Solving Inequalities

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August 7, 2024

Introductory Example

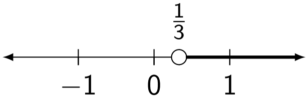
Example. *Find all numbers x such that $3x + 8 > 9$.*

Basic Strategy. Solve for x as if there were an equal sign instead of an inequality. If you multiply (or divide) by a negative number, flip the inequality.

$$3x + 8 > 9 \implies 3x > 1 \implies \boxed{x > \frac{1}{3}}$$

In interval form, our solution is $\boxed{\left(\frac{1}{3}, \infty\right).}$

Illustration.



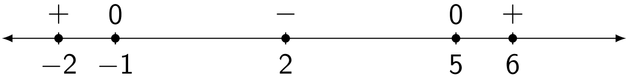
Using a Sign Chart

Example. *Solve the inequality* $x^2 - 4x - 5 < 0$.

Rearrange and factor:

$$x^2 - 4x - 5 < 0 \implies (x-5)(x+1) < 0.$$

We now set up a sign chart. The numbers $x = -1$ and $x = 5$ are labeled with 0, and the sign of $(x-5)(x+1)$ is marked at selected x-values around these numbers:

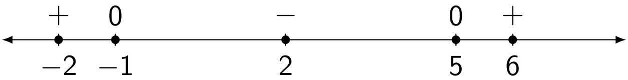


The product is negative on the interval $\boxed{(-1, 5).}$

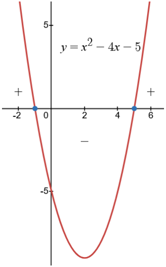
Illustration

To help visualize the sign chart in the previous example, consider the graph of that function $y = x^2 - 4x - 5$.

Sign chart:



Graph:



Notice that the region $(-1, 5)$ is precisely where the function dips below the x-axis.

Two Inequalities

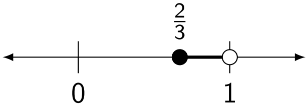
Example. *Solve for x:* $2x < 1 + x \leq 7x - 3$.

Here we have two inequalities, which means

$$2x < 1 + x \quad \textbf{and} \quad 1 + x \leq 7x - 3.$$

Then $2x < 1 + x \implies x < 1$ and $1 + x \leq 7x - 3 \implies x \geq \frac{2}{3}$.

The interval satisfying both of these is $\left[\frac{2}{3}, 1\right)$.



Last Example

Example. *Solve the inequality* $\frac{2}{x-3} < 4$.

Our instinct might be to multiply both sides by $x-3$. However, if $x-3<0$, the direction of the inequality will change.

- $x-3>0$: $2 < 4(x-3) \implies 14 < 4x \implies x > 7/2$
- $x-3<0$: $2 > 4(x-3) \implies 14 > 4x \implies x < 7/2$

In the first case, we have $x > 3$ **and** $x > 7/2$, i.e. $x > 7/2$.
In the second case, we require $x < 3$ **and** $x < 7/2$, i.e. $x < 3$.

The solution is then $\boxed{(-\infty, 3) \cup (7/2, \infty).}$