Solving Equations with Square Roots and Fractions

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Some Important Rules

- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)(x-y) = x^2 - y^2$
- $\sqrt{xy} = \sqrt{x}\sqrt{y}, \qquad x, y \geq 0$
- $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}, \qquad x \geq 0,\ y > 0$
- $\frac{x+y}{z} = \frac{x}{z} + \frac{y}{z}, \qquad z \neq 0$

Beware the Shortcut

Certain "simplifications" are simply untrue. Here is one of the most egregious examples.

$$\sqrt{x^2 + y^2} \neq x + y$$

Why? What are you really saying if you claim the two are equal?
Squaring both sides:

$$x^2 + y^2 \quad \text{vs.} \quad (x + y)^2$$

$$x^2 + y^2 \quad \text{vs.} \quad x^2 + 2xy + y^2$$

This is only true if x or y is 0.

An Equation with a Square Root

Example. *Solve for x:* $\sqrt{4x+7} = x + 2$.

<u>Solution</u>. Square both sides, being careful to square the sum $x + 2$ correctly:

$$\begin{aligned} 4x + 7 = (x+2)^2 &= x^2 + 4x + 4 \\ \implies x^2 - 3 &= 0 \\ \implies x &= \pm\sqrt{3}. \end{aligned}$$

Two Square Roots

Example. *Solve for x:* $\sqrt{x} + \sqrt{x+5} = 3$.

<u>Solution</u>.

$$\sqrt{x} + \sqrt{x+5} = 3 \implies \sqrt{x} = 3 - \sqrt{x+5}$$

$$\implies x = 9 - 6\sqrt{x+5} + x + 5$$

$$\implies 0 = -6\sqrt{x+5} + 14$$

$$\implies x + 5 = \left(\frac{7}{3}\right)^2$$

$$\implies x = \frac{4}{9}$$

It is always worth confirming that the final answer works in the original equation. Occasionally, squaring will add an extraneous solution.

Three Square Roots

Example. *Solve for x:* $\sqrt{x} + \sqrt{2x+1} = \sqrt{7x-3}$.

Solution. Squaring both sides,

$$\sqrt{x}+\sqrt{2x+1} = \sqrt{7x-3} \implies x+2\sqrt{x}\sqrt{2x+1}+2x+1 = 7x-3$$

$$\implies 4x-4 = 2\sqrt{2x^2+x}$$

$$\implies (2x-2)^2 = 2x^2+x$$

$$\implies 2x^2-9x+4=0$$

$$\implies (2x-1)(x-4)=0$$

$$\implies x = \frac{1}{2}, 4.$$

By inspection, we see that $x = \frac{1}{2}$ is not a solution to the original equation, hence the solution is $x = 4$.

Multiplying by the Conjugate

Example. *Simplify the expression:* $\dfrac{1}{\sqrt{7}-\sqrt{3}}$.

Solution. The conjugate of $a+b$ is $a-b$. When we multiply a conjugate pair, we get $(a+b)(a-b) = a^2 - b^2$. This can be useful in simplifying expressions where one or more terms has a square root.

$$\begin{aligned} \frac{1}{\sqrt{7}-\sqrt{3}} &= \frac{1}{\sqrt{7}-\sqrt{3}} \cdot \frac{\sqrt{7}+\sqrt{3}}{\sqrt{7}+\sqrt{3}} \\ &= \frac{\sqrt{7}+\sqrt{3}}{7-3} \\ &= \boxed{\frac{1}{4}(\sqrt{7}+\sqrt{3})} \end{aligned}$$

Common Denominators

Example. *Solve for x:* $\dfrac{4}{x-1} - \dfrac{5}{x+1} = 3.$

Solution.

$$\frac{4}{x-1} - \frac{5}{x+1} = 3 \implies \frac{4(x+1)}{x^2-1} - \frac{5(x-1)}{x^2-1} = \frac{3(x^2-1)}{x^2-1}.$$

Comparing numerators, we have

$$4x + 4 - 5x + 5 = 3x^2 - 3 \implies 3x^2 + x - 12 = 0$$

Using the quadratic formula, we have $x = \dfrac{-1 \pm \sqrt{145}}{6}$.

Another Pitfall

Although $\dfrac{x+y}{z} = \dfrac{x}{z} + \dfrac{y}{z}, \quad z \neq 0$ (i.e. each term in the numerator can be put into a different fraction), a sum in the denominator cannot be split:

$$\frac{x}{y+z} \neq \frac{x}{y} + \frac{x}{z}.$$

You can plug in simple values to convince yourself of this: $x = y = z = 1$ gives us $\frac{1}{2} \stackrel{?}{=} 1 + 1$, which is clearly false.

Simplifying Fractions

Example. *Write as a single fraction:* $\dfrac{\dfrac{1}{x+1} - \dfrac{2}{x+2}}{\dfrac{3}{x+3}}$.

<u>Solution</u>. Finding a common denominator, we have

$$\begin{aligned} \frac{\dfrac{(x+2)-2(x+1)}{x^2+3x+2}}{\dfrac{3}{x+3}} &= \left(\frac{-x}{x^2+3x+2}\right)\left(\frac{x+3}{3}\right) \\ &= \frac{-x^2-3x}{3(x^2+3x+2)} \end{aligned}$$