# Handbook of Applied Mathematics

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# Contents

# Part 1

# Mathematical Foundation

# Chapter 1

# Analysis

## 1.1 Calculus

### 1.1.1 Mean value theorem

**Theorem 1.1. (Rolle's theorem)** Given $n \geq 2$ and $f \in C^{n-1}([a,b])$ with $f^{(n)}(x)$ exists at each point of $(a,b)$, suppose that $f(x_0) = \cdots f(x_n) = 0$ for $a \leq x_0 < \cdots < x_n \leq b$, then there is a point $\xi \in (a,b)$ such that $f^{(n)}(\xi) = 0$.

**Theorem 1.2. (Lagrange's mean value theorem)** Given $f \in C^1([a,b])$, then there exists $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

**Theorem 1.3. (Cauchy's mean value theorem)** Given $f, g \in C^1([a,b])$, then there exists $\xi \in (a,b)$ such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

If $g(a) \neq g(b)$ and $g(\xi) \neq 0$, this is equivalent to

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

**Theorem 1.4. (First mean value theorems for definite integrals)** Given $f \in C([a,b])$ and $g$ integrable and does not change sign on $[a,b]$, then there exists $\xi$ in $(a,b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = f(\xi)\int_a^b g(x)\mathrm{d}x.$$

**Theorem 1.5. (Second mean value theorems for definite integrals)** Given $f$ a integrable function and $g$ a positive monotonically decreasing function, then there exists $\xi$ in $(a,b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x.$$

If $g$ is a positive monotonically increasing function, then there exists $\xi$ in $(a,b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = g(b)\int_\xi^b f(x)\mathrm{d}x.$$

If $g$ is a monotonically function, then there exists $\xi$ in $(a,b)$ such that

$$\int_a^b f(x)g(x)\mathrm{d}x = g(a)\int_a^\xi f(x)\mathrm{d}x + g(b)\int_\xi^b f(x)\mathrm{d}x.$$

### 1.1.2 Series

**Definition 1.6.** A series $\sum_{n=1}^{\infty} a_n$ is **absolute convergent** if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

**Theorem 1.7.** If a series is absolute convergent, then any reordering of it converges to the same limit.

**Theorem 1.8. (n-th term test)** If $\lim_{n\to\infty} a_n \neq 0$, then the series divergent.

**Theorem 1.9. (Direct comparison test)** If $\sum_{n=1}^{\infty} b_n$ is convergent and exists $N > 0$, for all $n > N$, $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ is convergent; if $\sum_{n=1}^{\infty} b_n$ is divergent and exists $N > 0$, for all $n > N$, $0 \leq b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

**Theorem 1.10. (Limit comparison test)** Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $a_n \geq 0, b_n > 0$. Then if $\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty)$, then either both series converge or both series diverge.

**Theorem 1.11. (Ratio test)** Given $\sum_{n=1}^{\infty} a_n$ and

$$R = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|, r = \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|,$$

if $R < 1$, then the series converges absolutely; if $r > 1$, then the series diverges.

**Theorem 1.12. (Root test)** Given $\sum_{n=1}^{\infty} a_n$ and

$$R = \limsup_{n\to\infty} (|a_n|)^{\frac{1}{n}},$$

if $R < 1$, then the series converges absolutely; if $R > 1$, then the series diverges.

**Theorem 1.13. (Integral test)** Given $\sum_{n=1}^{\infty} f(n)$ where $f$ is monotone decreasing, then the series converges iff the improper integral

$$\int_1^{\infty} f(x)\mathrm{d}x$$

is finite. In particular,

$$\int_1^{\infty} f(x)\mathrm{d}x \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x)\mathrm{d}x$$

**Theorem 1.14. (Alternating series test)** Given $\sum_{n=1}^{\infty} (-1)^n a_n$ where $a_n$ are all positive or negative, then the series converges if $|a_n|$ decreases monotonically and $\lim_{n\to\infty} a_n = 0$.

### 1.1.3 Multivariable calculus

**Theorem 1.15. (Green's theorem)** Let $\Omega$ be the region in a plane with $\partial\Omega$ a positively oriented, piecewise smooth, simple closed curve. If $P$ and $Q$ are functions of $(x, y)$ defined on an open region containing $\Omega$ and have continuous partial derivatives there, then

$$\oint_{\partial\Omega} (P\mathrm{d}x + Q\mathrm{d}y) = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

where the path of integration along $C$ is anticlockwise.

**Theorem 1.16. (Stokes' theorem)** Let $\Omega$ be a smooth oriented surface in $\mathbb{R}^3$ with $\partial\Omega$ a piecewise smooth, simple closed curve. If $\mathbf{F}(x, y, z) = \big(F_x(x, y, z), F_y(x, y, z), F_z(x, y, z)\big)$ is defined and has continuous first order partial derivatives in a region containing $\Omega$, then

$$\iint_{\Omega} (\nabla \times \mathbf{F}) \cdot \mathrm{d}S(x) = \oint_{\partial\Omega} \mathbf{F} \cdot \mathrm{d}x$$

**Theorem 1.17. (Gauss-Green theorem (Divergence theorem))** For a bounded open set $\Omega \in \mathbb{R}^n$ that $\partial\Omega \in C^1$ and a function $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), ..., F_n(\mathbf{x})) : \overline{\Omega} \to \mathbb{R}^n$ satisfies $\mathbf{F}(\mathbf{x}) \in C^1(\Omega) \cap C\left(\overline{\Omega}\right)$,

$$\int_{\Omega} \mathrm{div}\ \mathbf{F}(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} \mathrm{d}S(x),$$

where $\mathbf{n}$ is outward pointing unit normal vector at $\partial\Omega$.

**Definition 1.18.** An **implicit function** is a function of the form

$$F(x_1, ..., x_n) = 0,$$

where $x_1, ..., x_n$ are variables.

**Theorem 1.19.** Let $F(\mathbf{x}, \mathbf{y}) : \mathbb{R}^{n+m} \to \mathbb{R}^m$ be a differentiable function of two variables, and $(\mathbf{x}_0, \mathbf{y}_0)$ the point that $F(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. If the Jacobian matrix

$$J_{F,\mathbf{y}}(\mathbf{x}_0, \mathbf{y}_0) = \left( \frac{\partial F_i}{\partial y_j}(\mathbf{x}_0, \mathbf{y}_0) \right)$$

is invertible, then there exists an open set $\Omega \subseteq \mathbb{R}^n$ containing $\mathbf{x}_0$ such that there exists a unique function $f : \Omega \to \mathbb{R}^m$ such that $f(\mathbf{x}_0) = \mathbf{y}_0$ and $F(\mathbf{x}, f(\mathbf{y})) = \mathbf{0}$ for all $\mathbf{x} \in \Omega$.
Moreover, $f$ is continuously differentiable and, denoting the left-hand panel of the Jacobian matrix shown in the previous section as

$$J_{F,\mathbf{x}}(\mathbf{x}_0, \mathbf{y}_0) = \left( \frac{\partial F_i}{\partial x_j}(\mathbf{x}_0, \mathbf{y}_0) \right),$$

the Jacobian matrix of partial derivatives of $f$ in $\Omega$ is given by

$$\left( \frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right)_{m \times n} = -\left( J_{F,\mathbf{y}}(\mathbf{x}, f(\mathbf{x})) \right)_{m \times m}^{-1} \left( J_{F,\mathbf{x}}(\mathbf{x}, f(\mathbf{x})) \right)_{m \times n}.$$

## 1.2 Important Inequalities

### 1.2.1 Fundamental inequality

**Theorem 1.20. (Fundamental inequality)**

$$\forall x, y \in \mathbb{R}^+, \frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}, \text{ equality holds iff } a = b.$$

### 1.2.2 Triangle inequality

**Theorem 1.21. (Triangle inequality)**

$$a, b \in \mathbb{C}, \quad ||a| - |b|| \leq |a \pm b| \leq |a| + |b|,$$
$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^n, |\|\mathbf{a}\| - \|\mathbf{b}\|| \leq \|\mathbf{a} \pm \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

### 1.2.3 Bernoulli inequality

**Theorem 1.22. (Bernoulli inequality)**

$$\begin{aligned}
&\forall x \in (-1,+\infty), \forall a \in [1,+\infty), && (1+x)^a \geq 1+ax,\\
&\forall x \in (-1,+\infty), \forall a \in (0,1), && (1+x)^a \leq 1+ax,\\
&\forall x \in (-1,+\infty), \forall a \in (-1,0), && (1+x)^a \geq 1+ax,\\
&\forall x_i \in \mathbb{R}, i \in \{1,...,n\}, && \prod_{i=1}^{n}(1+x_i) \geq 1+\sum_{i=1}^{n} x_i,\\
&\forall y \geq x > 0, && (1+x)^y \geq (1+y)^x.
\end{aligned}$$

### 1.2.4 Jensen's inequality

**Theorem 1.23. (Jensen's inequality)** For a real convex function $f(x) : [a,b] \to \mathbb{R}$, numbers $x_1 ..., x_n \in [a,b]$ and weights $a_1, ..., a_n$, the Jensen's inequality can be start as

$$\frac{\sum_{i=1}^{n} a_i f(x_i)}{\sum_{i=1}^{n} a_i} \geq f\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right).$$

And for concave function $f$,

$$\frac{\sum_{i=1}^{n} a_i f(x_i)}{\sum_{i=1}^{n} a_i} \leq f\left(\frac{\sum_{i=1}^{n} a_i x_i}{\sum_{i=1}^{n} a_i}\right).$$

Equality holds iff $x_1 = \cdots = x_n$ or $f$ is linear on $[a,b]$.

### 1.2.5 Cauchy–Schwarz inequality

**Theorem 1.24. (Cauchy–Schwarz inequality)**
**Discrete form.** For real numbers $a_1, ...a_n, b_1, ...b_n \in \mathbb{R}, n \geq 2$

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \geq \left(\sum_{i=1}^{n} a_i b_i\right).$$

Equality holds iff $\frac{a_1}{b_1} = \cdots = \frac{a_n}{b_n}$ or $a_i = 0$ or $b_i = 0$.

**Inner product form.** For a inner product space $V$ with a norm induced by the inner product,

$$\forall \mathbf{a}, \mathbf{b} \in V \ \|\mathbf{a}\| \cdot \|\mathbf{b}\| \geq |\langle \mathbf{a}, \mathbf{b} \rangle|.$$

Equality holds iff $\exists k \in \mathbb{R}, \text{ s.t. } k\mathbf{a} = \mathbf{b} \ \text{ or } \mathbf{a} = k\mathbf{b}$.

**Probability form.** For random variables $X$ and $Y$,

$$\sqrt{E(X^2)} \cdot \sqrt{E(Y^2)} \geq |E(XY)|.$$

Equality holds iff $\exists k \in \mathbb{R}, \text{ s.t. } kX = Y \ \text{ or } X = kY$.

**Integral form.** For integrable functions $f, g \in L^2(\Omega)$,

$$\int_\Omega f^2(x)\mathrm{d}x + \int_\Omega g^2(x)\mathrm{d}x \geq \left(\int_\Omega f(x)g(x)\mathrm{d}x\right)^2.$$

Equality holds iff $\exists k \in \mathbb{R}, \text{ s.t. } kf(x) = g(x) \ \text{ or } f(x) = kg(x)$.

### 1.2.6 Hölder's inequality

**Theorem 1.25. (Hölder's inequality)**
**Discrete form.** For real numbers $a_1, ...a_n, b_1, ...b_n \in \mathbb{R}, n \geq 2$ and $p, q \in [1,+\infty)$ that $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$,

$$\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} \geq \left(\sum_{i=1}^{n} a_i b_i\right).$$

Equality holds iff $\exists c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0,\ \text{s.t. } c_1 a_i^p = c_2 b_i^q$.

**Integral form.** For functions $f \in L^p(\Omega), g \in L^q(\Omega)$ and $p, q \in [1, +\infty)$ that $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$,

$$\left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_\Omega |g(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \geq \int_\Omega f(x)g(x)\mathrm{d}x.$$

### 1.2.7 Young's inequality

**Theorem 1.26. (Young's inequality)** For $p, q \in [1, +\infty)$ that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\forall a, b \in \mathbb{R}^*, \frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Equality holds iff $a^p = b^q$.

### 1.2.8 Minkowski inequality

**Theorem 1.27. (Minkowski inequality)** For a metric space $S$,

$$\forall f, g \in L^p(S), p \in [1, +\infty], \|f\|_p + \|g\|_p \geq \|f + g\|_p.$$

For $p \in (1, +\infty)$, equality holds iff $\exists k \geq 0,\ \text{s.t. } f = kg$ or $kf = g$.

## 1.3 Special Functions

### 1.3.1 Gaussian function

**Definition 1.28.** A **Gaussian function**, or a Gaussian, is a function of the form

$$f(x) = a \exp\left(-\frac{(x-b)^2}{2c^2}\right),$$

where $a \in \mathbb{R}^+$ is the height of the curve's peak, $b \in \mathbb{R}$ is the position of the center of the peak and $c \in \mathbb{R}^+$ is the standard deviation or the Gaussian root mean square width.

**Theorem 1.29.** The integral of a Gaussian is

$$\int_{-\infty}^{+\infty} a \exp\left(-\frac{(x-b)^2}{2c^2}\right) \mathrm{d}x = ac\sqrt{2\pi}.$$

**Definition 1.30.** A **normal distribution** or a **Gaussian distribution** is a continuous probability distribution of the form

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\Big((x-\mu)^2\Big)(2\sigma^2)\Big),$$

where $\mu$ is the mean and $\sigma$ is the standard deviation.

### 1.3.2 Dirac delta function

**Definition 1.31.** The **Dirac delta function** centered at $\overline{x}$ is

$$\delta(x - \overline{x}) = \lim_{\varepsilon \to 0} f_{\overline{x},\varepsilon}(x - \overline{x}),$$

where $f_{\overline{x},\varepsilon}$ is a normal distribution with its mean at $\overline{x}$ and its standard deviation as $\varepsilon$.

**Theorem 1.32.** The Dirac delta function satisfies

$$\delta(x-\overline{x}) = \begin{cases} +\infty, & x = \overline{x} \\ 0, & x \neq \overline{x} \end{cases} \quad \int_{-\infty}^{x} \delta(x-\overline{x})\mathrm{d}x = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $H(x) = \int_{-\infty}^{x} \delta(x-\overline{x})\mathrm{d}x$ is called **Heaviside function** or **step function**.

**Theorem 1.33.** If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then

$$\int_{-\infty}^{+\infty} \delta(x-\overline{x}) f(x)\mathrm{d}x = f(\overline{x}).$$

## 1.3.3 Gamma function

**Definition 1.34.** The **Gamma function** defined on $\mathbb{C}$ is

$$\Gamma(z) = \int_{0}^{+\infty} t^{z-1} e^{-t} \mathrm{d}t,$$

where $\mathrm{Re}\ (z) > 0$.

**Theorem 1.35.** The Gamma function satisfies

$$\forall x \in \mathbb{C},\ \ \Gamma(x+1) = x\Gamma(x),$$
$$\forall n \in \mathbb{N}^*, \Gamma(n) = (n-1)!.$$

**Theorem 1.36.** The Gamma function satisfies

$$\forall x \in (0,1), \Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)},$$

which implies

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

## 1.3.4 Beta Function

**Definition 1.37.** For $p, q \in \mathbb{R}^+$, the **Beta function** is defined as

$$B(p,q) = \int_{0}^{1} x^{p-1}(1-x)^{q-1} \mathrm{d}x.$$

**Theorem 1.38.** The Beta function satisfies

$$\forall p, q \in \mathbb{R}^+, B(p,q) = B(q,p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

**Theorem 1.39.** The Beta function satisfies

$$\forall p > 0, \forall q > 1, B(p,q) = \frac{q-1}{p+q-1} B(p, q-1),$$
$$\forall p > 1, \forall q > 0, B(p,q) = \frac{p-1}{p+q-1} B(p-1, q),$$
$$\forall p > 1, \forall q > 1, B(p,q) = \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)} B(p-1, q-1).$$

# Chapter 2

# Algebra

## 2.1 Linear Space

**Definition 2.1. (Linear Space)** A **linear space** over a field $\mathbb{F}$ is a nonempty set $V$ with a addition and a scalar multiplication that satisfies

(1) Associativity of addition: $\forall \mathbf{x}, \mathbf{y} \in V, \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$,
(2) Commutativity of addition: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$,
(3) Identity element of addition: $\exists \mathbf{0} \in V, \forall \mathbf{x}, \mathbf{x} + \mathbf{0} = \mathbf{x}$,
(4) Inverse elements of addition: $\forall \mathbf{x} \in V, \exists \mathbf{y} \in V, \text{ s.t. } \mathbf{x} + \mathbf{y} = 0$,
(5) Compatibility of multiplication: $\forall \mathbf{x} \in V, a, b \in \mathbb{F}, (ab)\mathbf{x} = a(b\mathbf{x})$,
(6) Identity element of multiplication: $\exists 1 \in \mathbb{F}, \forall \mathbf{x} \in V, 1\mathbf{x} = \mathbf{x}$,
(7) Distributivity: $\forall \mathbf{x} \in V, a, b \in \mathbb{F}, (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$,
(8) Distributivity: $\forall \mathbf{x}, \mathbf{y} \in V, a \in \mathbb{F}, a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + b\mathbf{y}$.

**Notation 2.2.** The **dimension** of a linear space $V$ is written as $\dim(V)$.

**Definition 2.3.** Denoted by $V_1, ..., V_n$ linear spaces over a field $\mathbb{F}$, the **product of linear spaces** is defined as

$$V_1 \times \cdots \times V_n = \{(\mathbf{v}_1, ..., \mathbf{v}_n) : \mathbf{v}_1 \in V_1, ..., \mathbf{v}_n \in V_n\},$$

which is also a linear space over $\mathbb{F}$.

**Definition 2.4.** Given a linear space $V$, a subspace $U \subset V$ and $\mathbf{v} \in V$, the **coset** (or **affine subset**) is defined as

$$\overline{\mathbf{v}} = \{\mathbf{w} \in V : \mathbf{w} = \mathbf{v} + \mathbf{u}, \mathbf{u} \in U\}.$$

**Definition 2.5.** Given a linear space $V$ and a subspace $U \subset V$, the **quotient space** is defined as

$$V/U = \{\mathbf{v} + U : \mathbf{v} \in V\}.$$

### 2.1.1 Linear map

**Definition 2.6.** Denoted by $V$ and $W$ the linear spaces over a field $\mathbb{F}$, a function $f : V \to W$ is called a linear map between $V$ and $W$ if it satisfies

(1) Additivity: $\forall \mathbf{x}, \mathbf{y} \in V, f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$;
(2) Homogeneity: $\forall \mathbf{x} \in V, \forall k \in \mathbb{F}, f(k\mathbf{x}) = kf(\mathbf{x})$.

**Notation 2.7.** Denoted by $\mathcal{L}(V, W)$ the set of all linear maps between $V$ and $W$ (it also be written as $\mathcal{L}(V)$ if $V = W$).

**Theorem 2.8.** For linear space $V, W$ over a field $\mathbb{F}$ and linear maps $f, g \in \mathcal{L}(V, W)$, if we define

$$\forall \mathbf{x} \in V, \forall k \in \mathbb{F}, (f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) \text{ and } (kf)(\mathbf{x}) = kf(\mathbf{x}),$$

then $\mathcal{L}(V, W)$ is a linear space.

**Theorem 2.9.** For a linear map $f \in \mathcal{L}(V, W)$, $f(\mathbf{0}) = f(0\mathbf{v}) = 0f(\mathbf{v}) = 0$.

**Theorem 2.10.** Given $\mathbf{v}_1, ...\mathbf{v}_n$ the basis of linear space $V$ and $\mathbf{w}_1, ...\mathbf{w}_n$ the basis of linear space $W$, then there exists the only linear map $f \in \mathcal{L}(V, W)$ such that

$$\forall i \in \{1, ..., n\}, f(\mathbf{v}_i) = \mathbf{w}_i.$$

**Definition 2.11.** For a linear map $f \in \mathcal{L}(V, W)$, the **kernal** (or **null space**) of $f$ is defined as

$$\ker(f) = \{\mathbf{v} \in V : f(\mathbf{v}) = \mathbf{0}\},$$

where $\ker(f)$ is a subspace of $V$ and the number $\dim(\ker(f))$ is the **nullity** of $f$ which also written as $\text{nullity}(f)$

**Definition 2.12.** For a linear map $f \in \mathcal{L}(V, W)$, the **image** of $f$ is defined as

$$\text{im}(f) = \{\mathbf{w} \in W : \mathbf{w} = f(\mathbf{v}), \mathbf{v} \in V\},$$

where $\text{im}(f)$ is a subspace of $W$ and the number $\dim(\text{im}(f))$ is the **dimension** (or **rank**) of $f$ which also written as $\text{rank}(f)$

**Theorem 2.13. (Rank–nullity theorem)** For a linear map $f \in \mathcal{L}(V, W)$,

$$\dim(\ker(f)) + \dim(\text{im}(f)) = \dim(V).$$

**Definition 2.14.** A **isomorphism** is a invertible linear map.

**Definition 2.15.** Two linear spaces are called **isomorphic** if there exists a invertible linear map between them.

**Theorem 2.16.** Two linear spaces $V, W$ over a field $\mathbb{F}$ are isomorphic iff $\dim(V) = \dim(W)$.

**Theorem 2.17.** For a linear space $V$ that $\dim(V) < +\infty$ and a linear map $f \in \mathcal{L}(V)$, the following statements are equivalent:

(1) $f$ is invertible;
(2) $f$ is injective;
(3) $f$ is surjective.

## 2.2 Metric Space

**Definition 2.18. (Metric)** For a nonempty set $X$, the **metric** is a function $d : X \times X \to \mathbb{R}$ that satisfies

(1) Positive definiteness: $\forall \mathbf{x}, \mathbf{y} \in X, d(\mathbf{x}, \mathbf{y}) \geq 0, d(\mathbf{x}, \mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y}$,
(2) Symmetry: $\forall \mathbf{x}, \mathbf{y} \in X, d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
(3) Triangle inequality: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})$,

**Definition 2.19. (Metric space)** A **metric space** is a set $X$ provided with a metric.

**Notation 2.20. (Neighbourhood)** For a metric space $X$, the **neighbourhood** of $\mathbf{x} \in X$ with radius $\varepsilon > 0$ is defined as

$$U_X(\mathbf{x}, \varepsilon) = \{t : d(\mathbf{x}, t) < \varepsilon, t \in X\}.$$

**Notation 2.21. (Punctured neighbourhood)** For a metric space $X$, the **punctured neighbourhood** of $\mathbf{x} \in X$ with radius $\varepsilon > 0$ is defined as

$$U_X^\circ(\mathbf{x}, \varepsilon) = U_X(\mathbf{x}, \varepsilon) \setminus \{\mathbf{x}\} = \{t : d(\mathbf{x}, t) < \varepsilon, t \in X \setminus \{\mathbf{x}\}\}.$$

### 2.2.1 Completeness & Compactness

**Theorem 2.22. (Cauchy's convergence test)** A sequence $\{\mathbf{x}_n\}$ in a metric space $X$ is convergent (or said a **cauchy sequence**) iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m, n > N, \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon.$$

**Definition 2.23. (Completeness)** A metric space $X$ is **complete** iff all cauchy sequence of $X$ is convergent in $X$.

**Theorem 2.24. (Supremum and infimum principle)** For a nonempty set $X$, if the upper/ lower bound of $X$ exists, then the supremum/infimum of $X$ exists.

**Theorem 2.25. (The monotone bounded convergence Theorem)** For a bounded sequence $\{\mathbf{x}_n\}$, if it is increased, then

$$\lim_{n\to\infty} \mathbf{x}_n = \sup\{\mathbf{x}_n : n \in \mathbb{N}\}.$$

If it is decreased, then

$$\lim_{n\to\infty} \mathbf{x}_n = \inf\{\mathbf{x}_n : n \in \mathbb{N}\}.$$

### 2.2.2 Cover

**Definition 2.26. (Cover)** For a metric space $S \subseteq X$, A **cover** of $S$ is a set of open sets $\{D_n\}$ satisfies

$$\forall \mathbf{x} \in X, \exists D_n, \text{ s.t. } \mathbf{x} \in D_n.$$

**Definition 2.27. (Compactness)** A metric space $X$ is called **compact** if every open cover of $X$ has a finite subcover.

### 2.2.3 Cantor's intersection Theorem

**Theorem 2.28. (Cantor's intersection Theorem)** For a decreasing sequence of nested non-empty compact, closed subsets $S_n \subseteq X, n \in \mathbb{N}$ of a metric space, if $\{S_n\}$ satisfies

$$S_0 \supset S_1, \cdots, \supset S_n \supset \cdots,$$

then

$$\bigcap_{k=0}^{\infty} S_k \neq \emptyset.$$

where there is only one point $\mathbf{x} \in \bigcap_{k=0}^{\infty} S_k$ for a complete metric space.

**Corollary 2.29.** For decreasing sequence of nested non-empty compact, closed subsets $S_n \in X, n \in \mathbb{N}$ of a complete metric space and $\{\mathbf{x}\} = \bigcap_{k=0}^{\infty} S_k$, then

$$\forall \varepsilon > 0, \exists N > 0, \text{ s.t. } \forall n > N, X_n \subset U_X(x, \varepsilon).$$

### 2.2.4 Cluster point

**Definition 2.30. (Cluster point)** For a metric space $S \subseteq X$, the **cluster point** of $S$ is the point $\mathbf{x} \in X$ satisfies

$$\forall \varepsilon > 0, U_X^{\circ}(\mathbf{x}, \varepsilon) \cup S \neq \emptyset.$$

**Theorem 2.31.** For a convergent sequence $\left\{\mathbf{x}_n : n \in \mathbb{N}, \forall i \neq j, \mathbf{x}_i \neq \mathbf{x}_j\right\} \subseteq X$, the point $x = \lim_{n\to\infty} \mathbf{x}_n$ is a cluster point of $X$.

**Theorem 2.32. (Bolzano–Weierstrass Theorem)** For a metric sapce $X$ and a bounded infinite subset $S \in X$, there exists at least one cluster point of $X$.

## 2.3 Normed Space

**Definition 2.33. (Norm)** For a linear space $V$ over a field $\mathbb{F}$, the **norm** is a function $\|\cdot\| : V \to \mathbb{F}$ that satisfies

(1) Positive definiteness: $\forall \mathbf{x} \in V, \|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0$,
(2) Absolute homogeneity: $\forall \mathbf{x} \in V, k \in \mathbb{F}, \|k\mathbf{x}\| = |k|\ \|\mathbf{x}\|$,
(3) Triangle inequality: $\forall \mathbf{x}, \mathbf{y} \in V, \|\mathbf{x}\| + \|\mathbf{y}\| \geq \|\mathbf{x} + \mathbf{y}\|$,

**Definition 2.34. (Normed space)** A **normed space** is a linear space $V$ over the the field $\mathbb{F}$ with a norm.

## 2.4 Inner Product Space

**Definition 2.35. (Inner product)** For a linear space $V$ over a field $\mathbb{F}$, the **inner product** on $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ that satisfies

(1) Positive definiteness: $\forall \mathbf{x} \in V, \langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = 0$,
(2) Conjugate symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$,
(3) Linearity in the first argument: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, a, b \in \mathbb{F}$, $\langle a\mathbf{x} + b\mathbf{z}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{z}, \mathbf{y} \rangle$.

**Definition 2.36. (Inner product space)** An **inner product space** is a linear space $V$ over the field $\mathbb{F}$ with an inner product.

**Theorem 2.37.** Given a inner product space $V$ and the norm defined as $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ satisfies

$$\forall \mathbf{x}, \mathbf{y} \in V, \|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\ \|\mathbf{x}\|^2 + 2\ \|\mathbf{y}\|^2.$$

### 2.4.1 Orthonormal system

**Definition 2.38.** A subset $W$ of an inner product space $V$ is called textsf{orthonormal} if

$$\forall \mathbf{u}, \mathbf{v} \in S, \langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} 0, & u \neq v \\ 1, & u = v. \end{cases}$$

**Definition 2.39.** The **Gram-Schmidt process** takes in a finite or infinite independent list $(\mathbf{u}_1, \mathbf{u}_2, ...)$ and output two other lists $(\mathbf{v}_1, \mathbf{v}_2, ...)$ and $(\mathbf{u}_1^*, \mathbf{u}_2^*, ...)$ by

$$\mathbf{v}_{n+1} = \mathbf{u}_{n+1} - \sum_{i=1}^{n} \langle \mathbf{u}_{n+1}, \mathbf{u}_k^* \rangle \mathbf{u}_k^*,$$

$$\mathbf{u}_{n+1}^* = \frac{\mathbf{v}_{n+1}}{\|\mathbf{v}_{n+1}\|},$$

with the recursion basis as $\mathbf{v}_1 = \mathbf{u}_1$.

**Definition 2.40.** Let $(\mathbf{u}_1^*, \mathbf{u}_2^*, ...)$ be a finite or infinite orthonormal list. The **orthogonal expansion** or **Fourier expansion** for an arbitrary $\mathbf{w}$ is the series

$$\sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^*,$$

where the constants $\langle \mathbf{w}, \mathbf{u}_i^* \rangle$ are known as the **Fourier coefficients** of $\mathbf{w}$ and the term $\langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^*$ is the **projection** of $\mathbf{w}$ on $\mathbf{u}_i^*$.

**Theorem 2.41. (Minimum properties of Fourier expansions)** Let $\mathbf{u}_1^*, \mathbf{u}_2^*, ...$ be an orthonormal system and let $\mathbf{w}$ be arbitrary. Then

$$\forall a_1, ..., a_n, \|\mathbf{w} - \sum_{i=1}^{n} \langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^* \| \leq \|\mathbf{w} - \sum_{i=1}^{n} a_i \mathbf{u}_i^* \|,$$

where $\|\mathbf{w} - \sum_{i=1}^{n} a_i \mathbf{u}_i^*\|$ is minimized only when $a_i = \langle \mathbf{w}, \mathbf{u}_i^* \rangle$.

**Theorem 2.42. (Bessel inequality)** Let $\mathbf{u}_1^*, \mathbf{u}_2^*, ...$ be an orthonormal system and let $\mathbf{w}$ be arbitrary. Then

$$\sum_{i=1}^{n} |\langle \mathbf{w}, \mathbf{u}_i^* \rangle| \leq \|\mathbf{w}\|^2.$$

## 2.5 Banach Space

**Definition 2.43. (Banach space)** A **Banach space** is a complete normed vector space.

## 2.6 Hilbert Space

**Definition 2.44. (Hilbert space)** A **Hilbert space** is a inner product space that is also ce with respect to the distance function induced by the inner product.a complete metric space.

## 2.7 Single Variable Polynomial

**Definition 2.45.** Denoted by $\mathbb{V}$ a linear space and $x$ the variable, a **(single variable) polynomial** over $\mathbb{V}$ is defined as

$$p_{n(x)} = \sum_{i=0}^{n} c_i x^i,$$

where $c_0, ..., c_n \in \mathbb{V}$ are constants that called the **coefficients of the polynomial**.

**Definition 2.46.** Given a polynomial $p(x) = \sum_{i=0}^{n} c_i x^i$ where $c_n \neq 0$, the degree of $p(x)$ is marked as $\deg(p(x)) = n$. In particular, the degree of zero polynomial $p(x) = 0$ is $\deg(0) = -\infty$.

**Theorem 2.47.** Denoted by $\mathbb{P}_n = \{p : \deg(p) \leq n\}$ the set of polynomials with degree no more than $n$ $(n \geq 0)$, and $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ the set contains all polynomials, then $\mathbb{P}_n$ is a linear space and satisfies

$$\{0\} = \mathbb{P}_0 \subset \mathbb{P}_1 \subset \cdots \subset \mathbb{P}_n \subset \cdots \mathbb{P}$$

**Theorem 2.48. (Vieta's formulas)** Given a polynomial $p \in \mathbb{P}_n$ with the coefficients being real or complex numbers, denoted by $x_1, ..., x_n$ the complex roots, then

$$\begin{cases} x_1 + \cdots + x_n &= -c_{n-1}, \\ \sum_{i=1}^{n} \sum_{j=i+1}^{n} x_i x_j &= c_{n-2}, \\ & \cdots \\ \prod_{i=1}^{n} x_i &= (-1)^n c_0, \end{cases}$$

where $c_n = 1$ WLOG.

## 2.8 Orthogonal Polynomial

**Definition 2.49.** Given a weight function $\rho(x) : [a, b] \to \mathbb{R}^+$, satisfies

$$\int_a^b \rho(x)\mathrm{d}x > 0, \int_a^b x^k \rho(x)\mathrm{d}x > 0 \ \text{ exists.}$$

The set of **orthogonal polynomials** on $[a, b]$ with the weight function $\rho(x)$ is defined as

$$\{p_i, i \in \mathbb{N}\} \subset L_\rho([a, b]) = \left\{ f(x) : \int_a^b f^2(x)\rho(x)\mathrm{d}x < \infty \right\}.$$

where $\{p_i, i \in \mathbb{N}\}$ are calculate from $\{x^n, n \in \mathbb{N}\}$ using the Gram-Schmidt process with the inner product

$$\forall f, g \in L_\rho([a, b]), \langle f, g \rangle = \int_a^b \rho(x) f(x) g(x)\mathrm{d}x.$$

**Theorem 2.50.** Orthogonal polynomials $p_{n-1}(x), p_n(x), p_{n+1}(x)$ satisfies

$$p_{n+1}(x) = (a_n + b_n x)p_n(x) + c_n p_{n-1}(x).$$

where $a_n, b_n, c_n$ are depends on $[a, b]$ and $\rho$.

**Theorem 2.51.** The orthogonal polynomial $p_n(x)$ on $[a, b]$ with the weight function $\rho(x)$ has $n$ roots on $(a, b)$.

### 2.8.1 Legendre polynomial

**Definition 2.52.** The **Legendre polynomial** is defined on $[-1, 1]$ with the weight function $\rho(x) = 1$.

**Theorem 2.53.** The Legendre polynomials $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_{-1}^{1} p_i(x)p_j(x)\mathrm{d}x = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j. \end{cases}$$

**Theorem 2.54.** The Legendre polynomial $p_{n-1}, p_n, p_{n+1}$ satisfies

$$p_{n+1}(x) = \frac{2n+1}{n+1} x p_n(x) - \frac{n}{n+1} p_{n-1}(x).$$

**Example 2.55.** The first three terms of Legendre polynomials is

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

### 2.8.2 Chebyshev polynomial of the first kind

**Definition 2.56.** The **Chebyshev polynomial of the first kind** is defined on $[-1,1]$ with the weight function $\rho(x)=\frac{1}{\sqrt{1-x^2}}$.

**Theorem 2.57.** The Chebyshev polynomials of the first kind $\{p_i(x), i\in\mathbb{N}\}$ satisfies

$$\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}p_i(x)p_j(x)\mathrm{d}x=\begin{cases}\pi & i=j=0\\ \frac{\pi}{2} & i=j\neq 0\\ 0 & i\neq j.\end{cases}$$

**Theorem 2.58.** The Chebyshev polynomial of the first kind $p_{n-1}, p_n, p_{n+1}$ satisfies

$$p_{n+1}(x)=2xp_n(x)-p_{n-1}(x).$$

**Example 2.59.** The first three terms of Chebyshev polynomials of the first kind is

$$p_0(x)=1,\quad p_1(x)=x,\quad p_2(x)=2x^2-1.$$

### 2.8.3 Chebyshev polynomial of the second kind

**Definition 2.60.** The **Chebyshev polynomial of the second kind** is defined on $[-1,1]$ with the weight function $\rho(x)=\sqrt{1-x^2}$.

**Theorem 2.61.** The Chebyshev polynomials of the second kind $\{p_i(x), i\in\mathbb{N}\}$ satisfies

$$\int_{-1}^{1}\sqrt{1-x^2}p_i(x)p_j(x)\mathrm{d}x=\begin{cases}\frac{\pi}{2}, & i=j\\ 0, & i\neq j.\end{cases}$$

**Theorem 2.62.** The Chebyshev polynomial of the second kind $p_{n-1}, p_n, p_{n+1}$ satisfies

$$p_{n+1}(x)=2xp_n(x)-p_{n-1}(x).$$

**Example 2.63.** The first three terms of Chebyshev polynomials of the second kind is

$$p_0(x)=1,\quad p_1(x)=2x,\quad p_2(x)=4x^2-1.$$

### 2.8.4 Laguerre polynomial

**Definition 2.64.** The **Laguerre polynomial** is defined on $[0,+\infty)$ with the weight function $\rho(x)=x^{\alpha}e^{-x}$.

**Theorem 2.65.** The Laguerre polynomial $\{p_i(x), i\in\mathbb{N}\}$ satisfies

$$\int_{0}^{+\infty}x^{\alpha}e^{-x}p_i(x)p_j(x)\mathrm{d}x=\begin{cases}\frac{\Gamma(n+\alpha+1)}{n!}, & i=j\\ 0, & i\neq j.\end{cases}$$

**Theorem 2.66.** For $\alpha=0$, the Laguerre polynomial $p_{n-1}, p_n, p_{n+1}$ satisfies

$$p_{n+1}(x)=(2n+1-x)p_n(x)-n^2p_{n-1}(x).$$

**Example 2.67.** For $\alpha=0$, the first three terms of Laguerre polynomial is

$$p_0(x)=1,\quad p_1(x)=-x+1,\quad p_2(x)=x^2-4x+2.$$

### 2.8.5 Hermite polynomial (probability theory form)

**Definition 2.68.** The **Hermite polynomial** is defined on $(-\infty, +\infty)$ with the weight function $\rho(x) = \left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{x^2}{2}}$.

**Theorem 2.69.** The Hermite polynomial $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_i(x) p_j(x) \mathrm{d}x = \begin{cases} n!, & i = j \\ 0, & i \neq j. \end{cases}$$

**Theorem 2.70.** For $\alpha = 0$, the Hermite polynomial $p_{n-1}, p_n, p_{n+1}$ satisfies

$$p_{n+1}(x) = xp_n(x) - np_{n-1}(x).$$

**Example 2.71.** For $\alpha = 0$, the first three terms of Hermite polynomial is

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - 1.$$

# Chapter 3

# Ordinary Differential Equation

**Definition 3.1.** Given a function $F$, an **explicit ordinary differential equation** of order $n$ takes the form

$$\mathbf{F}\big(\mathbf{u}^{(n-1)}, ..., \mathbf{u}', \mathbf{u}, t\big) = \mathbf{u}^{(n)},$$

an **implicit ordinary differential equation** of order $n$ takes the form

$$\mathbf{F}\big(\mathbf{u}^{(n)}, ..., \mathbf{u}', \mathbf{u}, t\big) = \mathbf{0},$$

**Definition 3.2.** An ODE is **autonomous** if it does not depend on the variable x.

**Definition 3.3.** A ODE is **linear** if can be written as

$$\sum_{i=0}^{n} A_i(t)\mathbf{u}^{(n)} + \mathbf{r}(t) = \mathbf{0},$$

where $A_i(t)$ and $r(t)$ are continuous functions of $t$.

**Definition 3.4.** A linear ODE is **homogeneous** if $\mathbf{r}(t) = 0$, and there is always the trivial solution $\mathbf{u} \equiv \mathbf{0}$.

**Definition 3.5.** An ODE is **separable** if can be written as

$$P_1(x)Q_1(y) = P_2(x)Q_2(y)\frac{\mathrm{d}y}{\mathrm{d}x}.$$

**Definition 3.6.** For initial value $(\mathbf{u}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, $T \geq t_0$ and $\mathbf{f} : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^n$, the **initial value problem** (IVP) is to find $u(t) \in C^1([t_0, T])$ satisfies

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(t_0) = \mathbf{u}_0.$$

**Theorem 3.7.** Given an IVP, denoted by $u_0 = u$, $u_i, i = 1, ..., n$ the $i$th derivative of $u$, then the ODE

$$\mathbf{F}\big(\mathbf{u}^{(n-1)}, ..., \mathbf{u}', \mathbf{u}, t\big) = \mathbf{u}^{(n)}$$

can be written as an IVP,

$$\begin{pmatrix} \mathbf{u}_0' \\ \vdots \\ \mathbf{u}_{n-2}' \\ \mathbf{u}_{n-1}' \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{n-1} \\ \mathbf{F}(\mathbf{u}_{n-1}, ..., \mathbf{u}_1, \mathbf{u}_0, t) \end{pmatrix}.$$

## 3.1 General Theory

**Theorem 3.8. (Peano existence theorem)** Given an IVP with an open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if $\mathbf{f}(\mathbf{u}, t) \in C(\Omega)$ and $(\mathbf{u}_0, t_0) \in \Omega$, then there is a local solution $\tilde{\mathbf{u}} : U \to \mathbb{R}^n$ satisfies the IVP, where $U$ is a neighbourhood of $t_0$ in $\mathbb{R}$.

**Theorem 3.9. (Picard–Lindelöf theorem)** Given an IVP with an open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if $\mathbf{f}(\mathbf{u}, t) : \Omega \to \mathbb{R}^n$ is continuous in $t$ and Lipschitz continuous in $\mathbf{u}$ and $(\mathbf{u}_0, t_0) \in \Omega$, then there is a unique local solution $\tilde{\mathbf{u}} : U \to \mathbb{R}^n$ satisfies the IVP, where $U$ is a neighbourhood of $t_0$ in $\mathbb{R}$.

**Theorem 3.10. (Comparison theorem)** Given two IVPs

$$\mathbf{u}_1' = \mathbf{f}_1(\mathbf{u}_1, t), \quad \mathbf{u}_1(t_0) = \mathbf{u}_0,$$

$$\mathbf{u}_2' = \mathbf{f}_2(\mathbf{u}_2, t), \quad \mathbf{u}_2(t_0) = \mathbf{u}_0,$$

and a open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if for all $(\mathbf{u}, t) \in \Omega$, $\mathbf{f}_1(\mathbf{u}, t) < \mathbf{f}_2(\mathbf{u}, t)$, then

$$\begin{cases} \mathbf{u}_1(t) > \mathbf{u}_2(t), & t > t_0, (\mathbf{u}_1(t), t), (\mathbf{u}_2(t), t) \in \Omega, \\ \mathbf{u}_1(t) < \mathbf{u}_2(t), & t < t_0, (\mathbf{u}_1(t), t), (\mathbf{u}_2(t), t) \in \Omega, \end{cases}$$

## 3.2 Exact solutions

**Example 3.11.** Given an initial point $(y_0, x_0)$, and a separable equation

$$P_1(x)Q_1(y) = P_2(x)Q_2(y)\frac{\mathrm{d}y}{\mathrm{d}x},$$

the solution of the equation is

$$\int_{x_0}^{x} \frac{P_1(t)}{P_2(t)}\mathrm{d}t = \int_{y_0}^{y} \frac{Q_2(t)}{Q_1(t)}\mathrm{d}t.$$

**Example 3.12.** Given an initial point $(y_0, x_0)$, and a first-order homogeneous equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F\left(\frac{y}{x}\right),$$

the solution of the equation is

$$\int_{x_0}^{x} \frac{1}{x}\mathrm{d}x = \int_{\frac{y_0}{x_0}}^{\frac{y}{x}} \frac{1}{F(t) - t}\mathrm{d}t.$$

**Example 3.13.** Given an initial point $(y_0, x_0)$, and a first-order separable equation

$$yM(xy) + xN(xy)\frac{\partial y}{\partial x} = 0,$$

the solution of the equation is

$$\int_{x_0}^{x} \frac{1}{x}\mathrm{d}x = \int_{y_0 x_0}^{yx} \frac{N(t)}{t(N(t) - M(t))}\mathrm{d}t,$$

where $C$ is a constant.

# Chapter 4

# Partial Differential Equation

## 4.1 Poisson Equation

## 4.2 Heat Equation

## 4.3 Wave Equation

# Chapter 5

# Probability Theory

## 5.1 Discrete random varibles

## 5.2 Continous random varibles

## 5.3 Characteristic functions

## 5.4 Probability limit theorems

# Chapter 6

# Stochastic Process

## 6.1 Poisson process

## 6.2 Markov chain

# Chapter 7

# Statistics

# Chapter 8

# Graph

## 8.1 Shortest Path

## 8.2 Matching

## 8.3 Network Flow

## 8.4 Tree

# Chapter 9

# Combinatorics

## 9.1 Generating function

## 9.2 Inclusion–exclusion principle

## 9.3 Special Numbers

### 9.3.1 Catalan number

### 9.3.2 Stirling number

# Part 2

# Scientific Computing

# Chapter 10

# Interpolation

## 10.1 Polynomial Interpolation

### 10.1.1 Lagrange formula

**Definition 10.1.** To interpolate given points $(x_0, f(x_0)), ..., (x_n, f(x_n))$, the Lagrange formula is

$$p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x),$$

where the **elementary Lagrange interpolation polynomial** (or **fundamental polynomial**) for pointwise interpolation $l_k(x)$ is

$$l_k(x) = \prod_{i=0}^{n} \frac{x - x_i}{x_k - x_i}.$$

In particular, for $n = 0, l_0(x) = 1$.

### 10.1.2 Newton formula

**Definition 10.2.** The $k$th divided difference ($k \in \mathbb{N}^+$) on the **table of divided differences**

$$\begin{array}{c|lllll}
x_0 & f[x_0] & & & \\
x_1 & f[x_1] & f[x_0, x_1] & & \\
x_2 & f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & \\
x_3 & f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] \\
... & ... & ... & ... & ...
\end{array}$$

where the **divided differences** satisfy

$$\begin{aligned}
f[x_0] &= f(x_0), \\
f[x_0, ..., x_k] &= \frac{f[x_1, ..., x_k] - f\left[x_0, ..., x_{\{k-1\}}\right]}{x_k - x_0}.
\end{aligned}$$

**Corollary 10.3.** Suppose $(i_0, ..., i_k)$ is a permutation of $(0, ..., k)$. Then

$$f[x_0, ..., x_k] = f\left[x_{i_0}, ..., x_{i_k}\right].$$

**Theorem 10.4.** For distinct points $x_0, ..., x_n$ and $x$, we have

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, ..., x_n] \prod_{i=0}^{n-1} (x - x_i) + f[x_0, ..., x_n, x] \prod_{i=0}^{n} (x - x_i).$$

**Definition 10.5.** The **Newton formula** for interpolating the points $(x_0, f(x_0)), ..., (x_n, f(x_n))$ is

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, ..., x_n] \prod_{i=0}^{n-1} (x - x_i).$$

### 10.1.3 Neville-Aitken algorithm

**Definition 10.6.** Denote $p_0^{[i]}(x) = f(x_i)$ for $i = 0, ..., n$. For all $k = 0, ..., n-1$ and $i = 0, ..., n-k-1$, define

$$p_{k+1}^{[i]}(x) = \frac{(x - x_i)p_k^{[i+1]}(x) - (x - x_{x+k+1})p_k^{[i]}(x)}{x_{i+k+1} - x_i}.$$

Then each $p_k^{[i]}(x)$ is the interpolating polynomial for the function $f$ at the points $x_i, ..., x_{\{i+k\}}$. In particular, $p_n^{[0]}(x)$ is the interpolating polynomial of degree $n$ for the function $f$ at the points $x_0, ..., x_n$.

### 10.1.4 Hermite interpolation

**Definition 10.7.** Given distinct points $x_0, ..., x_k$ in $[a, b]$, non-negative integers $m_0, ..., m_k$, and a function $f \in C^M[a, b]$ where $M = \max_{i=0,...,k}(m_i)$, the **Hermite interpolation problem** seeks a polynomial $p(x)$ of the lowest degree satisfies

$$\forall i \in \{0, ..., k\}, \forall \mu \in \{0, ..., m_i\}, p^{(\mu)}(x_i) = f^{(\mu)}(x_i).$$

**Definition 10.8. (Generalized divided difference)** Let $x_0, ..., x_k$ be $k+1$ pairwise distinct points with each $x_i$ repeated $m_i + 1$ times; write $N = k + \sum_{i=0}^{k} m_i$. The $N$th divided difference associated with these points is the cofficient of $x^N$ in the polynomial p that uniquely solves the Hermite interpolation problem.

**Corollary 10.9.** The $n$th divided difference at $n+1$ "confluent" (i.e. identical) points is

$$f[x_0, ..., x_0] = \frac{1}{n!} f^{(n)}(x_0),$$

where $x_0$ is repeated $n+1$ times on the left-hand side.

### 10.1.5 Approximation

**Definition 10.10.** Given condition functions $c_0, ..., c_k : \mathbb{P}_n \to \mathbb{R}^+$, the **Approximation problem** seeks a polynomial $p_n(x)$ of the given degree $n$ satisfies a unconstrained optimization

$$\min_{p_n \in \mathbb{P}_n} \sum_{i=0}^{k} c_i\left(p_n^{(m_i)}\right).$$

where condition function $c(p)$ includes but is not limited to

$$|p^{(m)}(x)|, \left(p_n^{(m)}(x)\right)^2, \int_a^b |p^{(m)}| \, \mathrm{d}x, \int_a^b \left(p^{(m)}\right)^2 \mathrm{d}x.$$

**Example 10.11.** For non-negative integers $m_0, ..., m_k$ and condition functions $c_i(p_n) = \left(p_n^{(m_i)}(x)\right)^2$, denote by

$$p_n(x) = \sum_{i=0}^{n} c_i x^i$$

the polynomial of the given degree $n$, then the $m$th derivative of $p_n$ is

$$p_n^{(m)}(x) = \sum_{i=m}^{n} \frac{i!}{(i-m)!} c_i x^{i-m}.$$

All above implies the least squares system

$$\begin{cases} p_n^{(m_0)}(x) = \sum_{i=m_0}^{n} \frac{i!}{(i-m_0)!} c_i x^{i-m_0} = 0, \\ \cdots \\ p_n^{(m_k)}(x) = \sum_{i=m_k}^{n} \frac{i!}{(i-m_k)!} c_i x^{i-m_k} = 0, \end{cases}$$

which can be solved by algorithms such as Householder transformation.

### 10.1.6 Error analysis

**Theorem 10.12.** Let $f \in C^n[a,b]$ and suppose that $f^{(n+1)}(x)$ exists at each point of $(a,b)$. Let $p_n(x) \in \mathbb{P}_n$ denote the unique polynomial that coincides with f at $x_0, ..., x_n$. Define

$$R_n(f;x) = f(x) - p_n(x),$$

as the **Cauchy remainder** of the polynomial interpolation.
If $a \le x_0 < \cdots < x_n \le b$, then there exists some $\xi \in (a,b)$ satisfies

$$R_n(f;x) = \frac{f^{\{(n+1)\}}(xi)}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$$

where the value of $\xi$ depends on $x, x_0, ..., x_n$ and $f$.

**Theorem 10.13.** For the Hermite interpolation problem, denote $N = k + \sum_{i=0}^{k} m_i$. Denote by $p_N(x) \in \mathbb{P}_N$ the unique solution of the problem. Suppose $f^{(N+1)}(x)$ exists in $(a,b)$. Then there exists some $\xi \in (a,b)$ satisfies

$$R_N(f;x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i=0}^{k} (x - x_i)^{m_i+1}.$$

## 10.2 Spline

**Definition 10.14.** Given nonnegative integers $n$, $k$, and a strictly increasing sequence $a = x_1 < \cdots < x_N = b$, the set of **spline** functions of degree $n$ and smoothness class $k$ relative to the partition $\{x_i\}$ is

$$\mathbb{S}_n^k = \left\{ s : s \in C^k[a,b]; \forall i \in \{1, ..., N-1\}, s \mid_{[x_i, x_{i+1}]} \in \mathbb{P}_n \right\},$$

where $x_i$ is the **knot** of the spline.

### 10.2.1 Cubic spline

**Definition 10.15. (Boundary conditions of splines)** The followings are common boundary conditions of cubic splines.

- The **complete cubic spline** $s$ satisfies $s'(a) = f'(a), s'(b) = f'(b)$;
- The **cubic spline with specified second derivatives** $s$ satisfies $s''(a) = f''(a), s''(b) = f''(b)$;
- The **natural cubic spline** $s$ satisfies $s''(a) = s''(b) = 0$;
- The **not-a-knot cubic spline** $s$ satisfies $s'''(x)$ exists at $x = x_2$ and $x = x_{N-1}$.
- The **periodic cubic spline** $s$ satisfies $s(a) = s(b), s'(a) = s'(b), s''(a) = s''(b)$.

**Theorem 10.16.** Denote $m_i = s'(x_i), M_i = s''(x_i)$ for $s \in \mathbb{S}_3^2$, then

$$\forall i = 2, 3, ..., N-1, \quad \lambda_i m_{i-1} + 2m_i + \mu_i m_i + 1 = 3\mu_i f[x_i, x_{i+1}] + 3\lambda_i f[x_{i-1}, x_i],$$

$$\forall i = 2, 3, ..., N-1, \quad \mu_i M_{i-1} + 2M_i + \lambda_i m_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}],$$

where

$$\mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}, \quad \lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}.$$

In particular, $m_i$ and $M_i$ should be replaced to the derivatives given at the boundary.

**Theorem 10.17.** Cubic spline $s \in \mathbb{S}_3^2$ from the linear system of $\lambda_i, \mu_i, m_i, M_i$ and the boundary conditions.

### 10.2.2 B-spline

**Definition 10.18.** **B-splines** are defined recursively by

$$B_i^{n+1}(x) = (x - x_{i-1})(x_{i+n} - x_{i-1})B_i^n(x) + \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i-1}} B_{i+1}^n(x),$$

where recursion base is the B-spline of degree zero

$$B_i^0(x) = \begin{cases} 1, & x \in (x_{i-1}, x_i], \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem 10.19.** The $\{B_i^n(x)\}$ forms a basis of $\mathbb{S}_n^{n-1}$.

**Definition 10.20.** For $N \in \mathbb{N}^*$, the **support** of a $B_i^n(x)$ is

$$\text{supp}\,\{B_i^n(x)\} = \overline{\{x \in \mathbb{R} : B_i^n(x) \neq 0\}} = [x_{i-1}, x_{i+n}].$$

**Theorem 10.21. (Integrals of B-splines)** The average of a B-spline over its support only depends on its degree,

$$\frac{1}{t_{i+n} - t_{i-1}} \int_{t_{i-1}}^{t_{i+n}} B_i^n(x)\mathrm{d}x = \frac{1}{n+1}.$$

**Theorem 10.22. (Derivatives of B-splines)** For $n \geq 2$, we have

$$\forall x \in \mathbb{R}, \quad \frac{\mathrm{d}}{\mathrm{d}x} B_i^n(x) = \frac{nB_i^{n-1}(x)}{x_{i+n-1} - x_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{x_{i+n} - x_i}.$$

For $n = 1$, it holds for all $x$ except $x_{i-1}, t_i, t_{i+1}$, where the derivative of $B_i^1(x)$ is not defined.

### 10.2.3 Error analysis

**Theorem 10.23.** Suppose a function $f \in C^4[a, b]$, is interpolated by a complete cubic spline or a cubic spline with specified second derivatives at its end points. Then

$$\forall m = 0, 1, 2, |f^{(m)}(x) - s^{(m)}(x)| \leq c_m h^{4-m} \max_{x \in [a,b]} |f^{(4)}(x)|,$$

where $c_0 = \frac{1}{16}, c_1 = c_2 = \frac{1}{2}$ and $h = \max_{i=1,\ldots,N-1} |x_{i+1} - x_i|$.

# Chapter 11

# Integration

**Definition 11.1.** A **weighted quadrature formula** $I_n(f)$ is a linear function

$$I_n(f) = \sum_{i=1}^{n} w_i f(x_i),$$

which approximates the integral of a function $f \in C[a, b]$,

$$I(f) = \int_a^b \rho(x) f(x) \mathrm{d}x,$$

where the weight function $\rho \in [a, b]$ satisfies $\forall x \in (a, b),\ \rho(x) > 0$. The points $\{x_i\}$ at which the integrand $f$ is evaluated are called nodes or abscissas, and the multipliers $\{w_i\}$ are called weights or coefficients.

**Definition 11.2.** A weighted quadrature formula has (polynomial) **degree of exactness** $d_E$ iff

$$\begin{aligned} &\forall f \in \mathbb{P}_{d_E}, \quad E_n(f) = 0, \\ &\exists g \in \mathbb{P}_{d_E+1}, \ \text{s.t. } E_n(g) \neq 0 \end{aligned}$$

where $\mathbb{P}_d$ denotes the set of polynomials with degree no more than $d$.

**Theorem 11.3.** A weighted quadrature formula $I_n(f)$ satisfies $d_E \leq 2n - 1$.

**Definition 11.4.** The **error** or **remainder** of $I_n(f)$ is

$$E_n(f) = I(f) - I_n(f),$$

where $I_n(f)$ is said to be convergent for $C[a, b]$ iff

$$\forall f \in C[a, b], \lim_{n \to +\infty} E_n(f) = 0.$$

**Theorem 11.5.** Let $x_1, ..., x_n$ be given as distinct nodes of $I_n(f)$. If $d_E \geq n - 1$, then its weights can be deduced as

$$\forall k \in \{1, ..., n\}, w_k = \int_a^b \rho(x) l_k(x) \mathrm{d}x,$$

where $l_k(x)$ is the elementary Lagrange interpolation polynomial for pointwise interpolation applied to the given nodes.

## 11.1 Newton-Cotes Formulas

**Definition 11.6.** A **Newton-Cotes formula** is a formula based on approximating $f(x)$ by interpolating it on uniformly spaced nodes $x_1, ..., x_n \in [a, b]$.

### 11.1.1 Midpoint rule

**Definition 11.7.** The **midpoint rule** is a formula based on approximating $f(x)$ by the constant $f\left(\frac{a+b}{2}\right)$.
For $\rho(x) \equiv 1$, it is simply

$$I_M(f) = (b-a)f\left(\frac{a+b}{2}\right).$$

**Theorem 11.8.** For $f \in C^2[a,b]$, with weight functino $\rho \equiv 1$, the error (remainder) of midpoint rule satisfies

$$\exists \xi \in [a,b], \text{ s.t. } E_M(f) = \frac{(b-a)^3}{24} f''(\xi).$$

**Corollary 11.9.** The midpoint rule has $d_E = 1$.

### 11.1.2 Trapezoidal rule

**Definition 11.10.** The **trapezoidal rule** is a formula based on approximating $f(x)$ by the straight line that connects $(a, f(a))$ and $(b, f(b))$.
For $\rho(x) \equiv 1$, it is simply

$$I_T(f) = \frac{b-a}{2}(f(a) + f(b)).$$

**Theorem 11.11.** For $f \in C^2[a,b]$, with weight functino $\rho \equiv 1$, the error (remainder) of trapezoidal rule satisfies

$$\exists \xi \in [a,b], \text{ s.t. } E_T(f) = -\frac{(b-a)^3}{12} f''(\xi).$$

**Corollary 11.12.** The trapezoidal rule has $d_E = 1$.

### 11.1.3 Simpson's rule

**Definition 11.13.** The **Simpson's rule** is a formula based on approximating $f(x)$ by the quadratic polynomial that goes through the points $(a, f(a))$, $\left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right)$ and $(b, f(b))$.
For $\rho(x) \equiv 1$, it is simply

$$I_S(f) = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

**Theorem 11.14.** For $f \in C^4[a,b]$, with weight functino $\rho \equiv 1$, the error (remainder) of Simpson's rule satisfies

$$\exists \xi \in [a,b], \text{ s.t. } E_T(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi).$$

**Corollary 11.15.** The Simpson's rule has $d_E = 3$.

## 11.2 Gauss Formulas

**Theorem 11.16.** For an interval $[a,b]$ and a weight function $\rho : [a,b] \to \mathbb{R}$, the nodes for gauss formula $I_n(f)$ is the root of the $n$th order orthogonal polynomial on $[a,b]$ with the weight function $\rho(x)$.

**Theorem 11.17.** A Gauss formula $I_n(f)$ has $d_E = 2n - 1$.

# Chapter 12

# Optimization

## 12.1 One-dimensional Line Search

**Definition 12.1.** Given a function $f : \mathbb{R}^n \to \mathbb{R}$, a initial point $\mathbf{x}$ and a direction $\mathbf{d}$, denoted by $\varphi(\alpha) = f(\mathbf{x} + \alpha\mathbf{d})$, a **one-dimensional line search** method solves the problem

$$\varphi(\alpha) = \min_{t\in\mathbb{R}^+} \varphi(t).$$

**Method 12.2. (Success-failure method)** For a one-dimensional line search problem, the **success-failure method** is an inexact one-dimensional line search method to solve the interval $[a, b] \in [0, +\infty)$ that exact solution $\alpha^* \in [a, b]$, where we

(1) Choose initial value $\alpha_0 \in [0, +\infty)$, $h_0 > 0$, $t > 0$(commonly choose $t = 2$), calculate $\varphi(\alpha_0)$ and let $k = 0$;
(2) Let $\alpha_{k+1} = \alpha_k + h_k$ and calculate $\varphi(\alpha_{k+1})$, if $\varphi(\alpha_{k+1}) < \varphi(\alpha_k)$, then go to (3), otherwise go to (4);
(3) Let $h_{k+1} = th_k$, $\alpha = \alpha_k$, $k = k + 1$, and go to (2);
(4) If $k = 0$, then let $h_k = -h_k$ and go to (2), otherwise stop and the solution $[a, b]$ satisfies

$$a = \min\{\alpha, \alpha_k\}, \quad b = \max\{\alpha, \alpha_k\}.$$

**Definition 12.3.** A general form of one-dimensional line search method is the following three steps:

(1) **Initialization**: given initial point $\mathbf{x}$ and acceptable error $\varepsilon > 0$, $\delta > 0$;
(2) **Iteration**: calculate the direction $\mathbf{d}$ and step size $\alpha$ that $f(\mathbf{x} + \alpha\mathbf{d}) = \min_{t\in\mathbb{R}^+} f(\mathbf{x} + t\mathbf{d})$ and let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
(3) **Stop condition**: if $\|\nabla f(\mathbf{x})\| \leq \varepsilon$ or $U_{\mathbb{R}^n}(x, \delta)$ includes the exact solution, then the current $\mathbf{x}$ is the solution.

where the iteration step are repeated until $\mathbf{x}$ satisfies the stop condition.

**Definition 12.4.** Given a method, denoted by $\{\mathbf{x}_k\}$ the sequence of the iteration and $\mathbf{x}^*$ the exact solution, the method is **(Q-)linear convergence** if

$$\lim_{k\to\infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} \in (0, 1),$$

the method is **(Q-)sublinear convergence** if

$$\lim_{k\to\infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} = 1,$$

the method is **(Q-)superlinear convergence** if

$$\lim_{k\to\infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} = 0.$$

For a superlinear convergence method, the method is $r$-order linear convergence if

$$\lim_{k\to\infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|^r} \in [0, +\infty),$$

where when $r = 2$ is called **(Q-)quadratic convergence**.

**Remark 12.5.** There is another **R-convergence** for judging a sequence which use another Q-convergence sequence as the boundary of $\{\|\mathbf{x}_k - x^*\|\}$, but is not needed here.

**Method 12.6. (Golden section method)** Given the initial point $\mathbf{x}$, an interval $[a, b]$ and $\delta > 0$,

- The iteration step is:
  (1) Calculate the two testing points $\lambda = a + (1-k)(b-a)$ and $\mu = a + k(b-a)$ where $k = \frac{\sqrt{5}-1}{2}$ is the golden ratio;
  (2) If $\varphi(\lambda) > \varphi(\mu)$, let $a = \lambda$, otherwise let $b = \mu$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

**Theorem 12.7.** The golden section method is a **linear convergent** method.

**Method 12.8. (Fibonacci method)** Given the initial point $\mathbf{x}$, an interval $[a, b]$ and $\delta > 0$,

- The $k$-th iteration step is:
  (1) Calculate the two testing points $\lambda = a + \frac{F_k}{F_{k+2}}(b-a)$ and $\mu = a + \frac{F_{k+1}}{F_{k+2}}(b-a)$ where $F_k$ is the $k$-th fibonacci number and $k$;
  (2) If $\varphi(\lambda) > \varphi(\mu)$, let $a = \lambda$, otherwise let $b = \mu$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

**Theorem 12.9.** The Fibonacci method is a **linear convergent** method.

**Method 12.10. (Bisection method)** Given the initial point $\mathbf{x}$, an interval $[a, b]$ and $\delta > 0$,

- The iteration step is:
  (1) Calculate the midpoint $m = \frac{a+b}{2}$ and $\varphi(m)$;
  (2) If $\nabla f(m) \cdot d < 0$, let $a = m$, otherwise let $b = m$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

**Theorem 12.11.** The bisection method is a **linear convergent** method.

**Method 12.12. (Newton's method)** Given the initial point $\mathbf{x}$ and $\varepsilon > 0$,

- The iteration step is:
  (1) Calculate $\left(\nabla^2 f(\mathbf{x})\right)^T \cdot \mathbf{d}$ and $(\nabla f(\mathbf{x}))^T \cdot \mathbf{d}$;
  (2) Let $\mathbf{x} = \mathbf{x} - \frac{(\nabla f(\mathbf{x}))^T \cdot \mathbf{d}}{(\nabla^2 f(\mathbf{x}))^T \cdot \mathbf{d}}$;
- The stop condition is $(\nabla f(\mathbf{x}))^T \cdot \mathbf{d} \leq \varepsilon$;
- The solution is $\mathbf{x}$.

**Theorem 12.13.** The Newton's method is a **quadratic convergent** method.

## 12.2 Unconstrained Optimization

**Definition 12.14.** Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$, a **unconstrained optimization** method solves the problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

by

(1) **Initialization**: given initial point $\mathbf{x}$ and acceptable error $\varepsilon > 0$, $\delta > 0$;
(2) **Iteration**: calculate the direction $\mathbf{d}$ and step size $\alpha$, then let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
(3) **Stop condition**: if $\|\nabla f(\mathbf{x})\| \leq \varepsilon$ or $U_{\mathbb{R}^n}(\mathbf{x}, \delta)$ includes the exact solution, then the current $\mathbf{x}$ is the solution.

**Method 12.15. (Gradient descent method)** Given the initial point $\mathbf{x}$ and $\varepsilon > 0$,

- The iteration step is:
  (1) Calculate $\mathbf{d} = -\nabla f(\mathbf{x})$ and step size $\alpha$ by a line search method;
  (2) Let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- The stop condition is $\|\nabla f(\mathbf{x})\| \leq \varepsilon$;
- The solution is $\mathbf{x}$.

**Theorem 12.16.** The gradient descent method is a **linear convergent** method.

**Method 12.17. (Quasi-Newton method)** Given the initial point $\mathbf{x}$, $\varepsilon > 0$ and a matrix $H \in \mathbb{R}^{n\times n}$ (usually the identity matrix),

- The $k$-th iteration step is:
  (1) Calculate $\mathbf{d}_k = -H_k\nabla f(\mathbf{x}_k)$ and step size $\alpha_k$ by a line search method;
  (2) Let $\mathbf{x}_k = \mathbf{x}_k + \alpha_k\mathbf{d}_k$ and $H_{k+1} = r_k(H_k)$ where the function $r_k$ is a **update** depends on $\mathbf{x}_k$, $\mathbf{x}_{k+1}$, $\nabla f(\mathbf{x}_k)$ and $\nabla f(\mathbf{x}_{k+1})$;
- The stop condition is $\|\nabla f(\mathbf{x}_k)\| \leq \varepsilon$;
- The solution is $\mathbf{x}_k$ that satisfies the stop condition.

**Definition 12.18.** Let $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ and $\mathbf{y}_k = \nabla f(\mathbf{x}_{k+1}) - \nabla f(\mathbf{x}_k)$, the **Symmetric Rank-1 update (SR1)** is

$$H_{k+1} = H_k + \frac{(\mathbf{s}_k - H_k\mathbf{y}_k)(\mathbf{s}_k - H_k\mathbf{y}_k)^T}{(\mathbf{s}_k - H_k\mathbf{y}_k)^T\mathbf{y}_k}.$$

The **DFP update** is a rank-2 update defined as

$$H_{k+1} = H_k + \frac{\mathbf{s}_k\mathbf{s}_k^T}{\mathbf{s}_k^T\mathbf{y}_k} - \frac{H_k\mathbf{y}_k\mathbf{y}_k^T H_k}{\mathbf{y}_k^T H_k\mathbf{y}_K}.$$

The **BFGS update** is a rank-2 update defined as

$$H_{k+1} = H_k + \left(1 + \frac{\mathbf{y}_k^T H_k\mathbf{y}_k}{\mathbf{s}_k^T\mathbf{y}_k}\right)\frac{\mathbf{s}_k\mathbf{s}_k^T}{\mathbf{s}_k^T\mathbf{y}_k} - \frac{\mathbf{s}_k\mathbf{y}_k^T H_k + H_k\mathbf{y}_k\mathbf{s}_k^T}{\mathbf{s}_k^T\mathbf{y}_K}.$$

**Theorem 12.19.** The Quasi-Newton method is a **superlinear convergent** method.

**Method 12.20. (Newton's method)** Given the initial point $\mathbf{x}$ and $\varepsilon > 0$,

- The iteration step is:
  (1) Calculate $\mathbf{d} = -(\nabla^2 f(\mathbf{x}))^{-1}\nabla f(\mathbf{x})$ and step size $\alpha$ by a line search method;
  (2) Let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- The stop condition is $\|\nabla f(\mathbf{x})\| \leq \varepsilon$;
- The solution is $\mathbf{x}$.

**Theorem 12.21.** The Newton's method is a **quadratic convergent** method.

# Chapter 13

# Initial Value Problem

**Notation 13.1.** To numerically solve the IVP, we are given initial condition $\mathbf{u}_0 = \mathbf{u}(t_0)$, and want to compute approximations $\{\mathbf{u}_k, k = 1, 2, ...\}$ such that

$$\mathbf{u}_k \approx \mathbf{u}(t_k),$$

where $k$ is the uniform time step size and $t_n = nk$.

## 13.1 Linear Multistep Method

**Definition 13.2.** For solving the IVP, an s-step **linear multistep method** (LMM) has the form

$$\sum_{j=0}^{s} \alpha_j \mathbf{u}_{n+j} = k \sum_{j=0}^{s} \beta \mathbf{f}\left(\mathbf{u}_{n+j}, t_{n+j}\right),$$

where $\alpha_s = 1$ is assumed WLOG.

**Definition 13.3.** An LMM is **explicit** if $\beta_s = 0$, otherwise it is **implicit**.

## 13.2 Runge-Kutta Method

**Definition 13.4.** An s-stage **Runge-Kutta method** (RK) is a one-step method of the form

$$\begin{aligned} \mathbf{y}_i &= \mathbf{f}\left(\mathbf{u}_n + k \sum_{j=1}^{s} a_{ij} \mathbf{y}_j, t_n + c_i k\right), \\ \mathbf{u}_{i+1} &= \mathbf{u}_i + k \sum_{j=1}^{s} b_j \mathbf{y}_j, \end{aligned}$$

where $i = 1, ..., s$ and $a_{ij}, b_j, c_i \in \mathbb{R}$.

**Definition 13.5.** The textsf{Butcher tableau} is one way to organize the coefficients of an RK method as follows

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

The matrix $A = \left(a_{ij}\right)_{s \times s}$ is called the RK matrix and $\mathbf{b} = \left(b_1, ..., b_s\right)^T, \mathbf{c} = \left(c_1, ..., c_s\right)^T$ are called the RK weights and the RK nodes.

**Definition 13.6.** An s-stage **collocation method** is a numerical method for solving the IVP, where we

(1) choose $s$ distinct collocation parameters $c_1, ..., c_s$,
(2) seek $s$-degree polynomial $p$ satisfying
$\forall i = 1, 2, ..., s, \quad \mathbf{p}(t_n) = \mathbf{u}_n \;$ and $\mathbf{p}'(t_n + c_i k) = \mathbf{f}(\mathbf{p}(t_n + c_i k), t_n + c_i k),$
(3) set $\mathbf{u}_{n+1} = \mathbf{p}(t_{n+1})$.

**Theorem 13.7.** The s-stage collocation method is an s-stage IRK method with

$$a_{ij} = \int_0^{c_i} l_j(\tau)\mathrm{d}\tau, \quad b_j = \int_0^1 l_j(\tau)\mathrm{d}\tau,$$

where $i, j = 1, ..., s$ and $l_k(\tau)$ is the elementary Lagrange interpolation polynomial.

# 13.3 Theoretical analysis

**Definition 13.8.** A function $\mathbf{f} : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}^n$ is **Lipschitz continuous** in its first variable over some domain

$$\Omega = \{(\mathbf{u}, t) : \|\mathbf{u} - \mathbf{u}_0\| \le a, t \in [0, T]\}$$

iff

$$\exists L \ge 0, \text{ s.t. } \forall(\mathbf{u}, t) \in \Omega, \quad \|\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{v}, t) \le \|\mathbf{u} - \mathbf{v}\|.$$

## 13.3.1 Error analysis

**Definition 13.9.** The **local truncation error** $\tau$ is the error caused by replacing continuous derivatives with numerical formulas.

**Definition 13.10.** A numerical formulas is **consistent** if $\lim\limits_{k \to 0} \tau = 0$.

## 13.3.2 Stability

**Definition 13.11.** The **region of absolute stability** (RAS) of a numerical method, applied to

$$\mathbf{u}' = \lambda\mathbf{u}, \quad \mathbf{u}_0 = \mathbf{u}(t_0),$$

is the region $\Omega$ that

$$\forall \mathbf{u}_0, \quad \forall \lambda k \in \Omega, \quad \lim_{n \to +\infty} \mathbf{u}_n = 0.$$

**Definition 13.12.** The **stability function** of a one-step method is a function $R : \mathbb{C} \to \mathbb{C}$ that satisfies

$$\mathbf{u}_{n+1} = R(z)\mathbf{u}_n$$

for the $\mathbf{u}' = \lambda\mathbf{u}$ where Re $(E(\lambda)) \le 0$ and $z = k\lambda$.

**Definition 13.13.** A numerical method is **stable** or **zero stable** iff its application to any IVP with $\mathbf{f}(\mathbf{u}, t)$ Lipschitz continuous in $\mathbf{u}$ and continuous in $t$ yields

$$\forall T > 0, \quad \lim_{k \to 0, Nk = t} \|\mathbf{u}_n\| < \infty.$$

**Definition 13.14.** A numerical method is **A($\boldsymbol{\alpha}$)-statble** if the region of absolute stability $\Omega$ satisfies

$$\{z \in \mathbb{C} : \pi - \alpha \le \arg(z) \le \pi + \alpha\} \subseteq \Omega.$$

**Definition 13.15.** A numerical method is **A-statble** if the region of absolute stability $\Omega$ satisfies

$$\{z \in \mathbb{C} : \text{Re } (z) \le 0\} \subseteq \Omega.$$

**Definition 13.16.** A one-step method is **L-stable** if it is A-stable, and its stability function satisfies

$$\lim_{z\to\infty} |R(z)| = 0.$$

**Definition 13.17.** An one-step method is **I-stable** iff its stability function satisfies
$$\forall y \in \mathbb{R}, |R(y\mathbf{i})| \le 1.$$

**Definition 13.18.** An one-step method is **B-stable** (or **contractive**) if for any contractive ODE system, every pair of its numerical solutions $\mathbf{u}_n$ and $\mathbf{v}_n$ satisfy
$$\forall n \in \mathbb{N}, \|u_{n+1} - v_{n+1}\| \le \|u_n - v_n\|.$$

**Definition 13.19.** An RK method is **algebraically stable** iff the RK weights $b_1, ..., b_s$ are nonnegative, the **algebraic stability matrix** $M = \left(b_i a_{ij} + b_i a_{ji} - b_i b_j\right)_{s\times s}$ is positive semidefinite.

**Theorem 13.20.** The order of accuracy of an implicit A-stable LMM satisfies $p \le 2$. An explicit LMM cannot be A-stable.

**Theorem 13.21.** No ERK method is A-stable.

**Theorem 13.22.** An RK method is A-stable if and only if it is I-stable and all poles of its stability function $R(z)$ have positive real parts.

**Theorem 13.23.** If an A-stable RK method with a nonsingular RK matrix $A$ is stiffly accurate, then it is L-stable.

**Theorem 13.24.** If an A-stable RK method with a nonsingular RK matrix $A$ satisfies
$$\forall i \in \{1, ..., s\}, \quad a_{i1} = b_i,$$
then it is L-stable.

**Theorem 13.25.** B-stable one-step methods are A-stable.

**Theorem 13.26.** An algebraically stable RK method is B-stable and A-stable.

### 13.3.3 Convergence

**Definition 13.27.** A numerical method is convergent iff its application to any IVP with $\mathbf{f}(\mathbf{u}, t)$ Lipschitz continuous in $\mathbf{u}$ and continuous in $t$ yields
$$\forall T > 0, \quad \lim_{k\to 0, nk=T} \mathbf{u}_n = \mathbf{u}(T).$$

**Theorem 13.28.** A numerical method is convergent iff it is consistent and stable.

## 13.4 Important Methods

### 13.4.1 Forward Euler's method

**Definition 13.29.** The **forward Euler's method** solves the IVP by
$$\mathbf{u}_{n+1} = \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_n, t_n).$$

**Theorem 13.30.** The region of absolute stability for forward Euler's method is

$$\{z \in \mathbb{C} : |1 + z| \leq 1\}.$$

### 13.4.2 Backward Euler's method

**Definition 13.31.** The **backward Euler's method** solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_{n+1}, t_{n+1}).$$

**Theorem 13.32.** The region of absolute stability for backward Euler's method is

$$\{z \in \mathbb{C} : |1 - z| \geq 1\}.$$

### 13.4.3 Trapezoidal method

**Definition 13.33.** The **trapezoidal method** solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{k}{2}(\mathbf{f}(\mathbf{u}_n, t_n) + \mathbf{f}(\mathbf{u}_{n+1}, t_{n+1})).$$

**Theorem 13.34.** The region of absolute stability for trapezoidal method is

$$\left\{z \in \mathbb{C} : \left|\frac{2 + z}{2 - z}\right| \geq 1\right\}.$$

### 13.4.4 Midpoint method (Leapfrog method)

**Definition 13.35.** The **midpoint method (Leapfrog method)** solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_{n-1} + 2k\mathbf{f}(\mathbf{u}_n, t_n).$$

**Theorem 13.36.** The region of absolute stability for midpoint method is

$$\left\{z \in \mathbb{C} : \left|z \pm \sqrt{1 + z^2}\right| \leq 1\right\} \stackrel{?}{=} \{0\}.$$

### 13.4.5 Heun's third-order RK method

**Definition 13.37.** The **Heun's third-order formula** is an ERK method of the form

$$\begin{cases} \mathbf{y}_1 &= \mathbf{f}(\mathbf{u}_n, t_n), \\ \mathbf{y}_2 &= \mathbf{f}\left(\mathbf{u}_n + \frac{k}{3}\mathbf{y}_1, t_n + \frac{k}{3}\right), \\ \mathbf{y}_3 &= \mathbf{f}\left(\mathbf{u}_n + \frac{2k}{3}\mathbf{y}_2, t_n + \frac{2k}{3}\right), \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + \frac{k}{4}(\mathbf{y}_1 + 3\mathbf{y}_3). \end{cases} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array}$$

### 13.4.6 Classical fourth-order RK method

**Definition 13.38.** The **classical fourth-order RK method** is an ERK method of the form

$$\begin{cases} \mathbf{y}_1 &= \mathbf{f}(\mathbf{u}_n, t_n), \\ \mathbf{y}_2 &= \mathbf{f}\Big(\mathbf{u}_n + \frac{k}{2}\mathbf{y}_1, t_n + \frac{k}{2}\Big), \\ \mathbf{y}_3 &= \mathbf{f}\Big(\mathbf{u}_n + \frac{k}{2}\mathbf{y}_2, t_n + \frac{k}{2}\Big), \\ \mathbf{y}_4 &= \mathbf{f}(\mathbf{u}_n + k\mathbf{y}_3, t_n + k), \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + \frac{k}{6}(\mathbf{y}_1 + 2\mathbf{y}_2 + 2\mathbf{y}_3 + \mathbf{y}_4). \end{cases} \qquad \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

### 13.4.7 TR-BDF2 method

**Definition 13.39.** The **TR-BDF2 method** is an one-step method of the form

$$\begin{cases} \mathbf{u}_* &= \mathbf{u}_n + \frac{k}{4}\Big(\mathbf{f}(\mathbf{u}_n, t_n) + \mathbf{f}\Big(\mathbf{u}_*, t_n + \frac{k}{2}\Big)\Big), \\ \mathbf{u}_{n+1} &= \frac{1}{3}(4\mathbf{u}_* - \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_{n+1}, t_{n+1})). \end{cases}$$

# Chapter 14

# Number Theory

## 14.1 Prime Number

**Definition 14.1.** A **prime number** (or a **prime**) is a natural number greater than 1 that is not a product of two smaller natural numbers.

**Definition 14.2.** A **composite number** (or a **composite**) is a natural number greater than 1 that is a product of two smaller natural numbers.

### 14.1.1 Primality testing

**Theorem 14.3.** For a integer $n \in \mathbb{N}$, if it is a product of two natural number $a$ and $b$ thar $a \leq b$, then

$$1 \leq a \leq \sqrt{n} \leq b \leq n.$$

**Method 14.4. (Trial division)** Given a integer $n$, the **trial division method** divides $n$ by each integer from 2 up to $\sqrt{n}$. Any such integer dividing $n$ evenly establishes $n$ as composite, otherwise it is prime.

**Theorem 14.5. (Fermat's little theorem)** For a prime number $p$ and a number $a$ that gcd $(a, p) = 1$, then $a^{p-1} \equiv 1 (\bmod\ p)$

**Method 14.6.** The **Miller-Rabin** algorithm is a method of primality testing, where given a number $n$, where we

(1) determine directly for small numbers such as $p = 2$.
(2) factorize the number $p = u \times 2^t$;
(3) choose a number $a$ that gcd $(a, p) = 1$, and calculate $a^u, a^{u\times 2}, a^{u\times 2^2}, ..., a^{u\times 2^{t-1}}$;
(4) if $a^u \equiv 1 (\bmod\ p)$, or $\exists a^{u\times k}, k < t$ that $a^{u\times k} \equiv p - 1 (\bmod\ p)$ then $p$ passes the test, otherwise, $p$ is a composite number;
(5) repeat above steps to eliminate error.

For numbers less than $2^{32}$, choose $a \in \{2, 7, 61\}$ is enough, for numbers less than $2^{\{64\}}$, choose $a \in \{2, 325, 9375, 28178, 450775, 9780504, 1795265022\}$ is enough.

### 14.1.2 Sieves

**Method 14.7. (Sieve of Eratosthenes)** Given a upper limit $n$, the **sieve of Eratosthenes** solves all the prime numbers up to $n$ by marking composite numbers, where we

(1) create a list of consecutive integers from 2 to $n$: $\{2, 3, 4, ..., n\}$;
(2) initially, let $p = 2$, the smallest prime number;
(3) enumerate the multiples of $p$ by counting in increments of $p$ from $2p$ to $n$, and mark them in the list;
(4) find the smallest number in the list greater than $p$ that is not marked;
(5) if there was no such number, the method is terminated and the numbers remaining not marked in the list are all the primes below $n$, otherwise let $p$ now equal the new number which is the next prime, and repeat from step (3).

# Part 3

# Machine Learning

# Chapter 15

# Regression

## 15.1 Linear Regression

**Definition 15.1.** Given a data set $\{(\mathbf{x}_i, y_i), i \in \{1, ..., m\}\}$ where $\mathbf{x}_i \in \mathbb{R}^n$, the linear regression seeks $\tilde{\mathbf{w}} \in \mathbb{R}^n$ and $\tilde{b} \in \mathbb{R}$ such that

$$f(\mathbf{x}_i) = \tilde{\mathbf{w}}^T \mathbf{x}_i + \tilde{b} \approx y_i.$$

In general, we choose mean square error to estimate the error between $f(\mathbf{x}_i)$ and $y_i$, which implies

$$\left(\tilde{\mathbf{w}}, \tilde{b}\right) = \underset{\mathbf{w}\in\mathbb{R}^n, b\in\mathbb{R}}{\arg\min} \sum_{i=1}^{m} \left(f(\mathbf{x}_i) - y_i\right)^2 = \underset{\mathbf{w}\in\mathbb{R}^n, b\in\mathbb{R}}{\arg\min} \sum_{i=1}^{m} \left(\mathbf{w}^T x + b - y_i\right)^2.$$

**Theorem 15.2.** Given a data set $\{(\mathbf{x}_i, y_i), i \in \{1, ..., m\}\}$ where $\mathbf{x}_i \in \mathbb{R}^n$, let

$$X = \begin{pmatrix} \mathbf{x}_1^T & 1 \\ \vdots & 1 \\ \mathbf{x}_m^T & 1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

if $X^T X$ is invertible, the solution of linear regression can be written as

$$\begin{pmatrix} \mathbf{w} \\ b \end{pmatrix} = \left(X^T X\right)^{-1} X^T \mathbf{y}.$$

# Chapter 16

# Decision Tree

# Chapter 17

# Support Vector Machine

# Chapter 18

# Cluster

# Chapter 19

# Neural Networks