PreCalculus-Limit at Infinity (Learning Target LF)

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Technique 1: "Guess"

Technique 1: "Guess" the limit works well sometimes and can be tried before applying any other specific technique.

Example

Find $\lim_{x\to\infty} e^{\frac{1}{x}}$

Solution: take x values very big, eg. $x = 1000, 10000, 100000, \cdots$, we can see that the power $\frac{1}{x} \to 0$, so the limit is like e^0, which is 1.

Similarly, by plugging in some values tending to $+\infty$ or $-\infty$, we can get

Note : $\operatorname{sgn}(a) = 1$ if $a > 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$.

1. $\lim_{x\to\infty} \mathbf{e}^x = \infty$ & $\lim_{x\to-\infty} \mathbf{e}^x = 0$
2. $\lim_{x\to\infty} \ln(x) = \infty$ & $\lim_{x\to 0^+} \ln(x) = -\infty$
3. If $r > 0$ then $\lim_{x\to\infty} \frac{b}{x^r} = 0$
4. If $r > 0$ and x^r is real for negative x then $\lim_{x\to-\infty} \frac{b}{x^r} = 0$
5. n even : $\lim_{x\to\pm\infty} x^n = \infty$
6. n odd : $\lim_{x\to\infty} x^n = \infty$ & $\lim_{x\to-\infty} x^n = -\infty$
7. n even : $\lim_{x\to\pm\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
8. n odd : $\lim_{x\to\infty} a\,x^n + \cdots + b\,x + c = \operatorname{sgn}(a)\infty$
9. n odd : $\lim_{x\to-\infty} a\,x^n + \cdots + c\,x + d = -\operatorname{sgn}(a)\infty$

Technique 2: Compare the Growth Speed

Find limits of the type of $\pm\frac{\infty}{\infty}$ by Technique 2

when $x \to \infty$, here is a list of functions in order of their rate of growth to $+\infty$, quickest to slowest:

$$x!, \cdots, 4^x, 3^x, e^x, 1.5^x, \cdots, x^4, x^3, x^2, x * logx, x, logx, \cdots, 3, 2, 1$$

by category, it is

$$factorial \gg exponential \gg algebraic \gg logarithmic \gg constant$$

Then $\lim_{x \to \pm\infty} \frac{n(x)}{d(x)} = \lim_{x \to \pm\infty} \frac{\text{dominant term of } n(x)}{\text{dominant term of } d(x)}$,

Example

$\lim_{x \to \infty} \frac{4^x - x^7 + 2x}{x! + x^{10} - 1} = \lim_{x \to \infty} \frac{4^x}{x!}$

Find limits of the type of $\pm\frac{\infty}{\infty}$ by Technique 2

Case 1: If the numerator $n(x)$ grows faster than the denominator $d(x)$, then $\lim_{x\to\pm\infty} \frac{n(x)}{d(x)} = +\infty$ or $-\infty$

Examples:

a. $\lim_{x\to\infty} \frac{3x^5+x^2-5}{6+x+7x^2} = \lim_{x\to\infty} \frac{3x^5}{7x^2} = \lim_{x\to\infty} \frac{3}{7}x^3 = \infty$

whereas $\lim_{x\to-\infty} \frac{3x^5+x^2-5}{6+x+7x^2} = \lim_{x\to-\infty} \frac{3}{7}x^3 = -\infty$

b. $\lim_{x\to\infty} \frac{x!-\log x+1}{2^x+3x^5-5\log x} = \lim_{x\to\infty} \frac{x!}{2^x} = +\infty$ [Note that $\lim_{x\to-\infty} \frac{x!-\log x+1}{2^x+3x^5-5\log x}$ is meaningless since $x!$ is not defined for negative values.]

c. $\lim_{x\to\infty} 2x^4 - 5x + 3 = +\infty$ (Note that $\lim_{x\to-\infty} 2x^4 - 5x + 3 = +\infty$ as well [why?])

Find limits of the type of $\pm\frac{\infty}{\infty}$ by Technique 2

Case 2: If the numerator $n(x)$ grows slower than the denominator $d(x)$, then $\lim_{x\to\pm\infty} \frac{n(x)}{d(x)} = 0$

Examples:

a. $\lim_{x\to\infty} \frac{5x^3-2x^2+7x-13}{12-2x+x^4} = \lim_{x\to\infty} \frac{5x^3}{x^4} = \lim_{x\to\infty} \frac{5}{x} = 0$

b. $\lim_{x\infty\to} \frac{6x\log x+5x-2}{2^x\mp 2x-14\log x} = \lim_{x\to\infty} \frac{6x\log x}{2^x} = 0$

Case 3: If the numerator $n(x)$ grows the same as the denominator $d(x)$, then $\lim_{x\to\pm\infty} \frac{n(x)}{d(x)} = \frac{\text{leading coefficients of } n(x)}{\text{leading coefficients of } d(x)}$

Examples:

a. $\lim_{x\to\infty} \frac{3x^3-2x^2+x-4}{4+2x-5x^3} = \lim_{x\to\infty} \frac{3x^3}{-5x^3} = \lim_{x\to\infty} \frac{3}{-5} = -\frac{3}{5}$

b. $\lim_{x\to\infty} \frac{\frac{1}{2}2^x}{\frac{3}{4}2^x-\log x+4} = \frac{1/2}{3/4} = \frac{2}{3}$

Technique 3: L'Hôpital's Rule

Find limits of the type of $\pm\frac{\infty}{\infty}$ by Technique 3: L'Hôpital's Rule

If the limit $\lim\limits_{x\to\pm\infty}\frac{f(x)}{g(x)}$ is the type of $\pm\frac{\infty}{\infty}$, and $\lim\limits_{x\to\pm\infty}\frac{f'(x)}{g'(x)}$ exist and $g'(x)\neq 0$, then

$$\lim_{x\to\pm\infty}\frac{f(x)}{g(x)}=\lim_{x\to\pm\infty}\frac{f'(x)}{g'(x)}$$

Example

Find the limit $\lim_{x\to\infty}\frac{x^2}{2^x}$

$$\lim_{x\to\infty}\frac{x^2}{2^x}=\frac{\infty}{\infty}$$

Using L'Hôpital's Rule:

$$\lim_{x\to\infty}\frac{x^2}{2^x}=\lim_{x\to\infty}\frac{2x}{2^x ln2}=\frac{2}{ln2}\lim_{x\to\infty}\frac{x}{2^x}=\frac{\infty}{\infty}$$

Using L'Hôpital's Rule again:

$$\frac{2}{ln2}\lim_{x\to\infty}\frac{x}{2^x}=\frac{2}{ln2}\lim_{x\to\infty}\frac{1}{2^x ln2}=\frac{2}{(ln2)^2}\lim_{x\to\infty}\frac{1}{2^x}=\frac{2}{(ln2)^2}*0=0$$