Navier-Stokes Equations and Their Connection to SPH Simulation

1 Navier-Stokes Equations

The Navier-Stokes equations are fundamental partial differential equations that describe the motion of fluid substances such as liquids and gases. These equations are based on Newton's second law of motion and consist of a set of non-linear equations that account for the forces acting on a fluid element.

The general form of the Navier-Stokes equations in three dimensions is:

1.1 Continuity Equation (Conservation of Mass)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1}$$

where ρ is the fluid density and $\mathbf{u}$ is the velocity vector.

1.2 Momentum Equation (Conservation of Momentum)

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) = -\nabla p + \nabla \cdot \tau + \mathbf{f} \tag{2}$$

Here, p is the pressure, τ represents the viscous stress tensor, and $\mathbf{f}$ represents external forces (such as gravity).

The viscous stress tensor τ is typically given by:

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda(\nabla \cdot \mathbf{u})\delta_{ij} \tag{3}$$

where μ is the dynamic viscosity, λ is the second coefficient of viscosity (often taken as zero for incompressible flows), and δ_{ij} is the Kronecker delta.

For incompressible flows (constant density ρ), the continuity equation simplifies to:

$$\nabla \cdot \mathbf{u} = 0 \tag{4}$$

2 Connection to SPH Simulation

Smoothed Particle Hydrodynamics (SPH) is a computational method used for simulating fluid flows and other physical phenomena. The connection between the Navier-Stokes equations and SPH lies in how SPH approximates the solutions to these fundamental equations.

2.1 SPH Overview

SPH is a mesh-free, Lagrangian method where the fluid is represented by a set of discrete particles. Each particle carries properties such as mass, position, velocity, density, and pressure. The interactions between particles are computed using kernel functions that provide a smooth approximation to the continuous fluid field.

2.2 SPH and Navier-Stokes Equations

In SPH simulations, the Navier-Stokes equations are discretized and solved using the particle representation. Here's how SPH incorporates the main components of the Navier-Stokes equations:

2.2.1 Continuity Equation (Conservation of Mass)

$$\frac{d\rho_i}{dt} = \sum_j m_j (\mathbf{v}_i - \mathbf{v}_j) \cdot \nabla W(\mathbf{r}_i - \mathbf{r}_j, h) \tag{5}$$

where ρ_i is the density of particle i, m_j is the mass of particle j, $\mathbf{v}_i$ and $\mathbf{v}_j$ are the velocities of particles i and j, and W is the smoothing kernel function with smoothing length h.

2.2.2 Momentum Equation (Conservation of Momentum)

$$\frac{d\mathbf{v}_i}{dt} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla W(\mathbf{r}_i - \mathbf{r}_j, h) + \mathbf{F}_{\text{visc}} + \mathbf{F}_{\text{ext}} \tag{6}$$

where p_i and p_j are the pressures of particles i and j, and $\mathbf{F}_{\text{visc}}$ and $\mathbf{F}_{\text{ext}}$ represent viscous and external forces, respectively.

2.2.3 Viscous Forces

The viscous forces in SPH can be modeled in various ways, often through terms that represent the relative velocities and distances between particles. One common approach is:

$$\mathbf{F}_{\text{visc}} = \sum_j m_j \nu \frac{(\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2 + \epsilon^2} \nabla W(\mathbf{r}_i - \mathbf{r}_j, h) \tag{7}$$

where ν is the kinematic viscosity and ϵ is a small number to prevent singularities.

3 Advantages of SPH

- **Mesh-Free Nature:** SPH does not require a fixed grid, allowing it to easily handle large deformations, complex boundaries, and free surface flows.
- **Lagrangian Framework:** The method tracks fluid motion with particles, making it naturally suited for simulating phenomena with large displacements and complex interfaces.

4 Applications

SPH is widely used in various fields, including astrophysics (e.g., star formation, galaxy collisions), engineering (e.g., fluid-structure interactions, breaking waves), and computer graphics (e.g., realistic water and smoke effects in animations and games).

5 Summary

SPH provides a practical and flexible way to approximate the solutions to the Navier-Stokes equations for fluid flow simulations. By representing fluids with particles and using kernel functions to approximate derivatives, SPH captures the essential physics of fluid dynamics while offering computational advantages in handling complex, dynamic systems.