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Interpretable Gaussian Processes for Stellar Light Curves

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1. THE SPOT EXPANSION

1.1. *Preliminaries*

We adopt the following expression for the spherical harmonic coefficient of degree l and order m in the expansion of a spot at $\theta = \varphi = 0$:

$$y_{lm}(r, \delta) = \begin{cases} 1 - \dfrac{\delta c r}{2(1 + cr)} & l = m = 0 \\ \\ -\dfrac{\delta c r \, (2 + cr)}{2\sqrt{2l+1}(1 + cr)^{l+1}} & l > 0, m = 0 \\ \\ 0 & m \neq 0 \end{cases} \tag{1}$$

where $\delta \in [-\infty, 1]$ is the spot contrast (the fractional decrease in the brightness at the center of the spot) and $r \in (0, 1]$ is the normalized spot radius. The quantity $c \in (0, \infty)$ is a normalization constant for the radius (more details below). The expression in Equation (1) is convenient because it satisfies four important properties:

1 The surface intensity monotonically increases away from the spot center

2 The surface intensity at the spot center is $1 - \delta$

3 The surface intensity at the antipode of the spot center is unity

4 The size of the spot increases monotonically with r

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These properties may be demonstrated by considering the expression for the surface intensity at a given point (θ, φ):

$$
\begin{aligned}
I(\theta,\varphi) &= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} y_{lm} Y_{lm}(\theta,\varphi) \\
&= \sum_{l=0}^{\infty} y_{l0}\sqrt{2l+1}P_l(\cos\theta)
\end{aligned}
\tag{2}
$$

where P_l is the Legendre polynomial of degree l and we have implicitly assumed a normalization such that the integral of our expansion over the unit sphere is 4π. Combining this with Equation (1) and rearranging, we may write

$$
I(\theta,\varphi) = 1 + \frac{\delta cr}{2} - \frac{\delta cr\,(2+cr)}{2(1+cr)} \sum_{l=0}^{\infty} \left(\frac{1}{1+cr}\right)^{l} P_l(\cos\theta)\,. \tag{3}
$$

The summation in Equation (3) has a closed-form expression in terms of the generating function of the Legendre polynomials:

$$
\sum_{l=0}^{\infty} t^l P_l(\cos\theta) = \frac{1}{\sqrt{1+t^2-2t\cos\theta}}\,, \tag{4}
$$

so we may express the intensity in the fairly simple form

$$
I(\theta,\varphi) = A - \frac{B}{\sqrt{C-\cos\theta}}\,, \tag{5}
$$

where

$$
\begin{aligned}
A &= 1 + \frac{\delta cr}{2} \\
B &= \delta cr(2+cr)\sqrt{\frac{1}{8+8cr}} \\
C &= \frac{1+(1+cr)^2}{2+2cr}
\end{aligned}
\tag{6}
$$

are positive constants.

Differentiating Equation (5) with respect to θ, we have

$$
\frac{\mathrm{d}I(\theta,\varphi)}{\mathrm{d}\theta} = -\frac{B\sin\theta}{2(C-\cos\theta)^{\frac{3}{2}}}\,, \tag{7}
$$

which is zero only for $\theta = 0$ (for which $I(\theta,\varphi)$ is minimized) and $\theta = \pi$ (for which it is maximized). The intensity therefore increases monotonically from the spot center to the antipode, as stated in **1**. The value at the minimum is

$$
\begin{aligned}
I_{\mathrm{min}} &= A - \frac{B}{\sqrt{C-1}} \\
&= 1 - \delta\,,
\end{aligned}
\tag{8}
$$

as stated in **2**, and the value at the maximum is

$$
\begin{aligned}
I_{\max} &= A - \frac{B}{\sqrt{C+1}} \\
&= 1\,,
\end{aligned}
\tag{9}
$$

as stated in **3**. Finally, to show **4**, let us compute the half width at half minimum $\Delta\theta$ of the intensity profile:

$$
1 - \left(A - \frac{B}{\sqrt{C - \cos\Delta\theta}}\right) = \frac{1}{2}\delta
\tag{10}
$$

Solving for $\Delta\theta$ yields

$$
\Delta\theta = \cos^{-1}\left[\frac{2 + 3cr(2+cr)}{2(1+cr)^3}\right]\,.
\tag{11}
$$

Differentiation with respect to r yields

$$
\frac{\mathrm{d}\Delta\theta}{\mathrm{d}r} = \frac{3c^2 r\,(2+cr)}{\left(1+cr\right)^4 \sqrt{\frac{c^2 r^2 (2+cr)^2 (1+2cr)(3+2cr)}{(1+cr)^6}}}\,,
\tag{12}
$$

which is positive definite for all $c, r > 0$.

1.2. *Distributions over r and δ*

We model the normalized spot radius r as a random variable drawn from a Beta distribution, whose PDF is given by

$$
p(r \,|\, \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}\,,
\tag{13}
$$

where Γ is the Gamma function and α and β are hyperparameters characterizing the shape of the distribution. Since we require the brghtness $b = 1 - \delta$ at the center of the spot to be non-negative, it is convenient to parametrize the distribution of b instead of δ. We model b as a random variable drawn from a log-normal distribution:

$$
p(b \,|\, \mu, \nu) = \frac{1}{b\sqrt{2\pi\nu}} \exp\left[-\frac{\left(\ln b - \mu\right)^2}{2\nu}\right]\,,
\tag{14}
$$

where μ and ν are the mean and variance of the log-normal, respectively. The joint distribution over r and b is therefore separable, by construction:

$$
p(r, b \,|\, \alpha, \beta, \mu, \nu) = p(r \,|\, \alpha, \beta)\, p(b \,|\, \mu, \nu)\,.
\tag{15}
$$

1.3. *First moment*

The first moment of the distribution of y_{lm} over r and b is

$$
\begin{aligned}
\mathrm{E}[y_{lm}] &= \int_0^\infty \int_0^1 y_{lm}(r, 1-b) p(r, 1-b \,|\, \alpha, \beta, \mu, \nu) \mathrm{d}r \, \mathrm{d}b \\
&= \int_0^\infty \mathrm{E}_r[y_{lm}] p(b \,|\, \mu, \nu) \mathrm{d}b \,,
\end{aligned}
\tag{16}
$$

where

$$
\mathrm{E}_r[y_{lm}] = \int_0^1 y_{lm}(r, 1-b) p(r \,|\, \alpha, \beta) \mathrm{d}r \,.
\tag{17}
$$

Expanding Equation (17), we obtain

$$
\mathrm{E}_r[y_{lm}] = \begin{cases}
\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) r^{\alpha-1}(1-r)^{\beta-1} \mathrm{d}r & l = m = 0 \\[2em]
-\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \dfrac{(1-b)cr\,(2+cr)\, r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l+1}(1+cr)^{l+1}} \mathrm{d}r & l > 0, m = 0 \\[2em]
0 & m \neq 0 \,.
\end{cases}
\tag{18}
$$

Thanks to the dependence of the spherical harmonic coefficients on only powers of r and $(1+cr)$, the integrals above may be expressed in closed form in terms of the hypergeometric function ${}_2F_1$:

$$
\mathrm{E}_r[y_{lm}] = \begin{cases}
1 - \dfrac{(1-b)\alpha c}{2(\alpha+\beta)} G_0^1 & l = m = 0 \\[2em]
-\dfrac{(1-b)\alpha c}{\sqrt{2l+1}(\alpha+\beta)} \left[G_l^1 + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l^2 \right] & l > 0, m = 0 \\[2em]
0 & m \neq 0
\end{cases}
\tag{19}
$$

where we define

$$
G_l^k \equiv {}_2F_1(l+1, \alpha+k; \alpha+\beta+k; -c) \,.
\tag{20}
$$

Note that for a given value of α, β, and c, we need only compute G_0^1, G_1^1, G_0^2, and G_1^2 directly, since the remaining terms may be obtained recursively:

$$G_l^k = \frac{\alpha+\beta+k-l}{2l} G_{l-2}^k - \frac{2\alpha+\beta+2k-3l}{2l} G_{l-1}^k \,. \tag{21}$$

We may now evaluate the integral in Equation (16):

$$\mathrm{E}[y_{lm}] = \begin{cases} \dfrac{1}{b\sqrt{2\pi\nu}} \displaystyle\int_0^\infty \left(1 - \frac{(1-b)\alpha c}{2(\alpha+\beta)} G_0^1\right) \\ \qquad \times \exp\left[-\dfrac{(\ln b-\mu)^2}{2\nu}\right] \mathrm{d}b & l = m = 0 \\ -\dfrac{1}{b\sqrt{2\pi\nu}} \displaystyle\int_0^\infty \frac{(1-b)\alpha c}{\sqrt{2l+1}(\alpha+\beta)} \\ \qquad \times \left[G_l^1 + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l^2\right] \\ \qquad \times \exp\left[-\dfrac{(\ln b-\mu)^2}{2\nu}\right] \mathrm{d}b & l > 0, m = 0 \\ 0 & m \neq 0 \end{cases} \tag{22}$$

Most of the terms in the integrands above are constants, so these integrals may also be expressed in closed form:

$$\mathrm{E}[y_{lm}] = \begin{cases} 1 - \dfrac{\gamma\alpha c}{2(\alpha+\beta)} G_0^1 & l = m = 0 \\ -\dfrac{\gamma\alpha c}{\sqrt{2l+1}(\alpha+\beta)} \left[G_l^1 + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l^2\right] & l > 0, m = 0 \\ 0 & m \neq 0 \end{cases} \tag{23}$$

where we define

$$\gamma \equiv 1 - \exp\left[\mu + \frac{1}{2}\nu\right] \,. \tag{24}$$

1.4. *Second moment*

The second moment of the distribution of y_{lm} over r and b is

$$\begin{aligned} \mathrm{E}[y_{lm}y_{l'm'}] &= \int_0^\infty \int_0^1 y_{lm}(r,1-b)y_{l'm'}(r,1-b)p(r,1-b \,|\, \alpha,\beta,\mu,\nu)\mathrm{d}r\,\mathrm{d}b \\ &= \int_0^\infty \mathrm{E}_r[y_{lm}y_{l'm'}]p(b \,|\, \mu,\nu)\mathrm{d}b\,, \end{aligned} \tag{25}$$

where

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \int_0^1 y_{lm}(r,1-b)y_{l'm'}(r,1-b)p(r \,|\, \alpha,\beta)\mathrm{d}r\,. \tag{26}$$

Expanding Equation (26), we obtain

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right)^2 r^{\alpha-1}(1-r)^{\beta-1}\mathrm{d}r & \begin{array}{l} l = l', \\ m = m' = 0 \end{array} \\[2ex] -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) \\ \qquad \times \dfrac{(1-b)cr\,(2+cr)\,r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l+1}(1+cr)^{l+1}}\mathrm{d}r & \begin{array}{l} l > 0, \\ l' = 0, \\ m = m' = 0 \end{array} \\[2ex] -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) \\ \qquad \times \dfrac{(1-b)cr\,(2+cr)\,r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l'+1}(1+cr)^{l'+1}}\mathrm{d}r & \begin{array}{l} l = 0, \\ l' > 0, \\ m = m' = 0 \end{array} \\[2ex] \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \dfrac{(1-b)^2c^2r^2\,(2+cr)^2\,r^{\alpha-1}(1-r)^{\beta-1}}{4\sqrt{(2l+1)(2l'+1)}(1+cr)^{l+l'+2}}\mathrm{d}r & \begin{array}{l} l > 0, \\ l' > 0, \\ m = m' = 0 \end{array} \\[2ex] 0 & m, m' \neq 0\,. \end{cases} \tag{27}$$

These integrals again reduce to closed form expressions:

$$
\mathrm{E}_r[y_{lm}y_{l'm'}] = \begin{cases}
1 + \dfrac{(\alpha+1)\,\alpha\,(1-b)^2\,c^2}{4\,(c+1)\,(\alpha+\beta)} & \\
\quad -\dfrac{\alpha\,(1-b)\,c}{\alpha+\beta} & l = l', \\
 & m = m' = 0 \\
\quad \times\left[G_0^1 + \dfrac{(\alpha+1)\,(1-b)\,c\,(\alpha+\beta+\alpha c+c)}{4\,(c+1)\,(\alpha+\beta+1)}G_0^2\right] & \\
 & \\
-\dfrac{\alpha(1-b)c}{2\sqrt{2l+1}(\alpha+\beta)} & \\
\quad \times\left[2G_l^1 + \dfrac{c(\alpha+1)}{\alpha+\beta+1}G_l^2\right. & l > 0, \\
 & l' = 0, \\
\qquad -\dfrac{c(1-b)(\alpha+1)}{\alpha+\beta+1}G_{l+1}^2 & m = m' = 0 \\
\qquad \left.-\dfrac{c^2(1-b)(\alpha+1)(\alpha+2)}{2(\alpha+\beta+1)(\alpha+\beta+2)}G_{l+1}^3\right] & \\
 & l = 0, \\
\text{same as previous case, but with } l \to l' & l' > 0, \\
 & m = m' = 0 \\
 & \\
\dfrac{\alpha(\alpha+1)(1-b)^2c^2}{(\alpha+\beta)(\alpha+\beta+1)\sqrt{(2l+1)(2p+1)}} & \\
\quad \times\left[G_{l+l'+1}^2 + \dfrac{c(\alpha+2)}{\alpha+\beta+2}G_{l+l'+1}^3\right. & l > 0, \\
 & l' > 0, \\
 & m = m' = 0 \\
\qquad \left.+\dfrac{c^2(\alpha+2)(\alpha+3)}{4(\alpha+\beta+2)(\alpha+\beta+3)}G_{l+l'+1}^4\right] & \\
 & \\
0 & m, m' \neq 0\,.
\end{cases} \tag{28}
$$