PreCalculus-Limit at Discontinuity (Learning Target LC)

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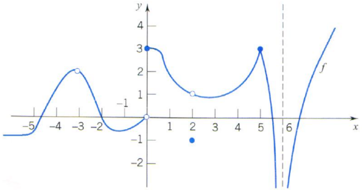
Continuity

Continuity: Continuity of a graph is loosely defined as the ability to draw a graph without having to lift your pencil.



$x = -3$ Discontinuous at this point *The value is not defined at -3* "Removable discontinuity"	$x = 0$ Discontinuous at this point *The limit of the left is not equal to the limit from the right* "Jump discontinuity"	$x = 2$ Discontinuous at this point *The limit from the left is equal to the right, but is not equal to the value of the function* "Removable discontinuity"
$x = 4$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 5$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 6$ Discontinuous at this point *The value of the limit is equal to negative infinity and therefore not defined* "Infinite discontinuity"

Limit for Continuous Functions

Technique : If $f(x)$ is continuous at a then $\lim\limits_{x \to a} f(x) = f(a)$

Partial list of continuous functions and the values of x for which they are continuous.

1. Polynomials for all x.
2. Rational function, except for x's that give division by zero.
3. $\sqrt[n]{x}$ (n odd) for all x.
4. $\sqrt[n]{x}$ (n even) for all $x \geq 0$.
5. $\mathbf{e}^x$ for all x.
6. $\ln x$ for $x > 0$.
7. $\cos(x)$ and $\sin(x)$ for all x.
8. $\tan(x)$ and $\sec(x)$ provided $x \neq \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \cdots$
9. $\cot(x)$ and $\csc(x)$ provided $x \neq \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots$

Examples:

(1). $\lim\limits_{x \to 5} \frac{x^2+1}{x-4} = \frac{5^2+1}{5-1} = 26$

(2). $\lim\limits_{x \to 2} \frac{2e^x}{\ln(x+1)} = \frac{2e^2}{\ln 3}$

(3). $\lim\limits_{x \to \pi} \sin(3x) = \sin(3\pi) = 0$

Limit for Discontinuous Functions - Guess

"Guess" the limit works well sometimes and can be tried before applying any specific technique.

Example

- Find $\lim_{x \to 0^+} \frac{e^x}{x^2}$

  Solution:. take x values great than 0, but very close to 0, eg. $x = 0.001, 0.0001, 0.00001, \cdots$, we can see that the numerator $e^x \to 1$, and the denominator $x^2 \to 0^+$, so the limit is like $\frac{1}{0^+}$, which is $+\infty$

- Find $\lim_{x \to 1^-} \frac{\ln(1-x)}{2e^{x-1}}$

  Solution:. take x values smaller than 1, but very close to 1, eg. $x = 0.9, 0.99, 0.999, \cdots$, we can see that the numerator $\ln(1-x) \to -\infty$, and the denominator $2e^{x-1} \to 2$, so the limit is like $\frac{-\infty}{2}$, which is $-\infty$

Limit for Discontinuous Functions

Technique 1: Factor and Cancel

$$\lim_{x\to 2}\frac{x^2+4x-12}{x^2-2x}=\lim_{x\to 2}\frac{(x-2)(x+6)}{x(x-2)}$$
$$=\lim_{x\to 2}\frac{x+6}{x}=\frac{8}{2}=4$$

Technique 2: Rationalize Numerator/Denominator

$$\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}=\lim_{x\to 9}\frac{3-\sqrt{x}}{x^2-81}\,\frac{3+\sqrt{x}}{3+\sqrt{x}}$$
$$=\lim_{x\to 9}\frac{9-x}{(x^2-81)(3+\sqrt{x})}=\lim_{x\to 9}\frac{-1}{(x+9)(3+\sqrt{x})}$$
$$=\frac{-1}{(18)(6)}=-\frac{1}{108}$$

Technique 3: Combine Rational Expressions

$$\lim_{h\to 0}\frac{1}{h}\left(\frac{1}{x+h}-\frac{1}{x}\right)=\lim_{h\to 0}\frac{1}{h}\left(\frac{x-(x+h)}{x(x+h)}\right)$$
$$=\lim_{h\to 0}\frac{1}{h}\left(\frac{-h}{x(x+h)}\right)=\lim_{h\to 0}\frac{-1}{x(x+h)}=-\frac{1}{x^2}$$

Technique 4: L'Hospital's Rule

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is in the form of $\frac{0}{0}$, then $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$ when this limit exists and $g'(x)\neq 0$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\frac{0}{0}=\lim_{x\to 0}\frac{e^x}{1}$$

$$\lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{e^x}{1}=\frac{1}{1}=1$$

Examples

Note: sometimes you need to do some algebraic manipulation to apply the above techniques.

Find $\lim_{x \to 0^+} (x * \ln x)$

Find $\lim_{x \to 2} (\frac{4}{x^2 - 4} - \frac{1}{x - 2})$