1. $a.true$

$f(n) = O(f(n))^2$

$f(n) \le c \cdot (f(n))^2$

$f(n) \rightarrow monotically\ increazing$

$f(n) \le f(n) \cdot f(n)\ c = 1, \forall n > n_0$

$b.false$

$if\ \ f(n) = O(N) \rightarrow 2^{f(n)} = O(2^n)$

$f(n) = 2n$

$2n \le c \cdot n, \forall c \ge 2$

$2^{2n} = 2^n \cdot 2^n \succ c \cdot 2^n, \forall c - \text{permanent}$

$2^{2n} \ne O(2^n)$

$c.true$

$\max(f(n), g(n)) = \Theta(f(n) + g(n))$

$\textbf{without loss of generality}\ \ we\ will\ assume\ f(n) \le g(n)$

$\max(f(n), g(n)) = g(n), c_1 = \frac{1}{2}, c_2 = 2$

$\frac{1}{2} \cdot (f(n) + g(n)) \le g(n) \le 2 \cdot (f(n) + g(n))$

$g(n) = \Theta(f(n) + g(n))$

$d.false$

$f(n) = \Omega(\log n)\ \&\ (f(n))^2 = \mathrm{O}(f(n))$

$\exists c\ f(n) \ge c \cdot \log n \rightarrow f(n) \ne \text{permanent}$

$(f(n))^2 = f(n) \cdot f(n) \succ d \cdot f(n), \forall d - \text{permanent}$

$(f(n))^2 \ne \mathrm{O}(f(n))$

$e.true$

$if\ \ f(g(n)) = \mathrm{O}(n)\ \&\ f(n) = \Omega(n) \rightarrow g(n) = \mathrm{O}(n)$

$\exists c\ f(g(n)) \le c \cdot n, \exists d\ f(n) \ge d \cdot n$

$f(g(n)) \le c \cdot n \le d \cdot n \le f(n)$

$f\ \ monotically\ increazing \rightarrow \exists e, g(n) \le e \cdot n,)$

$g(n) = \mathrm{O}(n)$

$2.\ a. (n+1)^5 = O(n)^5$

$(n+1)^5 < (n+n)^5 = (2n)^5 = 32n^5 < c \cdot n^5$

$\forall c > 32$

$b.\ 3n \cdot \log_2 n + 2n = \Omega(n \cdot \log_2 n)$

$3(n \log_2 n) + 2n > c(n \log_2 n)$

$\exists c = 1$

$c.\ n(2^n + 3^{n+1})^2 = O(n \cdot 9^n)$

$n(2^{2n} + 2 \cdot 2^n \cdot 3^{n+1} + 3^{2n+2}) < c \cdot n \cdot 3^{2n} / : n > 0$

$3^{2n}((\frac{2}{3})^{2n} + \frac{2^{n+1} \cdot 3^{n+1}}{3^{2n}} + \frac{3^{2n+2}}{3^{2n}}) < c \cdot n \cdot 3^{2n} / : 3^{2n} > 0$

$3^2 < c \cdot n$

$c, n \geq 3$

$d.\ \log_3 2^n = O(n)$

$n \log_3 2 \leq c \cdot n$

$\log_3 2 < 1, n > 1$

$c \geq 1$

$e.\ n^2 - 10n = \Omega(n^2)$

$n^2 - 10n \geq c \cdot n^2$

$c = \frac{1}{2}$

$n^2 - 10n \geq \frac{1}{2} n^2$

$n(\frac{1}{2} n - 10) \geq 0$

$n < 0 \,||\, n > 20$

$n \in \mathrm{N} \, so \, n > 20$

$f.\ (2^n + 3^{n+1})^2 = O(n \cdot 9^n)$

$(2^{2n} + 2 \cdot 2^n \cdot 3^{n+1} + 3^{2n+2}) \leq c \cdot n \cdot 3^{2n}$

$3^{2n}((\frac{2}{3})^{2n} + \frac{2^{n+1} \cdot 3^{n+1}}{3^{2n}} + \frac{3^{2n+2}}{3^{2n}}) \leq c \cdot n \cdot 3^{2n} / : 3^{2n} > 0$

$3^2 \leq c \cdot n$

$c, n \geq 3$

$$3.\ a. T(n) = T(\frac{2}{3}n) + T(\frac{1}{3}n)$$

$$\text{we will prove in induction } T(n) = \Theta(n)$$

$$base: n = 1 \quad T(1) = 1$$

$$assumption: \forall m < n \ \ T(m) = \Theta(m)$$

$$induction\ step: T(n) = T(\frac{2}{3}n) + T(\frac{1}{3}n) = \frac{2}{3}n + \frac{1}{3}n = n$$

$$b. T(n) = 6T(\frac{1}{2}n) + n^3$$

$$\text{with the master theory}:$$

$$a = 6 \ \ b = 2 \ \ f(n) = n^3$$

$$n^3 = \Omega n^{\log_2 6 + 0.1} \ e = 0.1 \ c = 1$$

$$6(\frac{1}{2}n)^3 < 2n^3 \quad c = 2$$

$$T(n) = \Theta(n^3)$$

$$c.T(n) = 5T(\frac{1}{2}n) + n^3 \log n$$

$$with\ the\ master\ theory:$$

$$a = 5\ \ b = 2\ \ f(n) = n^3 \log n$$

$$n^3 \log n = \Omega n^{\log_2 5+0.1}\ \ e = 0.1\ c = 1$$

$$5(\frac{1}{2}n)^3 \log(\frac{1}{2}n) < cn^3 \log n$$

$$\frac{5}{8}n^3(\log n - \log_2 2) < 2n^3 \log n$$

$$T(n) = \Theta(n^3 \log n)$$

$$f.T(n) = 2T(\frac{1}{2}n) + n^3$$

$$with\ the\ master\ theory:$$

$$a = 2\ \ b = 2\ \ f(n) = n^3$$

$$n^3 = \Omega n^{\log_2 2+0.1}\ \ e = 0.1$$

$$2(\frac{1}{2}n)^3 < 2n^3\ \ \ c = 2$$

$$T(n) = \Theta(n^3)$$

$$g.T(n) = 2T(\frac{1}{4}n) + 1$$

$$with\ the\ master\ theory:$$

$$a = 2\ \ b = 4\ \ f(n) = 1$$

$$1 = O(n^{\log_4 2-0.1})\ e = 0.1\ c = 1$$

$$T(n) = \Theta(\sqrt{n})$$

$h. T(n) = T(n-1) + n$

$we\ will\ prove\ in\ induction\ T(n) = \Theta(n^2)$

$base: n = 1 \quad T(1) = 1$

$assumption: \forall m < n \ \ t(m) = \Theta(m^2)$

$induction\ step: T(n) = T(n-1) + n = (n-1)^2 + n = n^2 - n + 1$

$$\frac{1}{3}n^2 \le n^2 - \frac{1}{2}n^2 + 1 \le n^2 - n + 1 \le n^2 \ \ c_1 = \frac{1}{3} \ \ c_2 = 1$$

$conclusion\ T(n) = \Theta(n^2)$

$i. T(n) = T(\sqrt{n}) + 1$

$we\ will\ mark: m = \log n \rightarrow n = 2^m$

$we\ will\ mark: S(m) = T(2^m)$

$$S(m) = S(\frac{m}{2}) + 1$$

$with\ the\ master\ theory:$

$a = 1 \ \ b = 2 \ \ f(n) = 1$

$1 \in \Theta(n^{\log_2 1}) \ \ c = 1$

$S(m) = \Theta(\log m)$

$T(n) = \Theta(\log(\log n))$

$4. T(n) = 16T(\frac{n}{4}) + f(n) \quad [a = 16, b = 4]$

$a.$ *there is no* $f(n)$ *that will be equal to* $\Theta(n\log n)$ *because the* min *imum value that could be is cons* tan *t and it is bigger then* $n\log n$ *by the master theory*

$b. f(n) = O(1)$, *cons* tan *t so by the master theory* : $n^{\log_4 16} = n^2 . c = 1$

$T(n) = \Theta(n^2)$

$c.$ *if* $f(n) = n^2$ *so by the master theory* $c1, c2 = 1$ *so* $f(n) = \Theta(f(n)\log n)$

so $f(n) = n^2 \log n$

$d.$ *if* $f(n) = n^4$, *so by the master theory* $n^4 \in \Omega(n^{\log_4 16+\varepsilon})\ c = 1\ \varepsilon = 1 \to n^4 \geq n^3$

$$16\frac{n^4}{16^3} = \frac{n^4}{64} \leq \frac{1}{2}n^4 \quad c = \frac{1}{2} \to T(n) = \Theta(n^4)$$

5. $a.$

$i = 1 \to j = (n...2)$

$i = 2 \to j = (n...3)$

$..i = k \to j = (n...k+1)$

$$T(n) = n-1+n-2+...+n-n = n^2 - 1(1+2+..+n) = n^2 - \frac{n(1+n)}{2} = \frac{n^2}{2} - \frac{n}{2}$$

$$c_1 = \frac{1}{8}\ n_0 = 2, \frac{n^2}{8} \leq \frac{n^2}{2} - \frac{n}{2} \leq 2n^2,\ c_2 = 2\ n_0 = 2$$

$T(n) = \Theta(n^2)$

$b.$ *this is a recursion function so we will decribe it's complexity* :

$T(n) = T(n-1) + c$, c *is acons* tan $t > 0$

$= T(n-2) + 2c = T(n-k) + kc = nc$

$$\frac{1}{c}n \leq cn \leq 2cn,\ c_1 = \frac{1}{c}\ c_2 = 2c \to T(n) = \Theta(n)$$

$c.$ *this is a recursion function so we will decribe it's complexity* :

$T(n) = T(n-1) + c_1\ n\%2 \neq 0 \,\|\, T(\frac{n}{2}) + c_2\ n\%2 = 0 \quad c_1, c_2$ *cons* tan *ts*

we will choose $n\%2 \neq 0$ *so* :

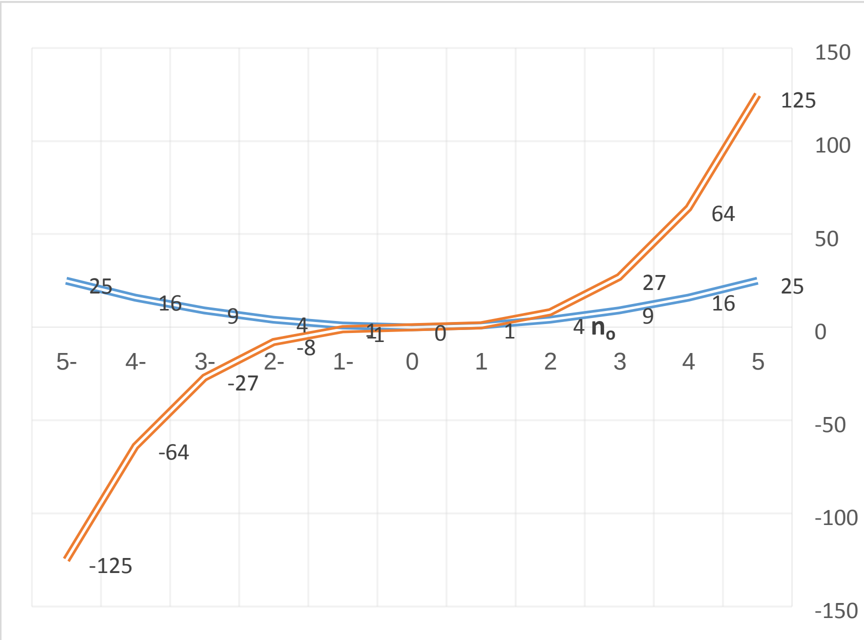
$$T(n) = T(n-1) + c_1 = T(\frac{n-1}{2}) + c_1 + c_2 = \textit{after } i \textit{ steps} = T(\frac{n-1}{2^i}) + ic_2$$

we will stop when $\frac{n}{2^i} = 1 \to i = \log n$

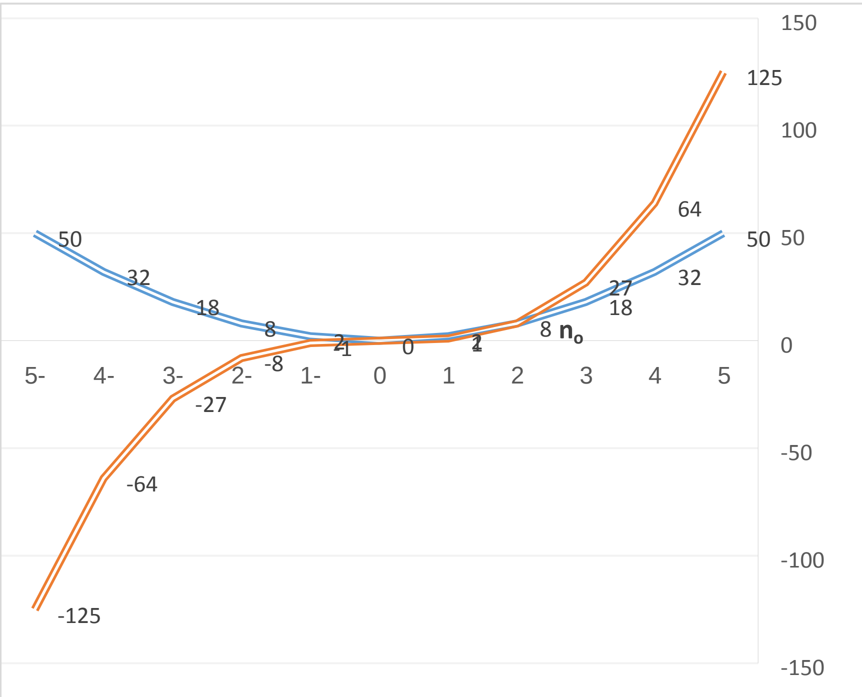
$T(n) = T(1) + \log n \cdot c_2 = \Theta \log n$

6.

a. $f_1(n) = n^2$, $g(n) = n^3$



b. $f_2(n) = 2n^2$, $g(n) = n^3$



$f_2(n)$, $f_1(n)$=O($g(n)$)

We will choose c=1 and n_0=3 $2n^2$<
n^3 for every n≥3

We will choose c=1 and n_0=2
n^2< n^3 for every n≥2 $f_2(n)$=θ($f_1(n)$)

We will choose c_1=1 and then $2n^2$≥ n^2
and c_2=3 and then $2n^2$≤ $3n^2$

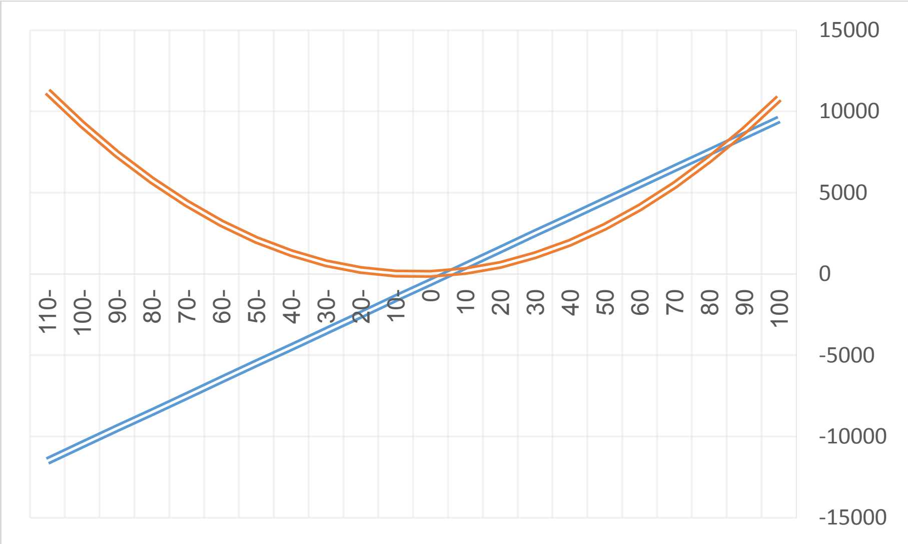
if c=1 then we will mathematically solve $2n^2 < n^3$

$0 < n^3 - 2n^2$ so $^2(n-2)>0$ and that only happens for n>2 so $n_0=3$

if c=1 then we will mathematically solve $n^2 < n^3$

$0 < n^3 - n^2$ so $^2(n-1)>0$ and that only happens for n>1 so $n_0=2$

c. $f_3(n)=100n-500$, $g_2(n)= n^2+8n+4$



$f_3(n)=O(g_2(n))$ we will choose c=100 $100n-500 \le 100(n^2+8n+4)$

simplify: $100n^2+700n+900 \ge 0$

So for every n>0 this is always true.