g., the first, does not need to be permuted as the distribution of **X** is invariant under reordering of the N subjects even without the null hypothesis.) Stochastic dependence within the rows (subjects) of **X** destroys this permutation symmetry. Informally, permutation symmetry under the null hypothesis means that each of the $N!^{P-1}$ permuted arrays $\mathbf{x}_\pi$ is equally likely. Because this includes the observed array **x**, obtained from the trivial identity permutation, each of the permuted data arrays is as likely as the actually observed array.

As an aside, it should be noted that only variables *within* columns are permuted on grounds of a shared marginal distribution. If the marginal distributions were assumed identical *across* columns, one would obtain full permutation symmetry across the whole array, unconstrained to columns. This assumption is unrealistic, but it is implicit if the data are assumed normally distributed.

We now consider the effects of permutations on statistics $T(\mathbf{x})$. The statistics of interest in this article are various types of eigenvalues and loadings. If the permutations π are somehow randomly distributed, a statistic $T(\mathbf{x})$ inherits a distribution via the mapping $\pi \rightarrow T(\mathbf{x}_\pi)$. If, in particular, the permutations are uniformly distributed, this is called the permutation or conditional null distribution of T given the data $\mathbf{x}$. The terminology is justified as the permutation distribution is indeed the conditional distribution given the set of observed values in each variable under the null assumption of full independence (Lehmann, 1986, chap. 2, sec. 4, example 7).

For actual computation of the conditional null distribution, one notices that a full enumeration of all permutations is infeasible for most realistic values of N and P. The number $N!^{P-1}$ of column index permutations π is a rapidly growing function of N and P. In actual computations one therefore resorts to Monte Carlo simulations in which an affordable number R of column permutations is sampled with replacement[1], the result being a so-called *approximate permutation distribution*. We will see that in many cases as few as $R = 9$ random permutations are sufficient to locate strong structure. For ambiguously weak effects or for reporting p-values in publications one may increase the number of random permutations to $R = 99$ or even $R = 499$.

The reason for using odd numbers of replications such as these is the following: the actual data together with the R simulated data sets form a pool of $R + 1$ data sets which are all equally likely under the null assumption. Based on this fact, it has become a convention to choose $R + 1$ rather than R as a round number. This facilitates ranking of the actual data as the n^{th} largest out of 10, 100 or 500 with regard to some test statistic which has been evaluated on the $R + 1$ data sets. If, for example, four permutations out of 499 produce a value which is greater than or equal the observed one, the p-value will be $(4 + 1) / (499 + 1) = 1\%$, because the observed value is the 5^{th} largest.

Permutation tests have the property that they achieve exact significance (up to some discreteness problems) under the full nonparametric null hypothesis of independence. This remains true if permutations are sampled (*approximate permutation tests*) rather than enumerated (*exact permutation tests*). The difference between two permutation tests with small and large numbers of Monte Carlo samples is that the former generally attains lower power. However, results by Dwass (1957) and T. W. Anderson (1982) state that for "reasonable alternatives" at the 5% level it suffices to use 99 Monte Carlo replications in order to attain at least 83% of the power which would be

[1] Sampling with replacement is necessary because it is computationally infeasible to keep track of previously sampled data permutations. Even for very small samples sizes N, the number of data permutations is so large compared to the number R of affordable permutations that the difference between sampling with and without replacement becomes negligible.

achieved if all column index permutations were enumerated rather than sampled. For 499 replications, the achieved power is at least 92%.

To the reader who wishes to use permutation methods, we strongly recommend the book by Noreen (1989) which gives an introduction to permutation testing and computer programs in BASIC, FORTRAN and PASCAL. Even if one is not planning to use any of these languages, the precision afforded by the program code makes it easy to implement these methods in other computing environments. In a slight adaptation of Fig. 2.2 of Noreen (1989), here is a high level outline of a program for computing permutation p-values of eigenvalues:

1. Initialization:
   a. Read data into an $N \times P$ matrix **x**.
   b. Compute the eigenvalues, and store them in a vector of length P.
   c. Initialize an integer vector **nge** of length P to zero (intended to count how often permutation eigenvalues exceed the observed ones).
   d. Allocate auxiliary data structures for permuted data, their eigenvalues, etc.
2. Repeat R times:
   a. Permute columns 2, ..., P of **x**, leaving the first column unchanged, using a separate random permutation for each column (see Noreen, 1989, for code).
   b. Obtain the eigenvalues of the permuted data.
   c. For $i = 1, ..., P$, increase the count **nge**[i] by one if the i^{th} eigenvalue is greater than or equal the observed ith eigenvalue.
3. Finally:
   a. Obtain p-values for each eigenvalue: (**nge**[i] + 1) / (R + 1), (i = 1, ..., P).

In the following sections, we apply the conditional null distributions to eigenvalues and loadings of three types of correlation matrices: raw (principal component analysis), R^2-adjusted (principal factor analysis), and resistant. With one exception, we illustrate the methods throughout the article with a real data set which consists of 15 questionnaire items and 28 subjects, a size which may be typical for a small exploratory study. The 15-item scale which measures "bargaining behavior/cooperativeness of opponent" has been previously used by Dwyer (1980, 1984), but the data at hand were obtained by the authors (Eyuboglu & Buja, in press). See Table 3 for a description of the items. In one case (Figure 2), we use another data set with 27 variables and 64 subjects. We chose a rather large number (R = 499) of Monte Carlo permutations. In an actual exploratory analysis, considerably fewer replications would be sufficient.

Eigenvalues from Principal Component Analysis

Table 1 shows the observed eigenvalues from principal component analysis (PCA) as well as permutation quantiles, permutation *p*-values, and Bartlett *p*-values. In Figure 1, the same eigenvalues and permutation quantiles are shown as profile plots.

Only the first two eigenvalues are significant at customary levels: whereas the first eigenvalue 5.54 is significant beyond any reasonable doubt, the second eigenvalue 2.46 with a *p*-value of 0.026 lies between the 99% and 95% quantiles, a nontrivial tail range of the permutation distribution. The third eigenvalue 1.81, which is very near the permutation median with a *p*-value of 0.524, may also be of interest. If one were to perform a permutation version of Horn's method with null medians rather than null means, one would consider this third eigenvalue a borderline case because all eigenvalues above their null medians would be retained.

The quantiles show again the severe effects of chance capitalization; for example, the medians of the two largest conditional null eigenvalues are 2.53 and 2.10, as compared to an underlying population eigenvalue 1.0. The permutation distributions are fairly narrow: the differences between the

Table 1
Sample Eigenvalues of 15-Item Data (28 Subjects) from Principal Component Analysis (PCA) with Permutation Null Quantiles and *P*-Values

Order	Observed Eigenvalue	Permutation Quantiles Median	75%	90%	95%	99%	Permutation *P*-Values	Bartlett *P*-Values
1	5.54	2.53	2.67	2.83	2.94	3.10	0.002	0.000
2	2.46	2.10	2.21	2.32	2.37	2.53	0.026	0.000
3	1.81	1.81	1.89	1.96	2.01	2.12	0.524	0.000
4	1.35	1.56	1.62	1.69	1.73	1.79	0.984	0.000
5	1.00	1.34	1.41	1.47	1.51	1.55	1.000	0.002
6	0.67	1.16	1.21	1.27	1.29	1.36	1.000	0.022
7	0.54	0.99	1.04	1.08	1.11	1.15	1.000	0.049
8	0.43	0.84	0.88	0.93	0.95	1.00	1.000	0.082
9	0.34	0.70	0.75	0.79	0.82	0.86	1.000	0.108
10	0.31	0.58	0.62	0.66	0.68	0.70	1.000	0.117
11	0.19	0.46	0.49	0.53	0.56	0.60	1.000	0.322
12	0.15	0.36	0.39	0.42	0.44	0.48	1.000	0.291
13	0.11	0.27	0.30	0.33	0.35	0.39	1.000	0.324
14	0.06	0.18	0.21	0.24	0.25	0.29	1.000	0.493
15	0.04	0.11	0.13	0.15	0.17	0.20	0.994	*****

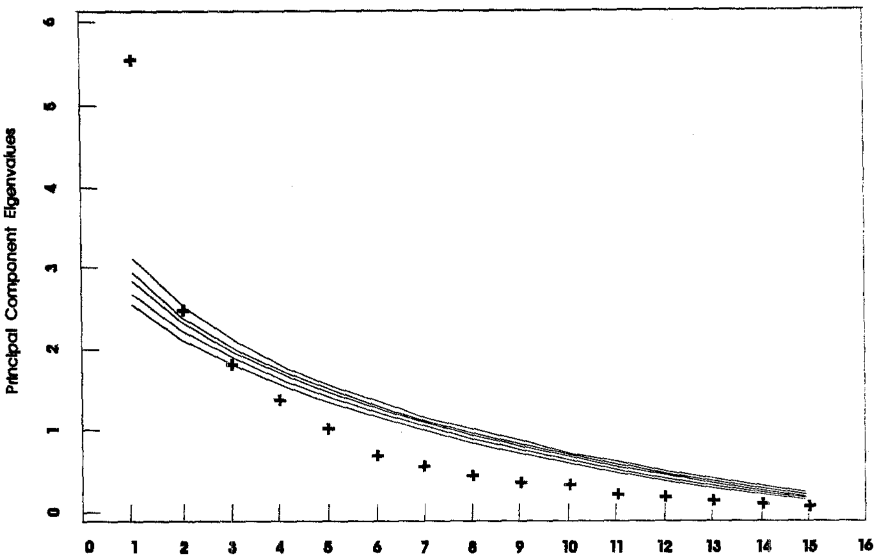

Figure 1: Observed eigenvalues (+) and null quantiles (from below: median, 75%, 90%, 95% and 99%)

median and the 95% quantiles are 0.41 and 0.27 for the largest and second largest eigenvalues, and the spreads narrow even further for higher orders. This observation is not specific to these data and holds even more so for larger N and P. Two important consequences are the following:

1. With computational savings in mind, it seems sufficient to start an analysis with as few as $R = 9$ random permutations in order to determine whether there are observed eigenvalues which are anywhere near the nontrivial range of their permutation distributions. In Figure 2 (next page) we plotted observed eigenvalues from our second real data set with 27 variables and 64 cases, together with 9 permutation profiles. Without further assessment it is clear that the first and second eigenvalue are highly significant whereas the third falls below all 9 permutation profiles.

2. For the same reason, the choice of significance level will generally make a difference only for a few components. In the 27-variable data of Figure 2, all conventional significance levels would give the same result. In the 15-item data (Figure 1), both the profiles of permutation medians and 95% quantiles cut across the observed profile between the second and third eigenvalue.

In practice one may flag the observed eigenvalues which fall in the median to 95% or 99% range as weak candidates and include them conditionally on the outcomes of an investigation of the loading patterns. The second and third eigenvalue of the 15-item data are borderline cases and probably deserve being flagged. This flagging procedure would be in line with Zwick and Velicer's

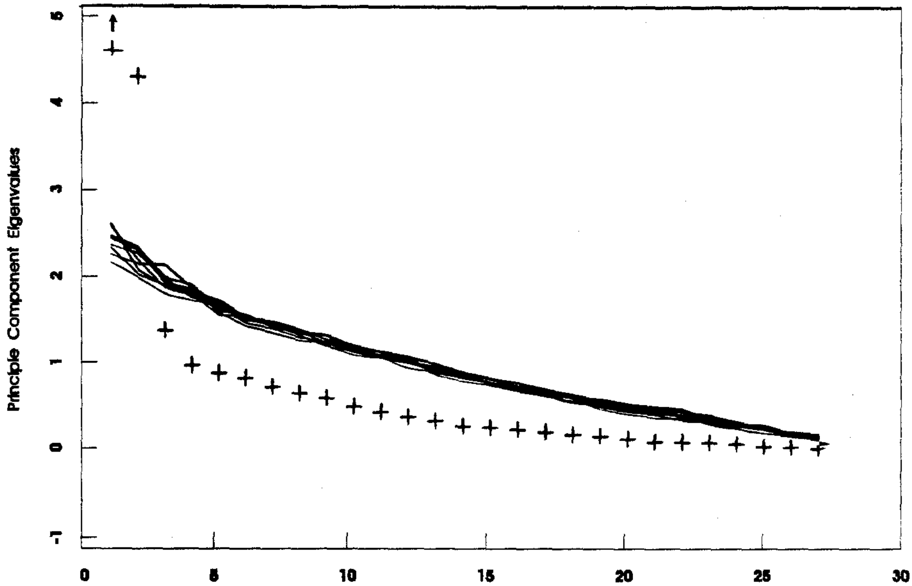

Figure 2: Observed eigenvalues (+) and nine permutation profiles (27-variables data)

(1986) observation that Horn's method tends to include what they call "minor components." A more conservative 1% or 5% significance level would somewhat correct this liberal tendency caused by an effective significance level of approximately 50% in Horn's PA.

The insignificant eigenvalues of the 15-item data fall far below their null medians for orders greater than 4 (see Table 1). This is a typical situation with a simple analytical explanation: if some eigenvalues are significantly greater than one, then others must be significantly less than one because the sum of the eigenvalues equals the total variance (which equals the number P of variables in the case of raw correlation matrices). Thus, the presence of large eigenvalues at the upper end has to be compensated for by depressed eigenvalues at the lower end[2]. An elementary consequence is the nonexistence of strict dominance of eigenvalue profiles: it is analytically impossible that one profile lies strictly above another profile for all orders of eigenvalues. In other words, crossing of eigenvalue profiles is necessary unless they are identical.

An important corollary is that tests based on eigenvalues lose power for increasing order because, under the summation constraint, the larger eigenvalues of low order are, the smaller and less significant subsequent eigenvalues of

[2] It should be noticed that permuting data values within items does not change the variance of the items. Thus the total variance of permuted data remains the same as for the observed data even if PCA is based on covariance matrices.

higher order become. This effect, however, does not seem to be undesirable. As the Monte Carlo evidence of Zwick and Velicer (1986) shows, it may be appropriate to introduce bias against high order eigenvalues in testing and selection procedures. In contrast, the Bartlett test applied to the tail end λ_k, λ_{k+1}, ..., λ_p is too liberal because its test statistic compensates for the effect of large low order eigenvalues λ_1, ..., λ_{k-1}. This is shown by the p-values for the Bartlett statistics in the last column of Table 1 where anywhere from 5 to 8 components would be selected according to this test.

Eigenvalues from Principal Factor Analysis and a Comparison with Principal Components

In principal factor analysis (PFA) the diagonal entries of the correlation matrix are replaced by estimates of the shared variance, usually the squared multiple correlation coefficients R_j^2 from the regression of the jth variable on all other variables. This is a procedure with conceptual problems which we ignore for now (see Harris, 1985, sec. 7.2 for a discussion).

Table 2 shows the observed PFA eigenvalues for the 15-item data along with upper permutation quantiles and p-values. The corresponding profile plot is shown in Figure 3.

Table 2
Sample Eigenvalues of 15-Item Data from Principal Factor Analysis (PFA) with Permutation Null Quantiles and P-Values

Order	Observed Eigenvalue	Permutation Quantiles: Median	75%	90%	95%	99%	Permutation P-Values
1	5.34	2.06	2.22	2.39	2.52	2.77	0.002
2	2.23	1.64	1.76	1.87	1.95	2.06	0.002
3	1.59	1.33	1.44	1.53	1.57	1.70	0.048
4	1.02	1.08	1.18	1.25	1.29	1.38	0.664
5	0.73	0.85	0.95	1.01	1.04	1.12	0.838
6	0.38	0.66	0.74	0.80	0.85	0.91	0.998
7	0.30	0.48	0.55	0.62	0.66	0.75	0.966
8	0.20	0.32	0.39	0.45	0.48	0.53	0.932
9	0.14	0.19	0.24	0.30	0.33	0.39	0.724
10	0.06	0.05	0.10	0.15	0.18	0.24	0.410
11	-0.04	-0.06	-0.02	0.03	0.05	0.09	0.382
12	-0.07	-0.14	-0.11	-0.08	-0.06	-0.02	0.076
13	-0.11	-0.21	-0.19	-0.17	-0.16	-0.13	0.006
14	-0.13	-0.27	-0.25	-0.23	-0.22	-0.20	0.002
15	-0.18	-0.31	-0.29	-0.27	-0.26	-0.24	0.002

Table 3
R_j^2-Values for the 15-Item Data

Description	Item	R_j^2
Feeble/Vigorous	1	0.80
Changeable/Stable	2	0.56
Dissonant/Harmonious	3	0.85
Discourteous/Courteous	4	0.73
Careless/Careful	5	0.71
Quarrelsome/Congenial	6	0.86
Selfish/Unselfish	7	0.76
Belligerent/Peaceful	8	0.84
Unpleasant/Pleasant	9	0.91
Weak/Strong	10	0.78
Foolish/Wise	11	0.65
Uncooperative/Cooperative	12	0.88
Excitable/Calm	13	0.84
Obstructive/Helpful	14	0.74
Bad/Good	15	0.55

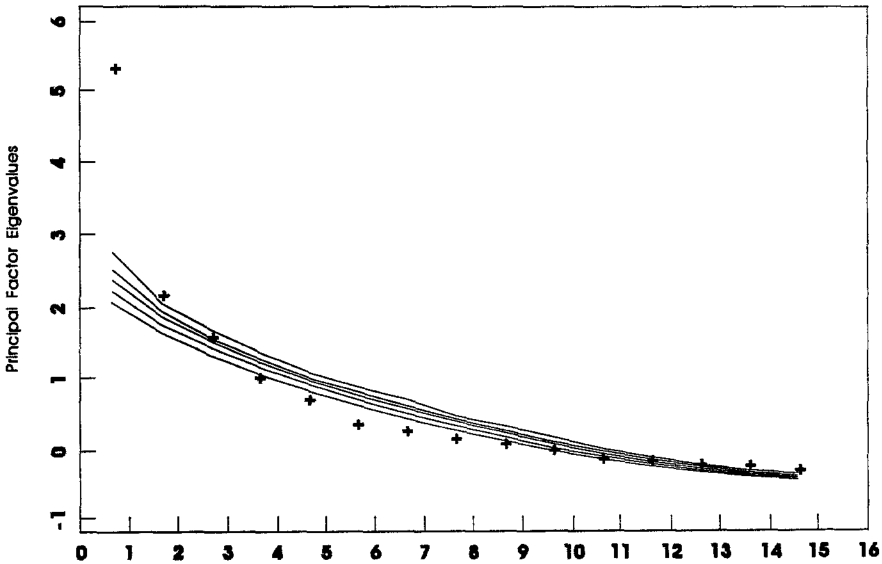

Figure 3: Observed eigenvalues (+) and null quantiles (from below: median, 75%, 90%, 95%, and 99%

An obvious difference from PCA is the occurrence of negative eigenvalues which prevents an interpretation of eigenvalues as variances. Although the dip below zero is not very large in absolute magnitude (-0.31 for the last median), the narrowness of the conditional null distributions implies that relevant upper quantiles fall below zero as well (-0.24 for the last 99% quantile). As a minor point, one notices that the smallest PFA eigenvalues are significantly greater than their permutation counterparts. This is an uninteresting nuisance due to greater convexity of the observed profile in comparison to the conditional null profiles.

More important are the inferential results of permutation PFA profiles: in contrast to PCA, the second eigenvalue is significant beyond conventional levels whereas the third eigenvalue is still significant at the 5% level. PFA null comparisons seem to suggest that three factors should be selected — a more liberal conclusion than from PCA. One can indeed show that PFA tends to include more factors than PCA, and the extent to which this is true can be quantified by the following rule of thumb: add to the observed PCA profile the quantity $\Sigma_j R_j^2/P$ - $(P - 1)/(N - 1)$ and inference drawn from the resulting situation will very closely resemble inference from PFA. For example, the average R^2 value for the 15 -item data is 0.76, and $(P - 1) / (N - 1)$ is 0.52. If one lifts the PCA profile by 0.24 = 0.76 - 0.52, one arrives at conclusions similar to those drawn from PFA. The above quantity is generally positive because correlations in actual data raise R^2 values above $(P - 1) / (N - 1)$ (which is just the null mean of R^2).

This simple rule can be derived quite easily from permutation theory by an argument similar to Box and Watson (1962)[3]. It casts a critical light on the application of parallel analysis to PFA if the recent literature on the favorable behavior of parallel analysis in PCA holds up. If it is true that even PA/PCA is somewhat on the liberal side by including occasional minor components, then PA/PFA may go further by selecting clearly negligible factors. The 15-item data (see Table 3) provide a glimpse of this behavior: the third factor should quite likely not be selected as the evidence of the following sections will

[3] See their Equation 21: It is easily seen that the expected null mean of R^2 is $(P - 1) / (N - 1)$, representing pure chance capitalization because the population R^2 is zero (null assumption = independence). Box and Watson (1962) permute only the predictors against the response, whereas we permute the predictors among themselves as well. The expectation is the same in our case because it differs from Box and Watson's only by an additional trivial averaging process with regard to the internal predictor permutations. The argument continues as follows: The expected average null R^2 equals the expected average null eigenvalue for PFA, whereas the average observed R^2 equals the average observed eigenvalue for PFA. The difference, $ave(R^2) - (P - 1) / (N - 1)$ represents a discrepancy in the overall levels of observed and null profiles. By contrast, in PCA, the average of the eigenvalues is identical +1 for observed and null profiles, and no discrepancy exists.

show, yet with PFA it gets significance at the 5% level. In the same vein, it was indicated above and will be further confirmed below that the second principal component is a good candidate for a minor component; in contrast, PFA assigns significance beyond conventional levels to its second factor.

The difference between PFA and PCA tends to diminish for fixed N and increasing P: the R_j^2 values inflate as their null average $(P - 1) / (N - 1)$ approaches +1. Furthermore, irrespective of N the influence of the diagonal on the eigen decomposition becomes smaller as P increases, because the proportional contribution of the $P^2 - P$ off-diagonal elements grows faster than that of the P diagonal elements of the correlation matrix.

Eigenvalues of Resistant Correlation Matrices

Resistant correlation matrices provide us with an example where tabulation of PA thresholds is no longer feasible. Monte Carlo simulation is necessary due to the presence of a tuning parameter for the level of resistance. A statistic is resistant if it is relatively immune to the presence of occasional outliers in data. Most traditional estimates, including raw sample correlations, are not resistant: a single extreme observation can force a correlation coefficient to take on any arbitrary value between -1 and +1. Several proposals for resistant correlations exist in the literature. We will use the one by Gnanadesikan and Kettenring (1972, see also Gnanadesikan 1977, p. 131ff) which is based on trimmed variances and an analytical re-expression of correlations in terms of variances. The method is computationally so inexpensive that it can be used easily in Monte Carlo simulations. As in most resistant estimation procedures, a parameter must be specified for the desired degree of resistance. This parameter, called trimming factor, is usually chosen on grounds of previous experience and/or initial exploration of the data at hand. Use of resistant estimates is often superior to outlier rejection exercised by the human analyst. These estimates can respond to multiple outliers, and they can reject individual values, as opposed to whole subjects (a subject may produce a bad response on just a few out of a large number of questionnaire items).

We found resistant correlation matrices to be a most useful tool for the analysis of data with small subsets of deviant values arising from misrecording, occasional failures in experimentation, or uncooperative or otherwise differing subjects. Such deviant behavior may show up as minor factors in standard PCA or PFA. By comparison, use of resistant estimates of correlation may either suppress these effects to some degree, or at least help to detect potential problems through a comparison of raw and resistant correlations. We give an illustration based on a small data set from the study mentioned earlier (Eyuboglu & Buja, in press). In a negotiation experiment, we obtained for each participant a measure of perceived power of his/her opponent. Based on a priori considerations, we hoped to find a negative correlation between A's perception of B's power and B's perception of A's power, amounting to a tendency among negotiators A and B to agree on power imbalance among them. The measurements obtained from 14 negotiating dyads are:

(-0.29, 2.62), (-0.94, 2.84), (-2.35, 3.43), (1.97, 0.84), (-0.94,-0.88), (0.84, 0.84), (0.84, 2.02), (-4.94, 4.90), (-0.29, 0.84), (-0.88, 1.12), (-2.35, 2.56), (-1.76, 0.84), (-5.82,-3.82), (-4.69, 3.43).

The raw correlation is -0.001. With a trimming factor of 1/14 (allowing for about 1 outlier in 14), the resistant correlation is -0.528. A scatterplot of the data reveals that the main reason for this discrepancy is the 13th dyad (-5.82,-3.82). Leaving it out gives a raw correlation of -0.663, whereas leaving out any other dyad gives raw correlations between -0.025 and +0.236. Thus, the discrepancy between raw and resistant correlations enabled us to detect the existence of a clear outlier. The method does not pinpoint the outlier, it only shows its presence. This may be sufficient for tipping off a data analyst who has to handle large numbers of variables (as was the case in our study). A minor flaw of resistant correlation matrices is that they are not necessarily Gramian (i.e., mildly negative eigenvalues may occur). This effect is generally not as strong as the one due to R_j^2-adjustment of the diagonal in PFA.

We turn to an examination of the eigenvalues obtained from resistant correlations of the 15-item data. The trimming factor was again chosen to be 1/14 (i.e, we allow the equivalent of 2 out of 28 observations to be automatically trimmed in the estimation of each correlation coefficient). The eigenvalues of the resistant correlation matrix, together with the conditional null quantiles and p-values, are listed in Table 4 (next page; see also Figure 4).

A comparison of Tables 1 and 4 shows that the null quantiles from resistant parallel analysis rise slightly above the values from raw parallel analysis, at least for the low orders which matter in practice. This can be partly explained by the trimming procedure which rejects roughly 2 out of 28 observations as potentially suspicious, thus reducing the effective sample size and raising the null distributions of low-order eigenvalues. Parallel analysis is therefore slightly more conservative, depending on the size of the trimming factor. It appears that resistant analysis leaves only the largest eigenvalue as clearly significant. The trimming procedure allowed the first eigenvalue to grow to 6.75 from 5.54, indicating that a small minority of data values may have obscured the real strength of the first component. The second eigenvalue of 2.44, although almost identical to that of Table 1 (2.46), lost significance with a p-value as high as 0.208 due to the rise in the overall level of the null quantiles.

Table 4
Sample Eigenvalues of 15-Item Data from Resistant Principal Component Analysis (RPCA) with Permutation Null Quantiles and P-Values

Order	Observed Eigenvalue	Median	75%	90%	95%	99%	Permutation P-Values
		Permutation Quantiles					
1	6.75	2.75	2.95	3.12	3.20	3.43	0.002
2	2.44	2.27	2.40	2.52	2.59	2.66	0.190
3	1.85	1.94	2.03	2.12	2.17	2.31	0.760
4	1.17	1.66	1.73	1.80	1.83	1.92	1.000
5	0.84	1.42	1.48	1.54	1.57	1.66	1.000
6	0.68	1.20	1.27	1.33	1.36	1.41	1.000
7	0.59	1.00	1.06	1.12	1.15	1.20	1.000
8	0.29	0.83	0.87	0.93	0.97	1.03	1.000
9	0.25	0.66	0.72	0.77	0.80	0.84	1.000
10	0.14	0.51	0.56	0.60	0.63	0.69	1.000
11	0.11	0.38	0.43	0.47	0.50	0.54	1.000
12	0.04	0.25	0.29	0.33	0.36	0.40	1.000
13	0.00	0.14	0.18	0.22	0.23	0.27	0.996
14	-0.06	0.03	0.07	0.10	0.12	0.15	0.950
15	-0.11	-0.09	-0.05	-0.01	0.00	0.05	0.658

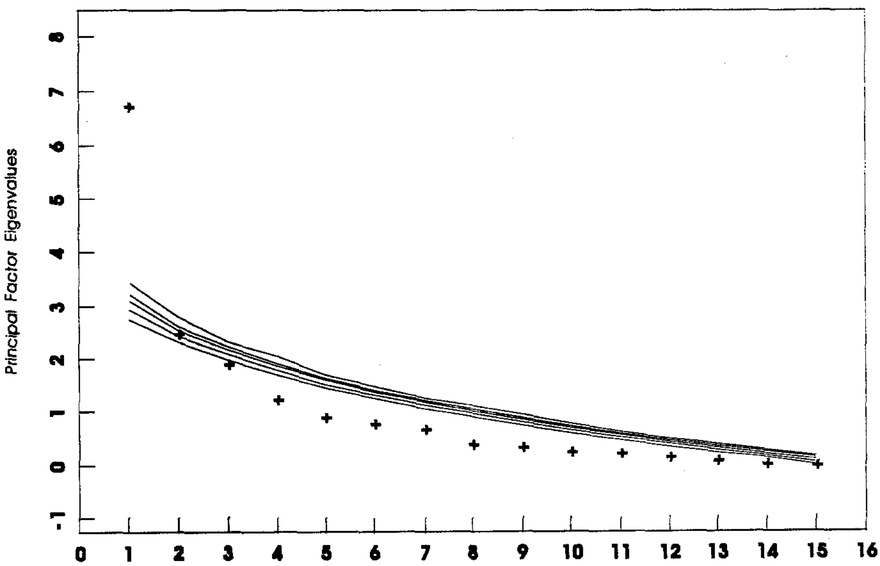

Figure 4: Observed eigenvalues (+) and null quantiles (from below: median, 75%, 90%, 95%, and 99%

The third eigenvalue, while increasing by a very small amount (to 1.85 from 1.81), nevertheless slipped below its null median.

The three eigenvalue analyses considered here — raw, R^2-adjusted, and resistant — gave slightly contradictory assessments of the second and third component of the 15-item data. We summarize as follows: The second and third eigenvalue behave borderline in the raw analysis, possibly indicating minor components if Zwick and Velicer's (1986) evidence is a guide. The R^2-adjusted analysis retains these two components, whereas the resistant analysis rejects them. Eigenvalues alone never tell the whole story, so we proceed now to the analysis of the loadings. As we will see, the second component has interpretable loading patterns, whereas the third one does not. It appears, therefore, that R^2-adjusted analysis can be overly liberal, whereas resistant analysis may favor major components while discounting minor ones.

Assessment of Loadings

If null distributions can be useful for the assessment of eigenvalues, the same should be expected for unrotated loadings. In the present article, all analyses deal with unrotated solutions, so that the issue of loadings versus scoring coefficients as a basis for interpreting components is moot. Table 5 (next page) shows loadings for the three largest principal components of the 15-item data. The patterns suggest that the first component is a weighted average of 12 out of the 15 items. The second component puts most weight on the remaining three items (1, 2, and 10). In pinpointing these three deviant items, the second component is interpretable, although it plays only a secondary role as a complement of the first component. Therefore, and based on the raw and resistant parallel analyses of the eigenvalues, it may be justifiable to declare the first component to be major, the second minor, and the third negligible. The latter does not call for interpretation as there are only occasional loadings with absolute value greater than, say, 0.5.

Applying parallel analysis to loadings, one should keep in mind that eigenvectors are determined only up to a change of sign[4]. Therefore, only two-sided (quasi-)inference should be applied to loadings. Table 5 shows permutation p-values for the absolute loadings of components 1-3. In spite of its overall strength the first component does not have many loadings which are significant at conventional levels. Only four have a p-value below 0.05, and only 8 out of 15 are around 0.10 or below. On the other hand, we find two loadings of the second component significant at the 0.01 level (the first and the tenth), whereas

[4] This should not be confused with the relative signs of loadings. Whether two variables load in the same direction or not is important for interpretation of components.

Table 5
Permutation P-Values and Average Null Quantiles for the Absolute Loadings of the Largest Three Principal Components (PCA) of the 15-Item Data

	First Component		Second Component		Third Component	
Item	Loading	P-Value	Loading	P-Value	Loading	P-Value
1	0.14	0.802	0.81	0.006	0.14	0.730
2	-0.12	0.826	0.51	0.180	-0.49	0.182
3	0.79	0.014	0.29	0.484	0.00	0.984
4	0.56	0.180	-0.21	0.672	0.16	0.686
5	0.61	0.112	0.22	0.574	0.37	0.316
6	0.67	0.086	-0.25	0.570	0.56	0.114
7	0.48	0.342	-0.45	0.286	-0.53	0.092
8	0.71	0.054	-0.43	0.288	0.17	0.658
9	0.83	0.004	-0.14	0.776	0.22	0.592
10	0.11	0.806	0.85	0.002	0.37	0.298
11	0.62	0.138	-0.02	0.964	-0.29	0.464
12	0.77	0.008	0.06	0.898	-0.47	0.228
13	0.64	0.106	0.35	0.410	-0.49	0.174
14	0.81	0.004	0.20	0.618	-0.09	0.824
15	0.52	0.292	-0.03	0.932	0.13	0.748
Average Quantiles						
50%	0.35		0.30		0.26	
90%	0.64		0.60		0.57	
95%	0.70		0.67		0.65	
99%	0.79		0.78		0.77	

its eigenvalue has a p-value of 0.026 only (Table 1). This example illustrates a potential divergence between inference based on a summary statistic (the eigenvalue) and inference based on its constituents (the loadings). We are not facing a real problem but an unavoidable possibility of which analysts must be aware. For an intuitive understanding, one may draw a comparison with regression: even if a multiple regression fit is highly significant, there is no necessity that even a single one of its regression coefficients is significant (provided that strong enough collinearity is present). The difference between regression and PCA/PFA is that in the latter the divergence can go either way: strong eigenvalues can arise from relatively weak but numerous loadings, and relatively weak eigenvalues can have strong loadings if they are few in number.

Another potential source of misunderstanding with quasi-inference is the following: loadings of large magnitude can suffer from strong sampling variability, even in the face of strong significance. The reason is that components with nearly identical population eigenvalues are ill-determined in the sense that their sample components will vary wildly. (See Cliff & Hamburger, 1967, pp. 435-436, for some related discussions.) Sampling stability of loadings is achieved only for well-separated population eigenvalues. This type of sampling behavior is well-expressed in asymptotic confidence intervals and regions for loading vectors, as found, for example, in Seber (1984, sec. 5.2.5, based on covariance matrices). Significance of loadings under the above null comparison therefore should not be mistaken as sampling stability. The only permissible conclusion is that loadings of a certain magnitude are unlikely to be due to chance alone. Yet, null quantiles such as those shown in the lower part of Table 5 may provide useful standards for comparison, particularly because they are *functions of sample size, number of variables, and order of the component.* In contrast, the conventional thresholds 0.5 and 0.8 for judging the size of a loading are blind to any of these parameters. It is interesting, for example, that according to Table 5, loadings for higher order components require lower magnitudes in order to achieve significance, a fact which reflects the greater degree of chance capitalization in low order components. See below for a discussion of loading quantile tables included in the Appendix.

The reader may have noticed that Table 5 features some counterintuitive p-values. For example, the third and the twelfth loading of the first component are 0.79 and 0.77, whereas their p-values are 0.014 and 0.008, respectively. One would expect larger magnitudes of loadings to be more significant (i.e., p-values to be smaller). Minor violations of inverse monotonicity as shown by these numbers should be expected on occasion because even for 499 Monte Carlo replications the extreme quantiles of the simulated distributions have relatively large variability. The situation can be improved by taking advantage of the parallel nature of computations for the P loadings. Rather than computing separate quantiles or p-values for each loading, one may pool the null computations of a given component across all variables and obtain a single null distribution for all loadings of this component. This is justified on grounds of the evidence presented in the next section; there we show that the influence of the marginal distribution on null distributions is negligible. These considerations justify the average null quantiles shown in the lower part of Table 5.

Irrelevance of the Marginal Distribution — a Justification for Tables

There is a question of how much the cutoff values of PA depend on the marginal distribution of the variables. If the dependence is not very pronounced, there is no special need to use data permutations for null quantiles. Horn's (1965) scheme of generating independent normal random deviates for a null comparison would be justified. To get some partial answers to this question we performed a simulation study in which we estimated null quantiles from four non-normal distributions which may approximate not too unrealistic departures from normality. We compared these quantiles with those obtained from normal deviates. The four non-normal distributions were:

1. A χ^2 with one degree of freedom, as an example of a continuous skewed distribution.

2. The third power of a standard normal deviate, as an example of a continuous symmetric distribution with heavy tails.

3. A seven point scale with probabilities 0.60, 0.20, 0.20 for the values 1, 2, 3, and probabilities 0.05 for the values 4 through 7; this is an example of a skewed discrete distribution with end point clustering.

4. A seven point scale with probability 0.05 for the values 1, 2, 6, 7, probability 0.1 for the values 3, 5, and probability 0.6 for the value 4; this is an example of a discrete symmetric distribution with center clustering.

Samples from the last two distributions were conditioned on nonzero sample variance so as to avoid nonexisting sample correlations. Each of the four distributions was used to generate null situations in place of the normal deviates of Horn's (1965) procedure. We chose four combinations of sample size and number of variables: $N = 20, P = 5; N = 20, P = 15; N = 100, P = 10;$ $N = 100, P = 50$ (i.e., a very small and a moderate sample size combined with a relatively small and large number of variables, respectively). This amounted to a 4×4 factorial design of four distributions by four (N, P) combinations. The number of replications was 1000 for each experiment. We compared the resulting null quantiles for $\alpha = 0.50, 0.75, 0.90$, and 0.95 with those of the corresponding normal simulations.

The results can be summarized quickly: the null quantiles in the non-normal scenarios stayed within ± 0.04 of those in the normal scenarios in all but three cases (of a total of 1280 numbers), and even the maximum deviation was only 0.07. No clear pattern in relation to the type of distribution, sample size and number of variables could be made out other than what would be expected anyway: that the largest discrepancies occurred for the outer quantiles of the largest eigenvalues when the $N{:}P$ ratio was small (i.e., $N = 20, P = 15$ and $N = 100, P = 50$). Although a discrepancy of 0.07 for 0.95 quantiles between normal and non-normal scenarios seems small, it may be large enough to turn a 0.95 quantile under normal assumptions into a 0.90 quantile under non-normal assumptions. However, this fact is secondary only in comparison to such decisions as (a) R^2-adjusted or resistant correlation matrices versus raw correlations, or (b) null quantiles for a = 0.90, 0.95, 0.99 versus null medians or null means as in Horn's original PA.

The discrepancy between normal and non-normal distributions is small enough that null situations based on normal deviates can be recommended as good approximations even in many non-normal situations. The role of permutation methods in PA therefore is limited to those cases where a simulation has to be carried out at any rate (e.g., when the presence of additional discretionary parameters, such as the trimming factor for resistant correlations, makes tabulation an unlikely possibility).

Tables for Null Quantiles of Eigenvalues and Loadings — Some Surprises

In view of the above results, tables can be extended from null means to null quantiles. We made a first step in this direction by tabulating null medians, 0.90, 0.95 and 0.99 null quantiles for components up to order ten in the Appendix. The null quantiles of the Appendix are for PCA (i.e., eigenvalues and unrotated loadings of raw correlation matrices).

The selection of sample sizes and numbers of variables in the Appendix may be dense enough to allow interpolation to many combinations of N and P which are not contained in these tables. The sample sizes are limited to $N \leq 200$ (i.e., small studies, in which the assessment problem is most urgent). The numbers shown are based on simulations with 2500 replications for each combination of N and P. The accuracy of medians and 0.90 and 0.95 quantiles is within 0.01, and for 0.99 quantiles within 0.02.

As mentioned in previous sections, these quantiles are functions of the sample size N, the number of variables P, and the order of the component. In the 15-item data we had noticed that the null quantiles of the loadings were a descending function of the order, a fact which can be explained by similar descending behavior of the eigenvalues. However, looking over the tables of loading quantiles in the Appendix, we find some surprises. For small numbers of variables, no monotone dependence on the order exists, and the nature of the trends differs for central and tail quantiles: for $P = 5$ and $N = 200$, the medians dip first and rise afterwards, whereas the 0.90 quantiles rise first and then dip. These patterns were so surprising to us that we decided to verify the numbers in a follow-up study. The results of the first study were fully confirmed by the second set of simulations, however.

The most important pattern in these tables is probably the decrease of the null quantiles for increasing number of variables. For larger numbers of subjects (e.g., $N = 200$), the null distribution of the loadings can be narrowed substantially by including a larger number of variables. It may be unpleasant to realize that, in the absence of all structure, one in ten loadings reaches above +0.7 or below -0.7 (approximately) for $N = 200$ and $P = 5$; by comparison, the 90% range narrows to about ± 0.4 for $P = 30$.

Summary

1. PA can be extended by making use of the full null distribution rather than only the null expectation as in classical PA. Using upper 5% or 1% null quantiles instead of null expectations may reduce some liberal tendencies of PA noted by Zwick and Velicer (1986). P-values derived from null distributions provide measures of strength for the individual eigenvalues.

2. PA may be applied usefully to other factoring methods besides PCA and PFA, an example being decompositions of resistant correlation matrices. For any such technique, PA gives insight into the amount of chance capitalization under the null assumption of unrelated variables — if applied in its original form based on null means. If the full null distribution in terms of null quantiles is used, one learns both about chance capitalization (bias) and variability (variance) under the null assumption. For the largest eigenvalues, the bias component, as measured by null median or mean minus one, is generally considerably larger than the variability, measured for instance by the difference between the 95% null quantile and the null median.

3. Permutation methods provide a nonparametric version of PA. The assumption of normally distributed variables inherent in classical PA thus can be avoided. However, there is evidence that PA features a degree of robustness of significance in that the null distributions are not sensitive to departures from normality. Therefore, inference derived under normal assumptions and from permutation methods may not differ by much. This confers approximate nonparametric properties to conventional PA and justifies tabulation of null quantiles of eigenvalues, or condensation in empirical formulas.

4. Permutation distributions allow us to compare PA for principal component and principal factor analysis. The difference is that the latter tends to accept a slightly higher number of dimensions. As a rough rule, an eigenvalue profile from principal factoring corresponds to a profile from principal components raised by the amount $\Sigma_j R_j^2/P - (P - 1)/(N - 1)$. This is the average R^2 adjusted for chance capitalization.

5. To some extent unrotated loadings can be assessed with quantile-based PA as well. For a given sample size, number of variables, and order of component, one sees what size loadings are likely to occur if the data are completely random.

Note Added in Proof

We are indebted to R. Stewart Longman who brought to our attention some literature on the use of permutation percentiles in PA of which we were unaware. The relevant references are: a presentation by R. A. Harshman and J. R. Reddon at the Annual Meeting of the Classification Society, May, 1983, which predates our work; and articles by R. S. Longman, A. A. Cota, R. R. Holden, G. C. Fekken (Multivariate Behavioral Research, 1989, 24(1), 59-69; Behavior Research Methods, Instruments, and Computers, 1992, 24(3), 493-496, among others), which is concurrent and independent of our work. Our article appeared in a first version as a 1988 Bellcore Technical Memorandum. We wish to apologize to authors of work on permutation percentiles in PA about which we did not know. Out work has merit on its own, (a) by showing robustness of significance of normal variate parallel analysis, (b) by establishing the consequent limited usefulness of permutation methods in practuce, (c) by comparing conventional and resistant principal components and principal factor analysis, and (d) by extending parallel analysis to the assessment of loadings. Not the least, we expect our tables to be of some use.

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Appendix
Medians for Principal Component Eigenvalues

N	*P*	1	2	3	4	5	6	7	8	9	10
20	5	1.67	1.24	0.96	0.69	0.42					
20	7	1.94	1.50	1.18	0.92	0.69	0.47	0.28			
20	10	2.30	1.83	1.49	1.21	0.97	0.76	0.57	0.41	0.27	0.14
20	15	2.85	2.34	1.95	1.65	1.39	1.15	0.95	0.77	0.60	0.46
20	20	3.32	2.77	2.38	2.05	1.76	1.50	1.28	1.07	0.89	0.73
30	5	1.54	1.21	0.97	0.75	0.52					
30	7	1.74	1.40	1.15	0.95	0.76	0.58	0.40			
30	10	2.02	1.67	1.41	1.19	1.00	0.83	0.67	0.53	0.39	0.26
30	15	2.44	2.06	1.78	1.54	1.34	1.16	1.00	0.85	0.72	0.59
30	20	2.82	2.41	2.11	1.86	1.65	1.46	1.28	1.13	0.98	0.85
40	5	1.46	1.18	0.98	0.79	0.58					
40	7	1.63	1.35	1.14	0.97	0.80	0.64	0.47			
40	10	1.87	1.57	1.36	1.17	1.01	0.86	0.72	0.60	0.47	0.34
40	15	2.21	1.90	1.67	1.47	1.31	1.16	1.02	0.89	0.77	0.67
40	20	2.52	2.19	1.95	1.75	1.57	1.42	1.27	1.14	1.01	0.90
50	5	1.40	1.16	0.99	0.81	0.63					
50	7	1.56	1.32	1.13	0.97	0.82	0.68	0.52			
50	10	1.76	1.51	1.32	1.16	1.02	0.89	0.76	0.64	0.53	0.40
50	15	2.06	1.80	1.59	1.43	1.28	1.15	1.03	0.92	0.81	0.71
50	20	2.33	2.05	1.84	1.67	1.52	1.38	1.25	1.13	1.03	0.93
50	30	2.81	2.51	2.29	2.10	1.93	1.79	1.65	1.53	1.41	1.30
75	5	1.33	1.13	0.99	0.86	0.70					
75	7	1.45	1.25	1.10	0.98	0.86	0.74	0.60			
75	10	1.61	1.41	1.26	1.14	1.02	0.92	0.81	0.71	0.61	0.49
75	15	1.84	1.64	1.48	1.35	1.24	1.13	1.03	0.95	0.86	0.78
75	20	2.05	1.84	1.68	1.54	1.42	1.32	1.22	1.13	1.04	0.96
75	30	2.41	2.19	2.02	1.88	1.75	1.64	1.54	1.44	1.36	1.27
75	50	3.04	2.80	2.62	2.47	2.33	2.21	2.10	1.99	1.89	1.80
100	5	1.28	1.11	0.99	0.88	0.73					
100	7	1.38	1.22	1.09	0.99	0.88	0.77	0.65			
100	10	1.52	1.35	1.23	1.12	1.02	0.93	0.84	0.75	0.66	0.56
100	15	1.71	1.54	1.41	1.30	1.20	1.12	1.03	0.95	0.88	0.81
100	20	1.89	1.71	1.58	1.47	1.37	1.28	1.20	1.12	1.05	0.98
100	30	2.19	2.01	1.87	1.75	1.65	1.56	1.47	1.39	1.32	1.25
100	50	2.71	2.51	2.37	2.24	2.13	2.03	1.94	1.86	1.77	1.70
200	5	1.20	1.08	1.00	0.91	0.81					
200	7	1.27	1.15	1.07	0.99	0.92	0.84	0.75			
200	10	1.36	1.25	1.16	1.09	1.02	0.96	0.89	0.83	0.76	0.68
200	15	1.49	1.38	1.29	1.22	1.15	1.09	1.04	0.98	0.93	0.87
200	20	1.60	1.49	1.41	1.33	1.27	1.21	1.15	1.10	1.05	1.00
200	30	1.80	1.68	1.60	1.52	1.46	1.40	1.34	1.29	1.24	1.19
200	50	2.12	2.01	1.92	1.84	1.77	1.71	1.65	1.60	1.55	1.50

Appendix
90% Quantiles for Principal Component Eigenvalues

N	P	1	2	3	4	5	6	7	8	9	10
20	5	1.96	1.41	1.08	0.83	0.57					
20	7	2.23	1.68	1.33	1.04	0.82	0.60	0.39			
20	10	2.61	2.04	1.65	1.35	1.10	0.89	0.69	0.52	0.36	0.22
20	15	3.22	2.58	2.15	1.81	1.53	1.29	1.07	0.88	0.71	0.56
20	20	3.75	3.05	2.60	2.24	1.93	1.66	1.41	1.20	1.01	0.84
30	5	1.76	1.34	1.07	0.86	0.65					
30	7	1.97	1.55	1.27	1.05	0.86	0.68	0.51			
30	10	2.29	1.83	1.53	1.30	1.10	0.93	0.77	0.62	0.48	0.35
30	15	2.74	2.24	1.92	1.67	1.46	1.26	1.10	0.94	0.80	0.68
30	20	3.14	2.61	2.28	2.01	1.77	1.57	1.38	1.23	1.07	0.94
40	5	1.65	1.29	1.07	0.89	0.70					
40	7	1.83	1.48	1.23	1.05	0.89	0.73	0.57			
40	10	2.08	1.71	1.47	1.27	1.10	0.95	0.81	0.68	0.55	0.42
40	15	2.45	2.06	1.79	1.58	1.40	1.24	1.10	0.97	0.85	0.74
40	20	2.78	2.36	2.09	1.87	1.68	1.51	1.36	1.22	1.09	0.97
50	5	1.57	1.26	1.06	0.90	0.73					
50	7	1.73	1.43	1.22	1.05	0.90	0.76	0.61			
50	10	1.95	1.63	1.41	1.24	1.09	0.96	0.84	0.72	0.60	0.48
50	15	2.26	1.93	1.71	1.52	1.37	1.23	1.10	0.99	0.88	0.78
50	20	2.56	2.20	1.96	1.77	1.60	1.46	1.33	1.21	1.10	0.99
50	30	3.04	2.66	2.42	2.22	2.04	1.87	1.73	1.60	1.49	1.37
75	5	1.45	1.21	1.05	0.92	0.78					
75	7	1.59	1.34	1.18	1.04	0.93	0.81	0.69			
75	10	1.76	1.51	1.34	1.20	1.08	0.98	0.87	0.78	0.67	0.57
75	15	2.00	1.75	1.57	1.42	1.31	1.19	1.09	1.00	0.91	0.83
75	20	2.21	1.95	1.77	1.62	1.50	1.38	1.28	1.18	1.10	1.02
75	30	2.59	2.31	2.13	1.97	1.83	1.72	1.61	1.51	1.42	1.32
75	50	3.23	2.94	2.74	2.57	2.42	2.29	2.17	2.06	1.96	1.86
100	5	1.39	1.19	1.04	0.94	0.81					
100	7	1.50	1.30	1.16	1.04	0.94	0.84	0.72			
100	10	1.65	1.44	1.29	1.18	1.08	0.98	0.89	0.81	0.72	0.63
100	15	1.84	1.63	1.49	1.37	1.27	1.18	1.09	1.01	0.93	0.86
100	20	2.02	1.81	1.66	1.54	1.44	1.34	1.25	1.17	1.09	1.02
100	30	2.34	2.11	1.96	1.83	1.71	1.62	1.53	1.45	1.37	1.29
100	50	2.87	2.63	2.46	2.33	2.21	2.10	2.01	1.92	1.83	1.75
200	5	1.27	1.13	1.03	0.96	0.87					
200	7	1.35	1.21	1.11	1.03	0.96	0.89	0.80			
200	10	1.44	1.31	1.21	1.13	1.06	0.99	0.93	0.87	0.80	0.73
200	15	1.58	1.44	1.34	1.26	1.20	1.13	1.07	1.01	0.96	0.91
200	20	1.69	1.55	1.46	1.38	1.31	1.24	1.18	1.13	1.08	1.03
200	30	1.88	1.75	1.65	1.57	1.50	1.44	1.38	1.32	1.27	1.22
200	50	2.22	2.08	1.97	1.89	1.82	1.76	1.69	1.64	1.58	1.53

Appendix

95% Quantiles for Principal Component Eigenvalues

N	P	1	2	3	4	5	6	7	8	9	10
20	5	2.06	1.46	1.11	0.86	0.61					
20	7	2.33	1.73	1.37	1.08	0.85	0.64	0.43			
20	10	2.73	2.11	1.71	1.39	1.14	0.92	0.72	0.55	0.39	0.24
20	15	3.35	2.66	2.21	1.86	1.58	1.33	1.11	0.91	0.74	0.59
20	20	3.88	3.13	2.67	2.29	1.98	1.70	1.45	1.24	1.05	0.87
30	5	1.83	1.38	1.10	0.89	0.69					
30	7	2.05	1.60	1.31	1.08	0.89	0.71	0.54			
30	10	2.38	1.88	1.57	1.34	1.13	0.95	0.80	0.65	0.51	0.37
30	15	2.85	2.30	1.97	1.71	1.49	1.29	1.13	0.97	0.82	0.70
30	20	3.23	2.68	2.33	2.05	1.80	1.60	1.41	1.25	1.10	0.96
40	5	1.71	1.34	1.09	0.92	0.73					
40	7	1.89	1.52	1.26	1.08	0.91	0.76	0.60			
40	10	2.15	1.76	1.50	1.29	1.13	0.97	0.83	0.70	0.58	0.44
40	15	2.52	2.10	1.83	1.61	1.43	1.27	1.13	0.99	0.88	0.76
40	20	2.85	2.42	2.14	1.91	1.70	1.53	1.38	1.24	1.11	1.00
50	5	1.61	1.29	1.08	0.92	0.76					
50	7	1.79	1.46	1.25	1.07	0.92	0.79	0.64			
50	10	2.01	1.67	1.44	1.27	1.12	0.98	0.86	0.74	0.62	0.50
50	15	2.33	1.97	1.74	1.55	1.39	1.25	1.12	1.01	0.90	0.80
50	20	2.62	2.25	1.99	1.80	1.63	1.48	1.35	1.23	1.11	1.01
50	30	3.11	2.72	2.46	2.26	2.07	1.90	1.76	1.63	1.51	1.39
75	5	1.49	1.24	1.07	0.94	0.81					
75	7	1.63	1.37	1.19	1.06	0.94	0.83	0.71			
75	10	1.81	1.54	1.37	1.22	1.10	0.99	0.89	0.79	0.69	0.59
75	15	2.05	1.78	1.59	1.45	1.32	1.21	1.11	1.01	0.93	0.85
75	20	2.25	1.98	1.80	1.65	1.52	1.40	1.30	1.20	1.11	1.03
75	30	2.64	2.35	2.15	2.00	1.85	1.74	1.63	1.53	1.43	1.34
75	50	3.30	2.98	2.77	2.59	2.45	2.31	2.19	2.08	1.98	1.88
100	5	1.43	1.21	1.06	0.95	0.83					
100	7	1.55	1.32	1.17	1.06	0.95	0.85	0.75			
100	10	1.69	1.47	1.31	1.19	1.10	1.00	0.91	0.82	0.73	0.64
100	15	1.88	1.66	1.51	1.39	1.28	1.19	1.10	1.02	0.94	0.87
100	20	2.07	1.84	1.68	1.56	1.45	1.35	1.26	1.19	1.11	1.03
100	30	2.38	2.14	1.98	1.85	1.74	1.63	1.55	1.46	1.38	1.31
100	50	2.91	2.67	2.49	2.35	2.23	2.12	2.02	1.93	1.85	1.76
200	5	1.30	1.15	1.04	0.97	0.88					
200	7	1.37	1.23	1.12	1.04	0.97	0.90	0.82			
200	10	1.47	1.32	1.22	1.14	1.07	1.00	0.94	0.88	0.81	0.74
200	15	1.61	1.46	1.36	1.27	1.21	1.14	1.08	1.02	0.97	0.91
200	20	1.71	1.57	1.47	1.39	1.32	1.25	1.20	1.14	1.09	1.04
200	30	1.91	1.77	1.67	1.59	1.51	1.45	1.39	1.33	1.28	1.23
200	50	2.25	2.10	1.99	1.91	1.84	1.77	1.71	1.65	1.60	1.54

Appendix

99% Quantiles for Principal Component Eigenvalues

N	*P*	1	2	3	4	5	6	7	8	9	10
20	5	2.26	1.55	1.18	0.93	0.69					
20	7	2.49	1.85	1.44	1.15	0.92	0.70	0.49			
20	10	2.92	2.24	1.80	1.46	1.20	0.98	0.79	0.60	0.45	0.29
20	15	3.56	2.82	2.34	1.96	1.68	1.40	1.16	0.98	0.80	0.65
20	20	4.20	3.30	2.78	2.39	2.06	1.79	1.52	1.32	1.10	0.92
30	5	1.97	1.46	1.14	0.93	0.75					
30	7	2.21	1.70	1.38	1.12	0.94	0.76	0.60			
30	10	2.56	1.96	1.64	1.40	1.18	1.01	0.85	0.70	0.55	0.41
30	15	3.04	2.41	2.04	1.78	1.54	1.35	1.17	1.02	0.87	0.75
30	20	3.43	2.78	2.44	2.12	1.87	1.66	1.47	1.30	1.16	1.01
40	5	1.83	1.40	1.14	0.96	0.78					
40	7	2.03	1.59	1.32	1.12	0.96	0.80	0.65			
40	10	2.29	1.85	1.56	1.35	1.17	1.02	0.88	0.75	0.62	0.49
40	15	2.66	2.21	1.90	1.67	1.47	1.32	1.17	1.03	0.92	0.80
40	20	2.99	2.50	2.22	1.97	1.75	1.57	1.42	1.28	1.16	1.04
50	5	1.73	1.35	1.13	0.96	0.81					
50	7	1.91	1.53	1.29	1.11	0.96	0.83	0.68			
50	10	2.13	1.75	1.49	1.31	1.16	1.01	0.89	0.78	0.66	0.54
50	15	2.45	2.06	1.80	1.61	1.44	1.29	1.16	1.04	0.94	0.82
50	20	2.78	2.34	2.07	1.86	1.67	1.53	1.39	1.27	1.15	1.04
50	30	3.25	2.83	2.53	2.32	2.13	1.95	1.81	1.67	1.55	1.43
75	5	1.59	1.30	1.11	0.97	0.84					
75	7	1.71	1.41	1.23	1.10	0.97	0.86	0.75			
75	10	1.90	1.59	1.42	1.26	1.14	1.02	0.92	0.82	0.73	0.63
75	15	2.14	1.84	1.64	1.49	1.36	1.24	1.14	1.04	0.96	0.87
75	20	2.37	2.05	1.84	1.69	1.56	1.44	1.34	1.23	1.14	1.06
75	30	2.75	2.43	2.22	2.05	1.90	1.78	1.66	1.57	1.47	1.38
75	50	3.40	3.06	2.83	2.65	2.49	2.36	2.23	2.12	2.02	1.91
100	5	1.49	1.25	1.09	0.98	0.86					
100	7	1.62	1.37	1.21	1.09	0.98	0.88	0.78			
100	10	1.76	1.52	1.35	1.23	1.12	1.02	0.94	0.85	0.77	0.67
100	15	1.97	1.71	1.55	1.43	1.31	1.22	1.13	1.04	0.97	0.89
100	20	2.16	1.90	1.73	1.59	1.48	1.38	1.29	1.21	1.13	1.06
100	30	2.48	2.19	2.03	1.89	1.77	1.67	1.58	1.49	1.41	1.33
100	50	3.02	2.74	2.55	2.41	2.28	2.15	2.06	1.97	1.88	1.79
200	5	1.35	1.18	1.07	0.98	0.90					
200	7	1.43	1.26	1.15	1.06	0.99	0.92	0.84			
200	10	1.52	1.36	1.25	1.16	1.09	1.02	0.95	0.90	0.83	0.77
200	15	1.66	1.49	1.39	1.30	1.23	1.17	1.10	1.04	0.98	0.93
200	20	1.76	1.60	1.50	1.41	1.34	1.27	1.21	1.16	1.10	1.06
200	30	1.98	1.81	1.70	1.61	1.54	1.47	1.41	1.35	1.30	1.25
200	50	2.31	2.15	2.03	1.94	1.86	1.79	1.73	1.67	1.62	1.56

Appendix
Medians for Absolute Values of Unrotated Principal Component Loadings

N	P	1	2	3	4	5	6	7	8	9	10
20	5	0.57	0.42	0.32	0.32	0.26					
20	7	0.49	0.39	0.30	0.26	0.24	0.21	0.17			
20	10	0.43	0.35	0.31	0.25	0.22	0.20	0.17	0.15	0.12	0.09
20	15	0.36	0.31	0.28	0.24	0.23	0.19	0.17	0.16	0.14	0.13
20	20	0.32	0.26	0.23	0.20	0.19	0.18	0.16	0.14	0.13	0.12
30	5	0.56	0.41	0.32	0.33	0.30					
30	7	0.47	0.38	0.30	0.27	0.26	0.23	0.19			
30	10	0.41	0.33	0.28	0.25	0.21	0.20	0.19	0.17	0.16	0.12
30	15	0.34	0.29	0.26	0.22	0.22	0.19	0.19	0.16	0.15	0.14
30	20	0.30	0.28	0.24	0.22	0.20	0.19	0.18	0.16	0.15	0.14
40	5	0.52	0.42	0.32	0.31	0.33					
40	7	0.44	0.37	0.31	0.27	0.26	0.24	0.22			
40	10	0.38	0.32	0.28	0.25	0.22	0.21	0.20	0.19	0.17	0.15
40	15	0.33	0.28	0.25	0.23	0.20	0.19	0.18	0.17	0.16	0.15
40	20	0.29	0.25	0.22	0.22	0.20	0.19	0.18	0.17	0.15	0.15
50	5	0.53	0.41	0.30	0.35	0.33					
50	7	0.44	0.37	0.30	0.26	0.27	0.25	0.24			
50	10	0.37	0.31	0.27	0.26	0.21	0.22	0.21	0.19	0.18	0.17
50	15	0.29	0.28	0.24	0.22	0.21	0.19	0.18	0.17	0.17	0.16
50	20	0.27	0.25	0.23	0.21	0.19	0.19	0.17	0.17	0.16	0.15
50	30	0.24	0.21	0.20	0.20	0.18	0.18	0.17	0.16	0.15	0.14
75	5	0.51	0.40	0.32	0.36	0.35					
75	7	0.42	0.35	0.30	0.26	0.27	0.27	0.26			
75	10	0.34	0.31	0.27	0.25	0.23	0.22	0.22	0.20	0.19	0.18
75	15	0.29	0.26	0.24	0.22	0.20	0.18	0.19	0.18	0.17	0.16
75	20	0.25	0.24	0.22	0.21	0.19	0.18	0.17	0.16	0.16	0.14
75	30	0.21	0.21	0.19	0.18	0.16	0.16	0.16	0.15	0.15	0.14
75	50	0.18	0.17	0.16	0.16	0.16	0.15	0.14	0.14	0.14	0.13
100	5	0.49	0.41	0.32	0.36	0.35					
100	7	0.41	0.34	0.30	0.27	0.27	0.28	0.27			
100	10	0.35	0.29	0.26	0.24	0.23	0.22	0.21	0.20	0.21	0.20
100	15	0.26	0.23	0.20	0.20	0.18	0.17	0.17	0.16	0.15	0.15
100	20	0.24	0.23	0.21	0.20	0.19	0.18	0.17	0.16	0.16	0.15
100	30	0.20	0.20	0.18	0.18	0.17	0.17	0.16	0.15	0.14	0.14
100	50	0.17	0.16	0.16	0.15	0.15	0.14	0.14	0.14	0.13	0.13
200	5	0.47	0.41	0.33	0.37	0.38					
200	7	0.39	0.34	0.29	0.27	0.27	0.28	0.29			
200	10	0.30	0.29	0.26	0.24	0.23	0.21	0.22	0.22	0.22	0.21
200	15	0.25	0.23	0.22	0.21	0.20	0.19	0.18	0.19	0.18	0.17
200	20	0.23	0.21	0.20	0.19	0.17	0.18	0.17	0.16	0.16	0.15
200	30	0.18	0.18	0.17	0.16	0.16	0.15	0.14	0.14	0.15	0.14
200	50	0.15	0.15	0.14	0.13	0.13	0.13	0.13	0.12	0.12	0.12

Appendix

90% Quantiles for Absolute Values of Unrotated Principal Component Loadings

N	*P*	1	2	3	4	5	6	7	8	9	10
20	5	0.81	0.78	0.74	0.56	0.44					
20	7	0.77	0.74	0.69	0.59	0.51	0.41	0.31			
20	10	0.72	0.68	0.63	0.57	0.51	0.45	0.39	0.33	0.27	0.19
20	15	0.67	0.64	0.60	0.55	0.52	0.44	0.42	0.38	0.34	0.29
20	20	0.64	0.60	0.56	0.53	0.49	0.45	0.42	0.37	0.34	0.30
30	5	0.78	0.77	0.73	0.60	0.48					
30	7	0.73	0.70	0.67	0.62	0.54	0.45	0.37			
30	10	0.69	0.66	0.62	0.56	0.54	0.48	0.42	0.37	0.33	0.26
30	15	0.64	0.59	0.57	0.52	0.49	0.45	0.43	0.39	0.36	0.33
30	20	0.59	0.57	0.53	0.49	0.47	0.44	0.42	0.40	0.36	0.35
40	5	0.77	0.77	0.74	0.62	0.50					
40	7	0.71	0.71	0.66	0.61	0.55	0.47	0.40			
40	10	0.66	0.64	0.61	0.57	0.53	0.49	0.45	0.40	0.34	0.29
40	15	0.61	0.57	0.55	0.51	0.49	0.45	0.45	0.40	0.37	0.35
40	20	0.56	0.54	0.51	0.50	0.46	0.44	0.43	0.39	0.37	0.35
50	5	0.75	0.76	0.73	0.62	0.52					
50	7	0.70	0.69	0.68	0.62	0.56	0.48	0.42			
50	10	0.65	0.62	0.60	0.59	0.51	0.48	0.45	0.41	0.37	0.32
50	15	0.58	0.56	0.54	0.52	0.48	0.46	0.43	0.41	0.39	0.37
50	20	0.55	0.53	0.49	0.49	0.45	0.43	0.42	0.40	0.37	0.36
50	30	0.50	0.46	0.47	0.44	0.42	0.42	0.37	0.37	0.35	0.34
75	5	0.73	0.73	0.75	0.63	0.54					
75	7	0.68	0.67	0.65	0.63	0.58	0.51	0.46			
75	10	0.62	0.61	0.59	0.56	0.52	0.49	0.48	0.42	0.39	0.35
75	15	0.56	0.54	0.51	0.49	0.47	0.44	0.45	0.41	0.39	0.37
75	20	0.52	0.50	0.48	0.46	0.44	0.43	0.40	0.38	0.38	0.35
75	30	0.46	0.45	0.43	0.41	0.38	0.39	0.37	0.36	0.35	0.35
75	50	0.40	0.38	0.38	0.37	0.35	0.34	0.33	0.32	0.33	0.32
100	5	0.72	0.74	0.76	0.65	0.56					
100	7	0.67	0.66	0.66	0.62	0.57	0.52	0.47			
100	10	0.60	0.59	0.58	0.55	0.53	0.50	0.47	0.45	0.42	0.38
100	15	0.53	0.52	0.48	0.47	0.45	0.44	0.43	0.41	0.40	0.39
100	20	0.49	0.47	0.45	0.45	0.44	0.41	0.40	0.39	0.37	0.37
100	30	0.44	0.42	0.40	0.40	0.39	0.38	0.37	0.36	0.35	0.33
100	50	0.38	0.36	0.35	0.34	0.34	0.34	0.33	0.32	0.30	0.31
200	5	0.69	0.72	0.75	0.66	0.58					
200	7	0.64	0.65	0.66	0.63	0.59	0.55	0.50			
200	10	0.56	0.57	0.56	0.56	0.53	0.50	0.49	0.46	0.45	0.41
200	15	0.50	0.48	0.49	0.47	0.47	0.45	0.42	0.44	0.42	0.39
200	20	0.46	0.44	0.44	0.42	0.41	0.41	0.39	0.38	0.37	0.37
200	30	0.40	0.40	0.38	0.37	0.37	0.36	0.34	0.35	0.34	0.33
200	50	0.33	0.34	0.32	0.32	0.31	0.30	0.30	0.30	0.29	0.28

Appendix

95% Quantiles for Absolute Values of Unrotated Principal Component Loadings

N	*P*	1	2	3	4	5	6	7	8	9	10
20	5	0.85	0.84	0.83	0.62	0.47					
20	7	0.81	0.79	0.77	0.69	0.58	0.45	0.35			
20	10	0.77	0.75	0.72	0.68	0.60	0.52	0.45	0.38	0.31	0.22
20	15	0.73	0.71	0.67	0.63	0.60	0.51	0.49	0.44	0.39	0.34
20	20	0.70	0.66	0.64	0.61	0.58	0.53	0.49	0.43	0.39	0.36
30	5	0.82	0.82	0.82	0.65	0.51					
30	7	0.78	0.76	0.76	0.69	0.60	0.51	0.41			
30	10	0.74	0.72	0.69	0.65	0.62	0.56	0.49	0.42	0.37	0.29
30	15	0.70	0.66	0.65	0.60	0.58	0.54	0.51	0.46	0.43	0.38
30	20	0.65	0.64	0.61	0.58	0.54	0.51	0.49	0.47	0.43	0.41
40	5	0.81	0.81	0.82	0.67	0.54					
40	7	0.76	0.76	0.74	0.72	0.62	0.52	0.43			
40	10	0.71	0.70	0.67	0.65	0.63	0.57	0.51	0.45	0.39	0.32
40	15	0.67	0.62	0.62	0.59	0.57	0.53	0.53	0.46	0.43	0.41
40	20	0.62	0.61	0.57	0.57	0.53	0.52	0.50	0.47	0.44	0.41
50	5	0.79	0.81	0.82	0.67	0.55					
50	7	0.74	0.74	0.74	0.71	0.61	0.53	0.46			
50	10	0.70	0.67	0.67	0.67	0.59	0.56	0.51	0.46	0.42	0.35
50	15	0.64	0.62	0.61	0.59	0.56	0.55	0.51	0.48	0.44	0.43
50	20	0.61	0.59	0.56	0.56	0.51	0.50	0.47	0.47	0.44	0.41
50	30	0.56	0.52	0.53	0.50	0.47	0.49	0.44	0.43	0.41	0.40
75	5	0.77	0.79	0.83	0.69	0.57					
75	7	0.73	0.72	0.72	0.72	0.63	0.57	0.49			
75	10	0.67	0.67	0.65	0.64	0.60	0.56	0.54	0.48	0.44	0.40
75	15	0.62	0.60	0.59	0.56	0.55	0.51	0.52	0.48	0.45	0.43
75	20	0.57	0.56	0.53	0.52	0.50	0.50	0.47	0.45	0.44	0.41
75	30	0.52	0.50	0.49	0.47	0.45	0.45	0.44	0.43	0.41	0.40
75	50	0.46	0.43	0.43	0.43	0.41	0.39	0.39	0.39	0.38	0.37
100	5	0.75	0.78	0.84	0.69	0.59					
100	7	0.71	0.72	0.73	0.72	0.63	0.57	0.51			
100	10	0.66	0.65	0.65	0.62	0.61	0.58	0.54	0.49	0.46	0.42
100	15	0.59	0.57	0.55	0.54	0.52	0.51	0.50	0.48	0.46	0.45
100	20	0.55	0.54	0.52	0.52	0.51	0.48	0.48	0.44	0.43	0.43
100	30	0.49	0.48	0.47	0.46	0.45	0.45	0.43	0.43	0.40	0.39
100	50	0.44	0.42	0.40	0.39	0.39	0.40	0.38	0.39	0.36	0.36
200	5	0.73	0.76	0.82	0.71	0.62					
200	7	0.67	0.70	0.73	0.72	0.66	0.60	0.54			
200	10	0.62	0.63	0.63	0.63	0.62	0.59	0.56	0.51	0.49	0.45
200	15	0.55	0.55	0.55	0.54	0.54	0.53	0.51	0.51	0.49	0.45
200	20	0.52	0.50	0.50	0.49	0.47	0.48	0.45	0.45	0.43	0.44
200	30	0.44	0.45	0.43	0.42	0.42	0.42	0.41	0.42	0.40	0.39
200	50	0.38	0.38	0.37	0.37	0.36	0.36	0.36	0.35	0.34	0.34

Appendix

99% Quantiles for Absolute Values of Unrotated Principal Component Loadings

N	*P*	1	2	3	4	5	6	7	8	9	10
20	5	0.89	0.90	0.93	0.70	0.54					
20	7	0.87	0.86	0.87	0.84	0.67	0.53	0.41			
20	10	0.85	0.83	0.81	0.81	0.76	0.64	0.54	0.45	0.36	0.28
20	15	0.81	0.79	0.78	0.77	0.71	0.65	0.61	0.57	0.49	0.41
20	20	0.80	0.76	0.75	0.72	0.72	0.67	0.61	0.55	0.52	0.46
30	5	0.86	0.89	0.93	0.73	0.57					
30	7	0.84	0.85	0.86	0.81	0.70	0.58	0.47			
30	10	0.81	0.81	0.79	0.78	0.74	0.66	0.60	0.50	0.43	0.35
30	15	0.79	0.75	0.76	0.72	0.72	0.70	0.64	0.55	0.55	0.47
30	20	0.74	0.74	0.73	0.70	0.66	0.66	0.61	0.58	0.54	0.52
40	5	0.86	0.88	0.94	0.76	0.59					
40	7	0.81	0.83	0.86	0.86	0.71	0.59	0.49			
40	10	0.78	0.79	0.77	0.77	0.76	0.69	0.61	0.53	0.46	0.38
40	15	0.74	0.72	0.72	0.71	0.69	0.66	0.66	0.57	0.52	0.51
40	20	0.71	0.70	0.68	0.67	0.66	0.63	0.62	0.59	0.56	0.53
50	5	0.84	0.87	0.92	0.75	0.61					
50	7	0.82	0.82	0.84	0.85	0.72	0.60	0.53			
50	10	0.78	0.78	0.77	0.79	0.75	0.69	0.62	0.54	0.49	0.41
50	15	0.73	0.72	0.71	0.70	0.70	0.67	0.63	0.61	0.54	0.50
50	20	0.70	0.69	0.66	0.67	0.62	0.64	0.61	0.58	0.55	0.52
50	30	0.65	0.64	0.65	0.61	0.59	0.59	0.56	0.55	0.52	0.50
75	5	0.82	0.85	0.93	0.76	0.63					
75	7	0.79	0.80	0.83	0.84	0.73	0.62	0.56			
75	10	0.75	0.75	0.73	0.76	0.76	0.69	0.64	0.58	0.50	0.46
75	15	0.70	0.69	0.69	0.68	0.67	0.63	0.63	0.60	0.59	0.53
75	20	0.66	0.66	0.65	0.62	0.63	0.63	0.58	0.58	0.54	0.51
75	30	0.61	0.59	0.59	0.60	0.56	0.57	0.57	0.54	0.52	0.52
75	50	0.55	0.53	0.54	0.55	0.51	0.53	0.49	0.48	0.48	0.47
100	5	0.80	0.86	0.93	0.77	0.65					
100	7	0.77	0.78	0.84	0.84	0.72	0.64	0.56			
100	10	0.72	0.72	0.76	0.75	0.75	0.71	0.65	0.57	0.53	0.48
100	15	0.68	0.68	0.65	0.65	0.67	0.64	0.63	0.63	0.58	0.55
100	20	0.64	0.64	0.62	0.62	0.62	0.62	0.59	0.57	0.58	0.54
100	30	0.59	0.56	0.57	0.55	0.56	0.56	0.54	0.54	0.53	0.49
100	50	0.53	0.51	0.50	0.49	0.49	0.49	0.48	0.48	0.46	0.46
200	5	0.77	0.84	0.92	0.79	0.66					
200	7	0.74	0.78	0.83	0.86	0.75	0.67	0.60			
200	10	0.69	0.69	0.72	0.75	0.74	0.75	0.67	0.62	0.56	0.51
200	15	0.64	0.64	0.64	0.66	0.64	0.64	0.66	0.64	0.60	0.56
200	20	0.59	0.60	0.61	0.59	0.58	0.58	0.58	0.57	0.57	0.54
200	30	0.53	0.53	0.53	0.52	0.54	0.51	0.50	0.51	0.50	0.50
200	50	0.48	0.47	0.45	0.44	0.44	0.45	0.45	0.42	0.42	0.44