Curves and Surfaces in CAD

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Part I.

Curves

1. Geometric fundamentals

2. Bézier Representation

Recall that the set of polynomials of degree n forms a vector space. A basis is

$$\{1, x, \ldots, x^n\}$$

.

Definition 2.1. *A representation ofa polyomial with respect to*

$$\{1, x, \ldots, x^n\}$$

is colled the monomial representation.

2.1. Bernstein Polynomial

$$1 = 1^n = (u + (1-u))^n = \sum_{i=0}^{n} \binom{n}{i} u^i (1-u)^{n-i}$$

Definition 2.2.

$$B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i}$$

is called a Bernstein Polynomial.

$B_i^n(u) = 0$ if $(i < 0)$ or $(i > n)$.

Lemma 2.1. $B_0^n \ldots B_n^n$ *are linear independent.*

Proof.

$$\begin{aligned} \sum b_i B_i^n &= \sum b_i \binom{n}{i} u^i (1-u)^{n-1} \\ &= \frac{0}{\frac{1}{(1-u)^n}} \\ &\Rightarrow \sum b_i (\frac{u}{1-u})^i = 0 \\ &\Rightarrow \sum b_i s^i = 0 \\ &\Rightarrow b_i = 0 \end{aligned}$$

$\square$

Theorem 2.1. *$B_0^n \dots B_n^n$ form a basis for polynomials of degree n. Proof follows from Lemma 2.1 and the fact that the dimension of the space of polynomials of degree n is $n+1$*

Lemma 2.2. *Symmetry*
$$B_i^n(0) = B_{n-1}^n(u)$$

Lemma 2.3.
$$B_i^n(0) = B_{n-1}^n(1) = \begin{cases} 1 & \textit{if } i = 0 \\ 0 & \textit{otherwise} \end{cases}$$

Lemma 2.4. *$B_0^n \dots B_0^n$ form a partition of unity ($\sum B_i^n = 1$)*

Lemma 2.5. *$B_i^n(u) > 0$, $u \in (0,1)$*

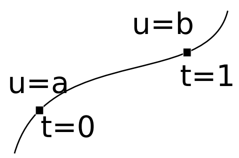
Lemma 2.6. *$B_i^{n+1}(u) = uB_{i-1}^n(u) + (1-u)B_i^n(u)$*

2.2. Bézier Representation

Definition 2.3. *A representation of a polynomial with respect to $B_0^n \dots B_n^n$ is calles the Bézier representation. Let $c(u) = \sum_{i=0}^n c_i B_i^n(u)$. c_i can be in $\mathbb{R}^n$. For practical reasons is $u \in [a,b]$.*
Note: $u(t) = a(1-t) + bT$.

$$b(t) := c(a(t))$$

, b has same the degree as c and represents the same polinomial, but with a different parametrisation.



$b(t)$ has a Bézierrepresentation of degree n.

$$b(t) = \sum_{i=0}^{n} b_i B_i^n(t)$$

Definition 2.4. *b_i is called a control point.*

Definition 2.5. *u is a global parameter, t is local.*

Definition 2.6. *The piecewise linear interpolant of the b_i is called the control polygone.*

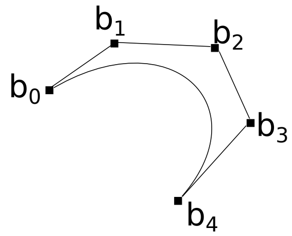


Figure 2.1.: control polygone

Lemma 2.7.

$$
\begin{aligned}
b(u) &= \sum_{i=0}^{n} b_i B_i^n(t) \\
&= \sum_{i=0}^{n} b_{n-i} B_{n-i}^n(1-t)
\end{aligned}
$$

Lemma 2.8 (end point interpolation)**.**

$$(a) = {}_0$$

3. Bézier techniques

Part II.

Surfaces