

Handbook of Applied Mathematics

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Part 1

Mathematical Foundation

Chapter 1

Analysis

1.1 Calculus

1.1.1 Mean value theorem

Theorem 1.1. (Rolle's theorem) Given $n \geq 2$ and $f \in C^{n-1}([a, b])$ with $f^{(n)}(x)$ exists at each point of (a, b) , suppose that $f(x_0) = \dots = f(x_n) = 0$ for $a \leq x_0 < \dots < x_n \leq b$, then there is a point $\xi \in (a, b)$ such that $f^{(n)}(\xi) = 0$.

Theorem 1.2. (Lagrange's mean value theorem) Given $f \in C^1([a, b])$, then there exists $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Theorem 1.3. (Cauchy's mean value theorem) Given $f, g \in C^1([a, b])$, then there exists $\xi \in (a, b)$ such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

If $g(a) \neq g(b)$ and $g'(\xi) \neq 0$, this is equivalent to

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 1.4. (First mean value theorems for definite integrals) Given $f \in C([a, b])$ and g integrable and does not change sign on $[a, b]$, then there exists ξ in (a, b) such that

$$\int_a^b f(x)g(x)dx = f(\xi) \int_a^b g(x)dx.$$

Theorem 1.5. (Second mean value theorems for definite integrals) Given f a integrable function and g a positive monotonically decreasing function, then there exists ξ in (a, b) such that

$$\int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx.$$

If g is a positive monotonically increasing function, then there exists ξ in (a, b) such that

$$\int_a^b f(x)g(x)dx = g(b) \int_\xi^b f(x)dx.$$

If g is a monotonically function, then there exists ξ in (a, b) such that

$$\int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx + g(b) \int_\xi^b f(x)dx.$$

1.1.2 Series

Definition 1.6. A series $\sum_{n=1}^\infty a_n$ is **absolute convergent** if the series of absolute values $\sum_{n=1}^\infty |a_n|$ converges.

Theorem 1.7. If a series is absolute convergent, then any reordering of it converges to the same limit.

Theorem 1.8. (n-th term test) If $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series divergent.

Theorem 1.9. (Direct comparison test) If $\sum_{n=1}^{\infty} b_n$ is convergent and exists $N > 0$, for all $n > N$, $0 \leq a_n \leq b_n$, then $\sum_{n=1}^{\infty} a_n$ is convergent; if $\sum_{n=1}^{\infty} b_n$ is divergent and exists $N > 0$, for all $n > N$, $0 \leq b_n \leq a_n$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem 1.10. (Limit comparison test) Given two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $a_n \geq 0, b_n > 0$. Then if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c \in (0, \infty)$, then either both series converge or both series diverge.

Theorem 1.11. (Ratio test) Given $\sum_{n=1}^{\infty} a_n$ and

$$R = \limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|, r = \liminf_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|,$$

if $R < 1$, then the series converges absolutely; if $r > 1$, then the series diverges.

Theorem 1.12. (Root test) Given $\sum_{n=1}^{\infty} a_n$ and

$$R = \limsup_{n \rightarrow \infty} (|a_n|)^{\frac{1}{n}},$$

if $R < 1$, then the series converges absolutely; if $R > 1$, then the series diverges.

Theorem 1.13. (Integral test) Given $\sum_{n=1}^{\infty} f(n)$ where f is monotone decreasing, then the series converges iff the improper integral

$$\int_1^{\infty} f(x) dx$$

is finite. In particular,

$$\int_1^{\infty} f(x) dx \leq \sum_{n=1}^{\infty} f(n) \leq f(1) + \int_1^{\infty} f(x) dx$$

Theorem 1.14. (Alternating series test) Given $\sum_{n=1}^{\infty} (-1)^n a_n$ where a_n are all positive or negative, then the series converges if $|a_n|$ decreases monotonically and $\lim_{n \rightarrow \infty} a_n = 0$.

1.1.3 Multivariable calculus

Theorem 1.15. (Green's theorem) Let Ω be the region in a plane with $\partial\Omega$ a positively oriented, piecewise smooth, simple closed curve. If P and Q are functions of (x, y) defined on an open region containing Ω and have continuous partial derivatives there, then

$$\oint_{\partial\Omega} (Pdx + Qdy) = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

where the path of integration along C is anticlockwise.

Theorem 1.16. (Stokes' theorem) Let Ω be a smooth oriented surface in \mathbb{R}^3 with $\partial\Omega$ a piecewise smooth, simple closed curve. If $\mathbf{F}(x, y, z) = (F_x(x, y, z), F_y(x, y, z), F_z(x, y, z))$ is defined and has continuous first order partial derivatives in a region containing Ω , then

$$\iint_{\Omega} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}(x) = \oint_{\partial\Omega} \mathbf{F} \cdot d\mathbf{x}$$

Theorem 1.17. (Gauss-Green theorem (Divergence theorem)) For a bounded open set $\Omega \in \mathbb{R}^n$ that $\partial\Omega \in C^1$ and a function $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_n(\mathbf{x})) : \overline{\Omega} \rightarrow \mathbb{R}^n$ satisfies $\mathbf{F}(\mathbf{x}) \in C^1(\Omega) \cap C(\overline{\Omega})$,

$$\int_{\Omega} \operatorname{div} \mathbf{F}(\mathbf{x}) d\mathbf{x} = \int_{\partial\Omega} \mathbf{F}(\mathbf{x}) \cdot \mathbf{n} dS(x),$$

where \mathbf{n} is outward pointing unit normal vector at $\partial\Omega$.

Definition 1.18. An **implicit function** is a function of the form

$$F(x_1, \dots, x_n) = 0,$$

where x_1, \dots, x_n are variables.

Theorem 1.19. Let $F(\mathbf{x}, \mathbf{y}) : \mathbb{R}^{n+m} \rightarrow \mathbb{R}^m$ be a differentiable function of two variables, and $(\mathbf{x}_0, \mathbf{y}_0)$ the point that $F(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$. If the Jacobian matrix

$$J_{F,\mathbf{y}}(\mathbf{x}_0, \mathbf{y}_0) = \left(\frac{\partial F_i}{\partial y_j}(\mathbf{x}_0, \mathbf{y}_0) \right)$$

is invertible, then there exists an open set $\Omega \subseteq \mathbb{R}^n$ containing \mathbf{x}_0 such that there exists a unique function $f : \Omega \rightarrow \mathbb{R}^m$ such that $f(\mathbf{x}_0) = \mathbf{y}_0$ and $F(\mathbf{x}, f(\mathbf{y})) = \mathbf{0}$ for all $\mathbf{x} \in \Omega$.

Moreover, f is continuously differentiable and, denoting the left-hand panel of the Jacobian matrix shown in the previous section as

$$J_{F,\mathbf{x}}(\mathbf{x}_0, \mathbf{y}_0) = \left(\frac{\partial F_i}{\partial x_j}(\mathbf{x}_0, \mathbf{y}_0) \right),$$

the Jacobian matrix of partial derivatives of f in Ω is given by

$$\left(\frac{\partial f_i}{\partial x_j}(\mathbf{x}) \right)_{m \times n} = - (J_{F,\mathbf{y}}(\mathbf{x}, f(\mathbf{x})))_{m \times m}^{-1} (J_{F,\mathbf{x}}(\mathbf{x}, f(\mathbf{x})))_{m \times n}.$$

1.2 Important Inequalities

1.2.1 Fundamental inequality

Theorem 1.20. (Fundamental inequality)

$$\forall x, y \in \mathbb{R}^+, \frac{2}{\frac{1}{a} + \frac{1}{b}} \leq \sqrt{ab} \leq \frac{a+b}{2} \leq \sqrt{\frac{a^2+b^2}{2}}, \text{ equality holds iff } a = b.$$

1.2.2 Triangle inequality

Theorem 1.21. (Triangle inequality)

$$a, b \in \mathbb{C}, \quad ||a| - |b|| \leq |a \pm b| \leq |a| + |b|,$$

$$\mathbf{a}, \mathbf{b} \in \mathbb{R}^n, \quad ||\mathbf{a}| - |\mathbf{b}|| \leq \|\mathbf{a} \pm \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|.$$

1.2.3 Bernoulli inequality

Theorem 1.22. (Bernoulli inequality)

$$\begin{aligned}
 &\forall x \in (-1, +\infty), \forall a \in [1, +\infty), (1+x)^a \geq 1+ax, \\
 &\forall x \in (-1, +\infty), \forall a \in (0, 1), (1+x)^a \leq 1+ax, \\
 &\forall x \in (-1, +\infty), \forall a \in (-1, 0), (1+x)^a \geq 1+ax, \\
 &\forall x_i \in \mathbb{R}, i \in \{1, \dots, n\}, \quad \prod_{i=1}^n (1+x_i) \geq 1 + \sum_{i=1}^n x_i, \\
 &\forall y \geq x > 0, \quad (1+x)^y \geq (1+y)^x.
 \end{aligned}$$

1.2.4 Jensen's inequality

Theorem 1.23. (Jensen's inequality) For a real convex function $f(x) : [a, b] \rightarrow \mathbb{R}$, numbers $x_1, \dots, x_n \in [a, b]$ and weights a_1, \dots, a_n , the Jensen's inequality can be start as

$$\frac{\sum_{i=1}^n a_i f(x_i)}{\sum_{i=1}^n a_i} \geq f\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i}\right).$$

And for concave function f ,

$$\frac{\sum_{i=1}^n a_i f(x_i)}{\sum_{i=1}^n a_i} \leq f\left(\frac{\sum_{i=1}^n a_i x_i}{\sum_{i=1}^n a_i}\right).$$

Equality holds iff $x_1 = \dots = x_n$ or f is linear on $[a, b]$.

1.2.5 Cauchy–Schwarz inequality

Theorem 1.24. (Cauchy–Schwarz inequality)

Discrete form. For real numbers $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}, n \geq 2$

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \geq \left(\sum_{i=1}^n a_i b_i\right)^2.$$

Equality holds iff $\frac{a_1}{b_1} = \dots = \frac{a_n}{b_n}$ or $a_i = 0$ or $b_i = 0$.

Inner product form. For a inner product space V with a norm induced by the inner product,

$$\forall \mathbf{a}, \mathbf{b} \in V \quad \|\mathbf{a}\| \cdot \|\mathbf{b}\| \geq |\langle \mathbf{a}, \mathbf{b} \rangle|.$$

Equality holds iff $\exists k \in \mathbb{R}$, s.t. $k\mathbf{a} = \mathbf{b}$ or $\mathbf{a} = k\mathbf{b}$.

Probability form. For random variables X and Y ,

$$\sqrt{E(X^2)} \cdot \sqrt{E(Y^2)} \geq |E(XY)|.$$

Equality holds iff $\exists k \in \mathbb{R}$, s.t. $kX = Y$ or $X = kY$.

Integral form. For integrable functions $f, g \in L^2(\Omega)$,

$$\int_{\Omega} f^2(x) dx + \int_{\Omega} g^2(x) dx \geq \left(\int_{\Omega} f(x)g(x) dx\right)^2.$$

Equality holds iff $\exists k \in \mathbb{R}$, s.t. $kf(x) = g(x)$ or $f(x) = kg(x)$.

1.2.6 Hölder's inequality

Theorem 1.25. (Hölder's inequality)

Discrete form. For real numbers $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}, n \geq 2$ and $p, q \in [1, +\infty)$ that $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$,

$$\left(\sum_{i=1}^n a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q\right)^{\frac{1}{q}} \geq \left(\sum_{i=1}^n a_i b_i\right).$$

Equality holds iff $\exists c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0$, s.t. $c_1 a_i^p = c_2 b_i^q$.

Integral form. For functions $f \in L^p(\Omega), g \in L^q(\Omega)$ and $p, q \in [1, +\infty)$ that $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$,

$$\left(\int_{\Omega} |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_{\Omega} |g(x)|^q dx\right)^{\frac{1}{q}} \geq \int_{\Omega} f(x)g(x)dx.$$

1.2.7 Young's inequality

Theorem 1.26. (Young's inequality) For $p, q \in [1, +\infty)$ that $\frac{1}{p} + \frac{1}{q} = 1$,

$$\forall a, b \in \mathbb{R}^*, \frac{a^p}{p} + \frac{b^q}{q} \geq ab.$$

Equality holds iff $a^p = b^q$.

1.2.8 Minkowski inequality

Theorem 1.27. (Minkowski inequality) For a metric space S ,

$$\forall f, g \in L^p(S), p \in [1, +\infty], \|f\|_p + \|g\|_p \geq \|f + g\|_p.$$

For $p \in (1, +\infty)$, equality holds iff $\exists k \geq 0$, s.t. $f = kg$ or $kf = g$.

1.3 Special Functions

1.3.1 Gaussian function

Definition 1.28. A **Gaussian function**, or a **Gaussian**, is a function of the form

$$f(x) = a \exp\left(-\frac{(x-b)^2}{2c^2}\right),$$

where $a \in \mathbb{R}^+$ is the height of the curve's peak, $b \in \mathbb{R}$ is the position of the center of the peak and $c \in \mathbb{R}^+$ is the standard deviation or the Gaussian root mean square width.

Theorem 1.29. The integral of a Gaussian is

$$\int_{-\infty}^{+\infty} a \exp\left(-\frac{(x-b)^2}{2c^2}\right) dx = ac\sqrt{2\pi}.$$

Definition 1.30. A **normal distribution** or a **Gaussian distribution** is a continuous probability distribution of the form

$$f_{\mu, \sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where μ is the mean and σ is the standard deviation.

1.3.2 Dirac delta function

Definition 1.31. The **Dirac delta function** centered at \bar{x} is

$$\delta(x - \bar{x}) = \lim_{\varepsilon \rightarrow 0} f_{\bar{x}, \varepsilon}(x - \bar{x}),$$

where $f_{\bar{x}, \varepsilon}$ is a normal distribution with its mean at \bar{x} and its standard deviation as ε .

Theorem 1.32. The Dirac delta function satisfies

$$\delta(x - \bar{x}) = \begin{cases} +\infty, & x = \bar{x} \\ 0, & x \neq \bar{x} \end{cases} \quad \int_{-\infty}^x \delta(x - \bar{x}) dx = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

where $H(x) = \int_{-\infty}^x \delta(x - \bar{x}) dx$ is called **Heaviside function** or **step function**.

Theorem 1.33. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then

$$\int_{-\infty}^{+\infty} \delta(x - \bar{x}) f(x) dx = f(\bar{x}).$$

1.3.3 Gamma function

Definition 1.34. The **Gamma function** defined on \mathbb{C} is

$$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt,$$

where $\operatorname{Re}(z) > 0$.

Theorem 1.35. The Gamma function satisfies

$$\begin{aligned} \forall x \in \mathbb{C}, \quad \Gamma(x+1) &= x\Gamma(x), \\ \forall n \in \mathbb{N}^*, \Gamma(n) &= (n-1)!. \end{aligned}$$

Theorem 1.36. The Gamma function satisfies

$$\forall x \in (0, 1), \Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)},$$

which implies

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

1.3.4 Beta Function

Definition 1.37. For $p, q \in \mathbb{R}^+$, the **Beta function** is defined as

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx.$$

Theorem 1.38. The Beta function satisfies

$$\forall p, q \in \mathbb{R}^+, B(p, q) = B(q, p) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Theorem 1.39. The Beta function satisfies

$$\begin{aligned} \forall p > 0, \forall q > 1, B(p, q) &= \frac{q-1}{p+q-1} B(p, q-1), \\ \forall p > 1, \forall q > 0, B(p, q) &= \frac{p-1}{p+q-1} B(p-1, q), \\ \forall p > 1, \forall q > 1, B(p, q) &= \frac{(p-1)(q-1)}{(p+q-1)(p+q-2)} B(p-1, q-1). \end{aligned}$$

Chapter 2

Algebra

2.1 Linear Space

Definition 2.1. (Linear Space) A linear space over a field \mathbb{F} is a nonempty set V with a addition and a scalar multiplication that satisfies

- (1) Associativity of addition: $\forall \mathbf{x}, \mathbf{y} \in V, \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$,
- (2) Commutativity of addition: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, (\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$,
- (3) Identity element of addition: $\exists \mathbf{0} \in V, \forall \mathbf{x}, \mathbf{x} + \mathbf{0} = \mathbf{x}$,
- (4) Inverse elements of addition: $\forall \mathbf{x} \in V, \exists \mathbf{y} \in V, \text{ s.t. } \mathbf{x} + \mathbf{y} = \mathbf{0}$,
- (5) Compatibility of multiplication: $\forall \mathbf{x} \in V, a, b \in \mathbb{F}, (ab)\mathbf{x} = a(b\mathbf{x})$,
- (6) Identity element of multiplication: $\exists 1 \in \mathbb{F}, \forall \mathbf{x} \in V, 1\mathbf{x} = \mathbf{x}$,
- (7) Distributivity: $\forall \mathbf{x} \in V, a, b \in \mathbb{F}, (a + b)\mathbf{x} = a\mathbf{x} + b\mathbf{x}$,
- (8) Distributivity: $\forall \mathbf{x}, \mathbf{y} \in V, a \in \mathbb{F}, a(\mathbf{x} + \mathbf{y}) = a\mathbf{x} + a\mathbf{y}$.

Notation 2.2. The **dimension** of a linear space V is written as $\dim(V)$.

Definition 2.3. Denoted by V_1, \dots, V_n linear spaces over a field \mathbb{F} , the **product of linear spaces** is defined as

$$V_1 \times \dots \times V_n = \{(\mathbf{v}_1, \dots, \mathbf{v}_n) : \mathbf{v}_1 \in V_1, \dots, \mathbf{v}_n \in V_n\},$$

which is also a linear space over \mathbb{F} .

Definition 2.4. Given a linear space V , a subspace $U \subset V$ and $\mathbf{v} \in V$, the **coset** (or **affine subset**) is defined as

$$\bar{\mathbf{v}} = \{\mathbf{w} \in V : \mathbf{w} = \mathbf{v} + \mathbf{u}, \mathbf{u} \in U\}.$$

Definition 2.5. Given a linear space V and a subspace $U \subset V$, the **quotient space** is defined as

$$V/U = \{\mathbf{v} + U : \mathbf{v} \in V\}.$$

2.1.1 Linear map

Definition 2.6. Denoted by V and W the linear spaces over a field \mathbb{F} , a function $f : V \rightarrow W$ is called a linear map between V and W if it satisfies

- (1) Additivity: $\forall \mathbf{x}, \mathbf{y} \in V, f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$;
- (2) Homogeneity: $\forall \mathbf{x} \in V, \forall k \in \mathbb{F}, f(k\mathbf{x}) = kf(\mathbf{x})$.

Notation 2.7. Denoted by $\mathcal{L}(V, W)$ the set of all linear maps between V and W (it also be written as $\mathcal{L}(V)$ if $V = W$).

Theorem 2.8. For linear space V, W over a field \mathbb{F} and linear maps $f, g \in \mathcal{L}(V, W)$, if we define

$$\forall \mathbf{x} \in V, \forall k \in \mathbb{F}, (f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x}) \text{ and } (kf)(\mathbf{x}) = kf(\mathbf{x}),$$

then $\mathcal{L}(V, W)$ is a linear space.

Theorem 2.9. For a linear map $f \in \mathcal{L}(V, W)$, $f(\mathbf{0}) = f(0\mathbf{v}) = 0f(\mathbf{v}) = \mathbf{0}$.

Theorem 2.10. Given $\mathbf{v}_1, \dots, \mathbf{v}_n$ the basis of linear space V and $\mathbf{w}_1, \dots, \mathbf{w}_n$ the basis of linear space W , then there exists the only linear map $f \in \mathcal{L}(V, W)$ such that

$$\forall i \in \{1, \dots, n\}, f(\mathbf{v}_i) = \mathbf{w}_i.$$

Definition 2.11. For a linear map $f \in \mathcal{L}(V, W)$, the **kernal** (or **null space**) of f is defined as

$$\ker(f) = \{\mathbf{v} \in V : f(\mathbf{v}) = \mathbf{0}\},$$

where $\ker(f)$ is a subspace of V and the number $\dim(\ker(f))$ is the **nullity** of f which also written as $\text{nullity}(f)$

Definition 2.12. For a linear map $f \in \mathcal{L}(V, W)$, the **image** of f is defined as

$$\text{im}(f) = \{\mathbf{w} \in W : \mathbf{w} = f(\mathbf{v}), \mathbf{v} \in V\},$$

where $\text{im}(f)$ is a subspace of W and the number $\dim(\text{im}(f))$ is the **dimension** (or **rank**) of f which also written as $\text{rank}(f)$

Theorem 2.13. (Rank–nullity theorem) For a linear map $f \in \mathcal{L}(V, W)$,

$$\dim(\ker(f)) + \dim(\text{im}(f)) = \dim(V).$$

Definition 2.14. A **isomorphism** is a invertible linear map.

Definition 2.15. Two linear spaces are called **isomorphic** if there exists a invertible linear map between them.

Theorem 2.16. Two linear spaces V, W over a field \mathbb{F} are isomorphic iff $\dim(V) = \dim(W)$.

Theorem 2.17. For a linear space V that $\dim(V) < +\infty$ and a linear map $f \in \mathcal{L}(V)$, the following statements are equivalent:

- (1) f is invertible;
- (2) f is injective;
- (3) f is surjective.

2.2 Metric Space

Definition 2.18. (Metric) For a nonempty set X , the **metric** is a function $d : X \times X \rightarrow \mathbb{R}$ that satisfies

- (1) Positive definiteness: $\forall \mathbf{x}, \mathbf{y} \in X, d(\mathbf{x}, \mathbf{y}) \geq 0, d(\mathbf{x}, \mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y}$,
- (2) Symmetry: $\forall \mathbf{x}, \mathbf{y} \in X, d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$,
- (3) Triangle inequality: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X, d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})$,

Definition 2.19. (Metric space) A **metric space** is a set X provided with a metric.

Notation 2.20. (Neighbourhood) For a metric space X , the **neighbourhood** of $\mathbf{x} \in X$ with radius $\varepsilon > 0$ is defined as

$$U_X(\mathbf{x}, \varepsilon) = \{t : d(\mathbf{x}, t) < \varepsilon, t \in X\}.$$

Notation 2.21. (Punctured neighbourhood) For a metric space X , the **punctured neighbourhood** of $\mathbf{x} \in X$ with radius $\varepsilon > 0$ is defined as

$$U_X^\circ(\mathbf{x}, \varepsilon) = U_X(\mathbf{x}, \varepsilon) \setminus \{\mathbf{x}\} = \{t : d(\mathbf{x}, t) < \varepsilon, t \in X \setminus \{\mathbf{x}\}\}.$$

2.2.1 Completeness & Compactness

Theorem 2.22. (Cauchy's convergence test) A sequence $\{\mathbf{x}_n\}$ in a metric space X is convergent (or said a **cauchy sequence**) iff

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall m, n > N, \|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon.$$

Definition 2.23. (Completeness) A metric space X is **complete** iff all cauchy sequence of X is convergent in X .

Theorem 2.24. (Supremum and infimum principle) For a nonempty set X , if the upper/lower bound of X exists, then the supremum/infimum of X exists.

Theorem 2.25. (The monotone bounded convergence Theorem) For a bounded sequence $\{\mathbf{x}_n\}$, if it is increased, then

$$\lim_{n \rightarrow \infty} \mathbf{x}_n = \sup\{\mathbf{x}_n : n \in \mathbb{N}\}.$$

If it is decreased, then

$$\lim_{n \rightarrow \infty} \mathbf{x}_n = \inf\{\mathbf{x}_n : n \in \mathbb{N}\}.$$

2.2.2 Cover

Definition 2.26. (Cover) For a metric space $S \subseteq X$, A **cover** of S is a set of open sets $\{D_n\}$ satisfies

$$\forall \mathbf{x} \in X, \exists D_n, \text{ s.t. } \mathbf{x} \in D_n.$$

Definition 2.27. (Compactness) A metric space X is called **compact** if every open cover of X has a finite subcover.

2.2.3 Cantor's intersection Theorem

Theorem 2.28. (Cantor's intersection Theorem) For a decreasing sequence of nested non-empty compact, closed subsets $S_n \subseteq X, n \in \mathbb{N}$ of a metric space, if $\{S_n\}$ satisfies

$$S_0 \supset S_1, \dots, \supset S_n \supset \dots,$$

then

$$\bigcap_{k=0}^{\infty} S_k \neq \emptyset.$$

where there is only one point $\mathbf{x} \in \bigcap_{k=0}^{\infty} S_k$ for a complete metric space.

Corollary 2.29. For decreasing sequence of nested non-empty compact, closed subsets $S_n \in X, n \in \mathbb{N}$ of a complete metric space and $\{\mathbf{x}\} = \bigcap_{k=0}^{\infty} S_k$, then

$$\forall \varepsilon > 0, \exists N > 0, \text{ s.t. } \forall n > N, X_n \subset U_X(x, \varepsilon).$$

2.2.4 Cluster point

Definition 2.30. (Cluster point) For a metric space $S \subseteq X$, the **cluster point** of S is the point $\mathbf{x} \in X$ satisfies

$$\forall \varepsilon > 0, U_X^\circ(\mathbf{x}, \varepsilon) \cap S \neq \emptyset.$$

Theorem 2.31. For a convergent sequence $\{\mathbf{x}_n : n \in \mathbb{N}, \forall i \neq j, \mathbf{x}_i \neq \mathbf{x}_j\} \subseteq X$, the point $x = \lim_{n \rightarrow \infty} \mathbf{x}_n$ is a cluster point of X .

Theorem 2.32. (Bolzano–Weierstrass Theorem) For a metric sapce X and a bounded infinite subset $S \in X$, there exists at least one cluster point of X .

2.3 Normed Space

Definition 2.33. (Norm) For a linear space V over a field \mathbb{F} , the **norm** is a function $\|\cdot\| : V \rightarrow \mathbb{F}$ that satisfies

- (1) Positive definiteness: $\forall \mathbf{x} \in V, \|\mathbf{x}\| \geq 0, \|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = 0$,
- (2) Absolute homogeneity: $\forall \mathbf{x} \in V, k \in \mathbb{F}, \|k\mathbf{x}\| = |k| \|\mathbf{x}\|$,
- (3) Triangle inequality: $\forall \mathbf{x}, \mathbf{y} \in V, \|\mathbf{x}\| + \|\mathbf{y}\| \geq \|\mathbf{x} + \mathbf{y}\|$,

Definition 2.34. (Normed space) A **normed space** is a linear space V over the the field \mathbb{F} with a norm.

2.4 Inner Product Space

Definition 2.35. (Inner product) For a linear space V over a field \mathbb{F} , the **inner product** on V is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ that satisfies

- (1) Positive definiteness: $\forall \mathbf{x} \in V, \langle \mathbf{x}, \mathbf{x} \rangle \geq 0, \langle \mathbf{x}, \mathbf{x} \rangle = 0 \Leftrightarrow \mathbf{x} = 0$,
- (2) Conjugate symmetry: $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$,
- (3) Linearity in the first argument: $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V, a, b \in \mathbb{F}, \langle a\mathbf{x} + b\mathbf{z}, \mathbf{y} \rangle = a\langle \mathbf{x}, \mathbf{y} \rangle + b\langle \mathbf{z}, \mathbf{y} \rangle$.

Definition 2.36. (Inner product space) An **inner product space** is a linear space V over the field \mathbb{F} with an inner product.

Theorem 2.37. Given a inner product space V and the norm defined as $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$ satisfies

$$\forall \mathbf{x}, \mathbf{y} \in V, \|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2 \|\mathbf{x}\|^2 + 2 \|\mathbf{y}\|^2.$$

2.4.1 Orthonormal system

Definition 2.38. A subset W of an inner product space V is called **orthonormal** if

$$\forall \mathbf{u}, \mathbf{v} \in S, \langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} 0, & u \neq v \\ 1, & u = v. \end{cases}$$

Definition 2.39. The **Gram-Schmidt process** takes in a finite or infinite independent list $(\mathbf{u}_1, \mathbf{u}_2, \dots)$ and output two other lists $(\mathbf{v}_1, \mathbf{v}_2, \dots)$ and $(\mathbf{u}_1^*, \mathbf{u}_2^*, \dots)$ by

$$\mathbf{v}_{n+1} = \mathbf{u}_{n+1} - \sum_{i=1}^n \langle \mathbf{u}_{n+1}, \mathbf{u}_i^* \rangle \mathbf{u}_i^*,$$

$$\mathbf{u}_{n+1}^* = \frac{\mathbf{v}_{n+1}}{\|\mathbf{v}_{n+1}\|},$$

with the recursion basis as $\mathbf{v}_1 = \mathbf{u}_1$.

Definition 2.40. Let $(\mathbf{u}_1^*, \mathbf{u}_2^*, \dots)$ be a finite or infinite orthonormal list. The **orthogonal expansion** or **Fourier expansion** for an arbitrary \mathbf{w} is the series

$$\sum_{i=1}^n \langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^*,$$

where the constants $\langle \mathbf{w}, \mathbf{u}_i^* \rangle$ are known as the **Fourier coefficients** of \mathbf{w} and the term $\langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^*$ is the **projection** of \mathbf{w} on \mathbf{u}_i^* .

Theorem 2.41. (Minimum properties of Fourier expansions) Let $\mathbf{u}_1^*, \mathbf{u}_2^*, \dots$ be an orthonormal system and let \mathbf{w} be arbitrary. Then

$$\forall a_1, \dots, a_n, \left\| \mathbf{w} - \sum_{i=1}^n \langle \mathbf{w}, \mathbf{u}_i^* \rangle \mathbf{u}_i^* \right\| \leq \left\| \mathbf{w} - \sum_{i=1}^n a_i \mathbf{u}_i^* \right\|,$$

where $\left\| \mathbf{w} - \sum_{i=1}^n a_i \mathbf{u}_i^* \right\|$ is minimized only when $a_i = \langle \mathbf{w}, \mathbf{u}_i^* \rangle$.

Theorem 2.42. (Bessel inequality) Let $\mathbf{u}_1^*, \mathbf{u}_2^*, \dots$ be an orthonormal system and let \mathbf{w} be arbitrary. Then

$$\sum_{i=1}^n |\langle \mathbf{w}, \mathbf{u}_i^* \rangle|^2 \leq \|\mathbf{w}\|^2.$$

2.5 Banach Space

Definition 2.43. (Banach space) A **Banach space** is a complete normed vector space.

2.6 Hilbert Space

Definition 2.44. (Hilbert space) A **Hilbert space** is a inner product space that is also ce with respect to the distance function induced by the inner product.a complete metric space.

2.7 Single Variable Polynomial

Definition 2.45. Denoted by \mathbb{V} a linear space and x the variable, a **(single variable) polynomial** over \mathbb{V} is defined as

$$p_{n(x)} = \sum_{i=0}^n c_i x^i,$$

where $c_0, \dots, c_n \in \mathbb{V}$ are constants that called the **coefficients of the polynomial**.

Definition 2.46. Given a polynomial $p(x) = \sum_{i=0}^n c_i x^i$ where $c_n \neq 0$, the degree of $p(x)$ is marked as $\deg(p(x)) = n$. In particular, the degree of zero polynomial $p(x) = 0$ is $\deg(0) = -\infty$.

Theorem 2.47. Denoted by $\mathbb{P}_n = \{p : \deg(p) \leq n\}$ the set of polynomials with degree no more than n ($n \geq 0$), and $\mathbb{P} = \bigcup_{n=0}^{\infty} \mathbb{P}_n$ the set contains all polynomials, then \mathbb{P}_n is a linear space and satisfies

$$\{0\} = \mathbb{P}_0 \subset \mathbb{P}_1 \subset \dots \subset \mathbb{P}_n \subset \dots \mathbb{P}$$

Theorem 2.48. (Vieta's formulas) Given a polynomial $p \in \mathbb{P}_n$ with the coefficients being real or complex numbers, denoted by x_1, \dots, x_n the complex roots, then

$$\begin{cases} x_1 + \cdots + x_n = -c_{n-1}, \\ \sum_{i=1}^n \sum_{j=i+1}^n x_i x_j = c_{n-2}, \\ \quad \quad \quad \dots \\ \prod_{i=1}^n x_i = (-1)^n c_0, \end{cases}$$

where $c_n = 1$ WLOG.

2.8 Orthogonal Polynomial

Definition 2.49. Given a weight function $\rho(x) : [a, b] \rightarrow \mathbb{R}^+$, satisfies

$$\int_a^b \rho(x) dx > 0, \int_a^b x^k \rho(x) dx > 0 \text{ exists.}$$

The set of **orthogonal polynomials** on $[a, b]$ with the weight function $\rho(x)$ is defined as

$$\{p_i, i \in \mathbb{N}\} \subset L_\rho([a, b]) = \left\{ f(x) : \int_a^b f^2(x) \rho(x) dx < \infty \right\}.$$

where $\{p_i, i \in \mathbb{N}\}$ are calculate from $\{x^n, n \in \mathbb{N}\}$ using the Gram-Schmidt process with the inner product

$$\forall f, g \in L_\rho([a, b]), \langle f, g \rangle = \int_a^b \rho(x) f(x) g(x) dx.$$

Theorem 2.50. Orthogonal polynomials $p_{n-1}(x), p_n(x), p_{n+1}(x)$ satisfies

$$p_{n+1}(x) = (a_n + b_n x) p_n(x) + c_n p_{n-1}(x).$$

where a_n, b_n, c_n are depends on $[a, b]$ and ρ .

Theorem 2.51. The orthogonal polynomial $p_n(x)$ on $[a, b]$ with the weight function $\rho(x)$ has n roots on (a, b) .

2.8.1 Legendre polynomial

Definition 2.52. The **Legendre polynomial** is defined on $[-1, 1]$ with the weight function $\rho(x) = 1$.

Theorem 2.53. The Legendre polynomials $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_{-1}^1 p_i(x) p_j(x) dx = \begin{cases} \frac{2}{2i+1}, & i = j \\ 0, & i \neq j. \end{cases}$$

Theorem 2.54. The Legendre polynomial p_{n-1}, p_n, p_{n+1} satisfies

$$p_{n+1}(x) = \frac{2n+1}{n+1} x p_n(x) - \frac{n}{n+1} p_{n-1}(x).$$

Example 2.55. The first three terms of Legendre polynomials is

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = \frac{3}{2}x^2 - \frac{1}{2}.$$

2.8.2 Chebyshev polynomial of the first kind

Definition 2.56. The **Chebyshev polynomial of the first kind** is defined on $[-1, 1]$ with the weight function $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.

Theorem 2.57. The Chebyshev polynomials of the first kind $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} p_i(x) p_j(x) dx = \begin{cases} \pi & i = j = 0 \\ \frac{\pi}{2} & i = j \neq 0 \\ 0 & i \neq j. \end{cases}$$

Theorem 2.58. The Chebyshev polynomial of the first kind p_{n-1}, p_n, p_{n+1} satisfies

$$p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x).$$

Example 2.59. The first three terms of Chebyshev polynomials of the first kind is

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = 2x^2 - 1.$$

2.8.3 Chebyshev polynomial of the second kind

Definition 2.60. The **Chebyshev polynomial of the second kind** is defined on $[-1, 1]$ with the weight function $\rho(x) = \sqrt{1-x^2}$.

Theorem 2.61. The Chebyshev polynomials of the second kind $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_{-1}^1 \sqrt{1-x^2} p_i(x) p_j(x) dx = \begin{cases} \frac{\pi}{2}, & i = j \\ 0, & i \neq j. \end{cases}$$

Theorem 2.62. The Chebyshev polynomial of the second kind p_{n-1}, p_n, p_{n+1} satisfies

$$p_{n+1}(x) = 2xp_n(x) - p_{n-1}(x).$$

Example 2.63. The first three terms of Chebyshev polynomials of the second kind is

$$p_0(x) = 1, \quad p_1(x) = 2x, \quad p_2(x) = 4x^2 - 1.$$

2.8.4 Laguerre polynomial

Definition 2.64. The **Laguerre polynomial** is defined on $[0, +\infty)$ with the weight function $\rho(x) = x^\alpha e^{-x}$.

Theorem 2.65. The Laguerre polynomial $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_0^{+\infty} x^\alpha e^{-x} p_i(x) p_j(x) dx = \begin{cases} \frac{\Gamma(n+\alpha+1)}{n!}, & i = j \\ 0, & i \neq j. \end{cases}$$

Theorem 2.66. For $\alpha = 0$, the Laguerre polynomial p_{n-1}, p_n, p_{n+1} satisfies

$$p_{n+1}(x) = (2n+1-x)p_n(x) - n^2 p_{n-1}(x).$$

Example 2.67. For $\alpha = 0$, the first three terms of Laguerre polynomial is

$$p_0(x) = 1, \quad p_1(x) = -x + 1, \quad p_2(x) = x^2 - 4x + 2.$$

2.8.5 Hermite polynomial (probability theory form)

Definition 2.68. The **Hermite polynomial** is defined on $(-\infty, +\infty)$ with the weight function $\rho(x) = \left(\frac{1}{\sqrt{2\pi}}\right)e^{-\frac{x^2}{2}}$.

Theorem 2.69. The Hermite polynomial $\{p_i(x), i \in \mathbb{N}\}$ satisfies

$$\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} p_i(x) p_j(x) dx = \begin{cases} n!, & i = j \\ 0, & i \neq j. \end{cases}$$

Theorem 2.70. For $\alpha = 0$, the Hermite polynomial p_{n-1}, p_n, p_{n+1} satisfies

$$p_{n+1}(x) = xp_n(x) - np_{n-1}(x).$$

Example 2.71. For $\alpha = 0$, the first three terms of Hermite polynomial is

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - 1.$$

Chapter 3

Ordinary Differential Equation

Definition 3.1. Given a function F , an **explicit ordinary differential equation** of order n takes the form

$$\mathbf{F}(\mathbf{u}^{(n-1)}, \dots, \mathbf{u}', \mathbf{u}, t) = \mathbf{u}^{(n)},$$

an **implicit ordinary differential equation** of order n takes the form

$$\mathbf{F}(\mathbf{u}^{(n)}, \dots, \mathbf{u}', \mathbf{u}, t) = \mathbf{0},$$

Definition 3.2. An ODE is **autonomous** if it does not depend on the variable x .

Definition 3.3. A ODE is **linear** if can be written as

$$\sum_{i=0}^n A_i(t) \mathbf{u}^{(i)} + \mathbf{r}(t) = \mathbf{0},$$

where $A_i(t)$ and $r(t)$ are continuous functions of t .

Definition 3.4. A linear ODE is **homogeneous** if $\mathbf{r}(t) = \mathbf{0}$, and there is always the trivial solution $\mathbf{u} \equiv \mathbf{0}$.

Definition 3.5. An ODE is **separable** if can be written as

$$P_1(x)Q_1(y) = P_2(x)Q_2(y) \frac{dy}{dx}.$$

Definition 3.6. For initial value $(\mathbf{u}_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, $T \geq t_0$ and $\mathbf{f} : \mathbb{R}^n \times [t_0, T] \rightarrow \mathbb{R}^n$, the **initial value problem (IVP)** is to find $\mathbf{u}(t) \in C^1([t_0, T])$ satisfies

$$\mathbf{u}' = \mathbf{f}(\mathbf{u}, t), \quad \mathbf{u}(t_0) = \mathbf{u}_0.$$

Theorem 3.7. Given an IVP, denoted by $u_0 = u$, $u_i, i = 1, \dots, n$ the i th derivative of u , then the ODE

$$\mathbf{F}(\mathbf{u}^{(n-1)}, \dots, \mathbf{u}', \mathbf{u}, t) = \mathbf{u}^{(n)}$$

can be written as an IVP,

$$\begin{pmatrix} \mathbf{u}'_0 \\ \vdots \\ \mathbf{u}'_{n-2} \\ \mathbf{u}'_{n-1} \end{pmatrix} = \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_{n-1} \\ \mathbf{F}(\mathbf{u}_{n-1}, \dots, \mathbf{u}_1, \mathbf{u}_0, t) \end{pmatrix}.$$

3.1 General Theory

Theorem 3.8. (Peano existence theorem) Given an IVP with an open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if $\mathbf{f}(\mathbf{u}, t) \in C(\Omega)$ and $(\mathbf{u}_0, t_0) \in \Omega$, then there is a local solution $\tilde{\mathbf{u}} : U \rightarrow \mathbb{R}^n$ satisfies the IVP, where U is a neighbourhood of t_0 in \mathbb{R} .

Theorem 3.9. (Picard–Lindelöf theorem) Given an IVP with an open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if $\mathbf{f}(\mathbf{u}, t) : \Omega \rightarrow \mathbb{R}^n$ is continuous in t and Lipschitz continuous in \mathbf{u} and $(\mathbf{u}_0, t_0) \in \Omega$, then there is a unique local solution $\tilde{\mathbf{u}} : U \rightarrow \mathbb{R}^n$ satisfies the IVP, where U is a neighbourhood of t_0 in \mathbb{R} .

Theorem 3.10. (Comparison theorem) Given two IVPs

$$\mathbf{u}'_1 = \mathbf{f}_1(\mathbf{u}_1, t), \quad \mathbf{u}_1(t_0) = \mathbf{u}_0,$$

$$\mathbf{u}'_2 = \mathbf{f}_2(\mathbf{u}_2, t), \quad \mathbf{u}_2(t_0) = \mathbf{u}_0,$$

and a open set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}$, if for all $(\mathbf{u}, t) \in \Omega$, $\mathbf{f}_1(\mathbf{u}, t) < \mathbf{f}_2(\mathbf{u}, t)$, then

$$\begin{cases} \mathbf{u}_1(t) > \mathbf{u}_2(t), & t > t_0, (\mathbf{u}_1(t), t), (\mathbf{u}_2(t), t) \in \Omega, \\ \mathbf{u}_1(t) < \mathbf{u}_2(t), & t < t_0, (\mathbf{u}_1(t), t), (\mathbf{u}_2(t), t) \in \Omega, \end{cases}$$

3.2 Exact solutions

Example 3.11. Given an initial point (y_0, x_0) , and a separable equation

$$P_1(x)Q_1(y) = P_2(x)Q_2(y)\frac{dy}{dx},$$

the solution of the equation is

$$\int_{x_0}^x \frac{P_1(t)}{P_2(t)} dt = \int_{y_0}^y \frac{Q_2(t)}{Q_1(t)} dt.$$

Example 3.12. Given an initial point (y_0, x_0) , and a first-order homogeneous equation

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right),$$

the solution of the equation is

$$\int_{x_0}^x \frac{1}{x} dx = \int_{\frac{y_0}{x_0}}^{\frac{y}{x}} \frac{1}{F(t) - t} dt.$$

Example 3.13. Given an initial point (y_0, x_0) , and a first-order separable equation

$$yM(xy) + xN(xy)\frac{\partial y}{\partial x} = 0,$$

the solution of the equation is

$$\int_{x_0}^x \frac{1}{x} dx = \int_{y_0 x_0}^{yx} \frac{N(t)}{t(N(t) - M(t))} dt,$$

where C is a constant.

Example 3.14. Given a n th-order, linear, inhomogeneous, constant coefficients equation

$$\sum_{i=0}^n a_i \frac{\partial^i y}{\partial x^i} = 0,$$

the solution of the equation is

$$\sum_{i=1}^k \left(\sum_{j=1}^{m_i} c_{ij} x^{j-1} \right) e^{\alpha_i x},$$

where $\{c_{ij}\}$ are constants and α_i is the root of

$$\sum_{i=0}^n a_i x^i = 0$$

that repeated m_i times.

3.3 Important ODEs

3.3.1 Bernoulli differential equation

Definition 3.15. The **Bernoulli differential equation** takes the form

$$y' + P(x)y = Q(x)y^n,$$

where $n \neq 0, 1$.

Theorem 3.16. The solution of the Bernoulli differential equation is

$$y = (z(x))^{\frac{1}{1-n}},$$

where $z(x)$ is the solution of

$$z' + (1-n)P(x)z + (1-n)Q(x) = 0.$$

3.3.2 Riccati equation

Definition 3.17. The **Riccati equation** takes the form

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2,$$

where $q_0(x) \neq 0, q_2(x) \neq 0$.

Theorem 3.18. If u is one particular solution of the Riccati equation, the general solution is obtained as $y = u + \frac{1}{v}$, where v satisfies

$$v' + (q_1(x) + 2q_2(x)u)v + q_2(x) = 0.$$

Chapter 4

Partial Differential Equation

Definition 4.1. A 2th order partial differential equation in \mathbb{R}^n takes the form

$$\sum_{i=0}^n \sum_{j=0}^n a_{ij}(\mathbf{x}) u_{x_i x_j} + \sum_{i=0}^n b_i(\mathbf{x}) u_{x_i} + c(\mathbf{x}) u(\mathbf{x}) = f(\mathbf{x}),$$

where $a_{ij}(\mathbf{x}) = a_{ji}(\mathbf{x})$.

Definition 4.2. Let $A(\mathbf{x}) = (a_{ij}(\mathbf{x}))_{n \times n}$ be a symmetric matrix, and $\lambda_1 \geq \dots \geq \lambda_n$ the eigenvalues of A at \mathbf{x}_0 , then

- The equation is **elliptic** at \mathbf{x}_0 if for $i = 1, \dots, n$, $\lambda_i < 0$
- The equation is **parabolic** at \mathbf{x}_0 if $\lambda_1 = 0$ and for $i = 2, \dots, n$, $\lambda_i < 0$;
- The equation is **hyperbolic** at \mathbf{x}_0 if $\lambda_1 > 0$ and for $i = 2, \dots, n$, $\lambda_i < 0$;

Definition 4.3. The boundary conditions for the unknown function y , constants c_0, c_1 specified by the boundary conditions, and known scalar functions g, h specified by the boundary conditions, where

- **Dirichlet boundary condition:** $y = g$;
- **Neumann boundary condition:** $\frac{\partial y}{\partial n} = g$;
- **Robin boundary condition:** $c_0 y + c_1 \frac{\partial y}{\partial n} = g$ where $c_0, c_1 \neq 0$;
- **Mixed boundary condition:** $y = g$ and $c_0 y + c_1 \frac{\partial y}{\partial n} = h$ where $c_0, c_1 \neq 0$;
- **Cauchy boundary condition:** $y = g$ and $\frac{\partial y}{\partial n} = h$.

4.1 Poisson's Equation

Definition 4.4. A Poisson's equation in \mathbb{R}^n takes the form

$$-\Delta u = f(\mathbf{x}),$$

where Δ is the Laplace operator, $u, f : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

4.2 Heat Equation

Definition 4.5. A Heat equation in $\mathbb{R}^n \times \mathbb{R}$ takes the form

$$\frac{\partial u}{\partial t} - a^2 \Delta u = f(\mathbf{x}, t),$$

where Δ is the Laplace operator on \mathbb{R}^n , $u, f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

4.3 Wave Equation

Definition 4.6. A Wave equation in $\mathbb{R}^n \times \mathbb{R}$ takes the form

$$\frac{\partial^2 u}{\partial t^2} - a^2 \Delta u = f(\mathbf{x}, t),$$

where Δ is the Laplace operator on \mathbb{R}^n , $u, f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

Chapter 5

Probability Theory

Definition 5.1. A **probability space** is a triple (Ω, \mathcal{F}, P) consisting of

- the sample space Ω : an arbitrary non-empty set;
- the σ -algebra $\mathcal{F} \subseteq 2^\Omega$: a set of subsets of Ω , called events, such that
 - \mathcal{F} contains the sample space: $\Omega \in \mathcal{F}$;
 - \mathcal{F} is closed under complements: if $A \in \mathcal{F}$, then also $(\Omega \setminus A) \in \mathcal{F}$;
 - \mathcal{F} is closed under countable unions: if $A_i \in \mathcal{F}, i = 1, \dots$, then also $(\cup_{i=1}^\infty A_i) \in \mathcal{F}$;
- the probability measure $P : \mathcal{F} \rightarrow [0, 1]$: a function such that
 - P is countably additive (also called σ -additive): if $\{A_i\}_{i=1}^\infty \subseteq \mathcal{F}$ is a countable collection of pairwise disjoint sets, then $P(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty P(A_i)$;
 - the measure of the entire sample space is equal to one: $P(\Omega) = 1$.

5.1 Characteristic functions

5.2 Probability limit theorems

Chapter 6

Stochastic Process

6.1 Poisson process

6.2 Markov chain

Chapter 7

Statistics

Chapter 8

Graph

8.1 Shortest Path

8.2 Matching

8.3 Network Flow

8.4 Tree

Chapter 9

Combinatorics

9.1 Generating function

9.2 Inclusion–exclusion principle

9.3 Special Numbers

9.3.1 Catalan number

9.3.2 Stirling number

Part 2

Scientific Computing

Chapter 10

Interpolation

10.1 Polynomial Interpolation

10.1.1 Lagrange formula

Definition 10.1. To interpolate given points $(x_0, f(x_0)), \dots, (x_n, f(x_n))$, the Lagrange formula is

$$p_n(x) = \sum_{i=0}^n f(x_i) l_i(x),$$

where the **elementary Lagrange interpolation polynomial** (or **fundamental polynomial**) for pointwise interpolation $l_k(x)$ is

$$l_k(x) = \prod_{i=0, i \neq k}^n \frac{x - x_i}{x_k - x_i}.$$

In particular, for $n = 0, l_0(x) = 1$.

10.1.2 Newton formula

Definition 10.2. The k th divided difference ($k \in \mathbb{N}^+$) on the **table of divided differences**

$$\begin{array}{l|llll} x_0 & f[x_0] & & & \\ x_1 & f[x_1] & f[x_0, x_1] & & \\ x_2 & f[x_2] & f[x_1, x_2] & f[x_0, x_1, x_2] & \\ x_3 & f[x_3] & f[x_2, x_3] & f[x_1, x_2, x_3] & f[x_0, x_1, x_2, x_3] \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

where the **divided differences** satisfy

$$\begin{aligned} f[x_0] &= f(x_0), \\ f[x_0, \dots, x_k] &= \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}. \end{aligned}$$

Corollary 10.3. Suppose (i_0, \dots, i_k) is a permutation of $(0, \dots, k)$. Then

$$f[x_0, \dots, x_k] = f[x_{i_0}, \dots, x_{i_k}].$$

Theorem 10.4. For distinct points x_0, \dots, x_n and x , we have

$$f(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, \dots, x_n] \prod_{i=0}^{n-1} (x - x_i) + f[x_0, \dots, x_n, x] \prod_{i=0}^n (x - x_i).$$

Definition 10.5. The **Newton formula** for interpolating the points $(x_0, f(x_0)), \dots, (x_n, f(x_n))$ is

$$p_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, \dots, x_n] \prod_{i=0}^{n-1} (x - x_i).$$

10.1.3 Neville-Aitken algorithm

Definition 10.6. Denote $p_0^{[i]}(x) = f(x_i)$ for $i = 0, \dots, n$. For all $k = 0, \dots, n-1$ and $i = 0, \dots, n-k-1$, define

$$p_{k+1}^{[i]}(x) = \frac{(x - x_i)p_k^{[i+1]}(x) - (x - x_{x+k+1})p_k^{[i]}(x)}{x_{i+k+1} - x_i}.$$

Then each $p_k^{[i]}(x)$ is the interpolating polynomial for the function f at the points $x_i, \dots, x_{\{i+k\}}$. In particular, $p_n^{[0]}(x)$ is the interpolating polynomial of degree n for the function f at the points x_0, \dots, x_n .

10.1.4 Hermite interpolation

Definition 10.7. Given distinct points x_0, \dots, x_k in $[a, b]$, non-negative integers m_0, \dots, m_k , and a function $f \in C^M[a, b]$ where $M = \max_{i=0, \dots, k} (m_i)$, the **Hermite interpolation problem** seeks a polynomial $p(x)$ of the lowest degree satisfies

$$\forall i \in \{0, \dots, k\}, \forall \mu \in \{0, \dots, m_i\}, p^{(\mu)}(x_i) = f^{(\mu)}(x_i).$$

Definition 10.8. (Generalized divided difference) Let x_0, \dots, x_k be $k+1$ pairwise distinct points with each x_i repeated $m_i + 1$ times; write $N = k + \sum_{i=0}^k m_i$. The N th divided difference associated with these points is the coefficient of x^N in the polynomial p that uniquely solves the Hermite interpolation problem.

Corollary 10.9. The n th divided difference at $n+1$ “confluent” (i.e. identical) points is

$$f[x_0, \dots, x_0] = \frac{1}{n!} f^{(n)}(x_0),$$

where x_0 is repeated $n+1$ times on the left-hand side.

10.1.5 Approximation

Definition 10.10. Given condition functions $c_0, \dots, c_k : \mathbb{P}_n \rightarrow \mathbb{R}^+$, the **Approximation problem** seeks a polynomial $p_n(x)$ of the given degree n satisfies a unconstrained optimization

$$\min_{p_n \in \mathbb{P}_n} \sum_{i=0}^k c_i(p_n^{(m_i)}).$$

where condition function $c(p)$ includes but is not limited to

$$|p^{(m)}(x)|, (p_n^{(m)}(x))^2, \int_a^b |p^{(m)}| \, dx, \int_a^b (p^{(m)})^2 \, dx.$$

Example 10.11. For non-negative integers m_0, \dots, m_k and condition functions $c_i(p_n) = (p_n^{(m_i)}(x))^2$, denote by

$$p_n(x) = \sum_{i=0}^n c_i x^i$$

the polynomial of the given degree n , then the m th derivative of p_n is

$$p_n^{(m)}(x) = \sum_{i=m}^n \frac{i!}{(i-m)!} c_i x^{i-m}.$$

All above implies the least squares system

$$\begin{cases} p_n^{(m_0)}(x) = \sum_{i=m_0}^n \frac{i!}{(i-m_0)!} c_i x^{i-m_0} = 0, \\ \dots \\ p_n^{(m_k)}(x) = \sum_{i=m_k}^n \frac{i!}{(i-m_k)!} c_i x^{i-m_k} = 0, \end{cases}$$

which can be solved by algorithms such as Householder transformation.

10.1.6 Error analysis

Theorem 10.12. Let $f \in C^n[a, b]$ and suppose that $f^{(n+1)}(x)$ exists at each point of (a, b) . Let $p_n(x) \in \mathbb{P}_n$ denote the unique polynomial that coincides with f at x_0, \dots, x_n . Define

$$R_n(f; x) = f(x) - p_n(x),$$

as the **Cauchy remainder** of the polynomial interpolation.

If $a \leq x_0 < \dots < x_n \leq b$, then there exists some $\xi \in (a, b)$ satisfies

$$R_n(f; x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{i=0}^n (x - x_i)$$

where the value of ξ depends on x, x_0, \dots, x_n and f .

Theorem 10.13. For the Hermite interpolation problem, denote $N = k + \sum_{i=0}^k m_i$. Denote by $p_N(x) \in \mathbb{P}_N$ the unique solution of the problem. Suppose $f^{(N+1)}(x)$ exists in (a, b) . Then there exists some $\xi \in (a, b)$ satisfies

$$R_N(f; x) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{i=0}^k (x - x_i)^{m_i+1}.$$

10.2 Spline

Definition 10.14. Given nonnegative integers n, k , and a strictly increasing sequence $a = x_1 < \dots < x_N = b$, the set of **spline** functions of degree n and smoothness class k relative to the partition $\{x_i\}$ is

$$\mathbb{S}_n^k = \left\{ s : s \in C^k[a, b]; \forall i \in \{1, \dots, N-1\}, s|_{[x_i, x_{i+1}]} \in \mathbb{P}_n \right\},$$

where x_i is the **knot** of the spline.

10.2.1 Cubic spline

Definition 10.15. (Boundary conditions of splines) The followings are common boundary conditions of cubic splines.

- The **complete cubic spline** s satisfies $s'(a) = f'(a), s'(b) = f'(b)$;
- The **cubic spline with specified second derivatives** s satisfies $s''(a) = f''(a), s''(b) = f''(b)$;
- The **natural cubic spline** s satisfies $s''(a) = s''(b) = 0$;
- The **not-a-knot cubic spline** s satisfies $s'''(x)$ exists at $x = x_2$ and $x = x_{N-1}$.
- The **periodic cubic spline** s satisfies $s(a) = s(b), s'(a) = s'(b), s''(a) = s''(b)$.

Theorem 10.16. Denote $m_i = s'(x_i), M_i = s''(x_i)$ for $s \in \mathbb{S}_3^2$, then

$$\forall i = 2, 3, \dots, N-1, \quad \lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} + 1 = 3\mu_i f[x_i, x_{i+1}] + 3\lambda_i f[x_{i-1}, x_i],$$

$$\forall i = 2, 3, \dots, N-1, \quad \mu_i M_{i-1} + 2M_i + \lambda_i m_{i+1} = 6f[x_{i-1}, x_i, x_{i+1}],$$

where

$$\mu_i = \frac{x_i - x_{i-1}}{x_{i+1} - x_{i-1}}, \quad \lambda_i = \frac{x_{i+1} - x_i}{x_{i+1} - x_{i-1}}.$$

In particular, m_i and M_i should be replaced to the derivatives given at the boundary.

Theorem 10.17. Cubic spline $s \in \mathbb{S}_3^2$ from the linear system of $\lambda_i, \mu_i, m_i, M_i$ and the boundary conditions.

10.2.2 B-spline

Definition 10.18. B-splines are defined recursively by

$$B_i^{n+1}(x) = (x - x_{i-1})(x_{i+n} - x_{i-1})B_i^n(x) + \frac{x_{i+n+1} - x}{x_{i+n+1} - x_{i-1}}B_{i+1}^n(x),$$

where recursion base is the B-spline of degree zero

$$B_i^0(x) = \begin{cases} 1, & x \in (x_{i-1}, x_i], \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 10.19. The $\{B_i^n(x)\}$ forms a basis of \mathbb{S}_n^{n-1} .

Definition 10.20. For $N \in \mathbb{N}^*$, the **support** of a $B_i^n(x)$ is

$$\text{supp } \{B_i^n(x)\} = \overline{\{x \in \mathbb{R} : B_i^n(x) \neq 0\}} = [x_{i-1}, x_{i+n}].$$

Theorem 10.21. (Integrals of B-splines) The average of a B-spline over its support only depends on its degree,

$$\frac{1}{t_{i+n} - t_{i-1}} \int_{t_{i-1}}^{t_{i+n}} B_i^n(x) dx = \frac{1}{n+1}.$$

Theorem 10.22. (Derivatives of B-splines) For $n \geq 2$, we have

$$\forall x \in \mathbb{R}, \quad \frac{d}{dx} B_i^n(x) = \frac{nB_i^{n-1}(x)}{x_{i+n-1} - x_{i-1}} - \frac{nB_{i+1}^{n-1}(x)}{x_{i+n} - x_i}.$$

For $n = 1$, it holds for all x except x_{i-1}, t_i, t_{i+1} , where the derivative of $B_i^1(x)$ is not defined.

10.2.3 Error analysis

Theorem 10.23. Suppose a function $f \in C^4[a, b]$, is interpolated by a complete cubic spline or a cubic spline with specified second derivatives at its end points. Then

$$\forall m = 0, 1, 2, |f^{(m)}(x) - s^{(m)}(x)| \leq c_m h^{4-m} \max_{x \in [a, b]} |f^{(4)}(x)|,$$

where $c_0 = \frac{1}{16}, c_1 = c_2 = \frac{1}{2}$ and $h = \max_{i=1, \dots, N-1} |x_{i+1} - x_i|$.

Chapter 11

Integration

Definition 11.1. A **weighted quadrature formula** $I_n(f)$ is a linear function

$$I_n(f) = \sum_{i=1}^n w_i f(x_i),$$

which approximates the integral of a function $f \in C[a, b]$,

$$I(f) = \int_a^b \rho(x) f(x) dx,$$

where the weight function $\rho \in [a, b]$ satisfies $\forall x \in (a, b), \rho(x) > 0$. The points $\{x_i\}$ at which the integrand f is evaluated are called nodes or abscissas, and the multipliers $\{w_i\}$ are called weights or coefficients.

Definition 11.2. A weighted quadrature formula has (polynomial) **degree of exactness** d_E iff

$$\begin{aligned} \forall f \in \mathbb{P}_{d_E}, \quad E_n(f) &= 0, \\ \exists g \in \mathbb{P}_{d_E+1}, \quad \text{s.t. } E_n(g) &\neq 0 \end{aligned}$$

where \mathbb{P}_d denotes the set of polynomials with degree no more than d .

Theorem 11.3. A weighted quadrature formula $I_n(f)$ satisfies $d_E \leq 2n - 1$.

Definition 11.4. The **error** or **remainder** of $I_n(f)$ is

$$E_n(f) = I(f) - I_n(f),$$

where $I_n(f)$ is said to be convergent for $C[a, b]$ iff

$$\forall f \in C[a, b], \quad \lim_{n \rightarrow +\infty} E_n(f) = 0.$$

Theorem 11.5. Let x_1, \dots, x_n be given as distinct nodes of $I_n(f)$. If $d_E \geq n - 1$, then its weights can be deduced as

$$\forall k \in \{1, \dots, n\}, w_k = \int_a^b \rho(x) l_k(x) dx,$$

where $l_k(x)$ is the elementary Lagrange interpolation polynomial for pointwise interpolation applied to the given nodes.

11.1 Newton-Cotes Formulas

Definition 11.6. A **Newton-Cotes formula** is a formula based on approximating $f(x)$ by interpolating it on uniformly spaced nodes $x_1, \dots, x_n \in [a, b]$.

11.1.1 Midpoint rule

Definition 11.7. The **midpoint rule** is a formula based on approximating $f(x)$ by the constant $f\left(\frac{a+b}{2}\right)$.

For $\rho(x) \equiv 1$, it is simply

$$I_M(f) = (b - a) f\left(\frac{a + b}{2}\right).$$

Theorem 11.8. For $f \in C^2[a, b]$, with weight function $\rho \equiv 1$, the error (remainder) of midpoint rule satisfies

$$\exists \xi \in [a, b], \text{ s.t. } E_M(f) = \frac{(b-a)^3}{24} f''(\xi).$$

Corollary 11.9. The midpoint rule has $d_E = 1$.

11.1.2 Trapezoidal rule

Definition 11.10. The **trapezoidal rule** is a formula based on approximating $f(x)$ by the straight line that connects $(a, f(a))$ and $(b, f(b))$.

For $\rho(x) \equiv 1$, it is simply

$$I_T(f) = \frac{b-a}{2}(f(a) + f(b)).$$

Theorem 11.11. For $f \in C^2[a, b]$, with weight function $\rho \equiv 1$, the error (remainder) of trapezoidal rule satisfies

$$\exists \xi \in [a, b], \text{ s.t. } E_T(f) = -\frac{(b-a)^3}{12} f''(\xi).$$

Corollary 11.12. The trapezoidal rule has $d_E = 1$.

11.1.3 Simpson's rule

Definition 11.13. The **Simpson's rule** is a formula based on approximating $f(x)$ by the quadratic polynomial that goes through the points $(a, f(a))$, $(\frac{a+b}{2}, f(\frac{a+b}{2}))$ and $(b, f(b))$.

For $\rho(x) \equiv 1$, it is simply

$$I_S(f) = \frac{b-a}{6} \left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right).$$

Theorem 11.14. For $f \in C^4[a, b]$, with weight function $\rho \equiv 1$, the error (remainder) of Simpson's rule satisfies

$$\exists \xi \in [a, b], \text{ s.t. } E_S(f) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi).$$

Corollary 11.15. The Simpson's rule has $d_E = 3$.

11.2 Gauss Formulas

Theorem 11.16. For an interval $[a, b]$ and a weight function $\rho : [a, b] \rightarrow \mathbb{R}$, the nodes for gauss formula $I_n(f)$ is the root of the n th order orthogonal polynomial on $[a, b]$ with the weight function $\rho(x)$.

Theorem 11.17. A Gauss formula $I_n(f)$ has $d_E = 2n - 1$.

Chapter 12

Optimization

12.1 One-dimensional Line Search

Definition 12.1. Given a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a initial point \mathbf{x} and a direction \mathbf{d} , denoted by $\varphi(\alpha) = f(\mathbf{x} + \alpha\mathbf{d})$, a **one-dimensional line search** method solves the problem

$$\varphi(\alpha) = \min_{t \in \mathbb{R}^+} \varphi(t).$$

Method 12.2. (Success-failure method) For a one-dimensional line search problem, the **success-failure method** is an inexact one-dimensional line search method to solve the interval $[a, b] \in [0, +\infty)$ that exact solution $\alpha^* \in [a, b]$, where we

- (1) Choose initial value $\alpha_0 \in [0, +\infty)$, $h_0 > 0$, $t > 0$ (commonly choose $t = 2$), calculate $\varphi(\alpha_0)$ and let $k = 0$;
- (2) Let $\alpha_{k+1} = \alpha_k + h_k$ and calculate $\varphi(\alpha_{k+1})$, if $\varphi(\alpha_{k+1}) < \varphi(\alpha_k)$, then go to (3), otherwise go to (4);
- (3) Let $h_{k+1} = th_k$, $\alpha = \alpha_k$, $k = k + 1$, and go to (2);
- (4) If $k = 0$, then let $h_k = -h_k$ and go to (2), otherwise stop and the solution $[a, b]$ satisfies $a = \min\{\alpha, \alpha_k\}$, $b = \max\{\alpha, \alpha_k\}$.

Definition 12.3. A general form of one-dimensional line search method is the following three steps:

- (1) **Initialization:** given initial point \mathbf{x} and acceptable error $\varepsilon > 0$, $\delta > 0$;
- (2) **Iteration:** calculate the direction \mathbf{d} and step size α that $f(\mathbf{x} + \alpha\mathbf{d}) = \min_{t \in \mathbb{R}^+} f(\mathbf{x} + t\mathbf{d})$ and let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- (3) **Stop condition:** if $\|\nabla f(\mathbf{x})\| \leq \varepsilon$ or $U_{\mathbb{R}^n}(\mathbf{x}, \delta)$ includes the exact solution, then the current \mathbf{x} is the solution.

where the iteration step are repeated until \mathbf{x} satisfies the stop condition.

Definition 12.4. Given a method, denoted by $\{\mathbf{x}_k\}$ the sequence of the iteration and \mathbf{x}^* the exact solution, the method is **(Q-)linear convergence** if

$$\lim_{k \rightarrow \infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} \in (0, 1),$$

the method is **(Q-)sublinear convergence** if

$$\lim_{k \rightarrow \infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} = 1,$$

the method is **(Q-)superlinear convergence** if

$$\lim_{k \rightarrow \infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|} = 0.$$

For a superlinear convergence method, the method is r -order linear convergence if

$$\lim_{k \rightarrow \infty} \frac{\|\mathbf{x}_{k+1} - \mathbf{x}^*\|}{\|\mathbf{x}_k - \mathbf{x}^*\|^r} \in [0, +\infty),$$

where when $r = 2$ is called **(Q-)quadratic convergence**.

Remark 12.5. There is another **R-convergence** for judging a sequence which use another Q-convergence sequence as the boundary of $\{\|\mathbf{x}_k - x^*\|\}$, but is not needed here.

Method 12.6. (Golden section method) Given the initial point \mathbf{x} , an interval $[a, b]$ and $\delta > 0$,

- The iteration step is:
 - (1) Calculate the two testing points $\lambda = a + (1 - k)(b - a)$ and $\mu = a + k(b - a)$ where $k = \frac{\sqrt{5}-1}{2}$ is the golden ratio;
 - (2) If $\varphi(\lambda) > \varphi(\mu)$, let $a = \lambda$, otherwise let $b = \mu$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

Theorem 12.7. The golden section method is a **linear convergent** method.

Method 12.8. (Fibonacci method) Given the initial point \mathbf{x} , an interval $[a, b]$ and $\delta > 0$,

- The k -th iteration step is:
 - (1) Calculate the two testing points $\lambda = a + \frac{F_k}{F_{k+2}}(b - a)$ and $\mu = a + \frac{F_{k+1}}{F_{k+2}}(b - a)$ where F_k is the k -th fibonacci number and k ;
 - (2) If $\varphi(\lambda) > \varphi(\mu)$, let $a = \lambda$, otherwise let $b = \mu$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

Theorem 12.9. The Fibonacci method is a **linear convergent** method.

Method 12.10. (Bisection method) Given the initial point \mathbf{x} , an interval $[a, b]$ and $\delta > 0$,

- The iteration step is:
 - (1) Calculate the midpoint $m = \frac{a+b}{2}$ and $\varphi(m)$;
 - (2) If $\nabla f(m) \cdot \mathbf{d} < 0$, let $a = m$, otherwise let $b = m$.
- The stop condition is $b - a \leq \delta$;
- The solution is $\mathbf{x} + \frac{a+b}{2}\mathbf{d}$.

Theorem 12.11. The bisection method is a **linear convergent** method.

Method 12.12. (Newton's method) Given the initial point \mathbf{x} and $\varepsilon > 0$,

- The iteration step is:
 - (1) Calculate $(\nabla^2 f(\mathbf{x}))^T \cdot \mathbf{d}$ and $(\nabla f(\mathbf{x}))^T \cdot \mathbf{d}$;
 - (2) Let $\mathbf{x} = \mathbf{x} - \frac{(\nabla f(\mathbf{x}))^T \cdot \mathbf{d}}{(\nabla^2 f(\mathbf{x}))^T \cdot \mathbf{d}}$;
- The stop condition is $(\nabla f(\mathbf{x}))^T \cdot \mathbf{d} \leq \varepsilon$;
- The solution is \mathbf{x} .

Theorem 12.13. The Newton's method is a **quadratic convergent** method.

12.2 Unconstrained Optimization

Definition 12.14. Given a convex function $f: \mathbb{R}^n \rightarrow \mathbb{R}$, a **unconstrained optimization** method solves the problem

$$\min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x})$$

by

- (1) **Initialization:** given initial point \mathbf{x} and acceptable error $\varepsilon > 0$, $\delta > 0$;
- (2) **Iteration:** calculate the direction \mathbf{d} and step size α , then let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- (3) **Stop condition:** if $\|\nabla f(\mathbf{x})\| \leq \varepsilon$ or $U_{\mathbb{R}^n}(\mathbf{x}, \delta)$ includes the exact solution, then the current \mathbf{x} is the solution.

Method 12.15. (Gradient descent method) Given the initial point \mathbf{x} and $\varepsilon > 0$,

- The iteration step is:
 - (1) Calculate $\mathbf{d} = -\nabla f(\mathbf{x})$ and step size α by a line search method;
 - (2) Let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- The stop condition is $\|\nabla f(\mathbf{x})\| \leq \varepsilon$;
- The solution is \mathbf{x} .

Theorem 12.16. The gradient descent method is a **linear convergent** method.

Method 12.17. (Quasi-Newton method) Given the initial point \mathbf{x} , $\varepsilon > 0$ and a matrix $H \in \mathbb{R}^{n \times n}$ (usually the identity matrix),

- The k -th iteration step is:
 - (1) Calculate $\mathbf{d}_k = -H_k \nabla f(\mathbf{x}_k)$ and step size α_k by a line search method;
 - (2) Let $\mathbf{x}_k = \mathbf{x}_k + \alpha_k \mathbf{d}_k$ and $H_{k+1} = r_k(H_k)$ where the function r_k is a **update** depends on \mathbf{x}_k , \mathbf{x}_{k+1} , $\nabla f(\mathbf{x}_k)$ and $\nabla f(\mathbf{x}_{k+1})$;
- The stop condition is $\|\nabla f(\mathbf{x}_k)\| \leq \varepsilon$;
- The solution is \mathbf{x}_k that satisfies the stop condition.

Definition 12.18. Let $\mathbf{s}_k = \mathbf{x}_{k+1} - \mathbf{x}_k$ and $\mathbf{y}_k = \nabla f(\mathbf{x}_{k+1}) - \nabla f(\mathbf{x}_k)$, the **Symmetric Rank-1 update (SR1)** is

$$H_{k+1} = H_k + \frac{(\mathbf{s}_k - H_k \mathbf{y}_k)(\mathbf{s}_k - H_k \mathbf{y}_k)^T}{(\mathbf{s}_k - H_k \mathbf{y}_k)^T \mathbf{y}_k}.$$

The **DFP update** is a rank-2 update defined as

$$H_{k+1} = H_k + \frac{\mathbf{s}_k \mathbf{s}_k^T}{\mathbf{s}_k^T \mathbf{y}_k} - \frac{H_k \mathbf{y}_k \mathbf{y}_k^T H_k}{\mathbf{y}_k^T H_k \mathbf{y}_k}.$$

The **BFGS update** is a rank-2 update defined as

$$H_{k+1} = H_k + \left(1 + \frac{\mathbf{y}_k^T H_k \mathbf{y}_k}{\mathbf{s}_k^T \mathbf{y}_k}\right) \frac{\mathbf{s}_k \mathbf{s}_k^T}{\mathbf{s}_k^T \mathbf{y}_k} - \frac{\mathbf{s}_k \mathbf{y}_k^T H_k + H_k \mathbf{y}_k \mathbf{s}_k^T}{\mathbf{s}_k^T \mathbf{y}_k}.$$

Theorem 12.19. The Quasi-Newton method is a **superlinear convergent** method.

Method 12.20. (Newton's method) Given the initial point \mathbf{x} and $\varepsilon > 0$,

- The iteration step is:
 - (1) Calculate $\mathbf{d} = -(\nabla^2 f(\mathbf{x}))^{-1} \nabla f(\mathbf{x})$ and step size α by a line search method;
 - (2) Let $\mathbf{x} = \mathbf{x} + \alpha\mathbf{d}$;
- The stop condition is $\|\nabla f(\mathbf{x})\| \leq \varepsilon$;
- The solution is \mathbf{x} .

Theorem 12.21. The Newton's method is a **quadratic convergent** method.

Chapter 13

Initial Value Problem

Notation 13.1. To numerically solve the IVP, we are given initial condition $\mathbf{u}_0 = \mathbf{u}(t_0)$, and want to compute approximations $\{\mathbf{u}_k, k = 1, 2, \dots\}$ such that

$$\mathbf{u}_k \approx \mathbf{u}(t_k),$$

where k is the uniform time step size and $t_n = nk$.

13.1 Linear Multistep Method

Definition 13.2. For solving the IVP, an s -step **linear multistep method** (LMM) has the form

$$\sum_{j=0}^s \alpha_j \mathbf{u}_{n+j} = k \sum_{j=0}^s \beta_j \mathbf{f}(\mathbf{u}_{n+j}, t_{n+j}),$$

where $\alpha_s = 1$ is assumed WLOG.

Definition 13.3. An LMM is **explicit** if $\beta_s = 0$, otherwise it is **implicit**.

13.2 Runge-Kutta Method

Definition 13.4. An s -stage **Runge-Kutta method** (RK) is a one-step method of the form

$$\begin{aligned} \mathbf{y}_i &= \mathbf{f} \left(\mathbf{u}_n + k \sum_{j=1}^s a_{ij} \mathbf{y}_j, t_n + c_i k \right), \\ \mathbf{u}_{i+1} &= \mathbf{u}_i + k \sum_{j=1}^s b_j \mathbf{y}_j, \end{aligned}$$

where $i = 1, \dots, s$ and $a_{ij}, b_j, c_i \in \mathbb{R}$.

Definition 13.5. The textsf{Butcher tableau} is one way to organize the coefficients of an RK method as follows

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1s} \\ \vdots & \vdots & & \vdots \\ c_s & a_{s1} & \cdots & a_{ss} \\ \hline & b_1 & \cdots & b_s \end{array}$$

The matrix $A = (a_{ij})_{s \times s}$ is called the RK matrix and $\mathbf{b} = (b_1, \dots, b_s)^T$, $\mathbf{c} = (c_1, \dots, c_s)^T$ are called the RK weights and the RK nodes.

Definition 13.6. An s -stage **collocation method** is a numerical method for solving the IVP, where we

- (1) choose s distinct collocation parameters c_1, \dots, c_s ,
- (2) seek s -degree polynomial p satisfying

$$\forall i = 1, 2, \dots, s, \quad \mathbf{p}(t_n) = \mathbf{u}_n \quad \text{and} \quad \mathbf{p}'(t_n + c_i k) = \mathbf{f}(\mathbf{p}(t_n + c_i k), t_n + c_i k),$$
- (3) set $\mathbf{u}_{n+1} = \mathbf{p}(t_{n+1})$.

Theorem 13.7. The s -stage collocation method is an s -stage IRK method with

$$a_{ij} = \int_0^{c_i} l_j(\tau) d\tau, \quad b_j = \int_0^1 l_j(\tau) d\tau,$$

where $i, j = 1, \dots, s$ and $l_k(\tau)$ is the elementary Lagrange interpolation polynomial.

13.3 Theoretical analysis

Definition 13.8. A function $\mathbf{f} : \mathbb{R}^n \times [0, +\infty) \rightarrow \mathbb{R}^n$ is **Lipschitz continuous** in its first variable over some domain

$$\Omega = \{(\mathbf{u}, t) : \|\mathbf{u} - \mathbf{u}_0\| \leq a, t \in [0, T]\}$$

iff

$$\exists L \geq 0, \text{ s.t. } \forall (\mathbf{u}, t) \in \Omega, \quad \|\mathbf{f}(\mathbf{u}, t) - \mathbf{f}(\mathbf{v}, t)\| \leq L \|\mathbf{u} - \mathbf{v}\|.$$

13.3.1 Error analysis

Definition 13.9. The **local truncation error** τ is the error caused by replacing continuous derivatives with numerical formulas.

Definition 13.10. A numerical formulas is **consistent** if $\lim_{k \rightarrow 0} \tau = 0$.

13.3.2 Stability

Definition 13.11. The **region of absolute stability** (RAS) of a numerical method, applied to

$$\mathbf{u}' = \lambda \mathbf{u}, \quad \mathbf{u}_0 = \mathbf{u}(t_0),$$

is the region Ω that

$$\forall \mathbf{u}_0, \quad \forall \lambda k \in \Omega, \quad \lim_{n \rightarrow +\infty} \mathbf{u}_n = 0.$$

Definition 13.12. The **stability function** of a one-step method is a function $R : \mathbb{C} \rightarrow \mathbb{C}$ that satisfies

$$\mathbf{u}_{n+1} = R(z) \mathbf{u}_n$$

for the $\mathbf{u}' = \lambda \mathbf{u}$ where $\text{Re}(E(\lambda)) \leq 0$ and $z = k\lambda$.

Definition 13.13. A numerical method is **stable** or **zero stable** iff its application to any IVP with $\mathbf{f}(\mathbf{u}, t)$ Lipschitz continuous in \mathbf{u} and continuous in t yields

$$\forall T > 0, \quad \lim_{k \rightarrow 0, Nk=t} \|\mathbf{u}_n\| < \infty.$$

Definition 13.14. A numerical method is **A(α)-stable** if the region of absolute stability Ω satisfies

$$\{z \in \mathbb{C} : \pi - \alpha \leq \arg(z) \leq \pi + \alpha\} \subseteq \Omega.$$

Definition 13.15. A numerical method is **A-stable** if the region of absolute stability Ω satisfies

$$\{z \in \mathbb{C} : \text{Re}(z) \leq 0\} \subseteq \Omega.$$

Definition 13.16. A one-step method is **L-stable** if it is A-stable, and its stability function satisfies

$$\lim_{z \rightarrow \infty} |R(z)| = 0.$$

Definition 13.17. An one-step method is **I-stable** iff its stability function satisfies

$$\forall y \in \mathbb{R}, |R(yi)| \leq 1.$$

Definition 13.18. An one-step method is **B-stable** (or **contractive**) if for any contractive ODE system, every pair of its numerical solutions \mathbf{u}_n and \mathbf{v}_n satisfy

$$\forall n \in \mathbb{N}, \|u_{n+1} - v_{n+1}\| \leq \|u_n - v_n\|.$$

Definition 13.19. An RK method is **algebraically stable** iff the RK weights b_1, \dots, b_s are nonnegative, the **algebraic stability matrix** $M = (b_i a_{ij} + b_i a_{ji} - b_i b_j)_{s \times s}$ is positive semidefinite.

Theorem 13.20. The order of accuracy of an implicit A-stable LMM satisfies $p \leq 2$. An explicit LMM cannot be A-stable.

Theorem 13.21. No ERK method is A-stable.

Theorem 13.22. An RK method is A-stable if and only if it is I-stable and all poles of its stability function $R(z)$ have positive real parts.

Theorem 13.23. If an A-stable RK method with a nonsingular RK matrix A is stiffly accurate, then it is L-stable.

Theorem 13.24. If an A-stable RK method with a nonsingular RK matrix A satisfies

$$\forall i \in \{1, \dots, s\}, \quad a_{i1} = b_i,$$

then it is L-stable.

Theorem 13.25. B-stable one-step methods are A-stable.

Theorem 13.26. An algebraically stable RK method is B-stable and A-stable.

13.3.3 Convergence

Definition 13.27. A numerical method is convergent iff its application to any IVP with $\mathbf{f}(\mathbf{u}, t)$ Lipschitz continuous in \mathbf{u} and continuous in t yields

$$\forall T > 0, \quad \lim_{k \rightarrow \infty, nk=T} \mathbf{u}_n = \mathbf{u}(T).$$

Theorem 13.28. A numerical method is convergent iff it is consistent and stable.

13.4 Important Methods

13.4.1 Forward Euler's method

Definition 13.29. The **forward Euler's method** solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_n, t_n).$$

Theorem 13.30. The region of absolute stability for forward Euler's method is

$$\{z \in \mathbb{C} : |1 + z| \leq 1\}.$$

13.4.2 Backward Euler's method

Definition 13.31. The backward Euler's method solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_{n+1}, t_{n+1}).$$

Theorem 13.32. The region of absolute stability for backward Euler's method is

$$\{z \in \mathbb{C} : |1 - z| \geq 1\}.$$

13.4.3 Trapezoidal method

Definition 13.33. The trapezoidal method solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_n + \frac{k}{2}(\mathbf{f}(\mathbf{u}_n, t_n) + \mathbf{f}(\mathbf{u}_{n+1}, t_{n+1})).$$

Theorem 13.34. The region of absolute stability for trapezoidal method is

$$\left\{ z \in \mathbb{C} : \left| \frac{2+z}{2-z} \right| \geq 1 \right\}.$$

13.4.4 Midpoint method (Leapfrog method)

Definition 13.35. The midpoint method (Leapfrog method) solves the IVP by

$$\mathbf{u}_{n+1} = \mathbf{u}_{n-1} + 2k\mathbf{f}(\mathbf{u}_n, t_n).$$

Theorem 13.36. The region of absolute stability for midpoint method is

$$\left\{ z \in \mathbb{C} : \left| z \pm \sqrt{1 + z^2} \right| \leq 1 \right\} \stackrel{?}{=} \{0\}.$$

13.4.5 Heun's third-order RK method

Definition 13.37. The Heun's third-order formula is an ERK method of the form

$$\begin{cases} \mathbf{y}_1 &= \mathbf{f}(\mathbf{u}_n, t_n), & 0 & \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \end{array} \right. \\ \mathbf{y}_2 &= \mathbf{f}\left(\mathbf{u}_n + \frac{k}{3}\mathbf{y}_1, t_n + \frac{k}{3}\right), & \frac{1}{3} & \\ \mathbf{y}_3 &= \mathbf{f}\left(\mathbf{u}_n + \frac{2k}{3}\mathbf{y}_2, t_n + \frac{2k}{3}\right), & \frac{2}{3} & \\ \mathbf{u}_{n+1} &= \mathbf{u}_n + \frac{k}{4}(\mathbf{y}_1 + 3\mathbf{y}_3). & \frac{1}{4} & \left| \begin{array}{ccc} 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{3} & 0 \end{array} \right. \end{cases}$$

13.4.6 Classical fourth-order RK method

Definition 13.38. The classical fourth-order RK method is an ERK method of the form

$$\begin{cases}
 \mathbf{y}_1 &= \mathbf{f}(\mathbf{u}_n, t_n), & 0 & \left| \begin{array}{cccc} 0 & 0 & 0 & 0 \end{array} \right. \\
 \mathbf{y}_2 &= \mathbf{f}\left(\mathbf{u}_n + \frac{k}{2}\mathbf{y}_1, t_n + \frac{k}{2}\right), & \frac{1}{2} & \left| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & 0 \end{array} \right. \\
 \mathbf{y}_3 &= \mathbf{f}\left(\mathbf{u}_n + \frac{k}{2}\mathbf{y}_2, t_n + \frac{k}{2}\right), & \frac{1}{2} & \left| \begin{array}{cccc} 0 & \frac{1}{2} & 0 & 0 \end{array} \right. \\
 \mathbf{y}_4 &= \mathbf{f}(\mathbf{u}_n + k\mathbf{y}_3, t_n + k), & 1 & \left| \begin{array}{cccc} 0 & 0 & 1 & 0 \end{array} \right. \\
 \mathbf{u}_{n+1} &= \mathbf{u}_n + \frac{k}{6}(\mathbf{y}_1 + 2\mathbf{y}_2 + 2\mathbf{y}_3 + \mathbf{y}_4). & \hline & \left| \begin{array}{cccc} \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \right.
 \end{cases}$$

13.4.7 TR-BDF2 method

Definition 13.39. The **TR-BDF2 method** is an one-step method of the form

$$\begin{cases}
 \mathbf{u}_* &= \mathbf{u}_n + \frac{k}{4}\left(\mathbf{f}(\mathbf{u}_n, t_n) + \mathbf{f}\left(\mathbf{u}_*, t_n + \frac{k}{2}\right)\right), \\
 \mathbf{u}_{n+1} &= \frac{1}{3}(4\mathbf{u}_* - \mathbf{u}_n + k\mathbf{f}(\mathbf{u}_{n+1}, t_{n+1})).
 \end{cases}$$

Chapter 14

Number Theory

14.1 Prime Number

Definition 14.1. A **prime number** (or a **prime**) is a natural number greater than 1 that is not a product of two smaller natural numbers.

Definition 14.2. A **composite number** (or a **composite**) is a natural number greater than 1 that is a product of two smaller natural numbers.

14.1.1 Primality testing

Theorem 14.3. For a integer $n \in \mathbb{N}$, if it is a product of two natural number a and b that $a \leq b$, then

$$1 \leq a \leq \sqrt{n} \leq b \leq n.$$

Method 14.4. (Trial division) Given a integer n , the **trial division method** divides n by each integer from 2 up to \sqrt{n} . Any such integer dividing n evenly establishes n as composite, otherwise it is prime.

Theorem 14.5. (Fermat's little theorem) For a prime number p and a number a that $\gcd(a, p) = 1$, then $a^{p-1} \equiv 1 \pmod{p}$

Method 14.6. The **Miller-Rabin** algorithm is a method of primality testing, where given a number n , where we

- (1) determine directly for small numbers such as $p = 2$.
- (2) factorize the number $p = u \times 2^t$;
- (3) choose a number a that $\gcd(a, p) = 1$, and calculate $a^u, a^{u \times 2}, a^{u \times 2^2}, \dots, a^{u \times 2^{t-1}}$;
- (4) if $a^u \equiv 1 \pmod{p}$, or $\exists a^{u \times k}, k < t$ that $a^{u \times k} \equiv p - 1 \pmod{p}$ then p passes the test, otherwise, p is a composite number;
- (5) repeat above steps to eliminate error.

For numbers less than 2^{32} , choose $a \in \{2, 7, 61\}$ is enough, for numbers less than 2^{64} , choose $a \in \{2, 325, 9375, 28178, 450775, 9780504, 1795265022\}$ is enough.

14.1.2 Sieves

Method 14.7. (Sieve of Eratosthenes) Given a upper limit n , the **sieve of Eratosthenes** solves all the prime numbers up to n by marking composite numbers, where we

- (1) create a list of consecutive integers from 2 to n : $\{2, 3, 4, \dots, n\}$;
- (2) initially, let $p = 2$, the smallest prime number;
- (3) enumerate the multiples of p by counting in increments of p from $2p$ to n , and mark them in the list;
- (4) find the smallest number in the list greater than p that is not marked;
- (5) if there was no such number, the method is terminated and the numbers remaining not marked in the list are all the primes below n , otherwise let p now equal the new number which is the next prime, and repeat from step (3).

Part 3

Machine Learning

Chapter 15

Regression

15.1 Linear Regression

Definition 15.1. Given a data set $\{(\mathbf{x}_i, y_i), i \in \{1, \dots, m\}\}$ where $\mathbf{x}_i \in \mathbb{R}^n$, the linear regression seeks $\tilde{\mathbf{w}} \in \mathbb{R}^n$ and $\tilde{b} \in \mathbb{R}$ such that

$$f(\mathbf{x}_i) = \tilde{\mathbf{w}}^T \mathbf{x}_i + \tilde{b} \approx y_i.$$

In general, we choose mean square error to estimate the error between $f(\mathbf{x}_i)$ and y_i , which implies

$$(\tilde{\mathbf{w}}, \tilde{b}) = \arg \min_{\mathbf{w} \in \mathbb{R}^n, b \in \mathbb{R}} \sum_{i=1}^m (f(\mathbf{x}_i) - y_i)^2 = \arg \min_{\mathbf{w} \in \mathbb{R}^n, b \in \mathbb{R}} \sum_{i=1}^m (\mathbf{w}^T \mathbf{x}_i + b - y_i)^2.$$

Theorem 15.2. Given a data set $\{(\mathbf{x}_i, y_i), i \in \{1, \dots, m\}\}$ where $\mathbf{x}_i \in \mathbb{R}^n$, let

$$X = \begin{pmatrix} \mathbf{x}_1^T & 1 \\ \vdots & 1 \\ \mathbf{x}_m^T & 1 \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix},$$

if $X^T X$ is invertible, the solution of linear regression can be written as

$$\begin{pmatrix} \mathbf{w} \\ b \end{pmatrix} = (X^T X)^{-1} X^T \mathbf{y}.$$

Chapter 16

Decision Tree

Chapter 17

Support Vector Machine

Chapter 18

Cluster

18.1 K-means

Definition 18.1. Given points $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$, **k-means clustering** aims to partition the points into $k \leq n$ sets $S = \{S_1, \dots, S_k\}$ satisfies

$$S = \arg \min_S \left\{ \sum_{i=1}^k \sum_{\mathbf{x} \in S_i} \|\mathbf{x} - \boldsymbol{\mu}_i\|^2 \right\},$$

where $\boldsymbol{\mu}_i$ is the mean (centroid) of points in S_i , i.e. denoted by $|S_i|$ the size of S_i ,

$$\boldsymbol{\mu}_i = \frac{1}{|S_i|} \sum_{\mathbf{x} \in S_i} \mathbf{x}.$$

Theorem 18.2. Denoted by $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n$ the points and $S = \{S_1, \dots, S_k\}$ sets given by K-means,

$$S = \arg \min_S \left\{ \sum_{i=1}^k \frac{1}{|S_i|} \sum_{\mathbf{x}, \mathbf{y} \in S_i} \|\mathbf{x} - \mathbf{y}\|^2 \right\}.$$

Method 18.3. (K-means clustering) Denoted by $S^{(t)} = \{S_1^{(t)}, \dots, S_k^{(t)}\}$ the sets given by k-means at t -th step and $\boldsymbol{\mu}_i^{(t)}$ the mean of $S_i^{(t)}$, the algorithm proceeds by

(1) **Assignment:** Assign each point to the cluster with the nearest mean,

$$S_i^{(t)} = \left\{ \mathbf{x}_p : \forall j \in \{1, \dots, k\}, \|\mathbf{x}_p - \boldsymbol{\mu}_i^{(t)}\|^2 \leq \|\mathbf{x}_p - \boldsymbol{\mu}_j^{(t)}\|^2 \right\};$$

(2) **Update:** Recalculate means (centroids) of each cluster,

$$\boldsymbol{\mu}_i^{(t)} = \frac{1}{|S_i^{(t)}|} \sum_{\mathbf{x} \in S_i^{(t)}} \mathbf{x}.$$

Chapter 19

Neural Networks