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Interpretable Gaussian Processes for Stellar Light Curves

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1. THE SPOT EXPANSION

1.1. *Preliminaries*

We adopt the following expression for the spherical harmonic coefficient of degree l and order m in the expansion of a spot at $\theta = \varphi = 0$:

$$
y_{lm}(r, \delta) = \begin{cases} 1 - \dfrac{\delta c r}{2(1 + cr)} & l = m = 0 \\ \\ -\dfrac{\delta c r\,(2 + cr)}{2\sqrt{2l+1}(1 + cr)^{l+1}} & l > 0, m = 0 \\ \\ 0 & m \neq 0 \end{cases} \tag{1}
$$

where $\delta \in [-\infty, 1]$ is the spot contrast (the fractional decrease in the brightness at the center of the spot) and $r \in (0, 1]$ is the normalized spot radius. The quantity $c \in (0, \infty)$ is a normalization constant for the radius (more details below). The expression in Equation (1) is convenient because it satisfies four important properties:

1 The surface intensity monotonically increases away from the spot center

2 The surface intensity at the spot center is $1 - \delta$

3 The surface intensity at the antipode of the spot center is unity

4 The size of the spot increases monotonically with r

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These properties may be demonstrated by considering the expression for the surface intensity at a given point (θ, φ):

$$
\begin{aligned}
I(\theta,\varphi) &= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} y_{lm} Y_{lm}(\theta,\varphi) \\
&= \sum_{l=0}^{\infty} y_{l0} \sqrt{2l+1} P_l(\cos\theta)
\end{aligned}
\tag{2}
$$

where P_l is the Legendre polynomial of degree l and we have implicitly assumed a normalization such that the integral of our expansion over the unit sphere is 4π. Combining this with Equation (1) and rearranging, we may write

$$
I(\theta,\varphi) = 1 + \frac{\delta cr}{2} - \frac{\delta cr\,(2+cr)}{2(1+cr)} \sum_{l=0}^{\infty} \left(\frac{1}{1+cr}\right)^l P_l(\cos\theta)\,. \tag{3}
$$

The summation in Equation (3) has a closed-form expression in terms of the generating function of the Legendre polynomials:

$$
\sum_{l=0}^{\infty} t^l P_l(\cos\theta) = \frac{1}{\sqrt{1+t^2-2t\cos\theta}}\,, \tag{4}
$$

so we may express the intensity in the fairly simple form

$$
I(\theta,\varphi) = A - \frac{B}{\sqrt{C-\cos\theta}}\,, \tag{5}
$$

where

$$
\begin{aligned}
A &= 1 + \frac{\delta cr}{2} \\
B &= \delta cr(2+cr)\sqrt{\frac{1}{8+8cr}} \\
C &= \frac{1+(1+cr)^2}{2+2cr}
\end{aligned}
\tag{6}
$$

are positive constants.

Differentiating Equation (5) with respect to θ, we have

$$
\frac{\mathrm{d}I(\theta,\varphi)}{\mathrm{d}\theta} = -\frac{B\sin\theta}{2(C-\cos\theta)^{\frac{3}{2}}}\,, \tag{7}
$$

which is zero only for $\theta = 0$ (for which $I(\theta,\varphi)$ is minimized) and $\theta = \pi$ (for which it is maximized). The intensity therefore increases monotonically from the spot center to the antipode, as stated in **1**. The value at the minimum is

$$
\begin{aligned}
I_{\text{min}} &= A - \frac{B}{\sqrt{C-1}} \\
&= 1 - \delta\,,
\end{aligned}
\tag{8}
$$

as stated in **2**, and the value at the maximum is

$$\begin{aligned} I_{\mathrm{max}} &= A - \frac{B}{\sqrt{C+1}} \\ &= 1\,, \end{aligned} \tag{9}$$

as stated in **3**. Finally, to show **4**, let us compute the half width at half minimum $\Delta\theta$ of the intensity profile:

$$1 - \left(A - \frac{B}{\sqrt{C - \cos\Delta\theta}} \right) = \frac{1}{2}\delta \tag{10}$$

Solving for $\Delta\theta$ yields

$$\Delta\theta = \cos^{-1}\left[\frac{2 + 3cr(2+cr)}{2(1+cr)^3}\right]\,. \tag{11}$$

Differentiation with respect to r yields

$$\frac{\mathrm{d}\Delta\theta}{\mathrm{d}r} = \frac{3c^2r\,(2+cr)}{(1+cr)^4\sqrt{\frac{c^2r^2(2+cr)^2(1+2cr)(3+2cr)}{(1+cr)^6}}}\,, \tag{12}$$

which is positive definite for all $c, r > 0$.

1.2. *Distributions over r and δ*

We model the normalized spot radius r as a random variable drawn from a Beta distribution, whose PDF is given by

$$p(r\,|\,\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} r^{\alpha-1}(1-r)^{\beta-1}\,, \tag{13}$$

where Γ is the Gamma function and α and β are hyperparameters characterizing the shape of the distribution. Since we require the brghtness $b = 1-\delta$ at the center of the spot to be non-negative, it is convenient to parametrize the distribution of b instead of δ. We model b as a random variable drawn from a log-normal distribution:

$$p(b\,|\,\mu,\nu) = \frac{1}{b\sqrt{2\pi\nu}}\exp\left[-\frac{(\ln b - \mu)^2}{2\nu}\right]\,, \tag{14}$$

where μ and ν are the mean and variance of the log-normal, respectively. The joint distribution over r and b is therefore separable, by construction:

$$p(r,b\,|\,\alpha,\beta,\mu,\nu) = p(r\,|\,\alpha,\beta)\,p(b\,|\,\mu,\nu)\,. \tag{15}$$

1.3. *First moment*

The first moment of the distribution of y_{lm} over r and b is

$$\begin{aligned} \mathrm{E}[y_{lm}] &= \int_0^\infty \int_0^1 y_{lm}(r, 1-b) p(r, 1-b \,|\, \alpha, \beta, \mu, \nu) \mathrm{d}r \, \mathrm{d}b \\ &= \int_0^\infty \mathrm{E}_r[y_{lm}] p(b \,|\, \mu, \nu) \mathrm{d}b \,, \end{aligned} \tag{16}$$

where

$$\mathrm{E}_r[y_{lm}] = \int_0^1 y_{lm}(r, 1-b) p(r \,|\, \alpha, \beta) \mathrm{d}r \,. \tag{17}$$

Expanding Equation (17), we obtain

$$\mathrm{E}_r[y_{lm}] = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) r^{\alpha-1}(1-r)^{\beta-1} \mathrm{d}r & l = m = 0 \\ -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int_0^1 \dfrac{(1-b)cr\,(2+cr)\, r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l+1}(1+cr)^{l+1}} \mathrm{d}r & l > 0, m = 0 \\ 0 & m \neq 0 \,. \end{cases} \tag{18}$$

Thanks to the dependence of the spherical harmonic coefficients on only powers of r and $(1+cr)$, the integrals above may be expressed in closed form in terms of the hypergeometric function ${}_2F_1$:

$$\mathrm{E}_r[y_{lm}] = \begin{cases} 1 - \dfrac{(1-b)\alpha c}{2(\alpha+\beta)} F_l & l = m = 0 \\ -\dfrac{(1-b)\alpha c}{\sqrt{2l+1}(\alpha+\beta)} \left[F_l + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l\right] & l > 0, m = 0 \\ 0 & m \neq 0 \end{cases} \tag{19}$$

where we define

$$\begin{aligned} F_l &\equiv {}_2F_1(l+1, \alpha+1; \alpha+\beta+1; -c) \\ G_l &\equiv {}_2F_1(l+1, \alpha+2; \alpha+\beta+2; -c) \,. \end{aligned} \tag{20}$$

Note that for a given value of α, β, and c, we need only compute F_0, F_1, G_0, and G_1 directly, since the remaining terms may be obtained recursively:

$$\begin{aligned}
F_l &= \frac{\alpha+\beta+1-l}{2l} F_{l-2} - \frac{2\alpha+\beta+2-3l}{2l} F_{l-1} \\
G_l &= \frac{\alpha+\beta+2-l}{2l} G_{l-2} - \frac{2\alpha+\beta+4-3l}{2l} G_{l-1}\,.
\end{aligned} \tag{21}$$

We may now evaluate the integral in Equation (16):

$$\mathrm{E}[y_{lm}] = \begin{cases}
\dfrac{1}{b\sqrt{2\pi\nu}} \displaystyle\int_0^\infty \left(1 - \frac{(1-b)\alpha c}{2(\alpha+\beta)} F_l\right) & \\
\qquad \times \exp\left[-\dfrac{(\ln b - \mu)^2}{2\nu}\right] \mathrm{d}b & l = m = 0 \\
-\dfrac{1}{b\sqrt{2\pi\nu}} \displaystyle\int_0^\infty \frac{(1-b)\alpha c}{\sqrt{2l+1}(\alpha+\beta)} & \\
\qquad \times \left[F_l + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l\right] & \\
\qquad \times \exp\left[-\dfrac{(\ln b - \mu)^2}{2\nu}\right] \mathrm{d}b & l > 0, m = 0 \\
0 & m \neq 0
\end{cases} \tag{22}$$

Most of the terms in the integrands above are constants, so these integrals may also be expressed in closed form:

$$\mathrm{E}[y_{lm}] = \begin{cases}
1 - \dfrac{\gamma\alpha c}{2(\alpha+\beta)} F_l & l = m = 0 \\
-\dfrac{\gamma\alpha c}{\sqrt{2l+1}(\alpha+\beta)} \left[F_l + \dfrac{(\alpha+1)c}{2(\alpha+\beta+1)} G_l\right] & l > 0, m = 0 \\
0 & m \neq 0
\end{cases} \tag{23}$$

where we define

$$\gamma \equiv 1 - \exp\left[\mu + \frac{1}{2}\nu\right]\,. \tag{24}$$

1.4. *Second moment*

The second moment of the distribution of y_{lm} over r and b is

$$\begin{aligned} \mathrm{E}[y_{lm}y_{l'm'}] &= \int\limits_0^\infty \int\limits_0^1 y_{lm}(r, 1-b) y_{l'm'}(r, 1-b) p(r, 1-b \,|\, \alpha, \beta, \mu, \nu) \mathrm{d}r \, \mathrm{d}b \\ &= \int\limits_0^\infty \mathrm{E}_r[y_{lm}y_{l'm'}] p(b \,|\, \mu, \nu) \mathrm{d}b \,, \end{aligned} \tag{25}$$

where

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \int\limits_0^1 y_{lm}(r, 1-b) y_{l'm'}(r, 1-b) p(r \,|\, \alpha, \beta) \mathrm{d}r \,. \tag{26}$$

Expanding Equation (26), we obtain

$$\mathrm{E}_r[y_{lm}y_{l'm'}] = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int\limits_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right)^2 r^{\alpha-1}(1-r)^{\beta-1} \mathrm{d}r & \begin{array}{l} l = l', \\ m = m' = 0 \end{array} \\ \\ -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int\limits_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) & \begin{array}{l} l > 0, \\ l' = 0, \\ m = m' = 0 \end{array} \\ \qquad \times \dfrac{(1-b)cr\,(2+cr)\, r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l+1}(1+cr)^{l+1}} \mathrm{d}r & \\ \\ -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int\limits_0^1 \left(1 - \dfrac{(1-b)cr}{2(1+cr)}\right) & \begin{array}{l} l = 0, \\ l' > 0, \\ m = m' = 0 \end{array} \\ \qquad \times \dfrac{(1-b)cr\,(2+cr)\, r^{\alpha-1}(1-r)^{\beta-1}}{2\sqrt{2l'+1}(1+cr)^{l'+1}} \mathrm{d}r & \\ \\ -\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \displaystyle\int\limits_0^1 \dfrac{(1-b)^2 c^2 r^2\,(2+cr)^2\, r^{\alpha-1}(1-r)^{\beta-1}}{4\sqrt{(2l+1)(2l'+1)}(1+cr)^{l+l'+2}} \mathrm{d}r & \begin{array}{l} l > 0 \\ l' > 0 \\ m = m' = 0 \end{array} \\ \\ 0 & m, m' \neq 0 \,. \end{cases} \tag{27}$$