Understands Inverse Functions

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Functions

A function is such that any input in the domain results in exactly one output in the range.

For example:

$f(x) = 2x$

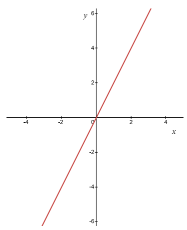
$f(x) = x^2$

$f(x) = e^x$

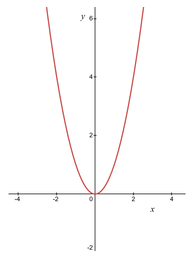
$f(x) = \sin x$

Function Graphs

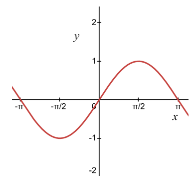
Functions can be graphed as $y = f(x)$.



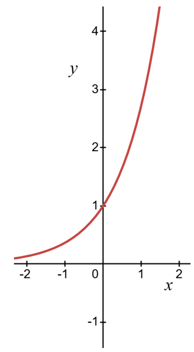
$y = 2x$



$y = x^2$



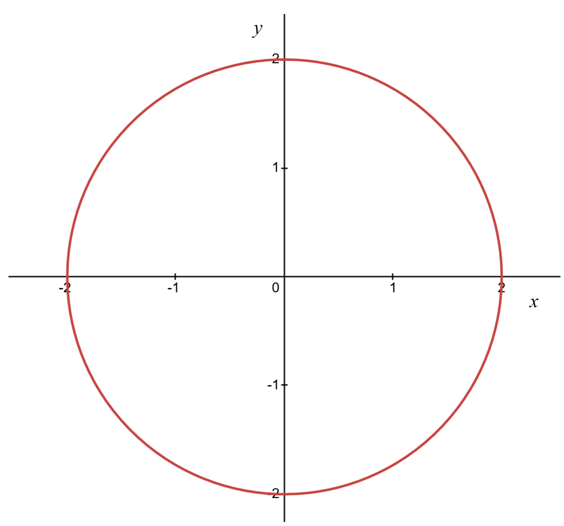
$y = \sin x$



$y = e^x$

Other Graphs

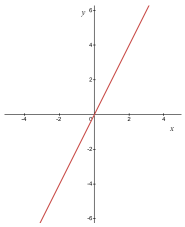
Not all graphs are graphs of functions.



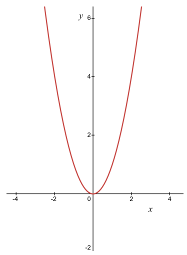
$x^2 + y^2 = 4$

Inverting

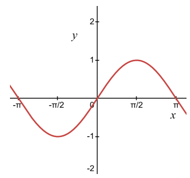
What if we know the output of a function, and we want to know the input?



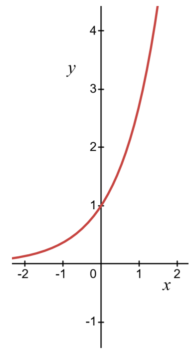
$y = 2x$



$y = x^2$



$y = \sin x$



$y = e^x$

If $y = \frac{1}{2}$, then what is $x =$?

One-to-One Functions

A function is one-to-one if

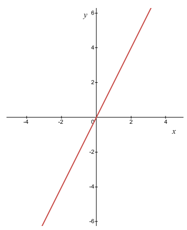
$f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$

Some functions are one-to-one and some are not.

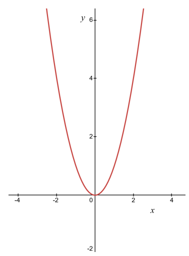
One-to-One?

Which functions are one-to-one?

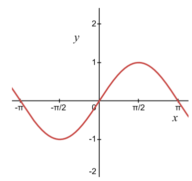
Use the horizontal line test.



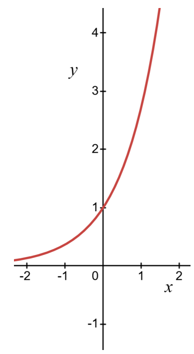
$y = 2x$



$y = x^2$



$y = \sin x$



$y = e^x$

Inverse Functions

One-to-One functions, $f(x)$, have inverse functions, $f^{-1}(x)$.

Definition: $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

Be careful. $f^{-1}(x)$ is the inverse of $f(x)$, while $(f(x))^{-1}$ is the reciprocal of $f(x)$.

Determining Inverse Functions

1. Write $y = f(x)$.
2. Interchange x and y.
3. Solve for y in terms of x, if possible.
4. Now, $y = f^{-1}(x)$.

Example: Find an equation for the inverse of $f(x) = \sqrt{x-5}$.

1. $y = \sqrt{x-5}$
2. $x = \sqrt{y-5}$
3. $x^2 = y - 5$
   $y = x^2 + 5$
4. $f^{-1}(x) = x^2 + 5$

Domain and Range

	Domain	**Range**
$f(x)$	A	B
$f^{-1}(x)$	B	A

For example:

	Domain	**Range**
$f(x) = \sqrt{x-5}$	$[5, \infty)$	$[0, \infty)$
$f^{-1}(x) = x^2 + 5$	$[0, \infty)$	$[5, \infty)$

Cancellation Equations

$f^{-1}(f(x)) = x$ for every x in the domain of $f(x)$.

$f(f^{-1}(x)) = x$ for every x in the domain of $f^{-1}(x)$.

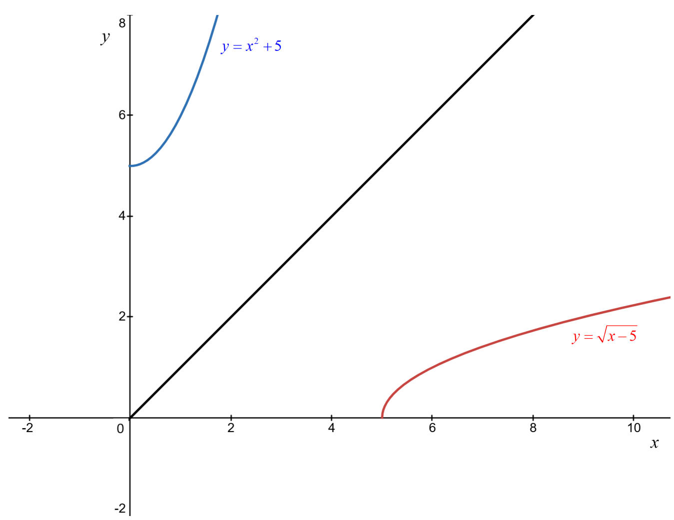
For example:

$(\sqrt{x-5})^2 + 5 = x$ for all x in $[5, \infty)$.

$\sqrt{(x^2+5)-5} = x$ for all x in $[0, \infty)$.

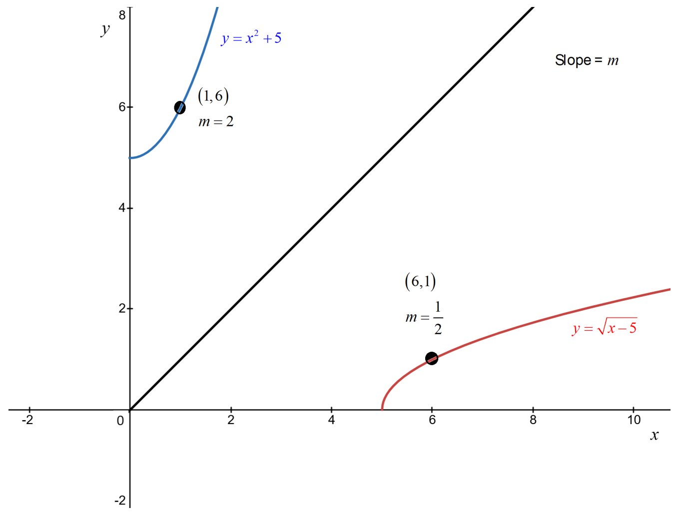
Graphing Inverse Functions

The graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the $y = x$ line.

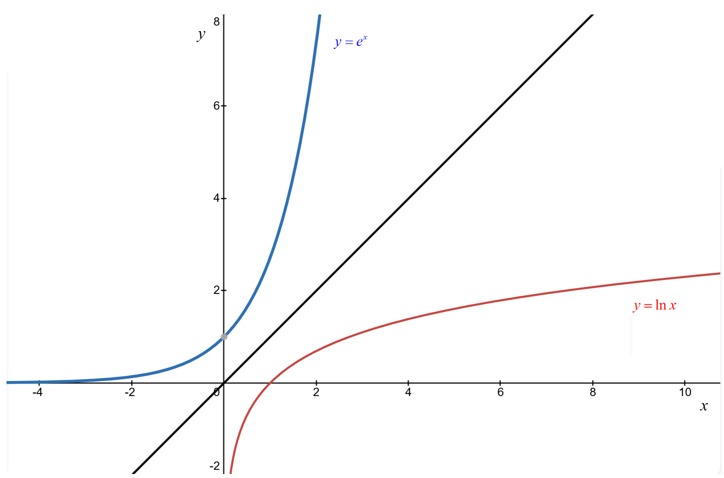


Slope

Slopes of corresponding points are reciprocals.



$\ln x$ and e^x



Cancellation Equations:

$$\ln(e^x) = x \text{ for all } x$$

$$e^{\ln x} = x \text{ for } x \in (0, \infty)$$

Example

Use Cancellation Equations to find the inverse function for $f(x) = \ln(x+2)$.

Solution:

$y = \ln(x+2)$ (Original function)

$x = \ln(y+2)$ (Interchange x and y.)

$e^x = e^{\ln(y+2)}$ (Apply exponential function to both sides.)

$e^x = y + 2$ (Simplify.)

$y = e^x - 2$ (Solve for y.)

$\boxed{f^{-1}(x) = e^x - 2}$ (Solution)

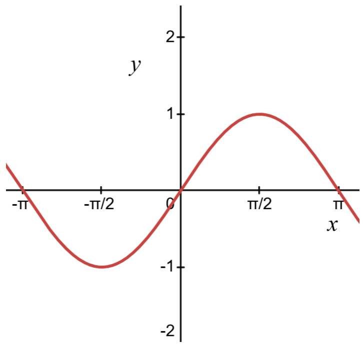
Trigonometric Functions

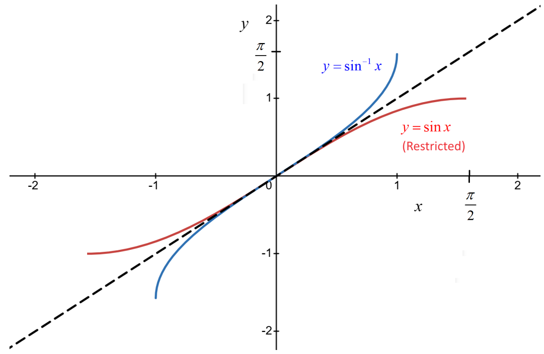
The trigonometric functions are not one-to-one.

So if we wish to have an inverse function, we must restrict the domain of the trig function to a portion that is one-to-one.

Inverse of $\sin x$

Restricted domain $[-\frac{\pi}{2}, \frac{\pi}{2}]$





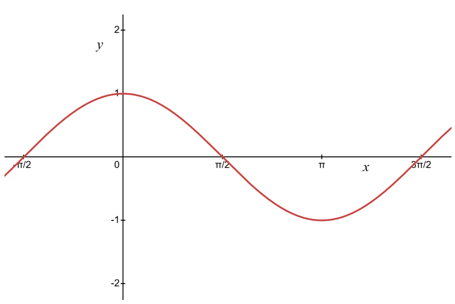
Cancellation equations:

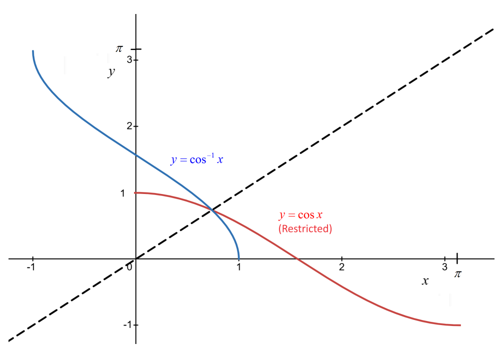
$$\sin(\sin^{-1}(x)) = x \text{ for } x \in [-1, 1]$$

$$\sin^{-1}(\sin(x)) = x \text{ for } x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Inverse of $\cos x$

Restricted domain $[0, \pi]$





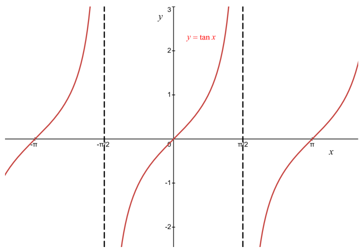
Cancellation equations:

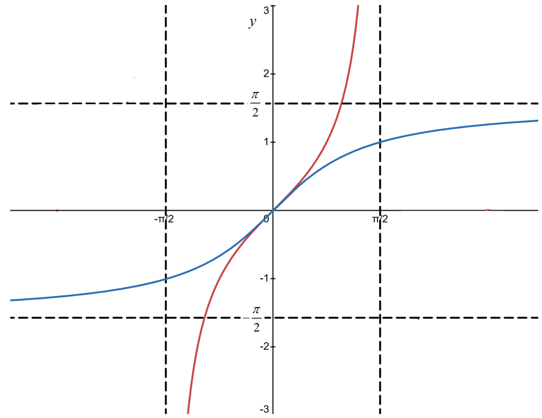
$$\cos(\cos^{-1}(x)) = x \text{ for } x \in [-1, 1]$$

$$\cos^{-1}(\cos(x)) = x \text{ for } x \in [0, \pi]$$

Inverse of $\tan x$

Restricted domain $(-\frac{\pi}{2}, \frac{\pi}{2})$





Cancellation equations:

$$\tan(\tan^{-1}(x)) = x \text{ for all } x.$$

$$\tan^{-1}(\tan(x)) = x \text{ for } x \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2})$$

$\sec x$, $\csc x$, $\cot x$

What are the restricted domains for $\sec x$, $\csc x$, and $\cot x$?