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Interpretable Gaussian Processes for Stellar Light Curves

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1. THE SPOT EXPANSION

1.1. *Preliminaries*

We adopt the following expression for the spherical harmonic coefficient of degree l and order m in the expansion of a spot at $\theta = \varphi = 0$:

$$
y_{lm}(r, \delta) = \begin{cases} 1 - \dfrac{\delta r'}{2(1 + r')} & l = m = 0 \\[2ex] -\dfrac{\delta r' \, (2 + r')}{2\sqrt{2l+1}(1 + r')^{l+1}} & l > 0, m = 0 \\[2ex] 0 & m \neq 0 \end{cases} \tag{1}
$$

where

$$
r' \equiv c_0 + c_1 r \tag{2}
$$

and $c_0, c_1 > 0$ are scaling constants (defined in more detail below). In the expressions above, $\delta \in (-\infty, 1]$ is the spot contrast (the fractional decrease in the brightness at the center of the spot) and $r \in [0, 1]$ is the normalized spot radius. The expression in Equation (1) is convenient because it satisfies five important properties:

1 The surface intensity is azimuthally symmetric about the spot center

2 The surface intensity monotonically increases away from the spot center

3 The surface intensity at the spot center is $1 - \delta$

4 The surface intensity at the antipode of the spot center is unity

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5 The size of the spot increases monotonically with r

These properties may be demonstrated by considering the expression for the surface intensity at a given point (θ, φ):

$$\begin{aligned} I(\theta, \varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} y_{lm} Y_{lm}(\theta, \varphi) \\ &= \sum_{l=0}^{\infty} y_{l0} \sqrt{2l+1} P_l(\cos\theta) \end{aligned} \tag{3}$$

where P_l is the Legendre polynomial of degree l and we have implicitly assumed a normalization such that the integral of our expansion over the unit sphere is 4π. Note that the expression above is independent of the azimuth φ, a consequence of the fact that all harmonics with $m \neq 0$ are zero; this expansion is therefore azimuthally symmetric about the spot center, as stated in **1**. Combining the above expression with Equation (1) and rearranging, we may write

$$I(\theta, \varphi) = I(\theta) = 1 + \frac{\delta r'}{2} - \frac{\delta r'\,(2 + r')}{2(1 + r')} \sum_{l=0}^{\infty} \left(\frac{1}{1 + r'} \right)^l P_l(\cos\theta)\,. \tag{4}$$

The summation in Equation (4) has a closed-form expression in terms of the generating function of the Legendre polynomials:

$$\sum_{l=0}^{\infty} t^l P_l(\cos\theta) = \frac{1}{\sqrt{1 + t^2 - 2t\cos\theta}}\,, \tag{5}$$

so we may express the intensity in the fairly simple form

$$I(\theta) = A - \frac{B}{\sqrt{C - \cos\theta}}\,, \tag{6}$$

where

$$\begin{aligned} A &= 1 + \frac{\delta r'}{2} \\ B &= \delta r'(2 + r') \sqrt{\frac{1}{8 + 8r'}} \\ C &= \frac{1 + (1 + r')^2}{2 + 2r'} \end{aligned} \tag{7}$$

are positive constants.

Differentiating Equation (6) with respect to θ, we have

$$\frac{\mathrm{d}I(\theta)}{\mathrm{d}\theta} = -\frac{B\sin\theta}{2(C - \cos\theta)^{\frac{3}{2}}}\,, \tag{8}$$

which is zero only for $\theta = 0$ (for which $I(\theta)$ is minimized) and $\theta = \pi$ (for which it is maximized). The intensity therefore increases monotonically from the spot center to the antipode, as stated in **2**. The value at the minimum is

$$\begin{aligned} I_{\mathrm{min}} &= A - \frac{B}{\sqrt{C-1}} \\ &= 1 - \delta\,, \end{aligned} \tag{9}$$

as stated in **3**, and the value at the maximum is

$$\begin{aligned} I_{\mathrm{max}} &= A - \frac{B}{\sqrt{C+1}} \\ &= 1\,, \end{aligned} \tag{10}$$

as stated in **4**. Finally, to show **5**, let us compute the half width at half minimum $\Delta\theta$ of the intensity profile:

$$1 - \left(A - \frac{B}{\sqrt{C - \cos\Delta\theta}} \right) = \frac{1}{2}\delta \tag{11}$$

Solving for $\Delta\theta$ yields

$$\Delta\theta = \cos^{-1}\left[\frac{2 + 3r'(2+r')}{2(1+r')^3} \right] . \tag{12}$$

Differentiation with respect to r' yields

$$\frac{\mathrm{d}\Delta\theta}{\mathrm{d}r'} = \frac{3r'\,(2+r')}{(1+r')^4\sqrt{\frac{r'^2(2+r')^2(1+2r')(3+2r')}{(1+r')^6}}}\,, \tag{13}$$

which is positive definite for all $r' > 0$.

1.2. *Distributions over r and δ*

We model the normalized spot radius r as a random variable drawn from a Beta distribution, whose PDF is given by

$$p(r \,|\, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}\,, \tag{14}$$

where Γ is the Gamma function and α and β are hyperparameters characterizing the shape of the distribution. Since we require the brightness

$$b \equiv 1 - \delta \tag{15}$$

at the center of the spot to be non-negative, it is convenient to treat $b \in [0, \infty)$ as our random variable instead of δ. We model b as a random variable drawn from a log-normal distribution:

$$p(b \,|\, \mu, \nu) = \frac{1}{b\sqrt{2\pi\nu}} \exp\left[-\frac{(\ln b - \mu)^2}{2\nu} \right]\,, \tag{16}$$

where μ and ν are the mean and variance of the log-normal, respectively. The joint distribution over r and b is therefore separable, by construction:

$$p(r, b \,|\, \alpha, \beta, \mu, \nu) = p(r \,|\, \alpha, \beta)\, p(b \,|\, \mu, \nu)\,. \tag{17}$$

1.3. *First moment*

The first moment of the distribution of y_{lm} over r and b is

$$\begin{aligned}
\mathrm{E}[y_{lm}] &= \int_0^\infty \int_0^1 y_{lm}(r, 1-b) p(r, 1-b \,|\, \alpha, \beta, \mu, \nu) \mathrm{d}r\, \mathrm{d}b \\
&= \int_0^\infty \mathrm{E}_r[y_{lm}] p(b \,|\, \mu, \nu) \mathrm{d}b\,,
\end{aligned} \tag{18}$$

where

$$\mathrm{E}_r[y_{lm}] = \int_0^1 y_{lm}(r, 1-b) p(r \,|\, \alpha, \beta) \mathrm{d}r\,. \tag{19}$$

Expanding Equation (19), we obtain

$$\mathrm{E}_r[y_{lm}] = \begin{cases}
\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)(c_0 + c_1 r)}{2(1 + c_0 + c_1 r)}\right) r^{\alpha-1}(1-r)^{\beta-1} \mathrm{d}r & l = m = 0 \\[2em]
-\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \dfrac{(1-b)(c_0 + c_1 r)\,(2 + c_0 + c_1 r)\, r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l+1}(1 + c_0 + c_1 r)^{l+1}} \mathrm{d}r & l > 0, m = 0 \\[2em]
0 & m \neq 0\,.
\end{cases} \tag{20}$$

Thanks to the dependence of the spherical harmonic coefficients on only powers of r and $(1 + c_0 + c_1 r)$, the integrals above may be expressed in closed form in terms of the hypergeometric function ${}_2F_1$:

$$\mathrm{E}_r[y_{lm}] = \begin{cases}
1 - \dfrac{(1-b)}{2(1+c_0)} \left[c_0 H_0^0 + c_1 H_0^1\right] & l = m = 0 \\[2em]
-\dfrac{(1-b)}{2\sqrt{2l+1}(1+c_0)^{l+1}} \left[c_0(2+c_0) H_l^0 + 2c_1(1+c_0) H_l^1 + c_1^2 H_l^2\right] & l > 0, m = 0 \\[2em]
0 & m \neq 0
\end{cases} \tag{21}$$

where we define

$$H_j^k \equiv \left(\prod_{n=0}^{k-1} \lambda_n\right) G_j^k\,. \tag{22}$$

with

$$\lambda_n \equiv \frac{\alpha + n}{\alpha + \beta + n} \tag{23}$$

and

$$G_j^k \equiv {}_2F_1\left(j+1, \alpha + k; \alpha + \beta + k; -\frac{c_1}{1+c_0}\right) . \tag{24}$$

Note that for a given value of α, β, and c, we need only compute G_0^0, G_1^0, G_0^1, and G_1^1 directly, since the remaining terms may be obtained recursively:

$$\begin{aligned} G_j^k &= \left[\frac{(\alpha+\beta+k-j)(1+c_0)}{j(1+c_0+c_1)}\right] G_{j-2}^k \\ &+ \left[-\frac{(\alpha+\beta-2j+k)(1+c_0)+(\alpha-j+k)c_1}{j(1+c_0+c_1)}\right] G_{j-1}^k \end{aligned} \tag{25}$$

$$\begin{aligned} G_j^k &= \left[\frac{(\alpha+\beta+k-2)(1+c_0)}{\lambda_{k-1}(\alpha+\beta-j+k-2)c_1}\right] G_j^{k-2} \\ &+ \left[-\frac{\alpha-(\alpha+k)c_1+\beta+k+(\alpha+\beta+k-2)c_0+(j+2)c_1-2}{\lambda_{k-1}(\alpha+\beta-j+k-2)c_1}\right] G_j^{k-1} . \end{aligned} \tag{26}$$

We may now evaluate the integral in Equation (18), with PDF given by Equation (16). Most of the terms in the integrands are constants, so the integrals are straightforward to evaluate and may also be expressed in closed form:

$$\mathrm{E}[y_{lm}] = \begin{cases} 1 - \frac{\gamma_1}{2(1+c_0)}\left[c_0 H_0^0 + c_1 H_0^1\right] & l = m = 0 \\ -\frac{\gamma_1}{2\sqrt{2l+1}(1+c_0)^{l+1}}\left[c_0(2+c_0)H_l^0 + 2c_1(1+c_0)H_l^1 + c_1^2 H_l^2\right] & l > 0, m = 0 \\ 0 & m \neq 0 \end{cases} \tag{27}$$

where we define

$$\gamma_1 \equiv 1 - \exp\left[\mu + \frac{1}{2}\nu\right] . \tag{28}$$

1.4. *Second moment*

The second moment of the distribution of y_{lm} over r and b is

$$\begin{aligned} \mathrm{E}[y_{lm} y_{l'm'}] &= \int_0^\infty \int_0^1 y_{lm}(r, 1-b) y_{l'm'}(r, 1-b) p(r, 1-b \,|\, \alpha, \beta, \mu, \nu) \mathrm{d}r\, \mathrm{d}b \\ &= \int_0^\infty \mathrm{E}_r[y_{lm} y_{l'm'}] p(b \,|\, \mu, \nu) \mathrm{d}b , \end{aligned} \tag{29}$$

where

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \int_0^1 y_{lm}(r, 1-b)y_{l'm'}(r, 1-b)p(r \,|\, \alpha, \beta)\mathrm{d}r\,. \tag{30}$$

Expanding Equation (30), we obtain

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)(c_0+c_1 r)}{2(1+c_0+c_1 r)}\right)^2 \\ \qquad \times\, r^{\alpha-1}(1-r)^{\beta-1}\mathrm{d}r & \begin{array}{l} l = l' = 0, \\ m = m' = 0 \end{array} \\[2em] -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)(c_0+c_1 r)}{2(1+c_0+c_1 r)}\right) \\ \qquad \times \left(\dfrac{(1-b)(c_0+c_1 r)\,(2+c_0+c_1 r)}{2\sqrt{2l+1}(1+c_0+c_1 r)^{l+1}}\right) & \begin{array}{l} l > 0, \\ l' = 0, \\ m = m' = 0 \end{array} \\ \qquad \times\, r^{\alpha-1}(1-r)^{\beta-1}\mathrm{d}r \\[2em] -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(\dfrac{(1-b)(c_0+c_1 r)\,(2+c_0+c_1 r)}{2\sqrt{2l+1}(1+c_0+c_1 r)^{l+1}}\right) \\ \qquad \times \left(\dfrac{(1-b)(c_0+c_1 r)\,(2+c_0+c_1 r)}{2\sqrt{2l'+1}(1+c_0+c_1 r)^{l'+1}}\right) & \begin{array}{l} l > 0, \\ l' > 0, \\ m = m' = 0 \end{array} \\ \qquad \times\, r^{\alpha-1}(1-r)^{\beta-1}\mathrm{d}r \\[2em] 0 & m, m' \neq 0\,, \end{cases} \tag{31}$$

where, by symmetry, the expression for $l' > 0$, $l = 0$, $m = m' = 0$ is the same as in the second case, provided we make the substitution $l \to l'$. These integrals again reduce to closed form expressions:

$$
\mathrm{E}_r[y_{lm}y_{l'm'}] = \begin{cases}
1+\dfrac{1-b}{1+c_0}\Bigg[-c_0H_0^0 - c_1H_0^1 \\ \qquad\qquad +\dfrac{1-b}{2(1+c_0)}\left(\dfrac{c_0^2}{2}H_1^0 + c_0c_1H_1^1 + \dfrac{c_1^2}{2}H_1^2\right)\Bigg] & \begin{array}{l} l=l'=0, \\ m=m'=0 \end{array} \\[2ex]
\dfrac{1-b}{2\sqrt{2l+1}(1+c_0)^{l+1}} \\ \quad\times\Bigg[c_0(2+c_0)H_l^0 + 2c_1(1+c_0)H_l^1 + c_1^2H_l^2 \\ \qquad -\dfrac{1-b}{2(1+c_0)}\Big(c_0^2(2+c_0)H_{l+1}^0 + c_0c_1(4+3c_0)H_{l+1}^1 \\ \qquad +(2+3c_0)c_1^2H_{l+1}^2 + c_1^3H_{l+1}^3\Big)\Bigg] & \begin{array}{l} l>0, \\ l'=0, \\ m=m'=0 \end{array} \\[2ex]
\dfrac{(1-b)^2}{4\sqrt{(2l+1)(2l'+1)}(1+c_0)^{l+l'+2}} \\ \quad\times\Bigg[c_0^2(2+c_0)^2H_{l+l'+1}^0 + 4c_0(1+c_0)(2+c_0)c_1H_{l+l'+1}^1 \\ \qquad +2(2+3c_0(2+c_0))c_1^2H_{l+l'+1}^2 + 4(1+c_0)c_1^3H_{l+l'+1}^3 \\ \qquad +c_1^4H_{l+l'+1}^4\Bigg] & \begin{array}{l} l>0, \\ l'>0, \\ m=m'=0 \end{array} \\[2ex]
0 & m,m'\neq 0\,.
\end{cases} \tag{32}
$$

The final step is to evaluate the integral in Equation (30), given the PDF in Equation (16):

$$
\mathrm{E}[y_{lm} y_{l'm'}] = \begin{cases}
1 + \dfrac{\gamma_1}{1+c_0}\Bigg[- c_0 H_0^0 - c_1 H_0^1 \\
\qquad\qquad + \dfrac{\gamma_2}{2\gamma_1(1+c_0)} \left(\dfrac{c_0^2}{2} H_1^0 + c_0 c_1 H_1^1 + \dfrac{c_1^2}{2} H_1^2 \right) \Bigg] & \begin{array}{l} l = l' = 0, \\ m = m' = 0 \end{array} \\
\dfrac{\gamma_1}{2\sqrt{2l+1}(1+c_0)^{l+1}} \\
\quad \times \Bigg[c_0(2+c_0) H_l^0 + 2c_1(1+c_0) H_l^1 + c_1^2 H_l^2 \\
\qquad - \dfrac{\gamma_2}{2\gamma_1(1+c_0)} \bigg(c_0^2(2+c_0) H_{l+1}^0 + c_0 c_1 (4+3c_0) H_{l+1}^1 & \begin{array}{l} l > 0, \\ l' = 0, \\ m = m' = 0 \end{array} \\
\qquad + (2+3c_0) c_1^2 H_{l+1}^2 + c_1^3 H_{l+1}^3 \bigg) \Bigg] \\
\dfrac{\gamma_2}{4\sqrt{(2l+1)(2l'+1)}(1+c_0)^{l+l'+2}} \\
\quad \times \Bigg[c_0^2 (2+c_0)^2 H_{l+l'+1}^0 + 4c_0(1+c_0)(2+c_0) c_1 H_{l+l'+1}^1 \\
\qquad + 2(2+3c_0(2+c_0)) c_1^2 H_{l+l'+1}^2 + 4(1+c_0) c_1^3 H_{l+l'+1}^3 & \begin{array}{l} l > 0, \\ l' > 0, \\ m = m' = 0 \end{array} \\
\qquad + c_1^4 H_{l+l'+1}^4 \Bigg] \\
0 & m, m' \neq 0 \,.
\end{cases} \tag{33}
$$

where we define

$$
\gamma_2 \equiv 1 - 2\exp\left[\mu + \frac{1}{2}\nu\right] + \exp\left[2\mu + 2\nu\right], \tag{34}
$$

and the expression for $l' > 0$, $l = 0$, $m = m' = 0$ is again the same as in the second case, provided we make the substitution $l \to l'$.