$B_i^n(u) = 0$ if $i < 0$ or $i > n$

Lem 2.1

$B_0^n \dots B_n^n$ are lin ~~dep~~ independent

Proof. $\sum b_i B_i^n = \sum b_i \binom{n}{i} u^i (1-u)^{n-i} = 0 / \frac{1}{(1-u)^n}$

$\Rightarrow \sum b_i \left(\frac{u}{1-u}\right)^i = 0$

$\overset{s := \frac{u}{1-u}}{\Rightarrow} \sum b_i s^i = 0$

$\Rightarrow b_i = 0$

Theorem 2.1

$B_0^n \dots B_n^n$ form a basis for polynomials of deg n

Proof follows from ~~the~~ ~~the~~ Lem 2.1 and the fact that the dim of the space of polynomials of deg n is $n+1$

Lem 2.2

$B_i^n(u) = B_{n-i}^n(u)$ (symmetry)

Lem 2.3

$B_i^n(0) = B_{n-i}^n(1) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{otherwise} \end{cases}$

Lem 2.4

$B_0^n \dots B_n^n$ ~~to~~ form a partition of unity $(\sum B_i^n = 1)$

Lem 2.5

$B_i^n > 0 \quad u \in (0,1)$

Lem 2.6

$B_i^{n+1}(u) = u B_{i-1}^n(u) + (1-u) B_i^n(u)$

2.2 Bezier Representation

Def. 2.3

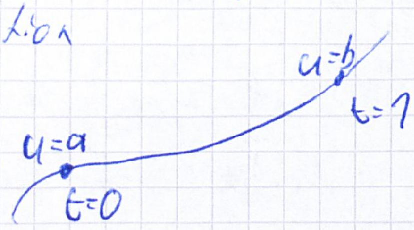
A representation of a polynomial with respect to $\{B_0^n \dots B_n^n\}$ is called the Bezier representation

Let $c(u) = \sum_{i=0}^{n} c_i B_i^n(u)$ (c_i can be in $\mathbb{R}^n$

for practical reasons $u \in [a,b]$

note ~~$u(t) = at + b(1-t)$~~ $u(t) = a(1-t) + bt$

$b(t) := c(u(t))$, b has the degree as c and represents the same polynomial but with a different parametrization



$b(t)$ has a Bézier rep. of deg. n

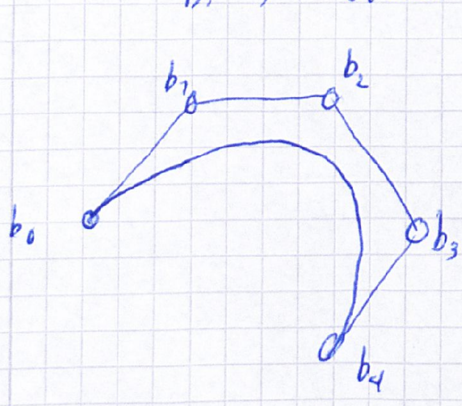
$b(t) = \sum_{i=0}^{n} b_i B_i^n(t)$

$(b(a)=i)$

Def 2.4 b_i is called a control point

Def. 2.5 u is global parameter; t is local

Def. 2.6 The piecewise linear interpolant of the b_i is called the control ~~line~~ polygone



Lem 2.7

$$b(u) = \sum_{i=0}^{n} b_i B_i^n(t)$$
$$= \sum_{i=0}^{n} b_{n-i} B_{n-i}^n(1-t)$$

Lem 2.8

$\mathbb{b}(a) = \mathbb{b}_0$ $(t=0)$ $\quad \mathbb{b}(b) = \mathbb{b}_n$ $(t=1)$ end point interpolation

Lem 2.9

Bézier representation is aff. invariant

$\mathbb{b}_0 \dots \mathbb{b}_n$ are the the control points of b

$\Rightarrow \phi(\mathbb{b}_0) \dots \phi(\mathbb{b}_n)$ are the control points of $\phi(b)$

Lem 2.10 convex hull

The line segment $b([a,b])$ lies in the convex hull of $\mathbb{b}_0 \dots \mathbb{b}_n$

Proof see Lem. 1.4 and 2.5

2.3 de Casteljau Alg

Input: $\mathbb{b}_0 \dots b_n, u$

Output: $\mathbb{b}(u)$

$\mathbb{b}_i^0 := \mathbb{b}_i$

for $k=0 \dots n-1$ {

for $i=0 \dots n-k$ {

$\mathbb{b}_i^{k+1} = (1-t)\mathbb{b}_i^k + t\,\mathbb{b}_{i+1}^k$

}

}

return $\mathbb{b}_0^n$

Lem 2.11 (de Casteljau Correctness)

Proof: $b(u) = \sum_{i=0}^{n} b_i^0 B_i^n(t)$

$$\overset{\text{Lem 2.6}}{=} \sum_{i=0}^{n} b_i^0 \left(t B_{i-1}^{n-1}(t) + (1-t) B_i^{n-1}(t)\right)$$

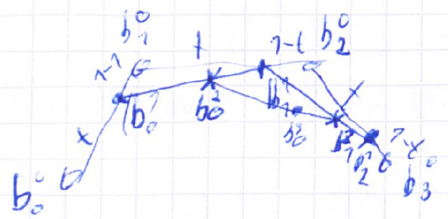
$$= \sum_{i=0}^{n-1} b_i^1 B_i^{n-1}(t)$$

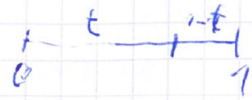
$$\vdots$$

$$= b_0^n \quad \blacksquare$$

$$\begin{array}{cccc}
b_0^0 & & & \\
b_1^0 & b_0^1 & & \\
b_2^0 & b_1^1 & & \\
\vdots & \vdots & \ddots & \\
b_n^0 & b_{n-1}^1 & \cdots & b_0^n
\end{array}$$

Key: $* \xrightarrow{1-t}$, $* \xrightarrow{t}$





Note that when $t \in [0,1]$ all aff comb are convex

$\Rightarrow$ de Casteljau is numerical stable

2.4. Derivatives

Lem 2.12:

$$\frac{d}{dt} B_i^n(t) = n\left(B_{i-1}^{n-1}(t) - B_i^{n-1}(t)\right)$$

Recall $u(t) = a(1-t) + bt \Rightarrow t = \frac{u-a}{b-a}$

Lem. 2.13

$$\frac{d}{du} b_u = \frac{n}{b-a} \sum_{i=0}^{n-1} \Delta b_i B_i^{n-1}(t) \quad \text{where}$$

$\Delta b_i = b_{i+1} - b_i$ (Δb_i is a vector)

Cor. 2.1

$$\mathbb{b}^{r}(u) = \frac{n!}{(n-r)!\,(b-a)} \sum_{i=0}^{n-r} \Delta^{r}\mathbb{b}_i\, B_i^{n-r}(t)$$

where $\Delta^r b_i = \Delta^{r-1}\mathbb{b}_{i+1} - \Delta^{r-1}\mathbb{b}_i$

Def 2.? $\Delta^r \mathbb{b}_i$ is called the forward differe

Lem 2.14 $\Delta^i \mathbb{b}_0 = \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} \mathbb{b}_k$

Lem 2.15 The derivates of $\mathbb{b}(t)$ at $t=0$ to order r depend only depents on the fi Bessier control point

Proof: $\mathbb{b}^r(0) = \Delta^n \mathbb{b}_0$, see lem. 2.14

In particular $\mathbb{b}_0, b_1$ span the tanget line

similary $\mathbb{b}_0, \mathbb{b}_1, \mathbb{b}_2$ span the osculating pl

Lem 2.16

$$\mathbb{b}^{(r)}(u) = \frac{n!}{(n-r)!\,(b-a)^r}\,\Delta^r(\mathbb{b}_0^{n-r})$$

Review: $\Delta^r \mathbb{b}_i = \Delta^{r-1} \mathbb{b}_{i+1} - \Delta^{r-1} \mathbb{b}_i$

Proof: Note that:

$$\begin{aligned}
\Delta b_i (1-t) &+ \Delta b_{i+1}\, t \\
&= (b_{i+1} - b_i)(1-t) + (b_{i+2} - b_{i+1})t \\
&= b_{i+2} t + b_{i+1}(1-t) - (b_{i+1} t + b_i (1-t)) \\
&= b^1_{i+1} - b^1_i \\
&= \Delta b^1_i \\
&= \Delta (b_{i+1} + b_i (1-t))
\end{aligned}$$

Thus, the forward differencing operator commutes with the steps of the de Casteljau algorithm. By induction, Δ^r as well

$$\Rightarrow\ b^{(r)}(u) = \frac{n!}{(n-r)!\,(b-a)^r} \sum_{i=0}^{n-r} \Delta^r b_i\, B_i^{n-r}(t)$$

$$= \frac{n!}{(n-r)!\,(b-a)^r}\,\Delta^r \left(\sum_{i=0}^{n-r} \mathbb{b}_i\, B_i^{n-r}(t) \right)$$

$$\overset{!}{=} \Delta^r \mathbb{b}_0^{n-r} \quad \blacksquare$$

special case: $r=1$

$$\mathbb{b}'(u) = \frac{n}{b-a}\,\Delta \mathbb{b}_0^{n-1}$$

$$= \frac{n}{b-a}(\mathbb{b}_1^{n-1} - \mathbb{b}_0^{n-1}$$

2.5 Integration

Lem 2.17 Let $\mathbb{b}(u) = \sum_{i=0}^{n} \mathbb{b}_i B_i^n(t)$, where $u = a(1-t) + bt$

Then $\mathbb{c}(u) := \int \mathbb{b}(u)\, du = \sum_{i=0}^{n+1} \mathbb{c}_i B_i^{n+1}(t)$

where $\mathbb{c}_i = \mathbb{c}_{i-1} + \frac{b-a}{n+1} \mathbb{b}_{i-1}$

$$\vdots$$
$$= \mathbb{c}_0 + \frac{b-a}{n+1} (\mathbb{b}_0 + \dots + \mathbb{b}_{i-1})$$

Pf.: $\frac{d}{du} \mathbb{c}(u) = \frac{n+1}{b-a} \sum_{i=0}^{n} \Delta \mathbb{c}_i B_i^n(t)$

$$\overset{\text{Def 2.7}}{=} \frac{n+1}{b-a} \sum_{i=0}^{n} (\mathbb{c}_{i+1} - \mathbb{c}_i) B_i^n(t)$$

$$\vdots$$
$$= \sum_{i=0}^{n} \mathbb{b}_i B_i^n(t) \quad \blacksquare$$

!! no interpretation in the affine space !!

Lem 2.18 $\int_a^b \mathbb{b}(u)\, du = \frac{b-a}{n+1} (\mathbb{b}_0 + \dots + \mathbb{b}_n)$

Pf.: $\int_a^b \mathbb{b}(u)\, du = \mathbb{c}(b) - \mathbb{c}(a) = \mathbb{c}_n - \mathbb{c}_0 = \dots \quad \blacksquare$

Lem 2.19

$$\int_0^1 B_i^n(t)\, dt = \frac{1}{n+1}$$

Pf: $B_i^n(t) = \sum_{j=0}^{n} b_j B_j^n(t)$, where $b_j = \begin{cases} 0, & j \neq i \\ 1, & \text{otherwise} \end{cases}$

the rest follows from Lem 2.18

recall: $B_i^n(t)$ is the Bernstein-polynomial

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$\{ B_0^n(t), \dots, B_n^n(t)$

2.6 Conversion to Bezier-representation

$\{1, t, \dots t^n\}$ monomial basis

Lem 2.20

Let $b(t) = \sum_{i=0}^{n} a_i \binom{n}{i} t^i$. Then

$b(t) = \sum_{j=0}^{n} b_j B_j^n(t)$, where $b_j = \sum_{i=0}^{n} \binom{j}{i} a_i$

Note that

$$\binom{n}{i} t^i = \binom{n}{i} t^i (1-t+t)^{n-i}$$

$$= \binom{n}{i} t^i \sum_{k=0}^{n-i} \binom{n-i}{k} t^k (1-t)^{n-i-k}$$

$$= \sum_{k=0}^{n-i} \binom{n}{i} \binom{n-i}{k} t^{i+k} (1-t)^{n-(i+k)}$$

$$= \sum_{k=0}^{n-i} \binom{i+k}{i} \binom{n}{i+k}$$

$$= \sum_{k=0}^{n-i} \binom{i+k}{i} B_{i+k}^n(t)$$

$$= \sum_{j=i}^{n} \binom{j}{i} B_{i+k}^n(t)$$

$$\Rightarrow b(t) = \sum_{i=0}^{n} a_i \binom{n}{i} t^i$$

$$= \sum_{i=0}^{n} a \sum_{j=0}^{n} \binom{j}{i} B_j^n(t) = \dots = \blacksquare$$

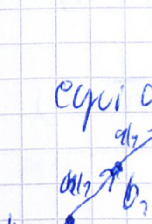
a_0 is a point, a_i is a vector, $i \neq 0$

$a_2, \dots, a_n = 0 \Rightarrow b(t)$ is linear and by Lem 2.

$b_j = a_0 + j \cdot a_1$

this implies that b_j are spaced equidistantly

with direction a_1 through a_0



Lem 2.21

$t = \sum_{i=0}^{n} \frac{i}{n} B_i^n(t)$

Pf: $a_1 = \frac{1}{n}$ $a_j = 0$ $j \neq 1$ (t is given in monomial representation)

by Lem 2.20 $b_j = \binom{j}{1} a_1 = \frac{j}{n}$

Lem 2.22 Linear Precision

If $\mathbb{b}_0, \dots, \mathbb{b}_n$ lie equidistantly on a line, then they define a linear polynomial $\mathbb{b}(t)$ such that

$\mathbb{b}(t) = (1-t)\,\mathbb{b}_0 + t\,\mathbb{b}_n$

Pf: $\mathbb{b}_i = \mathbb{b}_0 + \frac{i}{n}(\mathbb{b}_n - \mathbb{b}_0) = \frac{n-i}{n}\mathbb{b}_0 + \frac{i}{n}\mathbb{b}_n$

$\Rightarrow \mathbb{b}(t) = \sum \mathbb{b}_i B_i^n(t) = \sum \left(\frac{n-i}{n}\mathbb{b}_0 + \frac{i}{n}\mathbb{b}_n\right) B_i^n(t)$

$= \mathbb{b}_0 \sum \frac{n-i}{n} B_i^n(t) + \mathbb{b}_n \sum \frac{i}{n} B_i^n(t)$

$\overset{\text{Lem 2.21}}{=} \mathbb{b}_0 (1-t) + \mathbb{b}_n t$ $\blacksquare$

in Pf of Lem 2.22 we see that

$\mathbb{b}_i = \frac{n-i}{n}\mathbb{b}_0 + \frac{i}{n}\mathbb{b}_n = \mathbb{b}\left(\frac{i}{n}\right)$

Thus, we can write $\mathbb{b}(t) = \sum \mathbb{b}\left(\frac{i}{n}\right) B_i^n(t)$.

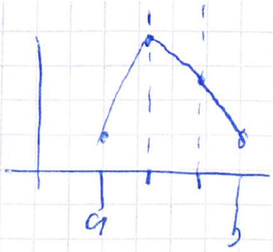
Lem 2.23: Let $\mathbb{b}(t) = \begin{bmatrix} t \\ b(t) \end{bmatrix}$, where $b(t)$

$b(t) = \sum b_i B_i^n(t)$. The Bezier points of $\mathbb{b}(t)$

are $\begin{bmatrix} i/n \\ b_i \end{bmatrix}$

Def 2.8: b_i is called the Bezier ordinate

Def 2.9 $\frac{i}{n}$ is called the Bezier abscissa



2.7 Conversion to monomial representation

Def. 2.10: The Taylor ~~expression~~ expansion of a real function $b(t)$ about a point $t=a$ is given by $b(t) = \sum_{i=0}^{\infty} b^{(i)}(a) \frac{(t-a)^i}{i!}$

Lem. 2.24: $b(u) = \sum b_i B_i^n(t)$, then

$$b(u) = \sum_{i=0}^{n} a_i u^i$$

$$a_i = \sum_{i=0}^{n} \binom{n}{i} \sum_{k=0}^{i} \binom{i}{k} (-1)^{i-k} b_k$$

Pf: $b(t)$ is pol. of deg $n \Rightarrow b^{(i)}(t) = 0 \quad i>n$;

$$\overset{\text{Lem 2.8, Cor 2.1}}{\Rightarrow} b(t) = \sum_{i=0}^{n} \frac{n!}{(n-i)!\,(b-a)^i} \Delta^i b_0 \frac{(t-a)^i}{i!}$$

$$= \sum_{i=0}^{n} \underbrace{\binom{n}{i} \Delta^i b_0}_{a_i} \left(\frac{t-a}{b-a}\right)^i$$

claim follows from Lem. 2.14

3 Bézier Techniques

3.1 Polar form (Different to the book)

Alg. 3.1 Generalized de Casteljau

Input: $b_0 \dots b_n$, $u_1 \dots u_n$

Output: $b[u_1, \dots, u_n]$

$u_i = a(1-t_i)$

$\Rightarrow t_i = \frac{u-a}{b-a}$

$b_i^0 = b_i$

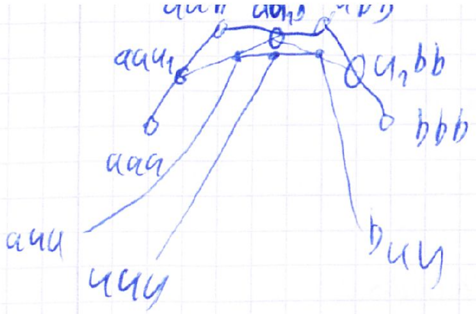
for $k=0$ to $n-1$ {

for $i=0$ to $n-k$ {

$b_i^{k+1} = (1-t_{k+1}) b_i^k + t_{k+1} b_{i+1}^n$

}

} return b_0^n



$\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u_1 \dots u_n] = \mathbb{b}_0^n$

Note that $\mathbb{b}_{\mathbb{b}_0 \dots b_n, a, b}$ is a polynomial

Lem 3.1 Diagonal

$\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u \dots u] = \mathbb{b}(u)$

Pf: if $u_1 = \dots = u_n = h$, Alg 3.1 reduces to Alg 2.1

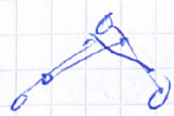
Lem 3.2 Symmetry

$\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[v_1 \dots v_n] = \mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u_1 \dots u_n]$

where $(v_1, \dots, v_n)$ is a permutation of $(u_1 \dots u_n)$

Pf: Homework Hint: A-Frame

Lem 3.3 (Multiaffinity)



$\mathbb{b}[(1-\alpha)u + \alpha v, u_2, \dots, u_n]$

$= (1-\alpha)\,\mathbb{b}[u, u_2, \dots u_n] + \alpha\,\mathbb{b}[v, u_2 \dots u_n]$

Proof: $\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}, [u_1 \dots u_{n-1}, (1-\alpha)u + \alpha u] = \mathbb{b}_0^n$

$= 1 - ((1-\alpha)u + \alpha v)\,\mathbb{b}_0^{n-1} + ((1-\alpha)u$

CAD

Lem 3.4

$$\mathbb{b}_i = \mathbb{b}_{\mathbb{b}_0 \dots b_n, a, b}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}] \qquad (t = \frac{u-a}{b-a})$$

Pf.: Then $t_1 = t_{n-i} = 0$

$$\Rightarrow \mathbb{b}_i^{n-i} = \mathbb{b}_i^{n-i-1} = \dots = \mathbb{b}_i^0 = \mathbb{b}_i \qquad (\mathbb{b}_i^{k+1} = (1-t)\mathbb{b}_i^k + t\,\mathbb{b}$$

Also, $t_{n-i+1} = \dots = t_n = 1$

$$\Rightarrow \mathbb{b}_{\mathbb{b}_0, \dots \mathbb{b}_n, a, b}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}] = \mathbb{b}_0^n = \dots = \mathbb{b}_i^{n-i} = \mathbb{b}_i \quad \blacksquare$$

Cor 3.1

$$\mathbb{b}_i^k = \mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a}[u_1 \dots u_k, \overset{n-i-k}{a \dots a}, \overset{i}{b_1 \dots b}]$$

Lem 3.5 Let $\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n$ be the Bézier points of $\mathbb{b}(u)$ over $[\hat{a}, \hat{b}]$. Then $\mathbb{b}_{\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}] = \mathbb{b}_i$

Pf.: If all param are equal, then Alg. 3.1 reduces to Alg 2.1

$$\sum_{i=0}^n \mathbb{b}_i B_i^n(t) \overset{\text{Lem 3.4}}{=} \sum \mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n}[a \dots a, b \dots b]\, B_i^n(t)$$

$$= \mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u \dots u]$$

$$\overset{\text{Lem 3.1}}{=} \mathbb{b}(u)$$

$$\overset{\text{Lem 3.1}}{=} \mathbb{b}_{\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}[u \dots u]$$

$$= \sum_{i=0}^n \mathbb{b}_{\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}]\, B_i^n(t)$$

By Lem 2.1, $\mathbb{b}_i = \mathbb{b}_{\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}]$ $\blacksquare$

Lem 3.6

Let $\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n$ be the BP of $\mathbb{b}(u)$ over $[\hat{a}, \hat{b}]$

Then $\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u_1 \dots u_n] = \mathbb{b}_{\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}[u_1 \dots u_n]$

Pf.: By Lem 3.2, 3.3, 3.4 we can write

$\mathbb{b}_{\mathbb{b}_0 \dots \mathbb{b}_n, a, b}[u_1 \dots u_n]$ is an lin. comb. of $\mathbb{b}_0, \dots, \mathbb{b}_n$

by substituting $u_i = (1-t_i)a + t_i b$ and

rearranging. By Lem 3.2, 3.3 we can also write

$\hat{\mathbb{b}}_0 \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}[u_1 \dots u_n]$ is a lin. comb. of $\mathbb{b}_{\hat{\mathbb{b}}_0, \dots \hat{\mathbb{b}}_n, \hat{a}, \hat{b}}$

with the same coefficients $[\overset{n-i}{a \dots a}, b \dots b]$

The claim follows from Lem 3.5 $\blacksquare$

Def 3.1 The n-variate, n-symmetric and n-affine polynomial $\mathbb{b}[u_1 \dots u_n]$ is called the polar form of $\mathbb{b}(u)$

Thm 3.1 (Main Theorem)

For every polynomial curve $\mathbb{b}(u)$ of degree $\leq n$ there ex. a unique polar form $\mathbb{b}[u_1 \dots u_n]$ with diagonal $\mathbb{b}[u \dots u] = \mathbb{b}(u)$.

Moreover, $\mathbb{b}_i = \mathbb{b}[\overset{n-i}{a \dots a}, b \dots b]$ are the Bézier points of $\mathbb{b}(u)$ over $[a b]$

Lem 3.7 Let $\mathbb{b}(u) = \sum_{i=0}^{n} c_i C_i(u)$ be a pol. of deg n and let $C_i[u_1 \dots u_n]$ be the polar form of $C_i(u)$. Then, $\mathbb{b}[u_1, \dots u_n] = \sum_{i=0}^{n} c_i C[u_1 \dots u_n]$ is the polar form of $\mathbb{b}(u)$

Lem 3.8
The polar form of $A_i^n(u) = \binom{n}{i} u^i$
is $A_i[u_1 \dots u_n] = \sum_{1 \le j_1 < \dots < j_i \le n} u_{j_1} \dots u_{j_i}$

Def 3.2
$\underbrace{\qquad}_{\text{elementary symmetric polynomial}}$

Lem 3.9 The polar form of $B_i^n(t)$
is $B_i^n[u_1 \dots u_n] = \sum_{\substack{j_1 < \dots < j_i \\ k_1 < \dots < k_{n-i}}} u_{j_1} \dots u_{j_i} (1-u_{k_1}) \dots (1-u_{k_{n-i}})$

$(j_1 \dots j_i, k_1 \dots k_{n-i})$ is a perm. of $(1 \dots n)$

3.2 Sym. Polynomial of the Derivative

Lem 3.10 $\mathbb{b}'(u) = \frac{n}{b-a} (\mathbb{b}[b, u \dots u] - \mathbb{b}[a, u \dots u])$

Pf.: $\mathbb{b}'(u) \overset{\text{Lem 2.16}}{=} \frac{n}{b-a} \Delta (\mathbb{b}_0^{n-1})$

$\overset{\text{Def 2.7}}{=} \frac{n}{b-a} (\mathbb{b}_1^{n-1} - \mathbb{b}_0^{n-1})$

$\overset{\text{Cor 3.1}}{=} \frac{n}{b-a} (\mathbb{b}[b, u \dots u] - \mathbb{b}[a, u \dots u])$ ∎

Cor 3.2
$\mathbb{b}'(u) = n (\mathbb{b}[1, u \dots u] - \mathbb{b}[0, u \dots u])$

Pf.: $\mathbb{b}'(u) \overset{\text{Lem 3.10}}{=} \frac{n}{b-a} (\mathbb{b}[\underset{(1-b)0+1b}{\underset{\parallel}{b}}, u \dots u] - \mathbb{b}[\underset{(1-a)0+1a}{\underset{\parallel}{a}}, u \dots u])$

$= \frac{n}{b-a} ((1-b) \mathbb{b}[0, u \dots u] + b \mathbb{b}[1, u \dots u]$
$- (1-a) \mathbb{b}[0, u \dots u] - a \mathbb{b}[1, u \dots u])$ □

$c(x) = b[x, u_2 \dots u_n]$

$c(x)$ is affine $\overset{\text{Lem 3.7}}{\Longrightarrow} \varphi(\delta) := b[\delta, u_2 \dots u_n]$

$$= b[b, u_2 \dots u_n] - b[a, u_2 \dots u_n]$$

is the underlying lin map where $\delta = b - a$ of $g(u) = b[u_1, u_2 \dots u_n]$.

$\emptyset(u) = Au + b$: $\emptyset(b) - \emptyset(a) = A(b-a)$

That's why we write

$b'(u) = n\, b[\varepsilon, u \dots u_{n}]$, where $\underset{}{\varepsilon} = \overset{\text{vector}}{1-0}$

Lem 3.11

$b'[u_2 \dots u_n] = n\,(b[1, u_2 \dots u_n] - b[0, u_2 \dots u_n])$

Pf: Check the properties

Cor 3.3

$b^{(r)}[u_{r+1} \dots u_n] = \frac{n!}{(n-r)!}\, b[\varepsilon^r, \dots, \varepsilon, u_{r+1}, \dots u_n]$, where

$b[\varepsilon^r, \dots, \varepsilon, u_{r+1} \dots u_n] = b[\varepsilon^{r-1} \dots \varepsilon, u_{r+1} \dots u_n] - b[\varepsilon^{r-1}, \dots \varepsilon, 0, u_{r+1} \dots u_n]$

Lem 3.12

$$\frac{\delta}{\delta u_1} b[u_1 \dots u_n] = \frac{1}{n} b'[u_2 \dots u_n]$$

Pf $\frac{\delta}{\delta u_1} b[u_1 \dots u_n] = \frac{\delta}{\delta u_1} b[(1-u_1)\,0 + u_1 1, u_2 \dots u_n])$

$$= \frac{\delta}{\delta u_1}((1-u_1)\, b[0, u_2 \dots u_n] + u_1 b[1, u_2 \dots u_n])$$

$$= b[1, u_2 \dots u_n] - b[0, u_2 \dots u_n]$$

$$= \frac{1}{n} b'[u_2 \dots u_n] \quad \blacksquare$$

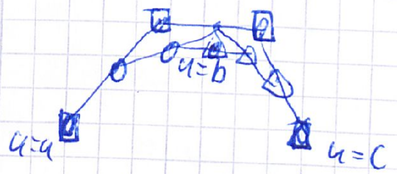
3.3 Subdivision

Consider $\mathbb{b}(u)$

$\mathbb{b}[a..a]$, $\mathbb{b}[a..a,b]$ ... $\mathbb{b}[b..b]$ are the Bézier points over $[a,b]$. Similarly, $\mathbb{b}[b..b]$, $\mathbb{b}[b..b,c]$ .. $\mathbb{b}[c..c]$ over $[b,c]$

By Cor. 3.1, $\mathbb{b}_0^k = \mathbb{b}[\overset{n-k}{a..a}, \overset{k}{b..b}]$ and

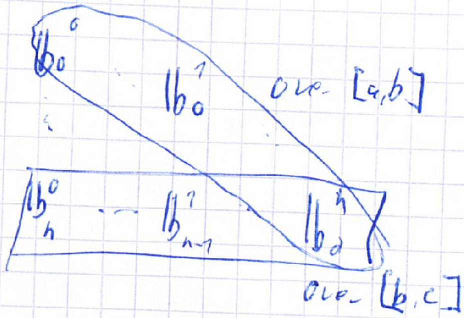
$\mathbb{b}_{n-k}^k = [\overset{n-k}{b..b}, \overset{k}{c..c}]$ (Evaluating de Casteljau over $[a,c]$ at $u=b$)



□ ∃ over $[a,c]$

o ∃ over $[a,b]$

△ ∃ over $[b,c]$



Def 3.3

The process of calculating the control points for a partitioning of $[a,b]$ is called subdivision

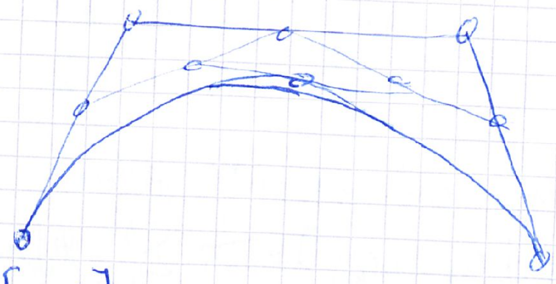
3.4 Convergence under subdivision

Def. 3.4 (Multivariate Taylor expansion)

The Taylor expansion of a multivariable real function $\mathbb{b}[u_1..u_n]$ about a point $(u_1...u_n) = (a_1..a_n)$ is given by

$$\mathbb{b}[u_1..u_n] = \sum_{i=0}^{\infty} \sum_{i_1+..+i_n=i} \frac{\partial^i}{\partial u_1^{i_1}..\partial u_n^{i_n}} \mathbb{b}[a_1..a_n] \frac{(u_1-a_1)^{i_1}}{i_1!} \cdots \frac{(u_n-a_n)^{i_n}}{i_n!}$$

[illegible]



$[a, c]$

Lem 3.13

Let $\mathbb{b}_0 \dots \mathbb{b}_n$ be a Bezier points of a polynomial $\mathbb{b}(u)$ over the interval $[c, c+nh]$ $(h<1)$. Let $c_i = c + ih$. There ex. a constant M not depending on c such that

$$\max_i \|\mathbb{b}(c_i) - \mathbb{b}_i\| \le M h^2$$

Pf.:

$$\mathbb{b}[u_1 \dots u_n] \overset{\text{Def. 3.4}}{=} \mathbb{b}[c_i \dots c_i] + \sum_{j=1}^{n} \frac{\partial}{\partial u_j} \mathbb{b}[c_i \dots c_i](u_j - c_i) + O(h^2)$$

$$\overset{\text{Lem 3.12}}{=} \mathbb{b}(c_i) + \frac{\partial}{\partial u_0} \mathbb{b}[c_i \dots c_i] \sum_{j=1}^{n} u_j - c_i + O(h^2)$$

The remainder is $O(h^2)$ because $h<1$ and $h^k < h^2 \ \forall k>2$

$$\mathbb{b}_i \overset{\text{Thm 3.4}}{=} \mathbb{b}[\overset{n-i}{c \dots c}, c+nh, \dots, c+nh]$$

$$= \mathbb{b}(c_i) + \frac{\partial}{\partial u_0} \mathbb{b}[c_i \dots c_i] \underbrace{\Big(\sum_{j=1}^{n-i} c - c_i + \sum_{j=n-i+1}^{n} (c+nh-c_i)\Big)}_{=0} + O(h^2)$$

∎

3.4 degree elevation

$\cancel{0x^3+x^2+x+1}$

Lem 3.18 Let $\mathbb{b}[u_1 \dots u_n]$ be the polar form of $\mathbb{b}(u)$. Then $\mathbb{c}[u_1 \dots u_{n+1}] = \frac{1}{n+1} \sum_{i=1}^{n+1} \mathbb{b}[u_1, \dots \hat{u_i} \dots u_{n+1}]$ is the polar form of $\mathbb{b}(u)$ of deg. $n+1$.

Lem 3.19

Let $\mathbb{b}(u) = \sum_{i=0}^{n} \mathbb{b}_i B_i^n(t)$. Then, $\mathbb{b}(u) = \sum_{i=0}^{n+1} c_i B_i^{n+1}(t)$

where $c_i = \frac{n+1-i}{n+1} \mathbb{b}_i + \frac{i}{n+1} \mathbb{b}_{i-1}$

Pf.: $c_i \overset{\text{Thm 3.1}}{=} \mathbb{c}[\overset{n+1-i}{a \dots a}, \overset{i}{b \dots b}]$

$= \frac{1}{n+1}\left((n+1-i)\, \mathbb{b}[\overset{n-i}{a \dots a}, \overset{i}{b \dots b}]\right.$

$\left. + i\, \mathbb{b}[\overset{n-(i-1)}{a \dots a}, \overset{i-1}{b \dots b}]\right)$

$\overset{\text{Thm 3.1}}{=} \frac{1}{n+1}\left((n+1-i)\mathbb{b}_i + i\mathbb{b}_{i-1}\right)$ ☐

Lem 3.20

Let $\mathbb{b}(u) = \sum_{i=0}^{n} \mathbb{b}_i B_i^n(t)$

Then, $\mathbb{b}(u) = \sum_{k=0}^{m} \mathbb{d}_k B_i^m(t)$, $m > n$

$\mathbb{d}_i = \sum_{i=0}^{n} \frac{\binom{n}{i}\binom{m-n}{k-i}}{\binom{m}{k}} \mathbb{b}_i$.

Pf (idea) multiply $\mathbb{b}(u)$ by $(1+1-t)^{m-n}$ ☐

Lem. 3.21 Let $\mathbb{b}(u) = \sum_{k=0}^{m} \mathbb{d}_k B_k^m(t)$

Then, $\max_{k=0}^{n} \| \mathbb{d}_k - \mathbb{b}(\frac{k}{n}) \| = O(\frac{1}{m})$

3.5 Curve Generation by subdivision

By Lem 3.13 setting $h = \frac{1}{2^k}$. We see that the Bezier Polygon over $[0, \frac{1}{2^k}, \frac{2}{2^k}, \dots 1]$ of a polynomial $\mathbb{b}(u)$ converges to the curve segment $\mathbb{b}[0,1]$ with order $O(h^2) = \frac{1}{4^k}$.

Algo 3.2: Plot Bezier

Input: $\mathbb{b}_0 \dots \mathbb{b}_n, k$

if $k = 0$

plot $\mathbb{b}_0 \dots \mathbb{b}_n$

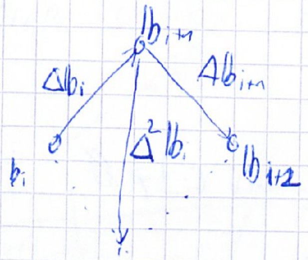
else

subdivide $\sum \mathbb{b}_i B_i^n$ at $\frac{1}{2}$ to get $a_0 \dots a_{2n}$

Plot Bezier $(a_0, \dots, a_n, k-1)$

Plot Bezier $(a_n, \dots a_{2n}, k-1)$

fi



Lem 3.14 $\Delta^2 \mathbb{b}_i = O(h^2)$

Pf: $\mathbb{b}^{(2)}(u) = \frac{n-1}{2h^2} \sum_{i=0}^{n-2} \Delta \mathbb{b}_i B_i^{n}(t)$

By Lem 3.13 $\frac{n-1}{2h^2} \Delta^2 \mathbb{b}_i = \mathbb{b}(c_i) = O(h^2)$

when $c_i = c + ih$. Then,

$\Delta^2 \mathbb{b}_i = \mathbb{b}(a + ih) \frac{2h^2}{n-1} \quad O(h^4) = O(h^2)$ $\square$

of cor 1.14 we terminate Alg. 3.2 when
$\|\Delta^2 b_i\| < \varepsilon$

note: one subdivision step requires $\frac{n(n+1)}{2}$
aff. combinations (Line 5)

Alg. 3.2 generating $2n$ new point in each instance ⇒ on average we need $\frac{n(n+1)}{2(2n)}$ aff. comb. per output point

3.6 Intersections

Def. 3.5 Let $b_0 \dots b_n \in \mathbb{R}^n$

$$\max\{b_0 \dots b_n\} = \begin{bmatrix} \max\{b_{00} \dots b_{n0}\} \\ \vdots \\ \max\{b_{0n} \dots b_{nn}\} \end{bmatrix}$$

Alg. 3.3 Intersect Bezier

Input: $b_0 \dots b_n, c_0 \dots c_n, \varepsilon$

if $[\min b_i, \max b_i] \cap [\min c_i, \max c_i] \neq \emptyset$ then
  if $\max \|\Delta^2 b_i\| > \varepsilon$, then
    subdivide $b_0 \dots b_n$ at $\frac{1}{2}$ to get $a_0 \dots a_{2n}$
    IntersectBezier $(a_0 \dots a_n, c_0 \dots c_n, \varepsilon)$
    IntersectBezier $(a_n \dots a_{2n}, c_0 \dots c_n, \varepsilon)$
  else if $\max \|\Delta c_i\| \geq \varepsilon$ then
    ''
  else
    intersect $b_0 b_n$ and $c_0 c_n$
fi

Alg. 3.3 is not the fastest (better Newton Method) but Alg. 3.3 can be used to find a good starting point

3.7 Variation Diminishing Property

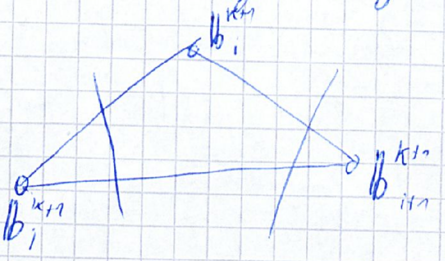
~~Rem~~ Def.: A hyperplane in aff. space of dim n is an aff. subspace of dim $n-1$

Ex.: A line is a hyperplane of $\mathbb{R}^2$
A plane " " of $\mathbb{R}^3$

Lem 3.14

Let H be a hyperplane in A and let $\mathbb{b}(u) \in A$ be a polynomial curve. The number of intersections $\mathbb{b}[0,1]$ with $H \leq$ num of intersections of the Bezier polygon over $[0,1]$ with H

Pf.: Alg. 2.1 (de Casteljau) can be viewed as a corner cutting algorithm



A hyperplane that intersects $\mathbb{b}_i^{k+1}\,\mathbb{b}_{i+1}^{k+1}$ necessarily intersects $\mathbb{b}_i^{k+1}\,\mathbb{b}_{i+1}^{k}\,\mathbb{b}_{i+1}^{k+1}$, but not the other way around

Therefor, the number of intersections of the Bezier polygone over $[0, t_1, \dots t_n, 1]$ with H can not be greater than the num. of inters. of the Bezier poly. over $[0,1]$ with H

In particular let t_i be such that $B(t_i) \in H$

Then the number of intersections of the Bezier polygone over $[0,1]$ with H is greater than or equal to the num. of intersections of $\mathbb{b}(u)$ with H □

Def 3.6

A planar curve is called convex if it intersects any hyperplane in at most 2 points or lies completely in the hyperplane

Lem 3.16

Bezier polygon convex $\Rightarrow$ curve is convex

Pf.: follows directly from Lem 3.15 □

Ex.
Converse is not true
counterexample: see fig. 3.7 in book

Lem 3.17

Let $\mathbb{b}(u) = \begin{bmatrix} u \\ b(u) \end{bmatrix}$ $b(u)$ is convex iff. $b''(u) \geq 0$
or $b''(u) \leq 0$

Pf.: The bezier polygon is convex iff.
$\Delta^2 b_i \geq 0$ or $\Delta^2 b_i \leq 0$

The claim follows from Lem 2.5

3.8 Simple C^r Joints

Thm 3.2 (Stark's Thm)

Let $\mathbb{a}_0, \ldots, \mathbb{a}_n$ and $\mathbb{b}_0 \ldots \mathbb{b}_n$ be the control polygons of $\mathbb{a}(u)$ over $[a,b]$ and $\mathbb{b}(u)$ over $[b,c]$. $\mathbb{a}(u)$ and $\mathbb{b}(u)$ have C^r contact at $u=b$ iff $\mathbb{a}_{n-r} \ldots \mathbb{a}_n$ and $\mathbb{b}_0 \ldots \mathbb{b}_r$ are Bezier polygons of some polynomial $\mathbb{p}(u)$ over $[a,b]$ and $[b,c]$